"Die Stabilitätstheorie des gedrückten Stabes," ZAMM **12** (1932), 351-368.

The stability of the compressed rod

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Summary. – In the present work, the buckling limit for circular and rectangular compressed rods will be calculated as an example of the theory of the stability of elastic equilibrium that **E. Trefftz** developed. The first section includes a sketch of the theory of elasticity for finite deformations. In particular, it will be shown that the expression for the elastic potential (internal elastic energy per unit volume) can be adapted from the theory of elasticity for small deformations when one substitutes the actual distortion quantities for finite deformations (i.e., the variations of the coefficients of the line element) for the linearized distortion quantities.

In the second part of the paper, the expression for the second variation of the internal energy, whose sign determines the stability, will be defined by the integral for the total internal elastic energy thus-obtained. The stability limit, i.e., the load limit beyond which the second variation can become negative, will then be determined from the standard methods of the calculus of variations. The third section includes the results of the numerical calculations, from which it emerges that the **Euler** formula that one ascertains by elementary methods in the study of the bending of beams is confirmed within the limits of computational accuracy (i.e., up to a fraction of a percent).

I would not like to neglect to thank Herrn Professor Dr. **Trefftz** warmly for the advice and encouragement that he contributed to this work.

Introduction. – The problem of elastic buckling ([1]) will be treated in what follows by taking the view that the essence of the buckling process is that of a problem in the theory of elasticity for finite deformations.

It will be assumed that the stresses that appear will lie within the domain of validity of the extension of **Hooke**'s law to finite deformations. The restriction to fixed body forces and surface tractions, i.e., ones that do not vary with the deformation, will also be imposed.

The method that is applied is the energetic one, when one considers the finitude of the deformations in all three coordinate directions, as **E. Trefftz** ([2]) developed it at the International Congress for Engineering Mechanics in Stockholm in 1930.

The following notations will be used:

$x^{(1)},\ x^{(2)},\ x^{(3)}$ the coordinates of a point

([1]) For the older work: *Enzyklopädie der mathematischen Wissenschaften*, Bd. IV, Leipzig 1907/08. For recent work: *Handbuch der Physik*, v. Geiger and Scheel, Bd. VI, Leipzig 1938.

([2]) **E. Trefftz**, "Über die Ableitung der Stabilitätstheorien des elastischen Gleichgewichts aus der Theorie endlicher Deformationen," Internationaler Kongreß für Technische Mechanik 1930, Stockholm, Teil III, pp. 44.

X_1, X_2, X_3	the components of the body force per unit volume in the undeformed body along the three axis directions
Ξ_1, Ξ_2, Ξ_3	the components of the surface traction per unit area in the undeformed body along the three axis directions
$u^{(1)}, u^{(2)}, u^{(3)}$	the components of the displacement along the three axis directions
$\delta u^{(1)}, \delta u^{(2)}, \delta u^{(3)}$	the components of the variation of the state from the equilibrium configuration (i.e., the perturbations)
E	the internal energy of the entire body

An elastic state is a state of stable equilibrium when for every finite displacement that is compatible with the geometric conditions the increase in internal energy is greater than the work done by external forces that is available, i.e., when:

$$\Delta E > \iiint \sum_\nu X_\nu \, \delta u^{(\nu)} \, dx^{(1)} \, dx^{(2)} \, dx^{(3)} + \iint \sum_\nu \Xi_\nu \, \delta u^{(\nu)} \, do \ .$$

Developing both sides of the equation in powers of $\delta u^{(\nu)}$ and their derivatives gives:

$$\Delta E = \delta E + \delta^2 E + \cdots > \iiint \sum_\nu X_\nu \, \delta u^{(\nu)} \, dx^{(1)} \, dx^{(2)} \, dx^{(3)} + \iint \sum_\nu \Xi_\nu \, \delta u^{(\nu)} \, do \ . \tag{1}$$

No powers of $\delta u^{(\nu)}$ appear on the right-hand side as a result of the restriction to fixed external forces X_ν and Ξ_ν.

Now should the left-hand side be greater than the right-hand side for arbitrary $\delta u^{(\nu)}$ that are compatible with the kinematical conditions, then the linear terns would have to vanish in their own right, i.e., one would need to have:

$$\delta E = \iiint \sum_\nu X_\nu \, \delta u^{(\nu)} \, dx^{(1)} \, dx^{(2)} \, dx^{(3)} + \iint \sum_\nu \Xi_\nu \, \delta u^{(\nu)} \, do \ .$$

The gist of this equation is referred to as the “principle of virtual displacements” and says that for every virtual displacement from the equilibrium configuration, the variation in internal energy will be equal to the work done by external forces.

Should the equilibrium state be stable, then the quadratic terms in the left-hand side of eq. (10) would have to exceed the right-hand side. As a result of the restriction to fixed external forces, the work done by external forces is exhausted by the linear terms. The stability condition then reduces to $\delta^2 E > 0$.

The stability limit is reached when the second variation of the internal energy vanishes for at least one system of displacements $\delta u^{(\nu)}$, so one will have $\delta^2 E = 0$.

Those completely-general Ansätze shall be applied to the stability problem of the theory of elasticity in what follows, and in particular to the determination of the buckling load for a compressed rod.

I. – The theory of elasticity for finite deformations.

1. The state of deformation. – “Substantial coordinates” will be used to fix the particles of an elastic body. They are assumed to be rectangular normal coordinates in the undeformed state and will be denoted by $x^{(\nu)}$. In the deformed state, each mass-particle is endowed with curvilinear coordinates.

An arbitrary mass-particle will have the coordinates $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ before the deformation. It occupies a point *P*, which will lead to the position vector $\mathfrak{x} = \sum_\nu x^{(\nu)} \mathfrak{E}_\nu$ in a fixed spatial axis-cross with the three perpendicular unit vectors $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$. A neighboring particle with the coordinates $x^{(1)} + dx^{(1)}$, $x^{(2)} + dx^{(2)}$, $x^{(3)} + dx^{(3)}$ will then assume a point *Q*. If $d\,\mathfrak{x}$ is the vector from *P* to *Q* then $d\,\mathfrak{x} = \sum_\nu dx^{(\nu)} \mathfrak{E}_\nu$, and the line element reads:

$$ds^2 = \sum_\nu \sum_\mu G_{\nu\mu}\, dx^{(\nu)}\, dx^{(\mu)} .$$

For the coefficients $G_{\nu\mu} = \mathfrak{E}_\nu \cdot \mathfrak{E}_\mu$ of the line element, one has the matrix:

$$\| G_{\nu\mu} \| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

for rectangular normal coordinates.

The body will be deformed by an elastic displacement $\mathfrak{u} = \sum u^{(\nu)} \mathfrak{E}_\nu$. The point *P* ($\mathfrak{x}$) will assume a new position $\bar{P}$ ($\mathfrak{r} = \mathfrak{x} + \mathfrak{u}$), and its neighboring point *Q* will assume the new position $\bar{Q}$ ($\mathfrak{r} + d\,\mathfrak{r} = \mathfrak{x} + d\,\mathfrak{r} + \mathfrak{u} + d\,\mathfrak{u}$). In that way, the following relation will exist between the components $\xi^{(\nu)}$ of the position vector $\mathfrak{r}$ after the deformation and the components $x^{(\nu)}$ of the position vector $\mathfrak{r}$ before the deformation:

$$\xi^{(\nu)} = x^{(\nu)} + u^{(\nu)}, \qquad d\xi^{(\nu)} = dx^{(\nu)} + du^{(\nu)}.$$

The line element after deformation reads:

$$d\sigma^2 = \sum_\nu \sum_\mu \Gamma_{\nu\mu}\, dx^{(\nu)}\, dx^{(\mu)} ,$$

in which one has:

$$\Gamma_{\nu\mu} = \sum_\kappa \frac{\partial \xi^{(\kappa)}}{\partial x^{(\nu)}} \frac{\partial \xi^{(\kappa)}}{\partial x^{(\mu)}} = G_{\nu\mu} + \frac{\partial u^{(\nu)}}{\partial x^{(\mu)}} + \frac{\partial u^{(\mu)}}{\partial x^{(\nu)}} + \sum_\kappa \frac{\partial u^{(\kappa)}}{\partial x^{(\nu)}} \frac{\partial u^{(\kappa)}}{\partial x^{(\mu)}} .$$

The line element ds^2 is then deformed into the line element $d\sigma^2$. Comparing the two will give the changes $\gamma_{\nu\mu} = \Gamma_{\nu\mu} - G_{\nu\mu}$ in the coefficients of the line element:

$$\gamma_{\nu\mu} = \frac{\partial u^{(\nu)}}{\partial x^{(\mu)}} + \frac{\partial u^{(\mu)}}{\partial x^{(\nu)}} + \sum_\kappa \frac{\partial u^{(\kappa)}}{\partial x^{(\nu)}} \frac{\partial u^{(\kappa)}}{\partial x^{(\mu)}} .$$

The $\gamma_{\nu\mu}$ are the "quantities of deformation" and describe the elongations and changes in angle that each volume element experiences. Due to the commutation rule $\gamma_{\nu\mu} = \gamma_{\mu\nu}$, there are only six distinct quantities of deformation. They define a tensor, and it is the symmetric deformation tensor that is associated with each point of the elastic body.

The line element $d\sigma^2$, when written out in detail in rectangular normal coordinates, then reads:

$$\begin{aligned} d\sigma^2 = {} & (1+\gamma_{11})\, dx^{(1)2} + (1+\gamma_{22})\, dx^{(2)2} + (1+\gamma_{33})\, dx^{(3)2} \\ & + 2\gamma_{12}\, dx^{(1)} dx^{(2)} + 2\gamma_{23}\, dx^{(2)} dx^{(3)} + 2\gamma_{31}\, dx^{(3)} dx^{(1)} , \end{aligned}$$

in which:

$$\left.\begin{aligned} \gamma_{11} &= 2\frac{\partial u^{(1)}}{\partial x^{(1)}} + \left(\frac{\partial u^{(1)}}{\partial x^{(1)}}\right)^2 + \left(\frac{\partial u^{(2)}}{\partial x^{(1)}}\right)^2 + \left(\frac{\partial u^{(3)}}{\partial x^{(1)}}\right)^2 , \\ \gamma_{12} &= \frac{\partial u^{(1)}}{\partial x^{(2)}} + \frac{\partial u^{(2)}}{\partial x^{(1)}} + \frac{\partial u^{(1)}}{\partial x^{(1)}} \frac{\partial u^{(1)}}{\partial x^{(2)}} + \frac{\partial u^{(2)}}{\partial x^{(1)}} \frac{\partial u^{(2)}}{\partial x^{(2)}} + \frac{\partial u^{(3)}}{\partial x^{(1)}} \frac{\partial u^{(3)}}{\partial x^{(2)}} , \end{aligned}\right\} \tag{2}$$

etc., and cyclic permutations.

