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On systems of functions of several variables

By

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When one lets F_0 denote a real function of n real variables $z_1, z_2, \ldots, z_n$, and lets F_{01}, $F_{02}, \ldots, F_{0n}$ denote its partial derivatives then one can apply the considerations to the system of n functions $(F_{01}, F_{02}, \ldots, F_{0n})$ that I discussed in my communication on 4 March of this year. That special system is worthy of particular interest because one will be led to it when one extends the theory of curvature of surfaces to functions of several variables. I shall preserve the terminology and notations of the article in the Monatsberichte on March of this year and set:

$$F_k = F_{0k} \qquad \text{(for } k = 1, 2, \ldots, n)$$

for the special case that will define the topic of the present article, and also for the sake of uniformity, replace F_0 with F_{00}, when it seems appropriate. Furthermore, as in my article on bilinear forms (Monastberichte of October 1866), I will let:

$$| A_{gh} |$$

denote the determinant that is defined by the $(n + 1)^2$ quantities:

$$A_{gh} \qquad (g \text{ and } h = 0, 1, 2, \ldots, n)$$

and let δ_{gh} denote “zero” or “one” according to whether $g \neq h$ or $g = h$, respectively. The *characteristic* of the system $(F_{01}, F_{02}, \ldots, F_{0n})$ is then expressed by:

$$\text{(K)} \qquad -\frac{1}{\varpi}\int | F_{gh} | \cdot \frac{dw}{S^{n+1}} .$$

In the determinant under the integral sign, the indices g and h assume all values from 0 to n, and F_{0h}, as well as F_{h0} (which coincides with F_h), means the derivative of F_0 with respect to z_h, whereas when g and h are non-zero, F_{gh} refers to the derivative of F_g with respect to z_h, or also the second derivative of F_0 with respect to z_g and z_h. If one sorts all systems of values (z) that satisfy the conditions:

$$F_0 < 0, \qquad F_{01} = F_{02} = \ldots = F_{0n} = 0$$

into two categories according to whether the Hessian determinant of F_0 has a positive or negative value then the integral (K), from the meaning of the word "characteristic," will give the amount by which the number of systems of values in the first category exceeds the number in the second category. On the other hand, the integral (K) has a well-known geometric meaning for the cases $n = 2$ and 3, and in fact, for $n = 3$, it means the *curvatura integra* that is extended over the entire closed surface, divided by 4π. Therefore, the agreement between the *curvatura integra* and the characteristic of the system (F_{01}, F_{02}, …, F_{0n}), divided by 4π has been proved, and in that way, a simple method for the determination of the total curvature of an arbitrary closed surface has been obtained.

The investigation of the integral above of the characteristic of a system (F_{01} , F_{02} , …, F_{0n}) has led me to carry over the theory of curvature to functions of n variables. In that investigation, I have found that the considerations that are useful for surfaces can be generalized in a very simple and elegant way and that the known analytical results will remain preserved entirely when one introduces n variables in place of the three space coordinates. I shall reserve communicating my detailed investigations for later, and here I will only give some provisional suggestions about them.

I shall say *planar* ν-fold manifold to mean one that is cut out from the total n-fold manifold by $(n - \nu)$ *linear* equations, which can then be defined by n equations:

$$z_k - z_k^0 = \sum_{i=1}^{\nu} c_{kl}\, u_i \qquad (k = 1, 2, \ldots, n)$$

when one introduces n new variables u. A *simple* planar manifold can then be represented by a *planar line* in the form:

$$z_k - z_k^0 = a_k\, t \qquad (k = 1, 2, \ldots, n),$$

where $\sum a_k^2 = 1$ and the variable t is, so to speak, the distance from the variable point (z) to the fixed point (z^0) . For two such lines, the expression:

$$\sum a_k\, a'_k \qquad (k = 1, 2, \ldots, n)$$

corresponds to the cosine of the difference between the directions of the two lines. Furthermore, as in my paper on 4 March of this year, the line:

$$z_k - z_k^0 = \frac{F_k}{S} \cdot p$$

is the normal to F_0 at the point (z^0), and p is the distance from the point (z) to the point (z^0). Obviously, the concept of the difference between the direction of a line and an (n – 1)-fold manifold can also be established with the help of that determination.

