General Continuum Theory of Dislocations And Proper Stresses ([†])

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Contents

This work ([*]) is both a review of the present-day situation in the general nonlinear continuum theory of stationary dislocations and internal stresses and a presentation of new results in that field.

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By the *general* theory, we mean the theory of a continuum whose deformed state is described by 15 functions of position and time (i.e., a theory of 15 degrees of freedom). Developing the views of GÜNTHER further, one has to look at such a continuum as a generalization of the Cosserat continuum (6 degrees of freedom), whose state is characterized by elastic displacements (3) and independent elastic rotations (3) of the geometric structure (in the case of crystals: of the lattice structure). Instead of displacements and rotations one can also use elastic deformations and structure curvatures in the geometric description, both of which have a tensor character. The extension of the Cosserat theory that leads to the general theory implies that both the deformations and the curvatures become incompatible; i.e., the possibility of deriving them from displacement and rotation fields is abandoned. The number of degrees of freedom increases to 6+9 = 15. The connection with dislocation theory and general differential geometry follows from the fact that the dislocation density is equivalent, on the one hand, to the Cosserat structure curvatures (NYE, GÜNTHER), and one the other hand, to the CARTAN's torsion (KONDO; BILBY, BULLOUGH & SMITH).

Instead of the geometric characterization of the state, it is possible to employ a static description, which likewise needs 15 functions: 6 functions for the (force-) stresses and 9 for what are called Cosserat torque stresses. In the present work it is shown that the torque stresses enable us to describe the *microscopically* fluctuating stresses of macroscopically continuous distributions of dislocations in a *macroscopic* way. When the constitutive law connecting stresses and torque stresses with deformations and curvatures is known, the geometric and static functions can be converted into each other.

The geometric part of the theory, which is given in Chapter II, is the main part of the present work. First of all, we present the theory in a new elementary form in which the usual methods of nonlinear continuum mechanics are taken over as much as possible. The concepts developed in the linear continuum theory of dislocations and internal stresses are completely incorporated within this nonlinear form. Secondly, we give a presentation of the theory that is based on differential geometry and partly follows from the ideas developed by KONDO and by BILBY, BULLOUGH & SMITH. However, we go further in the physical interpretation. The general (incompatible) Cosserat continuum proves to be identical with a medium whose geometric state is described by the Riemann-Cartan (= metric) affine connection Γ_{mlk} (15 degrees of freedom). A new feature is the introduction of the "matter tensor" that forms the right-hand side of the "Einstein equations of continuum mechanics," which are the generalization of the St. Venant compatibilitiy equations. The symmetric part of the matter tensor describes the inserted foreign matter (in crystals, e.g., distributions of interstitial atoms), whereas the antisymmetric part characterizes some kind of rotational matter that is connected with the ending of dislocations inside the body.

If the torque stresses are neglected then the integration of the Einstein equations leads to the determination of the state when one is given the actions (e.g., forces, dislocations, etc.). In the nonlinear theory, just as in the linear theory, the most important resource for the integration is the stress function tensor. The nonlinear integration problem can be linearized by an iteration procedure. The resulting linear summation problem has been solved previously, and will be discussed briefly. In Chapter III, we discuss the second boundary-value problem in more detail. The usefulness of SCHAEFER's stress functions is demonstrated by the example of a body bounded by two infinite planes: It is possible to reduce this problem for any distributions of surface terms to a pair of standard problems, one of potential theory, and the other, of bipotential theory.

Chapter IV contains an elementary account of the previously sketched concept of a para- or diaelastic continuum, which has practical importance for solid-state physics. The lattice defects appearing macroscopically as point defects can be characterized mechanically to a large extent as elastic dipoles (force couples) or polarizable substances. Very general and simple formulae hold for the potential energy of these defects in an elastic field and for the forces and torques exerted on the defect by the fields.

In the discussion at the end of this work, we point out the intimate relationships with the general theory of relativity. It is hoped that they will have favorable effects on the further development of both theories.

I. Introduction and overview

The range of problems that have been – or at least should have been – approached by the continuum mechanics of solids has expanded considerably in the last three years. This is consistent with the ineluctable ascent of solid-state physics to the status of the branch of physics that currently deals with the most applications, next to particle physics. The boom in solid-state physics began in the last twenty years since we learned to grow unitary lattice-oriented crystals (single crystals) and now understand that one must first explore the properties of these single crystals before one can think of understanding the results that have been previously associated with the polycrystalline state.

Some of the important stops along this road that are interesting to anyone with a mechanical outlook are:

a) The theoretical computation of the critical compressive stress of an *ideal* crystal by FRENKEL (1926) [**1**], which was shown by SCHMID and POLANYI (1929) [**2**] to be more than three powers of ten greater than the experimental value of the critical compressive stress for a *real* crystal at very low temperature.

b) DEHLINGER's [**3**] examination of the possibility that crystal defects are the sources of internal stress (1920). It then becomes clear why internal stresses are even possible in crystals to begin with.

c) The introduction of the ideas of dislocation theory into the theory of plastic deformation by TAYLOR, OROWAN, and POLANYI (1934) [**4**-**6**], which proved to be unusually fruitful, and its first great result was the removal of the aforementioned difficulties with the critical compressive stress.

d) The theoretical elasticity approach to singular dislocations by BURGERS (1939) [**7**], which led to a mathematically flawless definition of a dislocation.

e) The creation of a continuum theory of dislocations and internal stresses through the work of many other authors that will be mentioned in the sequel. As we explained before [**8**], this theory shall close the yawning gap between elasticity theory and phenomenological plasticity theory, and will likewise represent a bridge between the latter theory and the *atomic* theory of plasticity that is so important in solid-state physics.

The fundamental importance of crystal defects (lattice defects) in solids is generally acknowledged at present. Of the many things that are associated with matter transport in the solid state – phase transitions, fracture, plastic deformation, diffusion, et al. – there is not a single one that comes about in the absence of lattice defects. The role of lattice defects can therefore be discussed briefly as follows (cf., e.g., DEHLINGER [**9**]):

Only by the intermediary of lattice defects can the aforementioned phenomena follow from lower-order processes, and therefore happen at all. Thus, they come about in such a way that the free energy is as small as possible. Now the energy of the internal stress fields constitutes a very substantial share – if not the entirety – of the free energy, and the sources of the internal stress fields are precisely the lattice defects. Thus, they give rise to not only more or less strong electromagnetic (magnetic, resp.) effects, but also to very meaningful mechanical effects.

As a result of the development that we just sketched out, it emerges that the continuum mechanics of solids in its earlier scope, i.e., with its branches of elasticity theory and phenomenological plasticity theory, was far from addressing the new

problems. The most important ones that always appear in the continuum mechanics of solids fall into the following three types:

(1) If one is given arbitrary external influences on the body (forces, torques, temperature fluctuations, etc.) as functions of position and time, find the state of the body, also as a function of position and time.

(2) If one is given a source distribution of internal stresses – i.e., a dislocation density with the limiting case of a singular dislocation – find the state of the body (perhaps characterized by the internal stresses, lattice curvature, and elastic energy).

(3) If one is given an elastic field in a body, find the elastic energy of a particular lattice defect in this field (the force that this field exerts on the lattice defect, resp.). The forces between lattice defects also belong to this category.

Problem (1) simultaneously involves all three of the branches of the continuum mechanics of solids that were mentioned in e), and is currently strictly soluble only in special cases. Problems (3), and in particular (2), which are typical for the continuum theory of dislocations and internal stresses, often represent sub-problems of problem (1). Thus, they also frequently constitute the continuum-mechanical component of general physical questions that frequently go far beyond the scope of mechanics. The reader is therefore referred to the relevant literature [**10**-**14**].

In § 1 - § 3 we will seek to give a brief overview of new situation in the continuum mechanics of solids, in which we will, at the same time, have the opportunity of characterizing the shortcomings of the older continuum mechanics. Since we shall not go into the dynamical effects, it is clear that the foundations of the theory that we shall discuss here will, at the same time, also belong to the foundations of the yet-to-be-developed dynamical theory of dislocations and internal stresses. Frequent references will be made to the summary report of the author [**8**] on the *linear* approximation in the book *Kontinuumstheorie der Versetzungen und Eigenspannungen*. The present work shall therefore represent an extension to the *nonlinear* theory ([1]). It falls in line with the works of KRÖNER and SEEGER [**19**] that will appear shortly, in which the non-Riemannian geometry of KONDO [**20**, **21**], as well as BILBY, BULLOUGH, and SMITH [**22**-**26**] will be extended to a nonlinear elasticity theory of dislocations and internal stresses.

§1. General connections: geometry

The COSSERAT brothers, in their book that appeared in 1909 [**27**], have developed a theory of a body whose “points” can not only be elastically translated, but also elastically rotated, in a measurable way. Up to that time, one thought of a crystalline body as a field of applications, so it remained unclear how one might present, e.g., the state of a crystal whose building blocks had indeed been subjected to rotations, but not translations.

[1] If the reader feels that that the discussion of the methods of nonlinear elasticity theory in the present work is insufficient then let him confer the beautiful treatments of TRUESDELL and MURNAGHAN, as well as DOYLE and ERICKSEN [**15**-**18**].

It has been known for perhaps ten years that for certain specially-prepared crystals ([2]), which are free of loading stresses and internal stresses, that under Röntgenographic or even optical penetrating rays one can recognize a curvature to the lattice planes, i.e., the 'geometrical structure" of a crystal, which will correspond to a spatially-varying rotation of the smallest crystal domains. In many cases, one can perceive macroscopically constantly curved lattice planes that appear to be microscopically polygonized (CAHN, GUINIER, CRUSSARD, *et al.*, 1949 [**28**]).

Today, one can easily illustrate the emergence of a macroscopically stress-free rotation of the lattice structure of a mass element. As BURGERS and BRAGG [**7**, **29**] have found, there are certain arrangements of dislocations that are confined to a surface (viz., *grain-boundary arrangements*) and evoke no macroscopic stress fields, only changes in orientation between neighboring crystal domains. If a torque, say, acts on a mass element in order to rotate its orientation then this influence leaves a dislocation trail in a particular arrangement, which rotates the element and, at the same time, mediates the transition to the neighboring element as a grain boundary. The formation of dislocations, or, better yet, the rotation of the mass element that is equivalent to it, can be assumed to be elastic when the removal of the rotational moment leads to a reverse rotation that annihilates the dislocation trail. If such moments act on *all* of the mass elements of the body, although with varying strengths, then the position-varying rotation of the lattice structure will be given in such a way as to result in a macroscopically stress-free lattice curvature.

In 1953 NYE [**30**] had outlined a continuum theory of (macroscopically stress-free) lattice curvatures, yet it was GÜNTHER [**31**] who first recognized in 1958 that such a curved crystal, when considered continuum theoretically, represented a justification for the Cosserat continuum, i.e., the lattice rotations and curvatures that were described above have precisely the characteristics that were described by the COSSERATS. We refer to them as the *Cosserat-Nye rotations* and *curvatures*, or also the *structure rotations* and *curvatures.* Moreover, we call the curvatures that we just spoke of *compatible* since they are derived from a rotation field.