Those nonlinear equations go to the linear equations of the classical theory of elasticity when the products and squares of the $\frac{\partial u^{(\nu)}}{\partial x^{(\mu)}}$ can be neglected in comparison to the linear expressions in those terms.

2. The stress state and the equilibrium condition. – In order to describe the stress state at a mass-particle $(x^{(1)}, x^{(2)}, x^{(3)})$ in an elastic body, one considers the rectangular parallelepiped that is defined by the elements $dx^{(1)}$, $dx^{(2)}$, $dx^{(3)}$, which are parallel to the axes in the undeformed state. After the deformation, they will become a general parallelepiped with the edges $\frac{\partial \mathfrak{r}}{\partial x^{(1)}} dx^{(1)}$, $\frac{\partial \mathfrak{r}}{\partial x^{(2)}} dx^{(2)}$, $\frac{\partial \mathfrak{r}}{\partial x^{(3)}} dx^{(3)}$. The vectors $\mathfrak{e}_\mu = \frac{\partial \mathfrak{r}}{\partial x^{(\mu)}}$ that give the direction and expansion ratio after the deformation will be called "lattice vectors."

If the force $\mathfrak{k}^{(1)} dx^{(2)} dx^{(3)}$ acts upon the boundary face of the parallelepiped that lies in the direction of increasing $x^{(1)}$ then $\mathfrak{k}^{(1)}$ will be called the *stress vector* for the surfaces $x^{(1)}$ = const. Corresponding statements are true for the remaining surfaces. $\mathfrak{k}^{(\nu)}$ is the stress vector for the surface element $x^{(\nu)}$ = const., and it means a force per unit undeformed area.

Each of the three stress vectors can be decomposed along the lattice vectors:

$$\mathfrak{k}^{(\nu)} = \sum_{\nu,\mu} k^{\nu\mu}\, \mathfrak{e}_{\mu}\,, \tag{3}$$

which will produce nine stress components $k^{\nu\mu}$ that describe the stress state completely.

Equilibrium with respect to rotations around an arbitrary direction requires the vanishing of the sum of the moments of all stress forces that act upon the midpoint of the parallelepiped in question, so:

$$\sum M = \sum_{\nu} (\mathfrak{e}_{\nu} \times \mathfrak{k}^{(\nu)})\, dx^{(1)}\, dx^{(2)}\, dx^{(3)} = 0\,.$$

After introducing the reciprocal vectors $\mathfrak{e}^{1}$, $\mathfrak{e}^{2}$, $\mathfrak{e}^{3}$ to the lattice vectors $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ using the formula ([3]):

$$\mathfrak{e}^{(1)} = \frac{\mathfrak{e}_2 \times \mathfrak{e}_3}{[\mathfrak{e}_1\, \mathfrak{e}_2\, \mathfrak{e}_3]}\,, \qquad \text{and} \qquad \mathfrak{e}^{(\nu)} \cdot \mathfrak{e}_{(\mu)} = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu, \end{cases}$$

and when one recalls the component representation (3), it will follow that:

$$\mathfrak{e}^{(1)}\,(k^{23} - k^{32}) + \mathfrak{e}^{(2)}\,(k^{31} - k^{13}) + \mathfrak{e}^{(3)}\,(k^{12} - k^{21}) = 0\,.$$

That vector equation will be fulfilled only when:

$$k^{\nu\mu} = k^{\mu\nu}\,.$$

The **Cauchy** reciprocity law is then true for the $k^{\nu\mu}$. The number of stress quantities then drops from nine to six.

For equilibrium under a displacement in an arbitrary direction, one considers the force $\mathfrak{k}^{(1)}\, dx^{(2)} dx^{(3)}$ that acts on the surface element $x^{(1)}$ = const. of the parallelepiped. The force $\mathfrak{k}^{(1)}\, dx^{(2)} dx^{(3)} + \frac{\partial \mathfrak{k}^{(1)}}{\partial x^{(1)}}\, dx^{(2)} dx^{(3)}\, dx^{(1)}$ then acts upon the surface element $x^{(1)} + dx^{(1)}$ = const. The excess force for this surface-pair amounts to $\frac{\partial \mathfrak{k}^{(1)}}{\partial x^{(1)}}\, dx^{(2)} dx^{(3)}\, dx^{(1)}$. The excess forces for the remaining surface-pairs then follow from that by cyclic permutation. That will yield a resultant

([3]) **M. Lagally**, *Vorlesungen über Vectorrechnung*, Leipzig 1928.

$d\mathfrak{R}$ of the stress forces with a magnitude of $d\,\mathfrak{R} = \sum_\nu \frac{\partial \mathfrak{k}^{(\nu)}}{\partial x^{(\nu)}} dx^{(1)} dx^{(2)}\, dx^{(3)}$. In addition, the body force:

$$d\,\mathfrak{K} = \mathfrak{P}\, dx^{(1)} dx^{(2)}\, dx^{(3)} = \sum_\nu P^{(\nu)}\, \mathfrak{e}_\nu\, dx^{(1)} dx^{(2)}\, dx^{(3)}$$

acts upon the parallelepiped, in which $\mathfrak{P}$ is the body force per unit volume on the undeformed body, and $P^{(\nu)}$ is its component in the direction of the ν^{th} lattice vector. Equilibrium under displacements requires that:

$$d\,\mathfrak{R} + d\,\mathfrak{K} = 0\,,$$

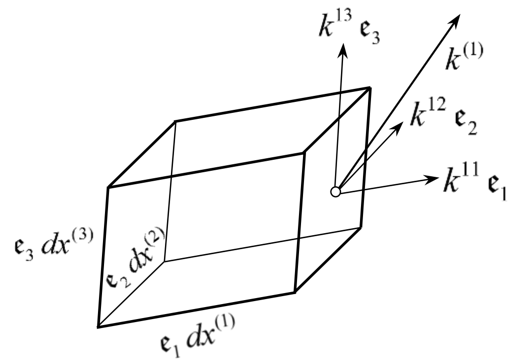

Figure 1.

i.e., after eliminating the product $dx^{(1)} dx^{(2)}\, dx^{(3)}$:

$$\sum_\nu \frac{\partial\, \mathfrak{k}^{(\nu)}}{\partial x^{(\nu)}} + \sum_\nu P^{(\nu)}\, \mathfrak{e}_\nu = 0\,.$$

With the use of formula (3), and after performing the differentiation, it will follow that:

$$\sum_\nu \sum_\mu \frac{\partial k^{\nu\mu}}{\partial x^{(\nu)}} \mathfrak{e}_\mu + \sum_\nu \sum_\mu k^{\nu\mu} \frac{\partial\, \mathfrak{e}_\mu}{\partial x^{(\nu)}} + \sum_\nu P^{(\nu)}\, \mathfrak{e}_\nu = 0\,.$$

Now, one has:

$$\frac{\partial\, \mathfrak{e}_\mu}{\partial x^{(\nu)}} = \sum_h \begin{Bmatrix} \mu\,\nu \\ h \end{Bmatrix} \mathfrak{e}_h\,,$$

in which $\begin{Bmatrix} \mu\,\nu \\ h \end{Bmatrix}$ are the **Christoffel** symbols of the second kind. Therefore:

$$\sum_\nu \sum_\mu \frac{\partial k^{\nu\mu}}{\partial x^{(\nu)}} \mathfrak{e}_\mu + \sum_\nu \sum_\mu \sum_h k^{\nu\mu} \begin{Bmatrix} \mu\,\nu \\ h \end{Bmatrix} \mathfrak{e}_h + \sum_\nu P^{(\nu)}\, \mathfrak{e}_\nu = 0\,.$$

Should all of the basis vectors be called $\mathfrak{e}_h$, then one can permute the indices accordingly. If one decomposes that vector equation into its components then that will give:

$$\sum_\nu \frac{\partial k^{\nu\mu}}{\partial x^{(\nu)}} + \sum_\nu \sum_h k^{\nu\mu} \begin{Bmatrix} \mu\,\nu \\ h \end{Bmatrix} + P^{(h)} = 0\,. \tag{4}$$

That equation yields a partial differential equation for each of the three directions $h = 1, 2, 3$. The middle term carries the curvature of the coordinate curve calculation, but the differentiations in (4) are with respect to the substantial coordinates, which are curvilinear in the deformed state.

3. Internal energy. – The considerations up to now associated each point of an elastic body with a state of deformation and stress. In order to represent the connection between the two of them, we shall consider the internal energy. Its existence follows for all reversible static processes on thermodynamic grounds. We shall use the following notations:

$\varepsilon = [\mathfrak{e}_1\ \mathfrak{e}_2\ \mathfrak{e}_3] = \sqrt{\lvert \Gamma_{\mu\nu} \rvert}$	unit volume in the lattice
$dV = \varepsilon\, dx^{(1)}\, dx^{(2)}\, dx^{(3)}$	volume of the infinitesimal parallelepiped
e	internal energy per unit volume of the undeformed body
$e\ \varepsilon$	internal per unit lattice volume
$d^2 V = e\, dV$	internal energy of the volume $d\,V$.

In order to represent the internal energy e as a function of the quantities of deformation $\gamma_{\nu\mu}$, one must look for invariants of the deformed state. One absolute invariant is:

$$\frac{1}{\lvert G_{\nu\mu} \rvert} \lvert \gamma_{\nu\mu} - \lambda\, G_{\nu\mu} \rvert$$

for all values of the parameter λ.

Developing the determinant in powers of λ will yield a function of degree three in λ. Since it is an invariant for all λ, the coefficients of the cubic form will likewise be invariants. That implies the three invariants:

$$I_1 = \frac{1}{\lvert G_{\nu\mu} \rvert}\{\gamma_{11}(G_{22}\, G_{33} - G_{23}^2) + \gamma_{22}(G_{33}\, G_{11} - G_{31}^2) + \gamma_{33}(G_{11}\, G_{22} - G_{12}^2)$$
$$+ 2\,\gamma_{12}\,(G_{31}\, G_{23} - G_{12}\, G_{33}) + 2\,\gamma_{23}\,(G_{12}\, G_{13} - G_{23}\, G_{11}) + 2\,\gamma_{31}\,(G_{23}\, G_{12} - G_{31}\, G_{22})\}\ ,$$

$$I_2 = \frac{1}{\lvert G_{\nu\mu} \rvert}\{G_{11}(\gamma_{22}\, \gamma_{33} - \gamma_{23}^2) + G_{22}(\gamma_{33}\, \gamma_{11} - \gamma_{31}^2) + G_{33}(\gamma_{11}\, \gamma_{22} - \gamma_{12}^2)$$
$$+ 2\, G_{12}\,(\gamma_{31}\, \gamma_{23} - \gamma_{12}\, \gamma_{33}) + 2\, G_{23}\,(\gamma_{12}\, \gamma_{13} - \gamma_{23}\, \gamma_{11}) + 2\, G_{31}\,(\gamma_{23}\, \gamma_{12} - \gamma_{31}\, \gamma_{22})\}\ ,$$

$$I_3 = \frac{\lvert \gamma_{\nu\mu} \rvert}{\lvert G_{\nu\mu} \rvert}\ .$$

I_1 is linear in the $\gamma_{\nu\mu}$, while I_2 is quadratic, and I_3 is cubic. Furthermore, I_2 emerges from I_1 by switching G with γ.