The (n – 1)-fold planar manifold that contacts the (n – 1)-fold manifold $F_0 = 0$ at the point (z^0) is:

$$\sum (z_k - z_k^0)\, F_k = 0 \qquad (k = 1, 2, \ldots, n),$$

where the variables z are replaced with the corresponding values z^0 in F_k , A *second* planar manifold that goes through the same contact point is:

$$(a, b) \qquad z_k - z_k^0 = a_k\, u + b_k\, v \qquad (k = 1, 2, \ldots, n),$$

in which u and v mean two variables. By a linear conversion of u and v, the values of a can chosen here in such a way that the line:

$$(a) \qquad z_k - z_k^0 = a_k\, t \qquad (k = 1, 2, \ldots, n)$$

lies in each contacting manifold, and that the line:

$$(b) \qquad z_k - z_k^0 = b_k\, t \qquad (k = 1, 2, \ldots, n)$$

is normal to the line (a). The following equation then come about:

$$\sum a_k\, F_k = 0, \quad \sum a_k\, b_k = 0,$$

while, as above, one has:

$$\sum a_k^2 = 1, \qquad \sum b_k^2 = 1,$$

moreover. If one imagines that a normal is determined to the line that is cut out of F_0 by the manifold (a, b) in the manifold (a, b), and indeed at the point (z^0), and if:

$$\rho\,(a, b)$$

is the distance along it from the point (z^0) to the point of intersection of the neighboring normal then that will be the corresponding radius of curvature, and the quantity ρ, which depends upon the coefficients a and b, will be given by the equation:

$$\rho\,(a, b) \cdot \sum a_i\, a_k\, F_{ik} = \sum b_k\, F_{ik} \qquad (i, k = 1, 2, \ldots, n).$$

Now, since the line:

$$z_k - z_k^0 = \frac{F_k}{S} \cdot p$$

represents the normal to F_0 at the point (z^0), the right-hand side of the equation above is nothing but S times the cosine of the difference between the directions of the normal and the line (b). Hence, when the line (b) coincides with the normal – i.e., so for a *normal section* (a, b) – one will have:

$$\rho(a, b) \cdot \sum a_i\, a_k\, F_{ik} = S,$$

and as a result, in analogy to Meusnier's theorem:

$$\rho(a, b) = \rho(a) \cdot \sum b_k\, f_k\,,$$

in which f_k has been written for the quotient F_k / S, to abbreviate.

If one now looks for those values of the function $\rho(a)$ for which its first derivatives all vanish [assuming that those derivatives can be defined, when one recalls the relations that exist between the quantities (a)] then it would be preferable to replace the n quantities a with $(n - 1)$ quantities α, which are defined by the following equations:

$$a_k = \sum c_{rk}\, \alpha_k \qquad (k = 1, 2, \ldots, n),$$

in which the summation over r extends over the values $1, 2, \ldots, n - 1$ (as it always will in what follows). The coefficients c_{rk} are then to be determined in such a way that the substitution:

$$x_k = \sum c_{rk}\, y_k$$

fulfills the conditions:

$$\sum F_k\, x_k = 0, \qquad \sum F_{ik}\, x_i\, x_k = \sum \lambda_r\, y_r^2\,, \qquad \sum x_k^2 = \sum y_r^2\,.$$

One then gets the following defining equations for the coefficients c :

$$\sum_k c_{rk}^2 = 1, \qquad \sum_h c_{rh}\, F_{0h} = 0, \qquad \sum_h c_{rh}\, F_{ih} = \lambda_r\, c_{ri}\,,$$

in which h assumes the values $0, 1, \ldots, n$, but i and k assume only the values $1, 2, \ldots, n$, and in which λ_r means any root of the equation:

$$|\, F_{gh} - \lambda \cdot \delta_{gh} \,| = 0 \qquad (g, h = 0, 1, 2, \ldots, n)\,.$$

Those roots are all real, and one easily convinces oneself of that fact when one replaces the last n^2 of the $(n + 1)^2$ quantities F_{gh} – so the quantities:

$$F_{ik} \qquad (i, k = 1, 2, \ldots, n),$$

with ϕ_k, which appear in the orthogonal transformation:

$$\sum F_{ik}\, x_i\, x_k = \sum \phi_r\, y_r^2\,, \qquad \sum x_k^2 = \sum y_r^2\,.$$