The *Cosserat-Nye tensor* $K^i{}_{,j}$ represents a suitable magnitude for the Cosserat-Nye curvature, since it gives the relative rotation of the lattice structure at two points that lie dx^i apart (thus $d\theta^i = K^i{}_{,j}\, dx^j$). Therefore, the curvatures are also completely characterized by the tensor $\alpha^i{}_{,j} = K^k_k \delta^i_j - K^i{}_{,j}\, dx^j$. $\alpha^i{}_{,j}$ is referred to as the *dislocation density tensor*; it gives the *Burgers vector* db_j of a dislocation that crosses a surface element dF_i (hence, $db_j = \alpha^i{}_{,j}\, dF_i$).

For the geometrical characterization of deformable bodies we further need the *elastic distortion tensor* β_{ij}, which gives the relative elastic displacement between two surface elements that lie dx^i from each other (hence, $ds_j = \beta_{ij}\, dx^i$). In the case of small distortions, the symmetric part of the distortion tensor is identical with the deformation tensor that we used in [**8**].

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[2] For example, by plastic bending. These macroscopic stress-free lattice curvatures must be noticeably different from the ones that are obtained from, say, *elastic* bending, which have stresses that are connected with the lattice curvatures. One obtains the polygonization that is mentioned below by gluing together such pieces. In this way the dislocations will be brought into a so-called grain-boundary arrangement, whose elastic energy is essentially negligible. The macroscopic lattice curvature is already available before polygonization.

We now think of the body as an ideal initial state that is brought into a deformed final state. We shall describe this geometrically by the tensors $\boldsymbol{\beta}$ and $\boldsymbol{K}$ ($\boldsymbol{\alpha}$, resp.). Now, should the initial state be compact, i.e., a connected continuum of pre-existing bodies, and should these properties persist throughout the deformation (hence, there are no tears or folds, since it is only in this case that we could speak of a continuum theory to begin with), then, as we shall show later, the asymmetric tensor $\boldsymbol{\delta}$ in the equation:

$$\text{Rot}\,\boldsymbol{\beta} - \boldsymbol{\alpha} = \boldsymbol{\delta} \tag{1}$$

must vanish, in general. It must be non-vanishing only when uses an a initial state comprised of an ideal crystal that is originally made up of "regular" atoms, but into which one has introduced "irregular matter" (we shall also say "extra matter") that remains in the final state, moreover. The tensor $\boldsymbol{\delta}$ is a measure of this extra matter, of which, the most important example is that of the so-called "foreign atoms" (interstitial or substitute atoms).

Eq. (1) allows us to understand essential feature of the general continuum theory of dislocations and internal stresses: The distortion tensor and the curvature tensor (the dislocation tensor, resp.) are regarded as "internal" quantities that directly describe the geometrical state of the continuum. On the other hand, the extra matter is regarded as an "external" influence that brings about distortions and curvatures. This corresponds perfectly to the picture that one must imagine for a real crystal, which the deformed state of our body certainly represents: One can think of a real crystal as an ideal crystal with nothing that has been added or subtracted from the outside, in which one has created dislocations. A real crystal that has been obtained in this way will include only regular atoms. One can then call the dislocations "*internal lattice fields.*" On the contrary, foreign atoms in an ideal crystal (or in a real crystal with dislocations) must be introduced from the outside, so we shall call them "*external lattice fields.*" The left-hand side of eq. (1) corresponds to internal quantities, and the right-hand side corresponds to external ones. We shall obtain an analogous representation in statics.

For finite deformations, eq. (1) is true only in the so-called *Lagrangian* description, i.e., one must understand dx^i to mean the relative length in the initial state [**8**]. We call the theorem that is formulated by means of eq. (1) the *fundamental geometrical theorem* (of general continuum mechanics). It becomes even more noteworthy when one assumes that the elastic distortion (deformation, resp.) and the structure curvature are incompatible, i.e., they are no longer derivable from a displacement (rotation, resp.) field. This represents a substantial extension of the original Cosserat theory. The number of functional degrees of freedom by which the body is geometrically described thus increases from six to fifteen (six elastic deformations, nine structure curvatures) ([3]).

The Lagrangian description is particularly unsuited to the important task of determining the state from the physical givens. One thus prefers to use the *Eulerian* representation for the geometry of the continuum. In geometrical hindsight, it is now most remarkable that the Eulerian formulation of the extended (i.e., incompatible) Cosserat theory is equivalent to the Riemann-Cartanian geometry of solids of KONDO [**20**, **21**], as well as BILBY, BULLOUGH, and SMITH [**22-26**]. In the opinion of the

[3] As one can show, the rotational part of h_{ij} follows from elastic deformations and the structure curvatures, so they contribute no degrees of freedom of their own.

author, there is no elegant formulation of the fundamental geometrical equations of the nonlinear problems of continuum mechanics, any more than there is in general differential geometry. The equations will not become simple by themselves, since this is completely inconsistent with the known complexity of nonlinear problems, or else the subsequent method of determination of the state from the physical givens would become correspondingly favorable.

We strongly emphasize that the geometrical aspects of the continuum theory of dislocations and internal stresses can be interpreted as a definitive departure from classical elasticity theory: in the latter, geometry is Euclidian and thus trivial. The geometrical part of the theory is therefore largely contained in the statement that one endows the body with an elastic displacement field that takes its points from the initial state to the final one. In mathematical language: as fundamental geometrical equations for elasticity, the compatibility conditions for elastic deformation are satisfied identically with the help of the displacement vector field.

By contrast, the geometry of a body with dislocations and impurities is principally non-Euclidian. It is not possible here to define an elastic displacement field (rotation field, resp.); the general equations (1) appear in place of the compatibility equations, which demands new methods of solution.

In conclusion, the Cosserat-Nye curvature is described by Cartanian geometry, and above all, the Cartanian notion of *torsion*, as will be thoroughly explained in chapter II. In particular, the identity between the Frank-Burgers paths that are used to define dislocations in dislocation theory and the well-known Cartan paths will be detailed. This identity defines a foundation of the differential geometrical statement of our continuum theory and says that the dislocation density (hence, also the Cosserat-Nye curvature) and the Cartanian torsion represent precisely the same phenomenon.

In this overview, we have brought the *state* of the medium to the foreground, whereas the previous *event* that led to this state is kept in the background [**8**]. We will make up for this in chapter II. For now, the following remark shall suffice: Any motion of dislocations gives rise to a plastic distortion of the medium that can be described by a distortion tensor, at least as far as its macroscopic character is concerned. This is immediately connected with the dislocation density $\boldsymbol{\alpha}$ through the relation:

$$\text{Rot}\,\boldsymbol{\beta}^P = -\,\boldsymbol{\alpha}, \tag{2}$$

which can also be regarded as the definition of the dislocation density. Thus, when one denotes the sum of the elastic and plastic deformations (in the Lagrangian description) by the total distortion $\boldsymbol{\beta}^G$ one writes, e.g., eq. (1) in the case where $\boldsymbol{\delta} = 0$:

$$\text{Rot}\,\boldsymbol{\beta}^P = 0. \tag{3}$$

In words: The total distortion of the body results in such a way that neither folds nor tears appear. Eq. (3) is a particularly simply formation of the fundamental geometrical theorem in the "reduced" (by way of $\boldsymbol{\delta} = 0$) theory. In particular, it implies the possibility of vanishing elastic distortion [**32**], i.e., macroscopically stress-free plastic deformation can result. This has considerable practical importance.

§2. General connections: statics

The difference between Cosserat statics and classical elastostatics resides in the appearance of the so-called *torque stresses* τ^{ij}. These are defined as the torques that a surface element is endowed with, as one might associate with a cut surface, when no rotations take place. As one knows, the *force stresses* – or simply *stresses* (σ^{ij}) – that one uses in elasticity theory are defined analogously: one creates forces when there is no translation.

As the bros. COSSERAT have shown, the following *fundamental static equations* (= equilibrium conditions for forces and moments) are valid in Eulerian notation ([4]):

$$\text{Div } \boldsymbol{\sigma} = -\mathfrak{F}, \tag{4}$$

$$\text{Div } \boldsymbol{\tau} + \vec{\sigma} = -\mathfrak{M}, \tag{5}$$

(cf., e.g., GÜNTHER [**31**]). Here $\mathfrak{F}$ and $\mathfrak{M}$ mean the densities of external forces and torques. Insofar as the (force-) stress tensor $\boldsymbol{\sigma}$, as well as the torque stress tensor $\boldsymbol{\tau}$, is, in general, asymmetric, $\vec{\sigma}$ is the vector that is equivalent to the anti-symmetric part of $\boldsymbol{\sigma}$.

It is clear that the torque stresses constitute an arrangement of geometrical quantities that described structure curvatures in the last section. We shall illustrate this fact in one example that comes up frequently in practice.

At sufficiently high temperatures, a crystal bar is plastically bent in such a manner that ultimately all of the dislocations are in a Nye – i.e., grain boundary – arrangement. Upon removing the external bending moments, this bar has no macroscopic stress fields, which therefore implies (macroscopically) constant lattice curvatures. As a result of the usual short-range distortional effect of the dislocations that exists, the energy content of this bar is greater than the energy content of a bar that was made the same with an ideal lattice construction. One can understand this to mean that torque stresses are associated with the lattice curvatures according to some material law. That is, if one makes a cut somewhere then dislocations will appear in the immediate neighborhood of the cut surface, as we show in Fig. 1.

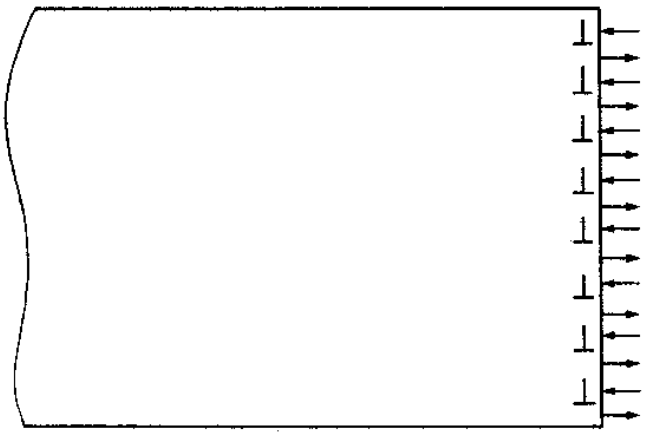

Fig. 1. The physical meaning of the (macroscopic) torques stresses.

It shows a fine grain boundary of step dislocations (symbol ⊥) near the cut surface. The dislocations of the body are a consequence of the reflecting effects of this surface

[4] As one knows, one of the great difficulties in nonlinear theory resides in the fact that the fundamental equations of statics take on a simple form only in Eulerian notation, whereas the fundamental equations of geometry take their simplest form in Lagrangian notation.

(i.e., by its internal stress fields) when one does not produce a compressive stress at any dislocation (outside the cut surface) that exerts a force in the opposite direction that would prevent the appearance of dislocations. The arrows at the cut surface give the type of effect that one finds there: *microscopically*, one considers forces; *macroscopically*, one considers torques. They insure the appearance (or also the removal) of dislocations at the cut surface without lattice rotations. We therefore have (for the first time here) obtained the important result:

The microscopic stress fields of the Nye dislocation arrangement may be macroscopically described by torque stress fields.

The way by which measurable torque stresses appear in the cut experiment that we just described persists in the absence of external influences; thus, in the usual elasticity-theoretic nomenclature they are referred to as internal or proper stresses. The fact that they persist originates in the inelastic behavior of dislocations: they need to be activated if they are to move. If the dislocations could move about the crystal easily then the removal of the external influences would immediately result in their absence, as well.