For the rectilinear normal coordinates, one has:

$$I_1 = \gamma_{11} + \gamma_{22} + \gamma_{33} \,,$$
$$I_2 = \gamma_{11}\,\gamma_{22} + \gamma_{22}\,\gamma_{33} + \gamma_{33}\,\gamma_{11} - \gamma_{12}^2 - \gamma_{23}^2 - \gamma_{31}^2 ,$$
$$I_3 = |\,\gamma_{\nu\mu}\,| \,.$$

Which Ansatz should be imposed upon e? From the wealth of possible Ansätze, the simplest one is the one that corresponds to classical theory, i.e., $\varepsilon\, e$ will be represented by the simplest quadratic invariant of the deformation state, and indeed:

$$e = \frac{\sqrt{|\,G_{\nu\mu}\,|}}{\sqrt{|\,\Gamma_{\nu\mu}\,|}} \left\{ \frac{\alpha}{2} I_1^2 - \beta\, I_2 \right\} .$$

The foregoing factor is an invariant since it is the ratio of the volumes of the parallelepiped in the undeformed and the deformed states. The consists α and β are the two independent elastic constants of the classical theory:

$$\alpha = \frac{G}{2}\frac{m-1}{m-2}, \qquad \beta = \frac{G}{2}.$$

Here, G means the shear modulus, and m is the **Poisson** number of the lateral contraction of the material.

The internal energy per unit lattice volume is then:

$$e\;\varepsilon = \sqrt{|\,G_{\nu\mu}\,|} \left\{ \frac{\alpha}{2} I_1^2 - \beta\, I_2 \right\} . \tag{5}$$

It follows from the condition that e must be positive that $2 \leq m \leq \infty$, as in the classical theory of elasticity. The Ansatz (5) is allowable in that domain, since ($e\;\varepsilon$) will be a positive-definite quadratic homogeneous form in the $\gamma_{\nu\mu}$.

For rectangular normal coordinates, one has:

$$e\;\varepsilon = \frac{\alpha}{2}(\gamma_{11} + \gamma_{22} + \gamma_{33})^2 - \beta\,(\gamma_{11}\,\gamma_{22} + \gamma_{22}\,\gamma_{33} + \gamma_{33}\,\gamma_{11} - \gamma_{12}^2 - \gamma_{23}^2 - \gamma_{31}^2) \;.$$

The infinitely-small deformations can be freely associated with the Ansatz (5) when the linearized expression for the quantities of deformations are employed.

4. The stress-extension equations and the extension of Hooke’s law to finite deformations. – The equilibrium conditions (4) alone are not sufficient to determine all quantities of stress and deformation. Relations between the forces that are acting and the deformations that they produce are required, in addition. As in the classical theory, the consideration of internal energy offers the possibility of deriving the stress-extension equations when one starts from the internal energy per unit lattice volume.

In order to derive it, an additional displacement $\delta\mathfrak{u}$ will be superimposed with the displacement $\mathfrak{u}$ that actually occurs, which will vary $\gamma_{\nu\mu}$ by $\delta\gamma_{\nu\mu}$. The increase in the internal energy is then equal to the work done on the volume element by the stress forces. The point A experiences the additional displacement $\delta\mathfrak{u}+\frac{\partial\,\delta\mathfrak{u}}{\partial x^{(1)}}\frac{dx^{(1)}}{2}$, and the point B experiences $\delta\mathfrak{u}-\frac{\partial\,\delta\mathfrak{u}}{\partial x^{(1)}}\frac{dx^{(1)}}{2}$. A variation in the internal energy is connected with that additional displacement whose magnitude is $\mathfrak{k}^{(1)}\cdot\frac{\partial\,\delta\mathfrak{u}}{\partial x^{(1)}}\,dx^{(1)}\,dx^{(2)}\,dx^{(3)}$. The variations of the internal energy for the remaining two directions are obtained from that by cyclic permutation. Their sum amounts to:

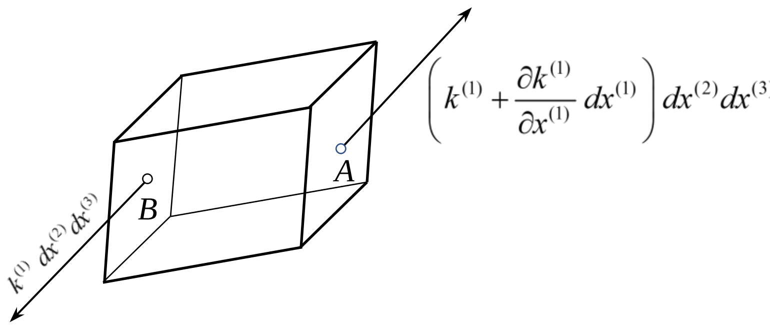

Figure 2.

$$\delta(e\ \varepsilon)=\sum_{\nu}\mathfrak{k}^{(\nu)}\cdot\frac{\partial\,\delta\mathfrak{u}}{\partial x^{(\nu)}}=\sum_{\nu}\sum_{\mu}k^{\nu\mu}\,\mathfrak{e}_{\mu}\cdot\frac{\partial\,\delta\mathfrak{u}}{\partial x^{(\nu)}}$$

per unit lattice volume. Now:

$$\mathfrak{e}_{\nu}=\mathfrak{E}_{\nu}+\frac{\partial\,\mathfrak{u}}{\partial x^{(\nu)}}\qquad\text{so}\qquad\delta\,\mathfrak{e}_{\nu}=\frac{\partial\,\delta\mathfrak{u}}{\partial x^{(\nu)}},\qquad\Gamma_{\nu\mu}=G_{\nu\mu}+\gamma_{\nu\mu}\qquad\text{so}\qquad\delta\Gamma_{\nu\mu}=\delta\gamma_{\nu\mu},$$

and since:

$$\Gamma_{\nu\mu}=\mathfrak{e}_{\nu}\cdot\mathfrak{e}_{\mu},\qquad\text{one will have}\qquad\delta\Gamma_{\nu\mu}=\mathfrak{e}_{\nu}\cdot\delta\,\mathfrak{e}_{\mu}+\mathfrak{e}_{\mu}\cdot\delta\,\mathfrak{e}_{\nu}.$$

That gives the detailed representation of the variation of the internal energy:

$$\delta(e\ \varepsilon)=\tfrac{1}{2}k^{11}\,\delta\gamma_{11}+\tfrac{1}{2}k^{22}\,\delta\gamma_{22}+\tfrac{1}{2}k^{33}\,\delta\gamma_{33}+k^{12}\,\delta\gamma_{12}+k^{23}\,\delta\gamma_{23}+k^{31}\,\delta\gamma_{31}\,. \tag{6}$$

On the other hand, from formula (5):

$$\delta(e\ \varepsilon)=\frac{\partial(e\varepsilon)}{\partial\gamma_{11}}\delta\gamma_{11}+\frac{\partial(e\varepsilon)}{\partial\gamma_{22}}\delta\gamma_{22}+\frac{\partial(e\varepsilon)}{\partial\gamma_{33}}\delta\gamma_{33}+\frac{\partial(e\varepsilon)}{\partial\gamma_{12}}\delta\gamma_{12}+\frac{\partial(e\varepsilon)}{\partial\gamma_{23}}\delta\gamma_{23}+\frac{\partial(e\varepsilon)}{\partial\gamma_{31}}\delta\gamma_{31}\ . \tag{6.a}$$

Comparing the two expressions will give the six equations:

$$\left.\begin{aligned} k^{11} &= \frac{\partial(e\,\varepsilon)}{\partial\gamma_{11}}, \quad k^{12} = \frac{\partial(e\,\varepsilon)}{\partial\gamma_{12}}, \\ k^{22} &= \frac{\partial(e\,\varepsilon)}{\partial\gamma_{22}}, \quad k^{23} = \frac{\partial(e\,\varepsilon)}{\partial\gamma_{23}}, \\ k^{33} &= \frac{\partial(e\,\varepsilon)}{\partial\gamma_{33}}, \quad k^{31} = \frac{\partial(e\,\varepsilon)}{\partial\gamma_{31}}. \end{aligned}\right\} \tag{7}$$

These are the stress-extension equations for the theory of finite deformations. They say that the stress quantities $k^{\nu\mu}$ can be obtained from the specific internal energy ($e\ \varepsilon$) by a partial differentiation with respect to the quantities of deformation $\gamma_{\nu\mu}$. They are also true in the form (7) when one does not start with normal coordinates.

From the Ansatz (5), the internal energy ($e\ \varepsilon$) is a homogeneous quadratic form in the quantities of deformation $\gamma_{\nu\mu}$. From (7), the $k^{\nu\mu}$ are linear homogeneous functions of the quantities of deformation and read:

$$\left.\begin{aligned} k^{11} &= \frac{2}{\sqrt{|G_{\nu\mu}|}}[\alpha\,I_1(G_{22}\,G_{33} - G_{23}^2) - \beta\,(G_{22}\,\gamma_{33} + G_{33}\,\gamma_{11} - 2G_{23}\,\gamma_{23})], \\ k^{12} &= \frac{2}{\sqrt{|G_{\nu\mu}|}}[\alpha\,I_1(G_{13}\,G_{23} - G_{12}\,G_{33}) - \beta\,(G_{23}\,\gamma_{13} + G_{13}\,\gamma_{23} - G_{33}\,\gamma_{12} - G_{12}\,\gamma_{33})], \end{aligned}\right\} \tag{8}$$

etc., and cyclic permutations.

Those equations are the extension of **Hooke**'s law to finite deformations.