Since one has:

$$\rho(a) \cdot \sum \lambda_r\, \alpha_r^2 = S, \qquad \sum \alpha_r^2 = 1,$$

from the determination that was given, all of the derivatives of $\rho(a)$ will vanish when all $(n-1)$ quantities α are equal to zero, except for just (α_r), and one will then set $\alpha_r^2 = 1$. Any of those special systems of values of the quantities a can then be given simply by the equations:

$$a_k = c_{rk} \qquad (k = 1, 2, \ldots, n),$$

and the corresponding value of ρ will be:

$$\frac{S}{\lambda_r}.$$

The radii of principal curvature will then be the $(n-1)$ values of the $\rho(a)$, namely:

$$\rho_r = \frac{S}{\lambda_r} \qquad (r = 1, 2, \ldots, n-1),$$

will then be determined by the equation:

$$|\,\rho \cdot F_{gh} - S \cdot \delta_{gh}\,| = 0,$$

so the two-fold planar manifold that belongs to any ρ_r, as the radius of curvature of the simple manifold that is cut out of F_0, will be:

$$z_k - z_k^0 = c_{rk}\, u + f_k\, v,$$

and the planar line in the contacting manifold that is cut from it will be:

$$z_k - z_k^0 = c_{rk}\, t\,.$$

All of those $(n-1)$ lines that correspond to different values of the index r are normal to each other; i. e., for any two lines:

$$z_k - z_k^0 = c_{rk}\, t\,, \qquad z_k - z_k^0 = c_{sk}\, t\,,$$

the relation:

$$\sum c_{rk}\, c_{sk} = 0$$

exists, and one have Euler's formula:

$$\frac{1}{\rho(a)} = \sum \frac{\alpha_r^2}{\rho_r}$$

analogously, in which α_r is defined by the equation:

$$\alpha_r = \sum_k c_{rk}\, a_k\,,$$

so it then represents the cosine of the difference between the directions of the two lines:

$$z_k - z_k^0 = c_{rk}\, t\,, \qquad z_k - z_k^0 = a_k\, t\,,$$

One sees from this how the quantities $\rho_1, \rho_2, \ldots, \rho_{n-1}$ correspond to the essential relations between the radii of principal curvature of surfaces, and therefore one also once more finds the following fundamental property of them:

There are $(n - 1)$ *points* (z) *on the normal:*

$$z_k - z_k^0 = f_k \cdot p$$

that are each cut by a neighboring normal, and the associated values of p *are the quantities* $\rho_1, \rho_2, \ldots, \rho_{n-1}$.

The negative reciprocal value of the product of the $(n - 1)$ values of ρ, which satisfy the equation:

$$|\,\rho \cdot F_{gh} - S \cdot \delta_{gh}\,| = 0,$$

is obviously identical to the expression that multiplies the element dw in the integral (K). The characteristic of the system:

$$(F_0, F_{01}, F_{02}, \ldots, F_{0n})$$

can then be represented by:

$$+\,\frac{1}{\varpi}\int \frac{dw}{\rho_1\,\rho_2 \cdots \rho_{n-1}}\,,$$

and the reciprocal value of the product $\rho_1\ \rho_2\ \ldots\ \rho_{n-1}$ corresponds to the Gaussian curvature, because when one extends **Gauss**'s determination of the curvature to $(n - 1)$-fold manifold $F_0 = 0$ and compares the normals to ones for which:

$$\sum z_k^2 = 1,$$

one will get just that expression that multiplies the element dw in the integral (K) for quantity that corresponds to the curvature when one takes its negative; i.e., the reciprocal value of the product $\rho_1\ \rho_2\ \ldots\ \rho_{n-1}$. It emerges from this that, in fact, for a function of n variables (F_0), the characteristic of the function F_0 and its system of n partial derivatives will correspond to the same number that gives the ratio of the *curvatura integra* of a closed surface to the area of a spherical surface. Since that characteristic is given by the excess of the points (z) that lie in the interior of F_0, for which:

$$F_{01} = F_{02} = \ldots = F_{0n} = 0, \qquad |\,F_{ik}\,| > 0,$$

over the ones for which:

$$F_{01} = F_{02} = \ldots = F_{0n} = 0, \qquad |\,F_{ik}\,| < 0,$$

when one varies the constant c for the various manifolds:

$$F_0 = c$$

that characteristic can change only when points for which:

$$F_{01} = F_{02} = \ldots = F_{0n} = 0$$

are included or excluded from the interior region ($F_0 < c$) under that variation.