Previously, we characterized the state of the deformed body *geometrically* by giving fifteen functions: the six elastic deformations and the nine structure curvatures. On the other hand, we can also recognize the state *statically* by giving fifteen functions: the six functions that represent the symmetric part of the stress tensor and the nine torque stresses ([5]). If one knows the material law that relates the stresses to the deformations and the torque stresses with the structure curvatures then one can express both states as functions of each other ([6]).

Only in the case of completely elastic deformation can one relate the state of the body to the external influences uniquely. This can also be described by fifteen functions; one finds the corresponding quantities on the right-hand side of eqs. (1), (4), and (5). They are: extra matter (9), external force (3), and torque (3). Concerning the physical reality of the latter, which is currently recognized, cf., e.g., [**33**], [**8**], pp. 87.

Above all, the previous discussion of the functional degrees of freedom of the state and the external influences should give an overview of the scope of the general theory. At the same time, they give us reason to introduce the expression: *the theory of fifteen degrees of freedom.* In the theory that follows, we shall bring this theory into a reduced form: Since the Cosserat theory of statics has not been sufficiently developed, and the "energy of curvature" is not generally very large, we will not consider the torque stresses, but instead we will completely focus on the fifteen geometrical degrees of freedom ([7]).

Before we present the *general* theory in §§ 7, 8 we will give a *reduced* theory in §§ 4-6, which we will also call the *theory of nine degrees of freedom:* This theory is characterized by the assumption that the extra matter $\boldsymbol{\delta}$ and the external torque $\mathfrak{M}$ vanish in order that the dislocation density $\boldsymbol{\alpha}$ be formally regarded as an external influence (and

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[5] As one can show, if one knows the symmetric part by way of the stresses and likewise, the antisymmetric part by way of the torque stresses then this state has no other degrees of freedom.

[6] The tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are coupled by eq. (5). In order to also couple them by means of a material law, one must do further analysis. It is certain that the new material constants cannot be computed from the Hookean constants alone.

[7] The torque stresses play a somewhat different role in certain one- and two-dimensional problems (beams, shells, plates) than the one that they play in dislocation theory. On this, cf., perhaps ERICKSEN and TRUESDELL [**34**], SCHAEFFER [**35**], GÜNTHER [**31**].

there as a given). In the notation that we previously introduced, the field equations of this theory read:

$$\text{Rot}\ \boldsymbol{\beta} = \boldsymbol{\alpha}, \qquad \text{Div}\ \boldsymbol{\sigma} = -\mathfrak{F}, \tag{6}$$

in which the nine degrees of freedom can be easily identified. For our material law, we shall suffice with the law that one uses in classical elasticity theory that the stresses are related to the deformations; the torques that are present in reality will not be considered at all. This reduced theory already allows for the solution of a great number of problems of practical importance and in recent times it has been found in an increasing volume of applications.

We shall now comment on the fundamental meaning of the stress functions in the (general and reduced) continuum theory of dislocations and internal stresses, which seem to be indispensable in the present state of the theory, and play the same role as the vector potential in electrodynamics.

We consider the contribution made by the external forces and torques in one calculation, which serves as a displacement (rotation, resp.) field, and, in so doing., we handle the case where the right-hand side of eqs. (4, 5) vanishes. These equations may then be satisfied with the help of a stress function Ansatz that was first given in full generality [which is also the generality that was assumed in eq. (5)] by GÜNTHER [**31**]. In the case where one ignores the torque stresses, *one* stress function tensor suffices, in which case eq. (4) is satisfied identically.

The problems that are posed in the physics of internal stresses are often essentially three-dimensional in character. Up until recently, stress functions were unused in the treatment of three-dimensional problems. This was changed fundamentally in the last year. It will be shown next, that in an infinitely extended elastic isotropic medium with given internal stress sources the stress functions are given by inhomogeneous biharmonic equations [**36**]. In a seminal work, SCHAEFER [**35**] has shown further that stress functions also endow the treatment of three-dimensional boundary-value problems with a thoroughly favorable aspect (to the extent that one can use the word “favorable” at all in light of the complexity of three-dimensional problems). One has to solve three Laplace equations that are coupled by the boundary conditions. In many cases, these equations can be decoupled. For the computation of internal stresses, this method has certain advantages over those used by PAPKOVITCH and NEUBER [**37**, **38**].

In view of the central significance of stress functions in the continuum theory of dislocations and internal stresses, it seems appropriate to dedicate a separate chapter to them (chapter III).

§3. Para- and Diaelasticity

Para- and diaelasticity are phenomena that are foreign to classical elasticity, and they are very closely connected with the atomic (crystalline, resp.) structure of our solid bodies. Therefore, we next discuss crystals.

Ideal crystals, which are composed of *one* type of atom, exhibit neither para- nor diaelasticity. By contrast, all real microscopically crystalline structured solids are diaelastic, and many are paraelastic, moreover. The deviation of the real crystal structure from the ideal one – i.e., the lattice defects – defines the basis for para- and diaelasticity.

The dislocations of foreign atoms belong to the most important class of lattice defects. If a crystal has many foreign atoms then one no longer speaks of them as lattice defects, but one speaks of mixed crystals, and, in the case of metals, also of alloys. Therefore, from our standpoint, there is no principal difference between, e.g., regarding the foreign atoms as extra matter and regarding them as alloy atoms. We would thus also like to deal with the lattice defects in what follows.

The objective of the theory of para- and diaelasticity is a subsequent treatment of the particular mechanical effects that are linked with lattice defects. Thus, the concepts and ideas are completely adapted from the established notions of electro- and magnetostatics. This will be so plainly expressed in the terminology that we shall introduce that we can thus renounce the analogy and discuss the subject by itself. A diaelastic body is thus perceived to be one in which the application of a mechanical stress produces "induced elastic dipoles," by which it is "elastically polarized." By contrast, a paraelastic body involves "permanent dipoles" that can be rotated in the direction of the elastic field. Permanent, and also induced, dipoles experience forces in elastic fields, which will drive their migration when their mobility is sufficiently large. This is why, e.g., understanding the migration of carbon atoms in iron as permanent elastic dipoles, has such great significance in technology. The theory of paraelasticity gives the possibility of exploring this migration quantitatively.

It will be proposed that the lattice defects, which appear to be macroscopically pointlike, can be mechanically characterized by their polarizability (dipole strength, resp.). Both quantities are accessible to experimental measurement. The mechanical properties of lattice defects are largely determined by these quantities. One obtains very simple and far-reaching formulas for the interaction of lattice defects with each other and with the internal stress fields of the crystal one obtains in this way, which already allows us to resolve a great number of important practical problems.

By far, the most important lattice defect is the dislocation. The mechanical theory of dislocations relates to the theory of para- and diaelasticity somewhat like the way that the theory of magnetic fields of stationary currents relates to the magnetostatics of material bodies. In particular, any pointlike lattice defect can be formally described as an (induced or permanent) infinitesimal displacement loop (more precisely: three mutually orthogonal dislocation loops), which corresponds to the Amperian assumption of magnetic dipoles as elementary current loops. This is discussed in detail in another place [**8**]. Thus, in the treatment of para- and diaelasticity in chapter IV we shall predominantly think of lattice defects that appear macroscopically pointlike.

II. The geometrical theory

The body that is explored in this work is the so-called Cosserat continuum, which differs from the bodies that usually appear in elasticity theory by the appearance of a measurable geometric structure. This takes the form: At each point of the medium one finds three non-coplanar, but not necessarily mutually orthogonal, distinguished directions. It is often useful to represent this continuum as a cubic crystal with vanishing lattice constants.

For the sake of simplicity, we shall choose the initial state to be the ideal state, in which the distinguished directions are parallel to each other over the entire body. Since it is not the directions themselves, but the *curvature* of the geometric structure that determines the state of the body, this particular choice of initial state represents no loss of generality in our formulas. In particular, this is also true without modification when the initial state is an undisturbed polycrystal.

In this chapter, we almost exclusively use the Eulerian notation, i.e., in terms of final coordinates, which is particular important for applications. We next (§ 4) give the reduced theory (nine degrees of freedom) that was defined in § 1. We will then go down the path that we did earlier in the linear theory. In the case of finite deformations, the required separation of the distortion into strain and rotation is no longer possible through the decomposition of the distortion tensor into symmetric and antisymmetric parts. Whenever we are called upon to make detailed geometrical considerations, we shall employ an artifice that seems somewhat unmotivated, but finds its justification in the differential geometric statement of the theory (§ 5). § 6 presents the detailed proof of the identity of the dislocation density and Cartanian torsion, which was regarded as a foundation in the general theory. In § 7, we qualitatively explain the concepts of the general theory, which get their mathematical form in §8. In § 9, we discuss, *inter alia*, the connection with the linear theory.

§4. The elementary statement of the reduced theory

In the reduced theory, we will consider three states for a medium:

The ideal – or initial – state ($\mathfrak{k}$),

The natural – or intermediate – state (κ),

The deformed – or final – state (k).

Let ($\mathfrak{k}$), (κ), (k) also symbolically denote the coordinate system that is used on the respective state. Let $dx^{\mathfrak{k}}$, dx^{κ}, dx^{k} be the distances between two points inside a mass element. We shall therefore deal with things around the same material points that ensue in all three states. The squares of their intervals are:

$$ds^2_{(\mathfrak{k})} = b_{\mathfrak{k}\mathfrak{l}}\, dx^{\mathfrak{k}}\, dx^{\mathfrak{l}}, \qquad ds^2_{(\kappa)} = g_{\kappa\lambda}\, dx^{\kappa} dx^{\lambda}, \qquad ds^2_{(k)} = a_{kl}\, dx^{k}\, dx^{l}, \tag{1}$$

(summation convention!), resp. The connection between the three states is given by the relations ([8]):

$$\begin{aligned} d\chi^{\kappa} &= A^{\kappa}_{\natural}\, dx^{\natural}, & dx^{\natural} &= A^{\natural}_{\kappa}\, dx^{\kappa}, \\ d\chi^{k} &= A^{k}_{\kappa}\, dx^{\kappa}, & dx^{\kappa} &= A^{\kappa}_{k}\, dx^{k}, \\ d\chi^{\natural} &= A^{\natural}_{k}\, dx^{k}, & dx^{k} &= A^{k}_{\natural}\, dx^{\natural}, \end{aligned} \tag{2}$$

in which the *A*-quantities refer to the transformations between the states. We shall refer to them as distortions (= combinations of strains and rotations) and they are almost precisely characterized. There are inversion relations between them:

$$\begin{aligned} A^{\kappa}_{\natural} A^{\natural}_{\kappa} &= \delta^{\natural}_{\natural}, \quad & A^{\kappa}_{\natural} A^{\natural}_{\kappa} &= \delta^{\kappa}_{\lambda}, \quad & A^{k}_{\kappa} A^{\kappa}_{l} &= \delta^{k}_{l}, \quad & A^{k}_{\kappa} A^{\lambda}_{k} &= \delta^{\lambda}_{\kappa}, \\ & & A^{\natural}_{k} A^{l}_{\natural} &= \delta^{l}_{k}, \quad & A^{\natural}_{\kappa} A^{k}_{\natural} &= \delta^{\natural}_{l}. \end{aligned} \tag{3}$$

We shall now clarify the three states. They are also used in the linear theory [**8**], in which one can find a greater degree of unity and more illustrations.