For rectangular normal coordinates, one has:

$$\left.\begin{aligned} k^{11} &= G\left(\gamma_{11} + \frac{\Phi}{m-2}\right), \quad k^{12} = G\,\gamma_{12}, \\ k^{22} &= G\left(\gamma_{22} + \frac{\Phi}{m-2}\right), \quad k^{23} = G\,\gamma_{23}, \\ k^{33} &= G\left(\gamma_{33} + \frac{\Phi}{m-2}\right), \quad k^{31} = G\,\gamma_{31}, \end{aligned}\right\} \tag{9}$$

in which:

$$\Phi = \gamma_{11} + \gamma_{22} + \gamma_{33}\,.$$

Equations (9) go to the classical stress-extension equations when one assumes infinitely-small deformations. With:

$$\frac{\partial u^{(1)}}{\partial x^{(1)}} = u_x\,, \qquad \frac{\partial u^{(2)}}{\partial x^{(2)}} = v_y\,, \qquad \frac{\partial u^{(3)}}{\partial x^{(3)}} = w_z\,,$$

one will then have:

$$\Phi = 2\,(u_x + v_y + w_z) = 2\,\Theta\,,$$

$$k^{11} = 2G\left(u_x + \frac{\Theta}{m-2}\right), \qquad k^{12} = G\,(u_y + v_x)\,,$$
$$k^{22} = 2G\left(v_y + \frac{\Theta}{m-2}\right), \qquad k^{23} = G\,(v_z + w_y)\,,$$
$$k^{33} = 2G\left(w_z + \frac{\Theta}{m-2}\right), \qquad k^{31} = G\,(w_x + u_z)\,.$$

Equations (9) imply the connection between the stress components and the quantities of deformations. If the values for the stress components $k^{\nu\mu}$ in (9) are substituted in the equilibrium conditions (4) then that will give the differential equations for the displacement components $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ that are required for the determination of equilibrium. Exactly as in the classical theory of elasticity, boundary conditions must be added to those differential equations that can refer to either the displacements or the surface tractions.

5. Stability of equilibrium. – In the foregoing, the equations were presented that would serve to determine the equilibrium state. For study of stability, it will be assumed that those equations have been integrated such that the stresses and displacements are known for the equilibrium state whose stability is under scrutiny. For the application of the theory, we will be content with a relatively-simple special case, viz., the compressed rod, whose stress and displacement state in equilibrium can be seen with no further analysis.

In order to be able to evaluate the stability, one must compare the internal energy that is contained in the body in the equilibrium state with the internal energy in a neighboring state. In addition to the equilibrium displacements $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, one must also consider the neighboring displacements $u^{(1)} + \delta u^{(1)}$, $u^{(2)} + \delta u^{(2)}$, $u^{(3)} + \delta u^{(3)}$, and develop the expression for the internal energy:

$$E = \iiint (e\,\varepsilon)\,dx^{(1)}\,dx^{(2)}\,dx^{(3)}$$

in powers of $\delta u^{(\nu)}$ (their derivatives, resp.) in the neighborhood of the equilibrium state.

From what was said in the introduction, the equilibrium state will be stable when the so-called second variation $\delta^2 E$, which includes quadratic terms in the aforementioned development, is always positive for every allowable (i.e., compatible with the geometric conditions) system of "perturbations" $\delta u^{(\nu)}$.

If the differentiations are denoted by subscripts, e.g., $u_i^{(\nu)} = \partial u^{(\nu)} / \partial x^{(i)}$, then the second variation will take the following form ([4]):

([4]) This expression was calculated for the first time by **Trefftz** in 1930 (from a personal communication).

$$\left.\begin{aligned}
\delta^2 E = \iiint \Big\{ & \tfrac{1}{2}\sum_h \left[4\alpha\,(1+u_h^{(h)})^2 + 2\beta\left(u_{h+1}^{(h)}\right)^2 + 2\beta\left(u_{h+1}^{(h)}\right)^2 + k^{hh} \right] \delta u_h^{(h)}\,\delta u_h^{(h)} \\
& + \tfrac{1}{2}\sum_{h,k} \left[2\beta\,(1+u_h^{(h)})^2 + 2\beta\left(u_{2(h+k)}^{(h)}\right)^2 + 4\alpha\left(u_k^{(h)}\right)^2 + k^{kk} \right] \delta u_k^{(h)}\,\delta u_k^{(h)} \\
& + \sum_{h,k} \left[2\,(\alpha-\beta)\,(1+u_h^{(h)})\,u_k^{(h)} + k^{kk} \right] \delta u_h^{(h)}\,\delta u_k^{(h)} \\
& + \sum_{h,k} \left[4\alpha\,(1+u_h^{(h)})\,u_h^{(k)} + 2\beta\,(1+u_k^{(k)})\,u_k^{(h)} + 2\beta\,u_{2(h+k)}^{(h)}\,u_{2(h+k)}^{(k)} \right] \delta u_k^{(h)}\,\delta u_h^{(k)} \\
& + \sum_{h,k} \left[4\,(\alpha-\beta)\,(1+u_h^{(h)})\,(1+u_k^{(k)}) + 2\beta\,u_k^{(h)}\,u_h^{(k)} \right] \delta u_h^{(h)}\,\delta u_k^{(k)} \\
& + \sum_{h,k,m} \left[4\,(\alpha-\beta)\,(1+u_h^{(h)})\,u_m^{(k)} + 2\beta\,u_k^{(h)}\,u_m^{(k)} \right] \delta u_h^{(h)}\,\delta u_m^{(k)} \\
& + \sum_{h,k,m} \left[2\,(2\alpha-\beta)\,u_m^{(h)}\,u_m^{(h)} + k^{mk} \right] \delta u_m^{(h)}\,\delta u_k^{(h)} \\
& + \sum_{h,k,m} \left[2\beta\,(1+u_h^{(h)})(1+u_k^{(k)}) + 4\,(\alpha-\beta)\,u_h^{(k)}\,u_k^{(h)} \right] \delta u_k^{(h)}\,\delta u_h^{(k)} \\
& + \sum_{h,k,m} \left[4\,(\alpha-\beta)\,u_m^{(h)}\,u_h^{(k)} + 2\beta\,(1+u_h^{(h)})\,u_m^{(k)} \right] \delta u_h^{(k)}\,\delta u_m^{(k)} \\
& + \sum_{h,k,m} \left[2\beta\,(1+u_h^{(h)})\,u_h^{(k)} + 4\alpha\,u_h^{(m)}\,u_h^{(k)} + 2\beta\,(1+u_k^{(k)})\,u_k^{(h)} \right] \delta u_h^{(k)}\,\delta u_h^{(m)} \Big\}\, dx^{(1)}\,dx^{(2)}\,dx^{(3)}.
\end{aligned}\right\} \quad (10)$$

The indices run through the values one to three, while indices that are denoted differently can never assume the same values. Values of the indices that are greater than three are reduced modulo three.

The problem then comes down to deciding whether allowable perturbations $\delta u^{(\nu)}$ can be found for which the second variation becomes negative, or if it is always positive. Thus, the displacements and stresses in the equilibrium state are now regarded as given, and what are sought are the "most dangerous" perturbations, i.e., the ones $\delta u^{(\nu)}$ that make the second variation as negative as possible.

6. Determining the stability limit. – From what was said in the introduction, in order to determine the stability limit of a state of elastic equilibrium, it is necessary to consider the second variation of the internal energy. That is obtained from the expression (10) when the components of the displacements and stresses that correspond to the particular problem are substituted in it. Since the quadratic form in the $\delta u^{(\nu)}$ and their derivatives under the integral can always be written as the difference of two positive-definite forms, it is convenient to write:

$$\delta^2 E = Q_1 - Q_2 ,$$

in which Q_1 and Q_2 are integrals of positive-definite quadratic forms in the derivatives of the $\delta u^{(\nu)}$.

In order to see whether the difference can become negative, one can write that as:

$$\delta^2 E = Q_2 (\lambda - 1) , \qquad \lambda = \frac{Q_1}{Q_2} .$$

The danger of $\delta^2 E$ becoming negative will become greater the smaller that λ becomes. The "most dangerous" displacement from the equilibrium configuration is the one for which λ is a minimum. Should λ be a minimum, then $\delta\lambda = \dfrac{Q_2\,\delta Q_1 - Q_1\,\delta Q_2}{Q_2^2}$ would have to vanish, i.e., one would need to have:

$$\delta Q_1 = \lambda\,\delta Q_2 , \tag{11}$$

for all allowable variations $\delta(\delta u^{(\nu)})$.

Eq. (11) has a solution $\lambda = \bar{\lambda}$ that can be greater or smaller than unity. The case of $\bar{\lambda} = 1$ yields the stability limit.

Since one is only asking what the stability might be, the problem can be simplified. Eq. (1) says that: For the "most dangerous" displacement $\delta u^{(1)}$, $\delta u^{(2)}$, $\delta u^{(3)}$ from the equilibrium configuration, one has:

$$\delta Q_1 = \lambda\,\delta Q_2$$

for all variations $\delta(\delta u^{(\nu)})$. Now, instead of determining the value of λ for given forces and then asking which forces will make $\lambda = 1$, one can also set $\lambda = 1$ directly. One then asks what the forces and associated displacements $\delta u^{(1)}$, $\delta u^{(2)}$, $\delta u^{(3)}$ from the equilibrium configuration would be for which one would have:

$$\delta Q_1 = \delta Q_2 , \tag{12}$$

for arbitrary $\delta(\delta u^{(\nu)})$. That equation is essentially the **Jacobi** criterion for the occurrence of a change in stability. It allows one to ascertain the stability limit for every state of elastic equilibrium.

It should be remarked that the stability limit can also be determined from the isoperimetric variational problem: Among all allowable variations of the displacements from the equilibrium state, the most dangerous will be the ones for which the integral Q_1 is a minimum under the auxiliary condition that $Q_2 = 1$. If λ is the **Lagrange** factor for the auxiliary condition then that will be identical to the equation:

$$\delta Q_1 = \lambda\,\delta Q_2$$

which coincides with (11).

The criterion for the stability limit that is expressed here in the language of the calculus of variations can be converted into a system of homogeneous differential equations and homogeneous boundary conditions for the most-dangerous variation. That homogeneous problem will have non-zero solutions only when the parameters (viz., loads) that characterize the equilibrium state assume well-defined critical values (buckling values).

That process shall be carried out in the example of a compressed rod in the following sections.

II. The stability of the compressed rod.

7. The equilibrium state. – The notations are adapted to the particular problem in order to carry out the **Trefftz** process for finding the stability limit of a compressed rod.

The coordinates are denoted by x, y, z. The coordinate system is arranged such that the z-axis coincides with the rod axis, and the remaining axes coincide with the principal axes of the cross-section.

The displacements from equilibrium are denoted by uppercase letters U, V, W. The variations of the U, V, W will be denoted by lowercase letters u, v, w from now on. When the variations of the u, v, w are used later on, they will be denoted by δu, δv, δw.

The rod in question shall be "compressed." That means that all points of the lowest cross-section and all points of the uppermost one experience the same vertical displacement, e.g., W shall be prescribed for the end cross-section. Thus, from now on, among all perturbations of the equilibrium state, the only allowable ones will be the ones for which $w = 0$ on the lower face of the rod, and also $\partial w / \partial x = 0$ and $\partial w / \partial y = 0$.