The geometric relationship between the characteristic and system of functions is not restricted to the special systems that were treated here. However, for the general system (F_0, F_1, F_2, …, F_n), the Kummer density will appear in place of the Gaussian curvature in the geometric relationship. Namely, if one considers the (n – 1)-fold infinite system of planar lines:

$$z_k - z_k^0 = \frac{F_k(z_1^0, z_2^0, \ldots, z_n^0)}{S(z_1^0, z_2^0, \ldots, z_n^0)} \cdot t \qquad (k = 1, 2, \ldots, n),$$

in which $F_0(z_1^0, z_2^0, \ldots, z_n^0) = 0$ and F_1, F_2, …, F_n mean any single-valued functions of the n variables z (so the assumption above that they agree with the derivatives of F_0 has been dropped), and every planar line of the system corresponds to a point (z^0) in the (n – 1)-fold manifold $F_0 = 0$, as well as a point (z):

$$\zeta_k = \frac{F_k(z_1^0, z_2^0, \ldots, z_n^0)}{S(z_1^0, z_2^0, \ldots, z_n^0)} \cdot t \qquad (k = 1, 2, \ldots, n)$$

of the (n – 1)-fold manifold:

$$\zeta_1^2 + \zeta_2^2 + \cdots + \zeta_n^2 = 1.$$

However, in the relation that arises between the two (n – 1)-fold manifolds:

$$F_0(z_1^0, z_2^0, \ldots, z_n^0) = 0, \qquad \sum \zeta_k^2 = 1,$$

the ratio of the absolute values of the elements of both manifolds will be expressed by:

$$\frac{1}{\mathfrak{S} \cdot S^n} \cdot | F_{gh} | \qquad (g, h = 0, 1, 2, \ldots, n),$$

in which one takes:

$$F_{g0} = F_g ,$$

$$F_{0k} = \frac{\partial F_0}{\partial z_k}, \qquad F_{ik} = \frac{\partial F_i}{\partial z_k}$$

for i, k = 1, 2, …, n, and:

$$\mathfrak{S}^2 = F_{01}^2 + F_{02}^2 + \cdots + F_{0n}^2 ,$$

$$S^2 = F_1^2 + F_2^2 + \cdots + F_n^2 ,$$

and one substitutes the variables z_0 that are coupled to each other by the equation:

$$F_0(z_1^0, z_2^0, \ldots, z_n^0) = 0$$

in the function F. Now, since one has:

$$R \cdot \mathfrak{S} = | F_{gh} | ,$$

with the meaning that I gave to R in my aforementioned paper in March of this year, that ratio of the elements will be represented by:

$$\frac{R}{S^n} \;;$$

i.e., by just the expression that multiplies the element dw of the manifold $F_0 = 0$ in the integral of the characteristic. Therefore, when one assigns the meaning of rectangular space coordinates to the variables z_1 , z_2 , z_3 in the case of $n = 3$, the element of the integral of the characteristic of a general system of functions:

$$(F_0 , F_1 , F_2 , F_3)$$

for the ray system that is defined by the equations:

$$z_k - z_k^0 = \frac{F_k(z_1^0, z_2^0, \ldots, z_n^0)}{S(z_1^0, z_2^0, \ldots, z_n^0)} \cdot t \qquad (k = 1, 2, 3),$$

$$F_0(z_1^0, z_2^0, z_3^0) = 0$$

will be *the element of the surface* ($F_0 = 0$) *projected onto the normal plane to the associated ray, multiplied by the Kummer density* or *the element of the surface* ($F_0 = 0$) *itself, multiplied by the density of the ray system that refers to it.* That will show that the theory of the characteristics of systems of functions is connected just as closely to the geometric theory as it is to potential theory, and one might probably recognize that this relationship between the original concept of the characteristic, which was developed from purely-analytical principles, and other known theories can be a probe for their authenticity, and therefore a proof that the introduction of that concept into science is entirely natural and necessary.

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