($\natural$): The ideal state shall correspond to a stress-free ideal crystal (with vanishing lattice constants). In this state the distinguished directions are the same everywhere throughout the body, so the structure curvature is null.

(κ): The natural state is a hypothetical state that is attained from the ideal state as follows: First of all, a stress-free *imprinted* or *plastic* distortion $A^{\kappa}_{\natural}$ is associated with each mass element of the body, which is realized by an arrangement of dislocations and their migration. For the duration of the distortion, the mass elements shall be enumerated and remain distinct from each other. If one performs an arbitrary variation $A^{\kappa}_{\natural}$ of an element to another element then the mass element will no longer pass to the other one in an unbroken manner, in general, under the distortion ([9]). The form of the individual mass elements will indeed vary under plastic deformation, but, however, its state and orientation will remain the same, which was explained in detail before [**8**]. It is important to remark that the distinguished directions are all, by definition, Euclidian parallel in the intermediate state, as well; otherwise, it would no longer enter into our analysis of the position of individual mass element. One considers a so-called anholonomic system for the coordinate system (κ) on this state; dx^{κ} then means something, even though there are no coordinates x^{κ} ([10]).

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[8] Although it seems very rash to use the same kernel symbol *A* for six different transformations, in fact, the indices are given in such a way that the change should be unambiguous.

[9] The equivalent situation is admissible when one regards the natural state as an aggregate of plastically distorted mass elements, viz., that the isolated mass elements are brought into a hypothetical non-Euclidian space in a stress-free manner, and in this space they no longer combine with each other in an unbroken fashion. With this assumption, we can eliminate the possibility that the mass elements will break apart upon collision during the distortion. One can regard the individual mass elements of the aggregate as (material) Euclidian spaces that are tangent to the corresponding points of the (material) non-Euclidian space of the total body.

[10] If one adopts the non-Euclidian standpoint that was taken in footnote 9 then one has a continuous body, and one can describe it by a (non-Euclidian) coordinate system x^{k}. One then calls this coordinate

(*k*): Finally, the deformed state is obtained form the ideal state when one executes the aforementioned migration of the dislocations in Euclidian space without intersections of the bodies. The constraint that one must remain in Euclidian space insures that plastic distortions $A_{\mathfrak{k}}^{\kappa}$ can be composed with elastic distortions A_{κ}^{k}. (In general, we ignore the particular choice of mass forces, which can be eliminated in a trivial way.)

In summation, the distortion of a mass element from an initial state to a final state is represented by the composition of a plastic distortion ($A_{\mathfrak{k}}^{\kappa}$) and an elastic one (A_{κ}^{k}):

$$A_{\mathfrak{k}}^{k} = A_{\mathfrak{k}}^{\kappa}\, A_{\kappa}^{k}. \tag{4}$$

The reciprocal distortions are compositions:

$$A_{k}^{\mathfrak{k}} = A_{\kappa}^{\mathfrak{k}}\, A_{k}^{\kappa}. \tag{5}$$

For the fundamental geometric law of the theory we propose: Neither tears nor folds can occur during the distortion $A_{\mathfrak{k}}^{k}$ from an initial state to a final one. In order for this to be true, it is necessary and sufficient that a macroscopically constant displacement field exists that takes the points of the initial state to those of the final state (so there is also a constant displacement field that takes them back to the initial state). The existence of the "forward displacement" implies (cf. footnote 10):

$$\partial_{\mathrm{m}} A_{\mathrm{l}}^{\kappa} - \partial_{\mathrm{l}} A_{\mathrm{m}}^{\kappa} = 0, \tag{6}$$

and likewise, the "reverse displacement" implies:

$$\partial_{m} A_{l}^{\mathfrak{k}} - \partial_{l} A_{m}^{\mathfrak{k}} = 0. \tag{7}$$

In eqs. (6) and (7), we have two statements for the fundamental geometrical law before us. The former refers to the initial state; the latter refers to the final state. We will soon discover more ways of formulating these laws.

Relations of the type (6, 7) are valid only for the total distortion, since it is only for this notion that the connection between the bodies is fundamentally true. The basis for a possible connecting disturbance in pure plastic distortion is the introduction of dislocations. There are essentially two definitions of dislocations: one differential and one integral. In the former case, the dislocation is defined as the boundary curve between a shifted (i.e., plastically displaced) region of a planar net and a non-shifted one, and is described by a shift vector and a tangent vector. In the latter case, the definition comes

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system *anholonomic relative to the system* $x^{\mathfrak{k}}$, because there are no transformations that take $x^{\mathfrak{k}}$ to x^{k} (cf. SCHOUTEN [**39**]). One has only the anholonomic Pfaffian transformation $dx^{\kappa} = A_{\mathfrak{k}}^{\kappa}\, dx^{\mathfrak{k}}$. The condition that x^{κ} be holonomic relative to $x^{\mathfrak{k}}$ obviously reads $\partial_{\mathrm{m}} A_{\mathrm{l}}^{\kappa} - \partial_{\mathrm{l}} A_{\mathrm{m}}^{\kappa} = 0$, which is also the condition for the mass elements to remain distinct under plastic distortion.

about with the help of the so-called Frank-Burgers loops around the dislocation. The differential definition is more appropriate to the instantaneous representation, whereas the integral definition is closer to the spirit of differential geometric considerations.

We will not comment on the details here, which were discussed in [**8**], but simply define the dislocation density $\alpha_{\mathrm{m}\mathfrak{l}}{}^{\kappa}$ ($\alpha^{\mathfrak{n}\kappa}$, resp.) to be the measure of the total current under pure plastic deformation, by way of:

$$\alpha_{\mathrm{m}\mathfrak{l}}{}^{\kappa} = -\,\varepsilon_{\mathrm{m}\mathfrak{l}\mathfrak{n}}\,\alpha^{\mathfrak{n}\kappa}/2 \equiv (\partial_{\mathrm{m}} A_{\mathfrak{l}}^{\kappa} - \partial_{\mathfrak{l}} A_{\mathrm{m}}^{\kappa})/2. \tag{8}$$

(In $\alpha^{\mathfrak{n}\kappa}$, $\mathfrak{n}$ refers to the direction of the curve and $-\kappa$ refers to the direction of the shift.) When this definition is applied to a singular dislocation it is identical with the definition of the boundary curve above ([11]). If one passes to the linear approximation, in which one sets $A_{\mathfrak{l}}^{\kappa} = \delta_{\mathfrak{l}}^{\kappa} + \beta^{P}{}_{\mathfrak{l}}^{\kappa}$, then, by neglecting the terms that are quadratic in the β^{P}s, one obtains the definition that was given in [**8**] in the form:

$$\boldsymbol{\alpha} \equiv -\,\mathrm{Rot}\,\boldsymbol{\beta}^{P}. \tag{9}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}^{P}$ are the (second rank) dislocation density tensor and the plastic deformation tensor, resp.

The form (8, 9) of the definition as the rotation of the plastic distortion is very incisive. As far as applications are concerned, (8) therefore has the drawback that the initial and intermediate states appear in the indices of $\alpha_{\mathrm{m}\mathfrak{l}}{}^{\kappa}$. We shall mostly use a representation that is expressed completely in terms of the final state (i.e., an Eulerian one). The conversion gives ([12]):

$$\alpha_{ml}{}^{\kappa} \equiv A_m^{\mathrm{m}} A_{\mathfrak{l}}^{l} A_{\kappa}^{k}\,\alpha_{\mathrm{m}\mathfrak{l}}{}^{\kappa} \equiv -\,A_{\mathfrak{k}}^{k}\,(A_l^{\kappa}\partial_m A_{\kappa}^{\mathfrak{k}} - A_m^{\kappa}\partial_l A_{\kappa}^{\mathfrak{k}})/2\,. \tag{10}$$

We will use this form for the definition when we now extend the statement (7) of the fundamental law by introducing the notion of dislocations. Thus, we substitute $A_l^{\mathfrak{k}}$, $A_m^{\mathfrak{k}}$ as in eq. (5) into eq. (7) and multiply by $A_{\mathfrak{k}}^{k}/2$, and obtain:

$$A_{\kappa}^{k}(\partial_m A_l^{\kappa} - \partial_l A_m^{\kappa})/2 + A_{\mathfrak{k}}^{k}(A_l^{\kappa}\partial_m A_{\kappa}^{\mathfrak{k}} - A_m^{\kappa}\partial_l A_{\kappa}^{\mathfrak{k}})/2 = 0, \tag{11}$$

and by means of (10) we obtain following statement of the fundamental geometric equations, as extended by the notion of dislocations:

$$A_{\kappa}^{k}(\partial_m A_l^{\kappa} - \partial_l A_m^{\kappa})/2 = \alpha_{ml}{}^{\kappa}. \tag{12}$$

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[11] Note: the boundary curve of a surface is its "vortex line."

[12] Write $A_{\kappa}^{k} = A_{\mathrm{k}}^{k} A_{\mathrm{k}}^{\kappa}$ and partial differentiate ($A_{\kappa}^{\mathfrak{k}}\partial_{\mathrm{m}} A_{\mathfrak{l}}^{\kappa} = -\,\partial_{\mathrm{m}} A_{\kappa}^{\mathfrak{k}} A_{\mathfrak{k}}^{\kappa}$, etc.). Furthermore, set $A_m^{\mathrm{m}}\partial_{\mathrm{m}} = \partial_m$ ([**39**], pp. 70).

Upon linearizing ($A^k_\kappa = \delta^k_\kappa + \beta^k_\kappa$, $A^\kappa_l = \delta^\kappa_l - \beta^\kappa_l$), it reads:

$$\text{Rot}\,\boldsymbol{\beta} = \boldsymbol{\alpha}.$$

The application of this equation to practical problems – in particular, the determination of the stresses for a given distribution of dislocations – is obstructed by the fact that the elastic *distortions* enter into eq. (12), whereas the elastic *deformations* are linked with the stresses by the material law. With the help of eq, (12), we succeed in deriving an equation that includes only deformations and dislocation densities, and proves to be suitable for the determination of the stresses. For this, we apply the trick that we mentioned in the introduction to this chapter. At an initial point, we have the identity:

$$\begin{aligned} B_{mlk} \equiv A_{k\kappa}\,\partial_m A^\kappa_l \equiv A_{k\kappa}(\partial_m A^\kappa_l + \partial_l A^\kappa_m)/2 + A_{k\kappa}(\partial_m A^\kappa_l - \partial_l A^\kappa_m)/2 - \\ - A_{l\kappa}(\partial_k A^\kappa_m - \partial_m A^\kappa_k)/2 + A_{l\kappa}(\partial_k A^\kappa_m - \partial_m A^\kappa_k)/2 + \\ + A_{m\kappa}(\partial_l A^\kappa_k - \partial_k A^\kappa_l)/2 - A_{m\kappa}(\partial_l A^\kappa_k - \partial_k A^\kappa_l)/2. \end{aligned} \tag{13}$$

From (12), the terms in the right-hand side are equal to:

$$\alpha_{mlk} + \alpha_{kml} - \alpha_{lkm} \equiv h_{mlk}. \tag{14}$$

We convert the first three of the six summands on the left-hand side by partial differentiation, e.g.:

$$A_{k\kappa}\,\partial_m A^\kappa_l = \partial_m(A_{k\kappa}\,A^\kappa_l) - A^\kappa_l\,\partial_m A_{k\kappa} = \partial_m g_{kl} - A_{l\kappa}\,\partial_m A^\kappa_k. \tag{15}$$