The rod will be compressed by a force $P = p \cdot f$ (f = cross-section) for a fixed lower end ($z = 0$) by a given distance $-\Delta L$. The displacements will then be:

$$U = a_x\, x , \qquad V = a_y\, y , \qquad W = a_z\, z ,$$

which is clear with no further discussion.

The connection between the elongations a_x, a_y, a_z and the pressure p is given by the stress-extension equations (9), which now read:

$$k^{xx} = G\left(\gamma_{xx} + \frac{\Phi}{m-2}\right) = 0 ,$$

$$k^{yy} = G\left(\gamma_{yy} + \frac{\Phi}{m-2}\right) = 0 ,$$

$$k^{zz} = G\left(\gamma_{zz} + \frac{\Phi}{m-2}\right) = -p .$$

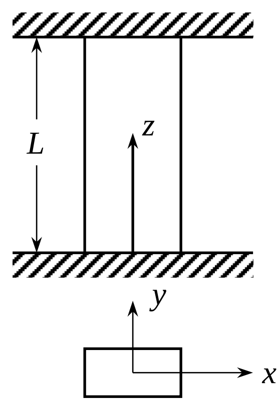

Figure 3.

One has:

$$\Phi = -\frac{m-2}{m+1}\frac{p}{G} , \qquad \gamma_{xx} = \gamma_{yy} = \frac{p}{(m+1)G} , \qquad \gamma_{zz} = \frac{m\,p}{(m+1)G} ,$$

which will make:

$$\gamma_{xx} = 2a_x + a_x^2 , \qquad a_x = \sqrt{1+\gamma_{xx}} - 1 ,$$

$$\gamma_{yy} = 2a_y + a_y^2 , \qquad a_y = \sqrt{1+\gamma_{yy}} - 1 ,$$

$$\gamma_{zz} = 2a_z + a_z^2 , \qquad a_z = \sqrt{1+\gamma_{zz}} - 1 .$$

The equilibrium conditions (3) are fulfilled automatically, as they are in any homogeneous state of stress.

8. The second variation of the internal energy. – The homogeneous equilibrium state loses its stability above a “critical” magnitude of the pressure p.

In order to find the stability limit, the second variation of the internal energy will be considered. It is given by:

$$\begin{aligned}\delta^2 E \;=\; & \iiint \{2a\,[(1+a_x)^2 u_x^2 + (1+a_y)^2 v_y^2 + (1+a_z)^2 w_z^2] \\ & + 4\,(\alpha - \beta)\,[(1+a_x)(1+a_y)\,u_x v_y + (1+a_y)(1+a_z)\,v_y w_z + (1+a_z)(1+a_x)\,w_z u_x] \\ & + \beta\,[(1+a_x)^2 (u_y^2 + u_z^2) + (1+a_y)^2 (v_x^2 + v_z^2) + (1+a_z)^2 (w_x^2 + w_y^2)] \\ & + [2\,(1+a_x)(1+a_y)\,v_x u_y + 2\,(1+a_y)(1+a_z)\,w_y v_z + 2\,(1+a_z)(1+a_x)\,v_z w_x]\}\,dx\,dy\,dz \\ & - \frac{p}{2} \iiint (u_z^2 + v_z^2 + w_z^2)\,dx\,dy\,dz\,.\end{aligned}$$

If the perturbations u, v, w are replaced with:

$$\bar{u} = (1+a_x)\,u\,, \qquad \bar{v} = (1+a_y)\,v\,, \qquad \bar{w} = (1+a_z)\,w\,,$$

and one sets $\alpha = \dfrac{G}{2}\dfrac{m-1}{m-2}$, $\beta = \dfrac{G}{2}$ then one will have:

$$\begin{aligned}\delta^2 E \;=\; & \iiint \left\{ \frac{G}{2}\frac{m-1}{m-2}(u_z^2 + v_z^2 + w_z^2) - 2G[4\bar{u}_x \bar{v}_y + 4\bar{v}_y \bar{w}_z \right. \\ & + 4\bar{w}_z \bar{u}_x - (\bar{u}_y + \bar{v}_x)^2 - (\bar{v}_z + \bar{w}_y)^2 - (\bar{w}_x + \bar{u}_z)^2]\}\,dx\,dy\,dz \\ & - \frac{p}{2} \iiint \left[\frac{\bar{u}_z^2}{(1+a_x)^2} + \frac{\bar{v}_z^2}{(1+a_y)^2} + \frac{\bar{w}_z^2}{(1+a_z)^2} \right] dx\,dy\,dz\,.\end{aligned}$$

The first integral is equal to precisely the deformation work that one defines in the classical theory of elasticity. It represents the deformation work that occurs under small displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ when they alone are present. The second integral includes the pressure p that is acting. In that way, the second variation of the internal energy of the compressed rod is written as the difference between two integrals over positive-definite forms in the derivatives of the $\bar{u}$, $\bar{v}$, $\bar{w}$, so:

$$\delta^2 E = Q_1 - Q_2\,.$$

9. The Jacobi equations. – From what was said in **6.**, the stability limit is reached for a state of elastic equilibrium when there is a “most dangerous” variation u, v, w for which one has:

$$\delta Q_1 = \delta Q_2$$

for every allowable variation δu, δv, δw, and for which Q_1 and Q_2 had the values above in the case of the compressed rod.

The fulfillment of eq, (12) for arbitrary variations of the u, v, w leads to a system of three partial differential equations. They represent the **Jacobi** equations that are assigned to the equilibrium problem as the variational problem (6).

In order to simply the derivation, the overbar on $\bar{u}$, $\bar{v}$, $\bar{w}$ will be once more dropped. The partial derivatives of the integrand of Q_1 with respect to the deformation quantities $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$, etc., will be denoted by the stresses σ_x, τ_{xy}, etc., in formal agreement with the classical theory. One has: $\sigma_x = 2G\left\{\frac{\partial u}{\partial x}+\frac{\Theta}{m-2}\right\}$, $\tau_{xy} = G\left\{\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right\}$, etc., in which $\Theta = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$.

Since Q_1 has the form of the **Hookean** deformation work, that will give the first variation of Q_1 precisely as before with $\Theta = u_x + v_y + w_z$:

$$\begin{aligned}
\delta Q_1 = {} & 2\iiint \delta u\, G\left(\Delta u+\frac{m}{m-2}\frac{\partial\Theta}{\partial x}\right)dx\,dy\,dz \\
& + 2\iiint \delta v\, G\left(\Delta v+\frac{m}{m-2}\frac{\partial\Theta}{\partial y}\right)dx\,dy\,dz \\
& + 2\iiint \delta w\, G\left(\Delta w+\frac{m}{m-2}\frac{\partial\Theta}{\partial z}\right)dx\,dy\,dz \\
& - \iint_{\text{sheath}} \{\delta u[\sigma_x\cos(n,x)+\tau_{xy}\cos(n,y)]+\delta v[\tau_{xy}\cos(n,x)+\sigma_y\cos(n,y)] \\
& + \delta w\,[\tau_{zx}\cos(n,x)+\tau_{zy}\cos(n,y)]\}\,do \\
& - \iint_{\text{upper face}} [\delta u\,\tau_{zx}+\delta v\,\tau_{zy}+\delta w\,\sigma_z]\,do + \iint_{\text{lower face}} [\delta u\,\tau_{zx}+\delta v\,\tau_{zy}+\delta w\,\sigma_z]\,do .
\end{aligned}$$

Moreover, one has:

$$\begin{aligned}
\delta Q_2 & \\
= {} & \frac{2p}{(1+a_x)^2}\iiint \delta u\frac{\partial^2 u}{\partial z^2}\,dx\,dy\,dz+\frac{2p}{(1+a_y)^2}\iiint \delta v\frac{\partial^2 v}{\partial z^2}\,dx\,dy\,dz+\frac{2p}{(1+a_z)^2}\iiint \delta w\frac{\partial^2 w}{\partial z^2}\,dx\,dy\,dz \\
& - \iint_{\text{upper face}}\left[\frac{p}{(1+a_x)^2}\frac{\partial u}{\partial z}\,\delta u+\frac{p}{(1+a_y)^2}\frac{\partial v}{\partial z}\,\delta v+\frac{p}{(1+a_z)^2}\frac{\partial w}{\partial z}\,\delta w\right]do \\
& + \iint_{\text{lower face}}\left[\frac{p}{(1+a_x)^2}\frac{\partial u}{\partial z}\,\delta u+\frac{p}{(1+a_y)^2}\frac{\partial v}{\partial z}\,\delta v+\frac{p}{(1+a_z)^2}\frac{\partial w}{\partial z}\,\delta w\right]do .
\end{aligned}$$

The equation $\delta Q_1 = \delta Q_2$ can exist for all allowable variations only when first of all the integrals in the spatial integral are equal to each other. It will then follow that:

$$\left.\begin{aligned} G\left(\Delta u + \frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right) &= \frac{p}{(1+a_x)^2}\frac{\partial^2 u}{\partial z^2}, \\ G\left(\Delta v + \frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right) &= \frac{p}{(1+a_y)^2}\frac{\partial^2 v}{\partial z^2}, \\ G\left(\Delta w + \frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right) &= \frac{p}{(1+a_z)^2}\frac{\partial^2 w}{\partial z^2}. \end{aligned}\right\} \tag{13}$$

Those equations are the **Jacobi** equations.

