“Sur un principe de minimum dans l’hydrodynamique des fluides visqueux,” C. R. Acad. Sci. Paris **213** (1941), 977-981.

On a minimum principle in the hydrodynamics of viscous fluids

Note by **JULIEN KRAVTCHENKO**, presented by Henri Villat.

Translated by D. H. Delphenich

There is great interest in reducing the boundary-value problems in the theory of partial differential equations to variational form. The problem consists of defining an expression that is as simple as possible and with respect to which the equation being studied will play the role of Euler’s equations of variation.

In the context of the equations of the hydrodynamics of viscous fluids, that problem has not been solved in the general case. Moreover, one can show ([1]) that it is impossible to define an integral functional that is quadratic in the components of velocity and whose solutions in question are the extrema.

Nonetheless, the reduction of the hydrodynamical problem (whose statement one will find specified later on) to a variational problem can be implemented by means of some restrictive hypotheses. We propose to implement that reduction in the case of the *slow and permanent* motion of *incompressible* viscous fluids that are subject to the action of an external force *that is derived from a uniform potential*.

I. – Refer space to a system of rectangular axes *OXYZ* and let μ denote the coefficient of viscosity, while ρ denotes the density (which is supposed to be *constant*) of a fluid mass that is animated with a *permanent* motion under the action of an external force that is derived from a uniform potential U (x, y, z) that is independent of time. If u (x, y, z), v (x, y, z), and w (x, y, z) denote the components along the axes of the velocity vector **V** (which is independent of time, by hypothesis), and p (x, y, z) denotes the mean pressure then the equations of *slow, permanent* motion of the *incompressible* viscous fluid in question are written:

$$(1)\qquad \left\{\begin{aligned} &\mu\,\Delta u+\frac{\partial}{\partial x}(\rho U-p)=0,\\ &\mu\,\Delta v+\frac{\partial}{\partial y}(\rho U-p)=0,\\ &\mu\,\Delta w+\frac{\partial}{\partial z}(\rho U-p)=0,\end{aligned}\right.$$

([1]) This result was presented to the Sorbonne (1940-1941).

which is a system to which one must add the continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,. \tag{2}$$

Let $\mathcal{D}$ be a simply-connected domain then that is bounded by the closed surface S. It will be assumed to be sufficiently regular that one can apply Green's theorem. The force function $U(x, y, z)$ is supposed to have continuous first derivatives in the closed domain $\mathcal{D} + S$. Having said that, the boundary-value problem that relates to the system (1) and (2) can be formulated as follows:

Find a system of solutions u, v, w, p of (1) *and* (2) *that are defined and continuous, along with the second derivatives, in $\mathcal{D}$ and take a set of values on S that are given in advance.*

That problem can be reduced to a minimum problem.

II. – Indeed, consider the set E of all functions u, v, w, and p of the variables x, y, z that are regular in $\mathcal{D}$ (i.e., they possess continuous second-order derivatives in $\mathcal{D}$), verify the condition (2), and take the respective values on S that are given in advance.

III. – Note that the function $p(x, y, z)$, which satisfies (1) and (2), is necessarily harmonic in $\mathcal{D}$. That function is then defined entirely in $\mathcal{D}$ by the given data of its values on the boundary S of $\mathcal{D}$, and one can consider it to be an extremal function of the Dirichlet integral:

$$I(p)=\iiint_{\mathcal{D}}\left[\left(\frac{\partial p}{\partial x}\right)^2+\left(\frac{\partial p}{\partial y}\right)^2+\left(\frac{\partial p}{\partial z}\right)^2\right]dx\,dy\,dz \tag{3}$$

over the set E (cf., section II). Hence, p is defined independently of the distribution of forces and that of the velocities in $\mathcal{D}$.

IV. – Now, pass on to the functions u, v, w. Introduce the functional:

$$J(u, v, w)=\iiint_{\mathcal{D}}\left[\Psi(u,v,w)-u\frac{\partial}{\partial x}(\rho U-p)-v\frac{\partial}{\partial y}(\rho U-p)-w\frac{\partial}{\partial z}(\rho U-p)\right]dx\,dy\,dz\,, \tag{4}$$

in which $\Psi(u, v, w)$ denotes Lord Rayleigh's dissipation function relative to the case of incompressible fluids:

$$(5) \qquad 2\,\Psi\,(u, v, w) = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\mu\left[\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial u}{\partial y}\right)^2\right],$$

and in which the function $(\rho\, U - p)$ can be considered to be given, from section III. Let us look for the extremum of $J\,(u, v, w)$ over the set E. If u_1, v_1, w_1 are the elements of E that realize an extremum then any other element of E can be put into the form:

$$u = u_1 + \xi\, u_2\,, \qquad u = u_1 + \xi\, u_2\,, \qquad u = u_1 + \xi\, u_2\,,$$

in which ξ, η, ζ are arbitrary parameters, and u_2, v_2, w_2 denote functions of x, y, z that are regular in $\mathcal{D}$, and are subject to verifying (2), and they are zero on S, otherwise arbitrary. It results from formulas (4) and (5) that the principal part δJ of the variation $J\,(u, v, w) - J\,(u_1, v_1, w_1)$ (when the parameters ξ, η, and ζ are small in absolute value) is given by the relation:

$$(6) \qquad \delta J = \xi\,\mu\Bigg\{\iiint_{\mathcal{D}}\left[2\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}u_1\right)+\frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial x}u_1+\frac{\partial u_1}{\partial y}u_2\right)+\frac{\partial}{\partial z}\left(\frac{\partial w_1}{\partial x}u_1+\frac{\partial u_1}{\partial z}u_2\right)\right]dx\,dy\,dz$$
$$-\iiint_{\mathcal{D}}\left[\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}\right)\right]u_2\;dx\,dy\,dz$$
$$-\iiint_{\mathcal{D}}\left[\Delta u_1+\frac{\partial}{\partial x}(\rho U - p)\right]u_2\;dx\,dy\,dz+\cdots\Bigg\},$$

in which the two unwritten terms are obtained by circular permutations of $x, y, z, u, v, w, \xi, \eta, \zeta$ in the term that was specified. The first integral on the right-hand side of (6) can transformed into a double integral that is extended over S. Now, the differential element in it will be identically zero since it contains the function u_2 as a factor, which is zero on S. Similarly, the second integral on the right-hand side of (6) is zero since its differential element is identically zero by virtue of (2). Under those conditions, since the parameters ξ, η, and ζ, and the functions u_2, v_2, w_2 are arbitrary, δJ can be identically zero only when the differential element in the third integral in (6) is identically zero. Now, from a fundamental lemma in the calculus of variations, that would demand [cf., (6)] that the functions u_1, v_1, w_1 must be solutions to (1). The system (1) then constitutes Euler's system of equations of variation for the functional J that is defined by (4) and (5), with the complementary condition (2).

Q. E. D.

We can then assert:

Any solution to the boundary-value problem that was stated in section I realizes an extremum for the functionals (3) *and* (4) *over the domain of variation E that was defined in section II.*