Here, we have defined g_{kl} by:

$$ds^2_{(\kappa)} \equiv g_{kl}\,dx^k\,dx^l, \tag{16}$$

and the relation that we used:

$$A_{k\kappa}\,A^\kappa_l = g_{kl}, \tag{17}$$

follows easily by equating (16) with (1) while using (2). Ultimately, what remains is ([13]):

$$B_{mlk} = (\partial_m g_{kl} + \partial_l g_{mk} - \partial_k g_{lm})/2 + h_{mlk} \equiv \Gamma_{mlk}. \tag{18}$$

As in [**19**], we now define the expressions:

$$\begin{aligned} g'_{mlk} &\equiv (\partial_m g_{kl} + \partial_l g_{mk} - \partial_k g_{lm})/2, \\ g_{mlk} &\equiv (\nabla_m g_{kl} + \nabla_l g_{mk} - \nabla_k g_{lm})/2. \end{aligned} \tag{19}$$

These definitions shall also be valid when one replaces g with a, b, or ε everywhere. Moreover, we use quantities a^{hk} and g^{hk}, which are defined by:

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[13] Eq. (18) may also be obtained from the fundamental equation by means of (17). If one assumes that the distortions that take (κ) to (k) can be decomposed into only elastic distortions (no quasi-plastic distortions, as in § 8) then one can (indeed, somewhat artificially) regard the relation (17) as representing the choice of connection and eq. (18) replaces eq. (12) as a complete expression for the fundamental law. We will pursue this further in the general theory (§ 8).

$$a^{hk}a_{hl} \equiv \delta^k_l\,, \qquad\qquad g^{hk}g_{hl} \equiv \delta^k_l\,. \tag{20}$$

Let ∇_n be the symbol for the covariant derivative in the deformed state relative to the metric a_{kl} ([14]). Ultimately, we define the Eulerian – i.e., in terms of the final state – deformations in the nonlinear theory in the usual way by:

$$ds^2_{(k)} - ds^2_{(\mathfrak{k})} \equiv 2\overset{G}{\varepsilon}_{kl}\, dx^k dx^l, \qquad ds^2_{(\kappa)} - ds^2_{(\mathfrak{k})} \equiv 2\overset{P}{\varepsilon}_{kl}\, dx^k dx^l, \qquad ds^2_{(k)} - ds^2_{(\kappa)} \equiv 2\varepsilon_{kl} dx^k dx^l. \tag{21}$$

We thus obtain:

$$\overset{G}{\varepsilon}_{kl} = (a_{kl} - b_{kl})/2, \qquad \overset{P}{\varepsilon}_{kl} = (g_{kl} - b_{kl})/2, \qquad \varepsilon_{kl} = (a_{kl} - g_{kl})/2 \tag{22}$$

for the total deformation, the plastic deformation, and the elastic deformation tensors, resp. Thus, we have:

$$g'_{mlk} = a'_{mlk} - 2\,\varepsilon'_{mlk}\,. \tag{23}$$

Now, let us return to eq. (18). One can easily calculate that the expression:

$$B_{ij} \equiv \tfrac{1}{2}\varepsilon^{jnm}\varepsilon^{jlk}(\partial_n B_{mlk} - g^{pq}B_{nkq}\,B_{mlp}) \tag{24}$$

vanishes identically, from which one has:

$$\Gamma^{ij} \equiv \tfrac{1}{2}\varepsilon^{jnm}\varepsilon^{jlk}(\partial_n \Gamma_{mlk} - g^{pq}\,\Gamma_{nkq}\,\Gamma_{mlp}) = 0. \tag{25}$$

It remains to be shown [**19**] that the antisymmetric part of these equations is identical with the following divergence condition, which follows from the definition of dislocation density:

$$\nabla_n \alpha^{n\kappa} = 0, \tag{26}$$

which says that dislocations cannot end inside of a continuum.

If one constructs the symmetric part of the tensor equation (25) ([15]) then, when one replaces ∂_n with ∇_n, as in footnote 13 ([16]), one obtains the *fundamental equations for determining the internal stresses in the reduced theory* [19]:

$$\Gamma^{(ij)} \equiv \tfrac{1}{2}\{\,\varepsilon^{jnm}\varepsilon^{jlk}\,[\nabla_n(-2\varepsilon_{mlk} + h_{mlk}) - \\ - g^{pq}(-2\varepsilon_{nkq} + h_{nkq})\,(-2\varepsilon_{mlp} + h_{mlp})]\}_{(ij)} = 0. \tag{27}$$

For $h_{mlk} = 0$ these are the well-known nonlinear compatibility conditions for elastic deformations [**17**, **40**]. In linearized form [**8**], the right-hand side of eq. (27) can be written:

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[14] For example, $\nabla_n P^{kl}_{..m} = \partial_n P^{kl}_{,,m} + a'_{np.}{}^{k} P^{pl}_{..m} + a'_{np.}{}^{l} P^{kp}_{..m} - a'_{nm.}{}^{p} P^{kl}_{..p}$.

[15] When we put two indices inside round (square, resp.) brackets, we intend that one should take the symmetric (anti-symmetric, resp.) part.

[16] If one chooses to describe the final state in Cartesian coordinates then one has $a'_{nk.}{}^{p} = 0$ and $\nabla_n = \partial_n$.

$$\text{Inc}\ \boldsymbol{\varepsilon} \equiv \nabla\times\boldsymbol{\varepsilon}\times\nabla = (\boldsymbol{\alpha}\times\nabla)^S \equiv \boldsymbol{\eta} \tag{28}$$

(for Inc, one should read: "the incompatibility in").

We now briefly suggest the use of these equations ([**8**, **19**], cf., also § 8)(*). With the help of the introduction of stress functions, the equilibrium conditions for the stresses are satisfied identically when one neglects bulk forces. By using the material law ε_{kl} (ε_{mlk}, resp.) will be expressed in terms of the stress functions in such a way that the fundamental equations of internal stresses become equations that include only stress functions and dislocation densities. The solution of these equations for given boundary conditions gives the stress functions, and thus gives the stresses and elastic deformations, as well. As we will show in § 5, if the physical significance of the h_{mlk} is the same as the Cosserat-Nye structure curvatures then it is trivially given for a given dislocation density. With that, the problem that was expressed by (2) on pp. 4 in the context of the reduced theory is solved.

At this point, we would like to leave behind the elementary statements of the theory and go into the differential geometric considerations.

§ 5. The differential geometric statement of the reduced theory

The following situation is typical for the application of general differential geometry in continuum mechanics: Let $B^{\mathfrak{k}}(1)$ be a vector that connects two points inside the mass element 1 in the initial state. Let $B^{\mathfrak{k}}(2)$ be a vector of the same length that is parallel to it at a neighboring point 2. We let B^k denote the vector that connects the same two points in the final state that $B^{\mathfrak{k}}$ does in the initial state, which makes it the image of $B^{\mathfrak{k}}$. If one can then give a value to dB^k for all neighboring elements in the final state then one also knows the total torsion that the body has experienced.

dB^k will be proportional to the vector B^k, and also to the distance dx^m between the points to which B^k is referred, i.e.:

$$dB^k = -\, b'_{ml.}{}^{k}\, B^l\, dx^m. \tag{29}$$

Analogously, if one considers a vector C^κ in the natural state, which we now regard as a (material) non-Euclidian space, as in footnote 9, 10, and which has an image C^k in the deformed state, then one obtains:

$$dC^k = -\, \Gamma_{ml.}{}^{k}\, C^l\, dx^m \tag{30}$$

or also ([17]):

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* *Ed. note:* Unfortunately, the quadratic elasticity law (54) that was used in [**19**] for the computation of the stress field of screw and step dislocations contained an error. Therefore, slight corrections are necessary in the formulas of §§ 5, 6 of [**19**] in order to make them agree with the treatment of the problems of screw and step dislocations with boundary conditions (cf., most likely, H. PFLEIDERER, et al., Z. Naturforschung **15**a (1960)).

[17] We will write the distinguished index in the connection (always the third one) as either an upper or lower one, as it suits us. For the raising and lowering of indices one must use the correct metric tensor, e.g., $b'_{mlk} = b_{kh}\, b'_{ml.}{}^{k}$, but $\Gamma_{mlk} = g_{kh}\, \Gamma_{ml.}{}^{h}$. Hence, we have $b_{kh} = A^{\mathrm{k}}_{k} A^{\mathrm{h}}_{h}\ b_{\mathrm{kh}}$, $g_{ki} = A^{\kappa}_{k}\, A^{\prime}_{i}\, g_{\kappa\tau}$. When in doubt, we shall give the metric tensor explicitly.

$$dC_k = -\Gamma_{mlk} C^l dx^m. \tag{31}$$

If one picks two vectors C^k at two neighboring points that are separated by dx^m and satisfy eq. (30), and are (non-Euclidian) parallel relative to $\Gamma_{ml.}{}^k$, then the covariant differential:

$$\delta C^k = dC^k + \Gamma_{ml.}{}^k C^l dx^m \tag{32}$$

vanishes.

We will see that Γ_{mlk}, just like the elastic deformations, implies the structure curvature, which characterizes the desired extent of the final state of the body. The quantities Γ_{mlk} (h'_{mlk}, resp.) are called a *linear* or *affine connection* (also an *affinity*). The affine connection represents the central quantities of the part of general differential geometry that we require. We shall now occupy ourselves with its properties.

One can already reach important conclusions about the shape of the body from the symmetry properties of Γ_{mlk} alone. Next, if the part $\Gamma_{m[lk]}$ of Γ_{mlk} that is anti-symmetric in l and k always leads to an infinitesimal rotation of the vector C^k while proceeding through dx^m then one must have $C^k dC_k = -\Gamma_{m[lk]} C^l C^k dx^m \equiv 0$. If one examines the distortion of a dreibein under parallel translation through dx^m relative to $\Gamma_{m[lk]}$ then one establishes that it likewise rotates without deformation. On the other hand, the part $\Gamma_{m(lk)}$ of Γ_{mlk} that is symmetric in l and k contributes a pure deformation of the dreibein. For a given connection there thus exists the simple possibility of separating the deformation and rotation parts additively, which therefore means that for the connection in question the *difference* between the deformations (rotations, resp.) of two neighboring elements is infinitesimal.

Since the connection Γ_{mlk} is a triply indexed quantity, it can possess principally 27 functional degrees of freedom in three dimensions. This number is reduced to 15 when one reduces to a metric space, that being the type of space in which the distance between two arbitrary points can be defined. Up until now, these are the only spaces that have appeared in continuum mechanics (see § 9).

In the textbooks on differential geometry, one learns that the most general metric connection Γ_{mlk} has the form:

$$\Gamma_{mlk} \equiv h'_{mlk} + h_{mlk}, \tag{33}$$

with the abbreviations:

$$h'_{mlk} \equiv (\partial_m g_{kl} + \partial_l g_{mk} - \partial_k g_{lm})/2, \tag{34}$$

$$h_{mlk} \equiv \Gamma_{[ml]k} + \Gamma_{[km]l} - \Gamma_{[lk]m}. \tag{35}$$

The meaning of g_{kl} then follows from:

$$ds^2_{(\kappa)} \equiv g_{kl} dx^k dx^l. \tag{36}$$

($ds_{(k)}$ is the distance between two points in the natural state, whose corresponding length in the deformed state is characterized by dx^k. Therefore:

$$\varepsilon_{kl} = -(a_{kl} - g_{kl})/2 \tag{37}$$

is the (Eulerian) deformation tensor that we already used in § 4.