The boundary condition that are implied by the surface integrals must be added to them.

a) The integral over the sheath, which is due to only δQ_1, must vanish. It then follows that the equations:

$$\begin{aligned} \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) &= 0, \\ \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) &= 0 \qquad [\cos(n, z) = 0], \\ \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) &= 0, \end{aligned}$$

must exist on the sheath, which say that the displacements u, v, w correspond to a force-free sheath.

b) The integrals of δQ_1 and δQ_2 that are taken over the end faces must be equal to each other. Now, since from **7.**, all allowable perturbations and their variations will give $w = 0$ on the two end faces, and therefore $\delta w = 0$ for all points, so one will also have $w_x = 0$ and $w_y = 0$, it will then follow that $\tau_{zx} = G\, u_z$ and $\tau_{xy} = G\, v_z$. Therefore, all that remains in the expression for the upper end face will be:

$$\iint \delta u\, G \frac{\partial u}{\partial z}\, dx\, dy = \iint \frac{p}{(1+a_x)^2} G \frac{\partial u}{\partial z}\, \delta u\, dx\, dy,$$

from which it will follow that for an arbitrary δu, one will have:

$$\frac{\partial u}{\partial z} = 0.$$

Corresponding statements are true for the upper end face, so one likewise has:

$$\frac{\partial v}{\partial z} = 0.$$

Together with the condition $w = 0$, the boundary conditions for the end faces read:

$$w = 0\,,\qquad \frac{\partial u}{\partial z} = 0\,,\qquad \frac{\partial v}{\partial z} = 0 \qquad \text{or} \qquad w = 0\,,\qquad \tau_{zx} = 0\,,\qquad \tau_{zy} = 0\,.$$

That finally gives: The "most dangerous" displacement u, v, w satisfies the homogeneous differential equations:

$$G\left(\Delta u + \frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right) = \frac{p}{(1+a_x)^2}\frac{\partial^2 u}{\partial z^2}\,,$$
$$G\left(\Delta v + \frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right) = \frac{p}{(1+a_y)^2}\frac{\partial^2 v}{\partial z^2}\,,$$
$$G\left(\Delta w + \frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right) = \frac{p}{(1+a_z)^2}\frac{\partial^2 w}{\partial z^2}\,,$$

with the homogeneous boundary conditions:

a) For the sheath:

$$\tau_{xy}\cos(n, y) + \sigma_x \cos(n, x) = 0\,,$$
$$\sigma_y \cos(n, y) + \tau_{xy}\cos(n, x) = 0\,,$$
$$\tau_{yz}\cos(n, y) + \tau_{xz}\cos(n, x) = 0\,,$$

b) For the end faces:

$$w = 0\,,\qquad u_z = 0\,,\qquad v_z = 0\,,$$

whereby the stresses are defined by the conventions of the classical theory, so:

$$\sigma_x = 2G\left(\frac{\partial u}{\partial x} + \frac{\Theta}{m-2}\right)\,,\qquad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\,.$$

Eq. (13) admits a simple interpretation that helps one to understand the connection between the extended theory and the elementary theory of approximations. When one starts from the equilibrium equations of the classical theory of elasticity:

$$G\left(\Delta u + \frac{m}{m-2}\frac{\partial \Theta}{\partial x}\right) + X = 0\,,$$
$$G\left(\Delta v + \frac{m}{m-2}\frac{\partial \Theta}{\partial y}\right) + Y = 0\,,$$
$$G\left(\Delta w + \frac{m}{m-2}\frac{\partial \Theta}{\partial z}\right) + Z = 0\,,$$

and substitutes the volume forces:

$$X = -\frac{p}{(1+a_x)^2}\,u_{zz}\,, \qquad Y = -\frac{p}{(1+a_y)^2}\,v_{zz}\,, \qquad Z = -\frac{p}{(1+a_z)^2}\,w_{zz}\,,$$

then that will give eq. (13). That says that the integration of those equations is identical to the problem of finding the equilibrium form of the rod when (for a suitable value of p) the loads per unit volume are proportional to the quantities u_{zz}, v_{zz}, w_{zz} .

When the volume load is integrated over the cross-section of the rod, that will give the loads per unit length of the rod:

$$\int X\,dF = -\frac{P}{(1+a_x)^2}\frac{d^2[u]}{dz^2}, \quad \int Y\,dF = -\frac{P}{(1+a_y)^2}\frac{d^2[v]}{dz^2}, \quad \int Z\,dF = -\frac{P}{(1+a_z)^2}\frac{d^2[w]}{dz^2},$$

in which $\frac{d^2[u]}{dz^2}$, etc., are mean values. If those mean values are replaced with the differential quotients $\frac{d^2u}{dz^2}$, $\frac{d^2v}{dz^2}$, that are taken for the beam centerline, and if the small quantities a in the denominator are neglected in comparison to unity then that will define the problem of finding the equilibrium form of a rod that is loaded perpendicular to the rod with forces $P\,\frac{d^2u}{dz^2}$, $-\,P\,\frac{d^2v}{dz^2}$ per unit length. That is precisely the gist of the equation $E\,I\,w^{\mathrm{IV}} = -\,P\,w^{\mathrm{II}}$ that follows from the elementary theory.

III. – Numerical results.

The **Jacobi** equations, together with the boundary condition on pp. 18, says that along with the initial state, there exists a neighboring equilibrium state. The displacements that take the initial state to the neighboring state are the solutions u, v, w, up to the factors $(1 + a_x)^2$, etc., which differ only slightly from unity. Since the **Jacobi** equations and the boundary conditions are homogeneous, one is dealing with an eigenvalue problem. The eigenvalue is the "critical" load p itself.

In this section, the eigenvalue problem will be solved for the circular and rectangular cross-sections. In the first case, it will be solved by an integration using a series development, while in the second case, it will be solved by means of the **Ritz** process.

10. The circular cross-section. – Eq. (13):

$$G\left(\Delta u + \frac{m}{m-2}\frac{\partial\Theta}{\partial x}\right) = \frac{p}{(1+a_x)^2}\frac{\partial^2 u}{\partial z^2},$$

$$G\left(\Delta v + \frac{m}{m-2}\frac{\partial\Theta}{\partial y}\right) = \frac{p}{(1+a_y)^2}\frac{\partial^2 v}{\partial z^2}, \quad \Theta = u_x + v_y + w_z,$$
$$G\left(\Delta w + \frac{m}{m-2}\frac{\partial\Theta}{\partial z}\right) = \frac{p}{(1+a_z)^2}\frac{\partial^2 w}{\partial z^2}$$

will be converted into cylindrical coordinates. They mean that one seeks the equilibrium form of a rod that is loaded with the forces (per unit volume):

$$-p\frac{\partial^2 u}{\partial z^2} = X, \quad -p\frac{\partial^2 v}{\partial z^2} = Y, \quad -p\frac{\partial^2 w}{\partial z^2} = Z,$$

in which the very small quantities a_x, a_y, a_z are neglected in comparison to unity.

Now, when cylindrical coordinates r, ϑ, z are introduced and the displacement in the radial direction is denoted by ρ, the one in the tangential direction by τ, and the one in the axial direction by w, the problem will become that of finding the equilibrium form of a rod that is loaded in those three directions with the volume forces $-p\frac{\partial^2\rho}{\partial z^2}$, $-p\frac{\partial^2\tau}{\partial z^2}$, $-p\frac{\partial^2 w}{\partial z^2}$. If σ_r, σ_ϑ, σ_z, $\tau_{r\vartheta}$, $\tau_{\vartheta z}$, τ_{rz} are the stress components, when referred to the cylindrical coordinates, then the equilibrium conditions will read ([5]):

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\vartheta}}{\partial\vartheta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\vartheta}{r} = p\frac{\partial^2\rho}{\partial z^2},$$

$$\frac{\partial\tau_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\vartheta}{\partial\vartheta} + \frac{\partial\tau_{\vartheta z}}{\partial z} + \frac{2\tau_{r\vartheta}}{r} = p\frac{\partial^2\tau}{\partial z^2},$$

$$\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\vartheta}}{\partial\vartheta} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = p\frac{\partial^2 w}{\partial z^2},$$

and the stress-extension equations:

$$\sigma_r = 2G\left(\frac{\partial\rho}{\partial r} + \frac{\Theta}{m-2}\right), \qquad \tau_{\vartheta z} = G\left(\frac{1}{r}\frac{\partial\rho}{\partial r} + \frac{\partial\tau}{\partial z}\right),$$

$$\sigma_\vartheta = 2G\left(\frac{1}{r}\frac{\partial\tau}{\partial\vartheta} + \frac{\rho}{r} + \frac{\Theta}{m-2}\right), \quad \tau_{zr} = G\left(\frac{\partial\rho}{\partial z} + \frac{\partial w}{\partial r}\right),$$

$$\sigma_z = 2G\left(\frac{\partial w}{\partial z} + \frac{\Theta}{m-2}\right), \qquad \tau_{\vartheta r} = G\left(\frac{\partial\tau}{\partial r} - \frac{\tau}{r} + \frac{1}{r}\frac{\partial\rho}{\partial r}\right),$$

([5]) Cf., *Handbuch der Physik*, Band VI, pp. 81.

$$\Theta = \frac{\partial\rho}{\partial r} + \frac{1}{r}\frac{\partial\tau}{\partial\vartheta} + \frac{\rho}{r} + \frac{\partial w}{\partial z}.$$

Eliminating the stresses in the equilibrium conditions by means of the stress-extension equations will yield the **Jacobi** equations in cylindrical coordinates. With:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

they read:

$$\left.\begin{aligned} &G\left[\Delta\rho + \frac{m}{m-2}\frac{\partial\Theta}{\partial r} - \frac{2}{r^2}\frac{\partial\tau}{\partial\vartheta} - \frac{\rho}{r^2}\right] = p\frac{\partial^2\rho}{\partial z^2}, \\ &G\left[\Delta\tau + \frac{m}{m-2}\frac{1}{r}\frac{\partial\Theta}{\partial r} + \frac{2}{r^2}\frac{\partial\rho}{\partial\vartheta} - \frac{\tau}{r^2}\right] = p\frac{\partial^2\tau}{\partial z^2}, \\ &G\left[\Delta w + \frac{m}{m-2}\frac{\partial\Theta}{\partial z}\right] = p\frac{\partial^2 w}{\partial z^2}. \end{aligned}\right\} \tag{14}$$

They are combined with the boundary conditions:

a) $\sigma_r = 0$, $\tau_{r\vartheta} = 0$, $\tau_{rz} = 0$ for the sheath of the cylinder, and

b) $w = 0$, $u_z = 0$, $v_z = 0$ for the end faces ($z = 0$, $z = L$).

One chooses the following Ansatz for the integration of equations (14):

$$\left.\begin{aligned} \rho &= P(r)\cos n\vartheta\cos\nu z, \\ \tau &= Q(r)\sin n\vartheta\cos\nu z, \\ w &= R(r)\cos n\vartheta\sin\nu z. \end{aligned}\right\} \tag{15}$$

It will satisfy the boundary conditions for the end faces when one takes:

$$\nu = \frac{2\pi}{L}. \tag{16}$$

The three ordinary differential equations for the functions $P(r)$, $Q(r)$, $R(r)$ will follow from (14):

$$\left.\begin{aligned}
&P'' + \frac{1}{r}P' - \left[1 + \frac{m-2}{2(m-1)}n^2 - \nu^2 (q-1)\frac{m-2}{2(m-1)}r^2\right]\frac{P}{r^2} \\
&\quad - \frac{3m-2}{2(m-1)}\,n\,\frac{Q}{r^2} + \frac{m-2}{2(m-1)}\frac{Q'}{r^2} + \frac{m\nu}{2(m-1)}R' = 0\,, \\
&Q'' + \frac{1}{r}Q' - \left[1 + \frac{2(m-1)}{m-2}n^2 - \nu^2 (q-1) r^2\right]\frac{Q}{r^2} \\
&\quad - \frac{3m-4}{m-2}\,n\,\frac{P}{r^2} - \frac{m\,n}{m-2}\frac{P'}{r^2} - \frac{m\,n\,\nu}{m-2}\frac{R}{r} = 0\,, \\
&R'' + \frac{1}{r}R' + \nu^2\left[q - \frac{2(m-1)}{m-2} - \frac{n^2}{r^2}\right]R \\
&\quad - \frac{m\nu}{m-2}P' - \frac{m\nu}{m-2}\frac{P}{r} - \frac{m\,n\,\nu}{m-2}\frac{Q}{r} = 0,
\end{aligned}\right\} \tag{17}$$

in which the primes mean derivatives with respect to r, and $q = p / G$.