The relative rotation between the mass elements, which is described by h_{mlk}, exists independently of the elastic deformations of the body, which can be completely described by the h'_{mlk} part of Γ_{mlk} by means of (37). In particular, the deformations can vanish, i.e., $\varepsilon'_{mlk} = 0$, while the rotations persist. If one considers the dreibein that gets translated by the relation (31) to be the *trièdre mobile* of the bros. COSSERAT then one immediately finds the relation: The h_{mlk} of Γ_{mlk} describes the relative rotation of the Cosserats (the structure curvature that results from it, resp.).

For CARTAN [**41**], the part $\Gamma_{[ml]k}$ of a connection Γ_{mlk} that is anti-symmetric in l and k is called the torsion. It has the transformation properties of a third rank tensor, although Γ_{mlk} itself transforms by a complicated formula.

We will derive the relation:

$$\Gamma_{[ml]k} \equiv \alpha_{mlk} \tag{38}$$

in § 6, i.e., the Cartanian torsion is identical with dislocation density. However, by (35), the previous relation (14) follows from this immediately. Consequently, the Cosserat structure tensor will be directly connected with the dislocations; they cannot be realized without dislocations. NYE [**30**] has described these curvatures with the help of the tensors:

$$K^n{}_{.m} \equiv - \varepsilon^{nlk} h_{mlk}/2, \tag{39}$$

which are equivalent to h_{mlk}, and were introduced by the definition:

$$d\theta^n \equiv K^n{}_{.m}\, dx^m, \tag{40}$$

in which $d\theta^n$ is the relative rotation between two neighboring elements. GÜNTHER has established the relationship between this and the work of Cosserat. It is very impressive to see the simple manner in which curvature appears in its modern differential geometric representation when it was so complicated in the book by the Cosserats. Since the structure curvature is given when the dislocation density is likewise given, the main problem is the determination of the internal stresses that are associated with the dislocations.

These are very closely connected with the h'_{mlk}, which are a measure of the elastic deformation. From the meaning of the h'_{mlk} we can therefore confine ourselves to the consideration of the expression:

$$- (\partial_m \varepsilon_{kl} + \partial_l \varepsilon_{mk} - \partial_k \varepsilon_{lm}). \tag{41}$$

The first summand is symmetric in k, l, and thus gives rise to a pure deformation, namely:

$$dC_k^{(\mathrm{def})} = C^l\, d\varepsilon_{kl}. \tag{42}$$

Therefore, the rest of the expression, namely, $- (\partial_l \varepsilon_{mk} - \partial_k \varepsilon_{lm})$, is anti-symmetric in k, l and gives rise to a pure rotation. It becomes:

$$dC_k^{(\mathrm{rot})} = C^l(\partial_l \varepsilon_{mk} - \partial_k \varepsilon_{lm})\, dx^m = \varepsilon_{lkh}\, C^l(\varepsilon^{fkh}\, \partial_f\, \varepsilon_{gm}\, dx^m), \tag{43}$$

in which the expression in brackets in the right-hand side is to be interpreted as a rotation of C^l around dx^m. (In the case where ε_{gm} is a displacement field – hence, $\varepsilon_{gm} = (\nabla_g s_m + \nabla_m s_g)/2$ – this expression becomes simply $d(\mathrm{rot}\, \mathfrak{s})$.)

The aforementioned rotations (the associated curvatures, resp.) are coupled directly with the elastic deformations; they vanish under relaxation (perhaps under fracture). Correspondingly, one certainly has load stresses in the geometrically simpler theory in which the deformations, as well as the rotations, are fixed. One must carefully distinguish the curvatures that are associated with (43) from the (macroscopically) stress-free structure curvatures that were discussed above.

If one covariant differentiates the connection $\Gamma_{ml.}^{\ \ k}$ (Γ_{mlk}, resp.) relative to the distinguished index k (and similarly for $\Gamma_{ml.}^{\ \ k}$), and if one takes the anti-symmetric part (symbol $[n\ m]$) of the result then one obtains the Riemann-Christoffel curvature tensor:

$$\Gamma_{nml.}^{\ \ \ k} \equiv 2[\partial_n \Gamma_{ml.}^{\ \ k} + \Gamma_{np.}^{\ \ k}\, \Gamma_{ml.}^{\ \ p}]_{[n\,m]}, \tag{44}$$

$$\Gamma_{nmlk} \equiv 2[\partial_n \Gamma_{mlm} - g^{pq}\Gamma_{nkp}\, \Gamma_{mlp}]_{[n\,m]}, \tag{45}$$

which is a very important quantity in the theory. In the case of a metric connection Γ_{nml} one has that Γ_{nmlk} is anti-symmetric in not only its first pair, but also its second, whereas it is symmetric in the exchange of n,m with l,k only under restrictive conditions. Due to the anti-symmetry thus obtained, the Einstein tensor Γ^{ij}, which is constructed by way of ([18]):

$$\Gamma^{ij} \equiv \tfrac{1}{4}\, \varepsilon^{jmn} \varepsilon^{ilk}\, \Gamma_{nmlk}, \tag{46}$$

is thus equivalent to the tensor Γ_{nmlk}. For Γ^{ij}, one has the divergence condition (SCHOUTEN [**39**], pp. 146):

$$\nabla_i \Gamma^{ij} = \varepsilon_{mlk}\, \alpha^{kl} \Gamma^{mj}, \tag{47}$$

which can also be written as a condition on Γ_{nmlk}, and is also called the Bianchi identity. The right-hand side of (47) vanishes in a linearized theory.

A well-known theorem of differential geometry says: If the curvature tensor Γ_{nmlk} vanishes then one has the following form for the connection Γ_{mlk}:

$$\Gamma_{mlk} = A_{k\kappa} \partial_m A_l^{\kappa}. \tag{48}$$

In the case where the identity (38) is valid, the anti-symmetric part of eq. (48) is the fundamental geometric equation ([19])(cf. (14) and (27)), and the equation:

$$\Gamma^{(ij)} = 0, \tag{49}$$

[18] This formula is valid only in three dimensions. The definition of the Einstein tensor that is valid for spaces of arbitrary dimensions is obtained from the so-called Ricci tensor; one can find this definition in any textbook on differential geometry and relativity theory.

[19] It is easy to show that one must still understand A_{λ}^{k} to be the elastic distortion, cf. §6.

in which we have replaced Γ_{mlk} by way of eq. (33, 34, 37), is the fundamental equation for the determination of proper stresses (cf., (14) and (27)). Once again, the equation:

$$\Gamma^{[ij]} = 0, \tag{50}$$

is the divergence condition for the dislocation density. Thus, the relationship between the elementary statements of the theory and the differential geometric ones is largely achieved.

Eq. (49) defines a coupling between the curvature part h_{mlk} and the deformation part g'_{mlk} of the connection Γ_{mlk}, which leads to stresses for a given dislocation density.

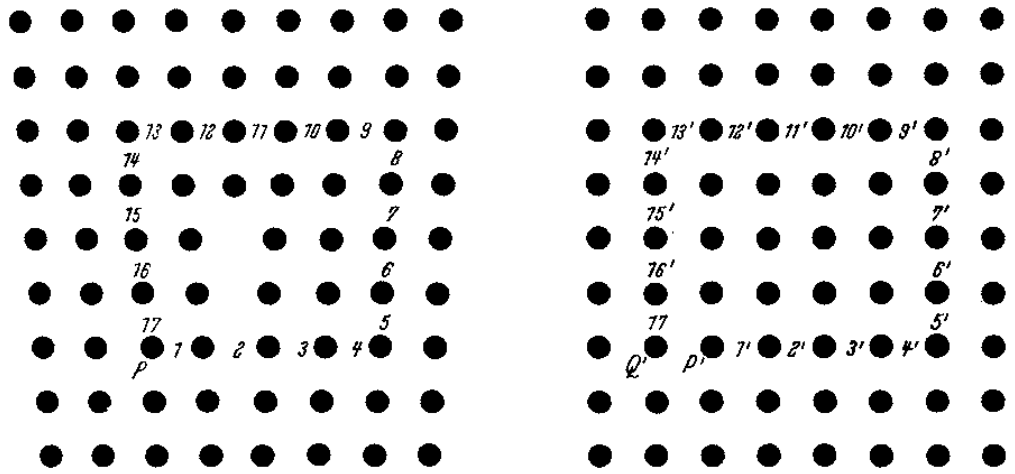
Apparently, eq. (48) (eq. (49, 50), resp.) appears to be a requirement that turns into the fundamental geometric equation (12). We must now ask what would give us the right to set Γ_{nmlk} (Γ^{ij}, resp.) equal to zero. This question did not turn up anywhere in the elementary treatment of the theory, since eq. (25) gave the evolution of the shape by itself in the presentation that was developed there. At this point, one recognizes that the differential geometric approach is the most general one, and it raises the question of the physical meaning of a non-vanishing curvature tensor (Einstein tensor, resp.). KONDO, in addressing this question [**20**, **21**], spoke of it in connection with "curvature defect points," as opposed to "torsion defect points" (dislocations), which begs the question of the physical meaning of the curvature defect points.

In contrast to KONDO (although it is really only a superficial distinction), BILBY and SMITH [**24**], on the basis of similar notions, were led to the relation $\Gamma_{nmlk} = 0$, which they interpreted as the physical condition for a crystal lattice to be defined at each point of the medium. We will discuss these important questions in § 6-9. In the meantime, in the next section, we shall show the identity between the dislocation density and Cartanian torsion, which defines the foundation of the differential theory of dislocations.

§ 6. The identity between dislocation density and Cartanian torsion ([20])

In this section we shall show that the Frank-Burgers loop [**42**] that is often used in the (integral) definition of a dislocation is identical with the Cartan loop that was known in differential geometry long before the notion of a dislocation was introduced. Therefore, we shall now briefly describe the Frank definition of a dislocation.

[20] For this section, cf., BILBY [**26**], in particular.



a b

Fig. 2a and 2b. The Frank definition of a dislocation in a crystal.

Fig. 2a shows a real crystal (*a*) with a dislocation (which corresponds to our deformed state). The numbers define a closed loop in it that begins and ends at the point *P* and encircles the dislocation. In the ideal crystal (*b*) in Fig. 2b (which has nothing to do with our initial state) let P' be a point of the lattice that corresponds to *P*, and follow the corresponding sequence of steps (primed numbers); each step in *a* then corresponds to a step in *b*. At the step that closes the loop in *a* there remains a remainder $\overline{Q'P'}$ between it and the lattice point P', the so-called closure defect. Frank then defined the dislocation with the help of the "Burgers vector."

We would now like to express the closure defect with the help of the connection. For this, we consider the real crystal *a* from a non-Euclidian standpoint in which we will pass over to the Cartan basis. It will be assumed that two vectors at two points that are dx^m apart differ from each other by:

$$dC^k = -\Gamma_{ml}^{\ k} C^l dx^m, \tag{51}$$

and are of equal length, as well as parallel to each other (BILBY, BULLOUGH, and SMITH [**23**]). From this standpoint, the real crystal becomes a non-Euclidian ideal crystal, i.e., its smallest lattice vectors are all the same length and are parallel when they have the same *k* (k = 1, 2, 3). (We do not need to omit the singular value at the center of the dislocation since the loop does not exist there.)

We now give a somewhat different description of the Frank loop that one can immediately recognize is equivalent to the previous one (for this, cf. SCHOUTEN [39], pp. 127). We imagine that the ideal crystal lies over the real crystal *b* in such a way that the point P' lies over the point *P*, and the ideal lattice *b* is tangent to the real lattice *a*. The first step of the loop makes the point 1′ cover 1 in such a way that *b* is now tangent to *a* at 1. We symbolically call this arrangement (1′ 1). The second step of the loop brings *b* into the arrangement (2′ 2), which is notated analogously to (1′ 1). We now wish to calculate the position of the point P' in the corresponding arrangement; indeed, we use a Cartesian coordinate system (κ) on the ideal lattice *a* at the respective location with the point of contact (e.g., n' in the arrangement ($n'n$)) as the origin. We then desire to know

the coordinates of the real lattice, instead of the coordinates x^k of the ideal lattice, to the greatest extent possible.

In the (1′ 1)-arrangement, P' has the coordinates $-\underset{1}{d}x^\kappa$ in the aforementioned Cartesian lattice. (The subscript 1 describes the number of the step.) We will see that it is generally permissible to replace $\underset{m}{d}x^\kappa$ with $\underset{n}{d}x^\kappa$ when one has $n = m$.

Under the following displacement of (1′ 1) to (2′ 2) the point 1′ in the Cartesian lattice with its origin at 2′ experiences a change of coordinates $-\underset{2}{d}x^\kappa$ ($-\underset{2}{d}x^k$, resp.) Under this displacement the vector $\overline{1'P'}$, i.e., $-\underset{1}{d}x^k$, is parallel translated to $-\underset{2}{d}x^k$, i.e., it has changed by $d\underset{1}{d}x^k = -\Gamma_{ml.}^{\ k}\ \underset{1}{d}x^l\underset{2}{d}x^m$. Therefore, the coordinates of P' (relative to the point 2′ = 2) are $-\underset{1}{d}x^k - \underset{2}{d}x^k + \Gamma_{ml.}^{\ k}\ \underset{1}{d}x^l\underset{2}{d}x^m$. During the next step, the latter vector is parallel translated over $-\underset{3}{d}x^k$ so one obtains the position vector of P' relative to 3′ as now being $-\underset{1}{d}x^k - \underset{2}{d}x^k - \underset{3}{d}x^k + \Gamma_{ml.}^{\ k}\underset{1}{d}x^l\underset{2}{d}x^m + \Gamma_{ml.}^{\ k}\underset{1}{d}x^l\underset{3}{d}x^m + \Gamma_{ml.}^{\ k}\underset{2}{d}x^l\underset{3}{d}x^m$. (The contributions of degree three in the dx^k have been omitted.)

It is clear how to proceed from here. Once we arrive at the arrangement ($Q'P$) we have traversed a complete Cartan loop. In order to calculate the closure defect $\overline{Q'P'}$ we must deal with a summation that appears to be complicated.

At this point, we make the transition to the continuum theory. We let the lattice constants go to zero and replace the individual dislocations with an arrangement of continuously distributed distributions of infinitesimal strength. We make the loop sufficiently small that we can regard $\Gamma_{ml.}^{\ k}$ as constant in the region in question (a conclusion that one often uses in continuum physics). We then write the closure defect $\overline{Q'P'}$ as:

$$\Delta b^k = \sum_n \underset{n}{d}x^k + \Gamma_{ml.}^{\ k}\left\{\begin{array}{r} \underset{1}{d}x^l\underset{2}{d}x^m + \underset{1}{d}x^l\underset{3}{d}x^m + \underset{1}{d}x^l\underset{4}{d}x^m + \cdots \\ + \underset{2}{d}x^l\underset{3}{d}x^m + \underset{2}{d}x^l\underset{4}{d}x^m + \cdots \\ + \underset{3}{d}x^l\underset{4}{d}x^m + \cdots \\ + \cdots \end{array}\right\}. \qquad (52)$$

Since the first summation is precisely $\sum_n \underset{n}{d}x^\kappa$, one recognizes at this point that no difference exists between the two sums, and to make one go to the other, one need only let the length of the step go to zero (one can perhaps consider the computation of the circumference of a circle with the help of a polygon whose number of vertices goes to infinity). Likewise, one sees that with the aforementioned replacement of $\underset{n}{d}x^\kappa$ with $\underset{n}{d}x^k$ the second sum in (52) can vary only by a term of third order in $\underset{n}{d}x^k$, since the difference between $\underset{n}{d}x^\kappa$ and $\underset{n}{d}x^k$ is only quadratic in $\underset{n}{d}x^k$. Therefore, the latter replacement of $\underset{n}{d}x^\kappa$ with $\underset{n}{d}x^k$ is completely justified.

The first sum in (52) vanishes, and the second one becomes precisely the surface element ΔF^{ml} that the loop bounds (sign consistent with the right-hand screw rule). Due

to the anti-symmetry of ΔF^{ml}, the symmetric part of $\Gamma_{ml.}{}^{k}$ obviously contributes nothing to the closure defect, and what remains is:

$$\Delta b^k \equiv \Gamma_{[ml].}{}^{k}\, \Delta F^{ml}. \tag{53}$$

Since:

$$\Delta b^k \equiv \alpha_{ml.}{}^{k}\, \Delta F^{ml} \equiv \alpha^{nk}\, \Delta F_n \tag{54}$$

is the continuum version of Frank's definition of dislocation we have proved the identity (38) between dislocation density and Cartanian torsion. Thus, we have not actually left the realm of distortion.

According to BILBY, BULLOUGH, and SMITH [**23**], one can (with somewhat less generality) easily express the closure defect in terms of elastic distortion. If one writes the coordinates of P' at the location $(n'n)$ as $-\sum_{i=1}^{n} d_i\, x^{\kappa} \mathfrak{e}_{\kappa}$, in which $\mathfrak{e}_{\kappa}$ is the Cartesian basis vector of the ideal lattice at the location $(n'n)$, $dx^{\kappa} = A_k^{\kappa}\, dx^k$. For a closed loop ($n' = Q'$) one then has:

$$b^{\kappa} \mathfrak{e}_{\kappa} = \overrightarrow{Q'P'} = -\sum_{i} d_i\, x^k A_k^{\kappa} \mathfrak{e}_{\kappa}\,. \tag{55}$$

Since $\mathfrak{e}_{\kappa}$ is Cartesian, we can put it outside of the sum, i.e., we can completely eliminate it from eq. (55). With the help of Stokes's theorem, the transition to continuum theory gives:

$$b^{\kappa} = -\oint dx^k A_k^{\kappa} = \tfrac{1}{2} \iint (\partial_m A_l^{\kappa} - \partial_l A_m^{\kappa})\, dF^{ml}, \tag{56}$$

and when the path encloses a sufficiently small surface element ΔF^{ml}:

$$\Delta b^{\kappa} = \tfrac{1}{2} (\partial_m A_l^{\kappa} - \partial_l A_m^{\kappa}) \Delta F^{ml}\,. \tag{57}$$

The image of Δb^{κ} in the deformed state $\Delta b^k \equiv A_{\kappa}^{k}\, \Delta b^{\kappa}$, which becomes:

$$\Delta b^k \equiv \alpha_{ml.}{}^{k} \Delta F^{ml}, \tag{58}$$

$$\alpha_{ml.}{}^{k} = A_{\kappa}^{k} (\partial_m A_l^{\kappa} - \partial_l A_m^{\kappa})/2. \tag{59}$$

One can regard this as the proof of the assertion in footnote 19 in connection with eq. (48).

In conclusion, we still have to mention a result that is derived in many textbooks on differential geometry or relativity theory and concerns the Riemannian curvature tensor. If one parallel displaces a vector C^k around the boundary of a surface element ΔF^{ml} once then the vector experiences a variation:

$$\Delta C^k = -\tfrac{1}{2} R_{nml.}{}^{k}\, C^l\, \Delta F^{nm}. \tag{60}$$

In our case of a metric connection, R_{nmlk} is anti-symmetric in l, k, so it follows that $\Delta C^k \perp C^l$. If one displaces a dreibein instead of a single vector then one sees that it will be rotated, but not deformed. Thus, one can easily put eq. (60) into the form:

$$\Delta\alpha_{lk} = -R_{nmlk}\,\Delta F^{nm} \tag{61}$$

or in the Einsteinian form (cf. [8], pp. 130) ([21]):

$$\Delta\omega^i = -R^{ij}\,\Delta F_j\,. \tag{62}$$

Therefore, ω_{lk} is the anti-symmetric tensor that describes the rotation and ω^j is the equivalent vector.

The relation (62) leads us to an intuitive meaning for the incompatibility tensor $\boldsymbol{\eta}$ – which is also an Einstein tensor (cf., eq. (86)) – that we used several times in the earlier linear theory of proper stresses. If one excises a surface element ΔF_j that bounds a ring out of a body with proper stresses and then cuts the ring then the final rotation is $\Delta\omega^i = -\eta^{ij}\Delta F_j$ (MORIGUTI [**43**], ESHELBY [44], KRÖNER [**8**], pp. 38).

From (61) it follows: Since the lattice orientation of a crystal, which we also assume has already been deformed, can be defined at any point (in which case, one no longer has a crystal), the Riemann tensor of a connection that describes the *lattice* distortion must vanish (BILBY and SMITH [**24**]).

§ 7. The concepts of the general theory

In § 4 we characterized the deformed state with the help of the elastic distortion tensor field A^k_κ. If one knows A^k_κ then one also knows $A^k_\kappa \partial_m A^m_l$, and from this, the deformations and structure curvatures also follow by way of eqs. (17) and (18). *A priori*, there is no need for reduction in order to be solved for the A^k_κ; thus, they involve nine functional degrees of freedom, in general. From dislocation theory it is known that dislocation lines cannot end in the interior of a body; we expressed this by the divergence condition (26). Thus, the dislocation density has six functional degrees of freedom. The number of degrees of freedom in the causes (dislocation densities, bulk forces) is therefore, as it must be, equal to the number of degrees of freedom in the effects (perhaps characterized by the elastic distortions), namely, nine.

By contrast, the general metric connection has fifteen functional degrees of freedom. The reduction to nine follows in the differential geometric theory by setting the Riemann tensor (Einstein tensor, resp.) equal to zero. From the standpoint of differential geometry, this step seems quite arbitrary, and one must therefore expect states in metric bodies that are generally different from those of § 4, 5. In order to describe these states, we have the connection Γ_{mlk}, which follows from the deformation components g'_{mlk} (six degrees of freedom) and the structure curvature components (nine degrees of freedom).

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[21] The account in [**8**] involves the assumption that R_{nmlk} does not have to be anti-symmetric in l, k. All of the remarks that relate to this must then be corrected if the connection is assumed to be metric. In particular, the distortions must be replaced with rotations in footnote 1, pp. 135 of [**8**].

The important question now is that of the causes of these general states, i.e., the physical meaning of tensor field B^{ij} that the one sets equal to the Einstein tensor Γ^{ij}.