With the introduction of cylindrical coordinates, we have then succeeded in reducing the system of partial differential equations (14) to a system of three ordinary second-order differential equations.

Equations (17) now imply the case of buckling, i.e., the lateral deflection of the solid, but hollow, cylindrical column for n = 1. When the value 10 / 3 has been substituted for the lateral contraction number, the equations will read:

$$\left.\begin{aligned}
&r^2 P'' + r\,P' + \tfrac{2}{7}\nu^2 (q-1) r^2 P - \tfrac{9}{7}P - \tfrac{9}{7}Q + \tfrac{5}{7}r\,Q' + \frac{5\nu}{7}r^2 R' = 0\,, \\
&r^2\,Q'' + r\,Q' + \nu^2 (q-1) r^2 Q - \tfrac{9}{2}Q - \tfrac{9}{2}P - \tfrac{5}{2}r\,P' - \frac{5\nu}{2}r\,R = 0\,, \\
&r^2\,R'' + r\,R' + \nu^2\left(q - \tfrac{7}{2}\right)R - R - \frac{5\nu}{2}r\,P - \frac{5\nu}{2}r^2 P' - \frac{5\nu}{2}r\,Q = 0\,.
\end{aligned}\right\} \tag{18}$$

The boundary conditions for the sheath simplify by means of the Ansatz (15) to the equations:

$$\left.\begin{aligned}
7\,r_a\,P' + 3(P+Q) + 3\nu\,r_a\,R &= 0\,, \\
P + Q - r_a\,Q' &= 0\,, \\
R' - \nu\,P &= 0,
\end{aligned}\right\} \tag{19}$$

to which one must add three analogous equations with r_i in place of r_a in the case of the hollow cylinder.

The complete integration of the system of equations (18) leads to six integration constants. That is contrasted with the six boundary conditions (three inside, three outside) in the case of a hollow cylinder. The missing three conditions are obtained from the requirement that the stresses

σ_r, $\tau_{r\vartheta}$, τ_{rz} must remain finite for $r = 0$. The functions P, Q, R must behave regularly for $r = 0$. That regularity requirement implies three further givens for determining the constants.

Equations (18) are integrated in the usual way by a power series development for the solid cylinder. When the integration constants are denoted by c_1, c_2, c_3, the successive calculation of the coefficients will yield the series:

$$\left.\begin{aligned}
P(r) = {} & \frac{c_1}{\nu}\left[1 \quad + \frac{147-22q}{7\cdot 192}(q-1)\nu^4 r^4 + \frac{434-343q+34q^2}{7\cdot 96\cdot 96}(q-1)\nu^6 r^6 + \cdots\right] \\
& + \nu c_2\left[\quad r^2 + \frac{154-29q}{7\cdot 24}\ \nu^2 r^4 + \frac{441-357q+41q^2}{7\cdot 12\cdot 96}\ \nu^4 r^6 + \cdots\right] \\
& + \ c_3\left[\qquad \frac{140-15q}{7\cdot 96}\ \nu^3 r^4 + \frac{420-315q+20q^2}{7\cdot 48\cdot 96}\ \nu^5 r^6 + \cdots\right], \\
-Q(r) = {} & \frac{c_1}{\nu}\left[1 + 2(q-1)\nu^2 r^2 + \frac{231-206q}{7\cdot 192}(q-1)\nu^4 r^4 + \frac{2450-4375q+2050q^2}{49\cdot 64\cdot 144}(q-1)\nu^6 r^6 + \cdots\right] \\
& + \nu c_2\left[\qquad 19r^2 + \frac{266-241q}{7\cdot 24}\ \nu^2 r^4 + \frac{2793-5061q+2393q^2}{49\cdot 32\cdot 144}\ \nu^4 r^6 + \cdots\right] \\
& + \ c_3\left[\qquad 5\nu r^2 + \frac{280-225q}{7\cdot 96}\ \nu^3 r^4 + \frac{2940-5355q+2540q^2}{49\cdot 32\cdot 144}\ \nu^5 r^6 + \cdots\right], \\
-R(r) = {} & \frac{c_1}{\nu}\left[\frac{5}{8}(q-1)\nu^3 r^3 + \frac{70-45q}{7\cdot 192}(q-1)\nu^5 r^5 + \frac{735-945q+335q^2}{49\cdot 96\cdot 96}(q-1)\nu^7 r^7 + \cdots\right] \\
& + \nu c_2\left[\qquad 5\nu r^3 + \frac{70-45q}{7\cdot 24}\ \nu^3 r^5 + \frac{735-945q+335q^2}{49\cdot 12\cdot 96}\ \nu^5 r^7 + \cdots\right] \\
& + \ c_2\left[-r + \frac{9+q}{8}\nu^2 r^3 + \frac{133-76q-7q^2}{7\cdot 192}\ \nu^4 r^5 + \frac{1421-1743q+523q^2+49q^3}{49\cdot 48\cdot 96}\ \nu^6 r^7 + \cdots\right],
\end{aligned}\right\} \quad (20)$$

In addition to those equations, the system (18) also possesses logarithmically singular solutions that will drop out because of the demand of regularization.

Upon multiplying them successively by trigonometric functions of ϑ and z according to the Ansatz (15), once q has been found, one will get the desired values for the "most-dangerous" displacements of the cylindrical rod from the equilibrium configuration. Generally, one the three arbitrary integration constants c_1, c_2, c_3 in those equations are determined from the boundary conditions (19). They read:

$$7r_a P' + 3(P+Q) + 3\nu\, r_a\, R = 0\,,$$

$$P + Q - r_a Q' = 0\,,$$

$$R' - \nu P = 0\,,$$

and say that the sheath is stress-free.

If the values in (20) with $r = r_a$ are substituted for P, Q, R then the boundary conditions will define a system of three homogeneous linear equations for the constants c_1, c_2, c_3. The quantities $\nu = 2\pi / L$ and the radius r_a appear in the powers with the same degrees such that one can set their product equal to $k = 2\pi\, r_a / L$. That system of three homogeneous linear equations can be solved for non-zero values of the constants only when the determinant D of the system vanishes. Since the determinant includes the unknown load magnitude q, the latter can be determined in such a way that the determinant will become zero.

For the following calculations, the series expansions for P, Q, R will be employed with the accuracy that is given in formulas (20). The laborious calculation of the determinant leads to the equation:

$$\frac{40}{3}\,q\,k^2 - \frac{546 - 1099\,q + 503\,q^2}{63}\,k^4 - \frac{12495 - 36043\,q + 31781\,q^2 - 8358\,q^3}{49\cdot 144}\,k^6 - \cdots = 0\,. \tag{21}$$

Carrying out those calculations is only possible numerically. Due to the complicated structure of eq. (21) with respect to q, the value of the load is prescribed in order to evaluate the series. Since $q = p / G$ and the shear modulus G is very large, q must be a very small quantity. With G = 800000 kg/cm^2, p = 800 kg/cm^2 and p = 1600 kg/cm^2 will give the values q = 1/1000 (q = 1/500. resp.) for the loads that lie in the domain of **Hooke**'s law.

The results of eq. (21) can be compared with the elementary theory, in which the buckling length L_k is given by:

$$L_k^2 = 4\pi^2\,\frac{E\,I}{P_k}\,.$$

The result deviates only slightly from the **Euler** formula. It is:

$$q = \frac{1}{1000}\,, \qquad L^2 = 4\pi^2\,\frac{E\,I}{P_\kappa}\,(1 - 0.0012)\,,$$
$$q = \frac{1}{500}\,, \qquad L^2 = 4\pi^2\,\frac{E\,I}{P_\kappa}\,(1 - 0.0027)\,.$$

The **Euler** formula is then confirmed by that, within the limits of computational accuracy.

11. The rectangular cross-section. – The rod has, in turn, the length L. The rectangle has sides of length $2a$ and $2b$ ($a > b$). The position and compression ratios are the same as before. The stability limits go back to the equation $\delta Q_1 = \delta Q_2$, in which Q_1 and Q_2 have the same meaning that was explained on page 16.

The **Ritz** process will be employed in order to solve the equation.

An initial Ansatz for the displacements u, v, w will be to choose:

$$u = A\,(x, y)\cos\,\nu z\,, \quad v = B\,(x, y)\cos\,\nu z\,, \quad w = C\,(x, y)\sin\,\nu z\,, \quad \nu = \frac{2\pi}{L}\,, \tag{22}$$

so the spatial problem will reduce to a planar problem. It fulfills the boundary conditions for the end faces, so one must have $w = 0$, $u_z = 0$, $v_z = 0$ for $z = 0$ and $z = L$. The integration of Q_1 and Q_2 over the length of the rod L gives:

$$\left.\begin{aligned} Q_1 &= \frac{G\,L}{2}\int_{-a}^{+a}\int_{-b}^{+b}\left\{\frac{m-1}{m-2}(A_x + B_y + \nu\,C)^2 - 2A_x\,B_y - 2\nu\,B_y\,C - 2\nu A_x\,C\right. \\ &\qquad \left. + \tfrac{1}{2}(A_y + B_x)^2 + \tfrac{1}{2}(C_y - \nu\,B)^2 + \tfrac{1}{2}(C_x - \nu\,A)^2\right\} dx\,dy \\ Q_2 &= \frac{\nu^2\,L}{2}\int_{-a}^{+a}\int_{-b}^{+b}(A^2 + B^2 + C^2)\,dx\,dy. \end{aligned}\right\} \tag{23}$$

One chooses the Ansatz for A, B, C:

$$\left.\begin{aligned} A &= \frac{\alpha_0}{\nu} + \alpha_1\,x + \alpha_2\,y + \alpha_{11}\,\nu\,x^2 + 2\alpha_{11}\,\nu\,x\,y + \alpha_{22}\,\nu\,y^2, \\ B &= \frac{\beta_0}{\nu} + \beta_1\,x + \beta_2\,y + \beta_{11}\,\nu\,x^2 + 2\beta_{11}\,\nu\,x\,y + \beta_{22}\,\nu\,y^2, \\ C &= \frac{\gamma_0}{\nu} + \gamma_1\,x + \gamma_2\,y + \gamma_{11}\,\nu\,x^2 + 2\gamma_{11}\,\nu\,x\,y + \gamma_{22}\,\nu\,y^2. \end{aligned}\right\} \tag{24}$$

If they are substituted in (23) then that will give:

$$\begin{aligned} Q_1 = {} & 2\,a\,b\,L\,G\Big\{\Big[\tfrac{7}{2}(\alpha_1^2 + \beta_2^2 + \gamma_0^2) + \tfrac{1}{2}(\alpha_0^2 + \beta_0^2 + \alpha_1^2 + \beta_1^2 + \gamma_1^2 + \gamma_2^2) \\ & + \tfrac{3}{2}(\alpha_1\,\beta_2 + \alpha_1\,\gamma_0 + \beta_2\,\gamma_0) + \alpha_2\,\beta_1 - \beta_0\,\gamma_2 - \alpha_0\,\gamma_1\Big] \\ & + \Big[\tfrac{1}{2}(\alpha_1^2 + \beta_1^2) + 7(\alpha_{11}^2 + \beta_{12}^2) + 2(\beta_{11}^2 + \gamma_{11}^2 + \alpha_{12}^2 + \gamma_{12}^2) + \tfrac{7}{4}\gamma_1^2 \\ & + \alpha_0\,\alpha_{11} + \beta_0\,\beta_{11} + \tfrac{7}{2}\gamma_0\,\gamma_{11} - \tfrac{1}{2}\alpha_1\,\gamma_{11} + \tfrac{3}{2}\beta_2\,\gamma_{11} + 2\,\alpha_{11}\,\gamma_1 \\ & + 6\,\alpha_{11}\,\beta_{12} + 3\,\gamma_1\,\beta_{12} + 4\,\alpha_{12}\,\beta_{12} - \beta_{12}\,\gamma_1 - 2\,\beta_1\,\gamma_{12}]\,\frac{\nu^2\,a^2}{3} \\ & + \Big[\tfrac{1}{2}(\alpha_2^2 + \beta_2^2) + 7(\alpha_{12}^2 + \beta_{22}^2) + 2(\alpha_{22}^2 + \gamma_{22}^2 + \beta_{12}^2 + \gamma_{12}^2) + \tfrac{7}{4}\gamma_2^2 \\ & + \alpha_0\,\alpha_{22} + \beta_0\,\beta_{22} + \tfrac{7}{2}\gamma_0\,\gamma_{22} - \tfrac{1}{2}\beta_2\,\gamma_{22} + \tfrac{3}{2}\alpha_1\,\gamma_{22} + 2\beta_{22}\,\gamma_2 \\ & + 6\,\alpha_{12}\,\beta_{22} + 3\,\gamma_2\,\alpha_{12} + 4\,\alpha_{22}\,\beta_{12} - \alpha_{22}\,\gamma_1 - 2\,\alpha_2\,\gamma_{12}]\,\frac{\nu^2\,b^2}{3} \\ & + \Big[2(\alpha_{12}^2 + \beta_{12}^2) + \alpha_{11}\,\alpha_{22} + \beta_{11}\,\beta_{22} + \tfrac{7}{2}(\gamma_{11}\,\gamma_{22} + 2\gamma_{12}^2)]\frac{\nu^4\,a^2\,b^2}{9} \\ & + \Big[\tfrac{1}{2}(\alpha_{11}^2 + \beta_{11}^2) + \tfrac{7}{4}\gamma_{11}^2]\frac{\nu^4\,a^4}{5} + \Big[\tfrac{1}{2}(\alpha_{22}^2 + \beta_{22}^2) + \tfrac{7}{4}\gamma_{22}^2]\frac{\nu^4\,b^4}{5}\ ; \end{aligned}$$

$$
\begin{aligned}
Q_2 = a\,b\,L\,p\{&[\alpha_0^2 + \beta_0^2 + \gamma_0^2] \\
&+ \left[\alpha_1^2 + \beta_1^2 + \gamma_1^2 + 2\,\alpha_0\,\alpha_{11} + 2\,\beta_0\,\beta_{11} + 2\,\gamma_0\,\gamma_{11}\right] \frac{\nu^2 a^2}{3} \\
&+ \left[\alpha_2^2 + \beta_2^2 + \gamma_2^2 + 2\,\alpha_0\,\alpha_{22} + 2\,\beta_0\,\beta_{22} + 2\,\gamma_0\,\gamma_{22}\right] \frac{\nu^2 b^2}{3} \\
&+ \left[4\,\alpha_{12}^2 + 4\,\beta_{12}^2 + 4\,\gamma_{12}^2 + 2\,\alpha_{11}\,\alpha_{22} + 2\,\beta_{11}\,\beta_{22} + 2\,\gamma_{11}\,\gamma_{22}\right] \frac{\nu^4 a^2 b^2}{9} \\
&+ [\alpha_{11}^2 + \beta_{11}^2 + \gamma_{11}^2] \frac{\nu^4 a^4}{5} + [\alpha_{22}^2 + \beta_{22}^2 + \gamma_{22}^2] \frac{\nu^4 a^4}{5} \Big\} .
\end{aligned}
$$

The integrals Q_1 and Q_2 are homogeneous quadratic functions of the eighteen coefficients. The equation for the stability limit $\delta Q_1 = \delta Q_1$ implies eighteen linear equations for the coefficients according to the formula $\dfrac{\partial Q_1}{\partial a_{i\kappa}} = \dfrac{\partial Q_2}{\partial a_{i\kappa}}$. The system of eighteen equations decomposes into four groups that include the following coefficients:

1. α_2 ; β_1 ; γ_{12}
2. α_1 ; β_2 ; γ_0 ; γ_{11} ; γ_{22}
3. α_0 ; α_{11} ; α_{22} ; β_{11} ; γ_1
4. β_0 ; β_{11} ; β_{22} ; α_{12} ; γ_2 .

The individual groups are systems of homogeneous linear equations for the coefficients that occur in them and will have non-zero solutions only when the determinants vanish. If we then select one group and set its determinant equal to zero, which will determine the buckling load, then we will get the coefficients of that group, up to a common factor. The coefficients of the other groups will be zero, since the determinants of the other groups will not vanish for the value of the buckling load that one ascertains. What is of immediate interest to us here is the determinant equation that yields the buckling load. We then ask which of the four groups we have to take. Naturally, it will be the one that gives the lowest buckling load, which is one that we can, however, recognize with no difficulty. If the rod deflects in the *x*-direction then all points of the cross-section will have roughly the same displacement in the *x*-direction. *u* cannot be an odd function of *x* then. Now, the only group that gives *u* as a function that is not odd is the third one. Thus, if we set the determinant of that group equal to zero then we must get the value of the buckling load that results from a deflection in the *x*-direction. The fourth group, which emerges from the third one by switching *x* and *y*, implies the buckling load for a deflection in the *y*-direction. The determinant equation is the same as it was for the third group when we switch *a* and *b*. Groups 1 and 2, which go to each other by a permutation, imply a type of crush limit, but those values have no physical significance, since that process lies outside the bounds of the extended **Hooke** law.

We write out the third group of equations in detail as:

$$18\alpha_{11} + 12n^2\alpha_{22} + [42 + 12n^2 + 4n^2k^2(1-q)]\beta_{12} + 9\gamma_1 = 0\,,$$
$$3(1-q)\alpha_0 + (1-q)k^2\alpha_{11} + (1-q)n^2k^2\alpha_{22} - 3\gamma_1 = 0\,,$$
$$15(1-q)\alpha_0 + 5(1-q)k^2\alpha_{11} + [9(1-q)n^2k^2 + 60]\alpha_{22} + 60\beta_{12} - 15\gamma_1 = 0\,,$$
$$15(1-q)\alpha_0 + [9(1-q)k^2 + 210]\alpha_{11} + 5(1-q)n^2k^2\alpha_{22} + 90\beta_{12} + 30\gamma_1 = 0\,,$$
$$-6\alpha_0 + 4k^2\alpha_{11} - 2n^2k^2\alpha_{22} + 6k^2\beta_{12} + [6 + (7-2q)k^2]\gamma_1 = 0\,,$$

in which we have set:

$$q = \frac{p}{G}, \qquad n = \frac{b}{a}, \qquad a\,\nu = \frac{2\pi a}{L} = k\,,$$

to abbreviate. The determinant of that system, when multiplied out, reads:

$$\left.\begin{aligned}
q^3 \quad & - \frac{15n^2k^2 + 9312k^2 + 51}{2n^2k^2}q^4 + \frac{26n^4k^4 + 291n^2k^2 + 432n^2k^2 + 93n^2 + 630}{4n^4k^4}q^3 \\
& - \left[\frac{100n^4k^6 + 2535n^4k^4 + 1296n^2k^4 + 22797n^2k^2 + 3060}{4n^4k^6}\right. \\
& \left. + \frac{1890n^4 + 14850n^2 + 28890}{4n^4k^8}\right]q^2 + \left[\frac{60n^4k^8 + 1518n^2k^2 + 1734n^2k^6}{4n^4k^8}\right. \\
& \left. + \frac{4896n^4k^4 + 38421n^2k^4 + 4230k^4 + 1890n^4k^2 + 14850n^2k^2 + 99090k^2 + 3060}{4n^4k^8}\right]q \\
& - \left[\frac{56n^4k^6 + 25681n^4k^4 + 6200n^4k^2 + 1320n^2k^4 + 39735n^2k^2 + 5600k^2 + 280800}{16n^4k^6}\right] = 0.
\end{aligned}\right\} \quad (25)$$

Since the coefficients of the powers of q alternate in sign, the equation can have only positive roots.

For the numerical calculation, we remark that q, as well as k, are very small. If we consider the limiting case of a very slender rod, i.e., we let k go to zero, then after multiplying by k^8, only the constant and linear terms will remain, which will give the equation:

$$13k^2 - 15q = 0$$

for determining q. That will imply the buckling load:

$$p_k = \frac{13}{15}Gk^2\,.$$

If we substitute:

$$G = \frac{m}{2(m+1)}E\,, \qquad m = \frac{10}{3}, \qquad k = \frac{2\pi\, a}{L}, \qquad \frac{I}{F} = \frac{a^2}{3}, \qquad P_k = p_k \cdot F$$

in that then we will get the **Euler** formula for the compressed rod precisely:

$$P_k = \frac{4\pi^2 E\, I}{l^2}\,.$$

That formula is therefore confirmed for the limiting case of a very slender rod. The lateral ratio n does not enter into it explicitly. Strictly speaking, a correction that depends upon n is required, but it will obviously not make a noticeable contribution in practical cases.