We point out that the equation that one obtains by setting the anti-symmetric part of the Einstein tensor equal to zero is identical with the divergence condition for the dislocation density for the deformed state. One will thus have to deal with the fact that dislocations can also cease to exist in the interior of a body. Indeed, one obviously must generally define the dislocation density as in § 4. It must be emphasized that the definition of a dislocation with the aid of the Cartan loop does not have to forbid its vanishing in a body.

Whereas the vanishing of the dislocation density also implies the vanishing of the anti-symmetric part of the Einstein tensor, this is not true for the symmetric part, in general. For that, we write:

$$\Gamma^{(ij)} = B^{(ij)}, \tag{63}$$

in which we must replace Γ_{mlk} with g'_{mlk}.

The $h^{(ij)}$ thus lead to an elastic deformation, and with that, to internal stresses. Therefore, no Cosserat-Nye structure curvatures appear, since we have $h_{mlk} = 0$. We would now like to show that one can realize such internal stresses by the insertion of matter into a body that is originally in the ideal state. That is essentially best explained in the context of crystals.

This matter exists as a sort of atom and is found in the ideal state. For example, we can identify it with the crystal in sec. 2b. If we insert matter into this crystal perhaps in the form of a new planar net that vanishes in the interior then we obtain the real crystal of sec. 2a; we have produced a dislocation line in the ideal crystal. However, this is precisely what we cannot do, since the dislocation will alter the structure, in the sense that that the Cosserat-Nye curvature h_{mlk} will no longer vanish everywhere.

We can therefore insert matter into the body in other ways, namely, as so-called interstices. Then we will elastically displace the regular atoms, from which the planar network will naturally also become curved. This is therefore bound up with the components g'_{mlk}, and the elastic deformations are closely coupled with the curvature, as we described by way of eq. (43). We thus have two kinds of matter to distinguish: *regular matter*, from which the ideal state is composed, and *extra matter*, which gives rise to elastic deformations of the body, and without which dislocations would appear. None of the regular planar nets of a body filled with extra matter can end in its interior.

The effect produced by the extra matter then consists of compressing the neighboring regular matter closer together, and thereby producing elastic deformations. Every single lattice atom can be regarded as an elastic dipole, as we will describe in greater detail in chap. IV. Such a dipole is a *symmetric* tensor ([22]). (It is symmetric because no torques appear in its neighborhood.) In the sequel, the extra matter will be described by the symmetric tensor field of a constant distribution of elastic dipoles.

We now separate the individual mass elements, so they can freely stretch; i.e., the stresses vanish, and the rest of the mass elements undergo a stress-free (this is also called

[22] A lattice atom does not generally push in all directions to the same degree, and from this one arrives at the tensorial nature of extra matter. The substitution of foreign atoms and holes works in a completely similar way. They are therefore included in the extra matter. In particular, the sign of this extra matter can also be negative at these lattice defects.

quasi-plastic) deformation $\overset{Q}{\varepsilon}_{\text{tl}}$ compared to the state with no extra matter. One can also use the symmetric tensor field of this quasi-plastic deformation to characterize the extra matter.

The inhomogeneous introduction of extra matter alters the regular crystal structure in such a way that the lattice constants will vary from place to place and one obtains the orientation from this. Such a lattice can exist stress-free in a particular non-Euclidian (Riemannian) space. If one were therefore compelled to take such a position in a Euclidian space, for example, that the regular atoms define the sites of an ideal lattice then this state would not be (force-) stress-free.

On the other hand, one can now think of a substance that, when introduced into an ideal crystal, has the effect that the body can only be torque-stress-free in a non-Euclidian (Cartanian) space (that it is not torque-stress-free at that position in an ideal crystal, as above). No detailed discussion of the manner by which this substance evolves in time exists, but one can think of a certain microscopically fluctuating distribution of interstitial masses, which corresponds to the aforementioned equivalence of macroscopic torque stresses and microscopic force stresses that was mentioned in the first chapter. We leave these questions open, and choose to provisionally characterize the given substance as *rotational matter*. It has six degrees of freedom (in contrast to macroscopic extra matter, which has three), and therefore only half of it is described by the anti-symmetric part $B^{[ij]}$ of B^{ij}.

The fundamental geometrical equation in chapter I is particularly suited for a qualitative discussion of rotational matter, which was written, in the Lagrangian notation:

$$\text{Rot } \boldsymbol{\beta} - \boldsymbol{\alpha} = \boldsymbol{\delta}.$$

If one decomposes this accordingly:

$$\boldsymbol{\delta} = \text{Rot } \overset{Q}{\boldsymbol{\varepsilon}} + \text{Rot } \overset{Q}{\boldsymbol{\omega}} + \text{Grad } \mathfrak{a},$$

in which $\overset{Q}{\boldsymbol{\varepsilon}}$ is the aforementioned quasi-plastic deformation tensor, which macroscopically describes the extra matter, then the rotation tensor field $\overset{Q}{\boldsymbol{\omega}}$ and the vector field $\mathfrak{a}$ represent the rotational matter. By the fact that $\text{Div } \boldsymbol{\alpha} = -\text{Div grad } \mathfrak{a}$, one has that the gradient part of $\boldsymbol{\delta}$ means the same thing as $B^{[ij]}$, whereas the $\text{Rot } \overset{Q}{\boldsymbol{\omega}}$ part of $\boldsymbol{\delta}$, as will follow from the calculations of the next section, does not contribute to the construction of the Einstein tensor.

When we extend the theory of nine degrees of freedom by the nine degrees of freedom of the extra matter then we should, we hope, arrive at a theory with eighteen degrees of freedom. In fact, this does not happen since in the former theory we artificially considered the dislocation densities as external influences, and therefore as givens. In reality, this is inadmissible. Extra matter and dislocation densities must be given independently of each other. In many cases, our conception of the general theory will involve the *matter tensor* B^{ij} (6), the dislocation density (6), and the bulk forces (3) being given independently of each other. We thus attribute only six degrees of freedom

to the dislocation density, since Div $\boldsymbol{\alpha}$ is precisely equivalent with $B^{[ij]}$. We thus arrive at the correct number of fifteen degrees of freedom.

Next, we will make an even further restriction by taking $B^{[ij]} = 0$. This corresponds to simultaneously requiring: Rotational matter first becomes interesting in a theory in which one also considers only the torque stresses.

The mathematical statement of the general theory that we shall give in the next section will be linked with the following assumption: the initial state is the ideal crystal (k). As before, we are brought into the intermediate state (κ) by means of a plastic distortion A^{κ}_{k}, which one may assume to be an aggregate of isolated mass elements or as a non-Euclidian state. We now introduce extra matter into the mass elements, which will alter the ideal crystal lattice from the outside. The new intermediate state is called (κ'), and we give the change in form of the mass element under the transition (κ) $\rightarrow$ (κ') the name $A^{\kappa'}_{k}$. As before, we pass from the state (κ') into the final state by an elastic distortion $A^{k}_{\kappa'}$.

§ 8. The mathematical statement of the general theory

We begin with some notations. We let b_{tl}, c_{kl}, $g_{\kappa'\lambda'}$, a_{kl}, denote the metric tensors of the four states (t), (κ), (κ'), (k), resp., or, e.g., if we need to measure the distance between two points, which is characterized by the mutual separation (dx^k) in the final state, we denote them by b_{kl}, c_{kl}, g_{kl}, a_{kl}. The tensors that are defined on the final state and represent the total deformation (t $\rightarrow$ k), the resulting lattice deformation ($\kappa \rightarrow k$), the elastic deformation ($\kappa' \rightarrow k$), the plastic deformation (t $\rightarrow$ κ), and the quasi-plastic deformation ($\kappa \rightarrow \kappa'$), are then ([23]):

$$\overset{G}{\varepsilon}_{kl} = (a_{kl} - b_{kl})/2, \quad \overset{R}{\varepsilon}_{kl} = (a_{kl} - c_{kl})/2, \qquad \varepsilon_{kl} = (a_{kl} - c_{kl})/2, \qquad (?) \qquad (64)$$

$$\overset{P}{\varepsilon}_{kl} = (c_{kl} - b_{kl})/2, \qquad \overset{Q}{\varepsilon}_{kl} = (g_{kl} - c_{kl})/2.$$

Definitions of the type (19) are valid for the three-index symbols. Each of the deformations (64) is a constituent of a distortion for which we previously employed the kernel A and the indices that corresponded to the intended transition.

Our first goal is now to describe the crystal *lattice* in the deformed state. Since the transition (t) $\rightarrow$ (κ) does not alter the crystal structure (cf. § 4), we must go on to (κ). Let $B^{\kappa}(1)$ and $B^{\kappa}(2)$ be perhaps two primitive lattice vectors at two neighboring mass elements, let their distance apart be dx^{μ}, and let them be Euclidian parallel and of equal length, so that $dB^{\kappa} = 0$, when (κ) is a Cartesian coordinate system. We now have that the distortion $A^{\kappa'}_{\kappa}$ takes (κ) to (κ'), and $A^{k}_{\kappa'}$ takes (κ') to (k). With both operations, one is

[23] On the grounds of the additivity of the deformation one can easily give a prescription for the measurement of, e.g., $\overset{Q}{\varepsilon}_{kl}$: It is the deformation that the mass element that was removed from the final state suffers when the extra matter is suddenly removed from it under constant application of the forces on its boundary.

treating a distortion of the crystal lattice; in the first case, a stress-free (quasi-plastic) one, and in the second case, an elastic distortion. As a result, the total distortion of the regular lattice is equal to:

$$A_{\kappa}^{k} = A_{\kappa'}^{\kappa} A_{\kappa}^{k'}. \tag{65}$$

If we compute the difference between the lattice vectors above, for which $dB^k = 0$ is true in the final state, then we obtain, by taking into account that $B^{\kappa} = A_l^{\kappa} B^l$, $B^k = A_{\kappa}^{k} B^{\kappa}$:

$$dB^{\kappa} = B^{\kappa} dA_{\kappa}^{k} = A_l^{\kappa} B^l (\partial_m A_{\kappa}^{k}) dx^m, \tag{66}$$
$$dB^{\kappa} = -A_{ml}{}^{k} B^l \, dx^m, \tag{67}$$
$$A_{ml}{}^{k} = A_{\kappa}^{k} \partial_m A_l^{\kappa}. \tag{68}$$

A_{mlk} is the connection that describes the resulting distortion of the lattice. It has a form that leads to a vanishing Riemann tensor, which means, from § 6, that the orientation of the crystal at every point of the body is also uniquely determined in the deformed state.

In principle, the lattice structure in the deformed state, and thus, the connection A_{mlk}, can be measured; e.g., by Röntgenography. The state of the body is not, however, completely determined by such a measurement, since it would not say how big the quasi-plastic and elastic parts of the distortion were. Thus, the internal stresses cannot be measured Röntgenographically in the presence of extra matter, either. Additionally, one can experimentally determine the decomposition that gives the elastic deformations, and thus, the state is completely specified. (We temporarily disregard the discussion of rotational matter in the last section.)

The lattice connection A_{mlk} may be brought into the form:

$$A_{mlk} = c'_{mlk} + h_{mlk}, \tag{69}$$

in the same manner as in § 4 or 5 (cf., e.g., eq. (13), et seq.), from which their subdivision into deformation and lattice curvature parts appears obvious. One now expresses the resulting lattice deformation, according to (64), as the sum of a quasi-plastic and an elastic deformation; i.e.:

$$c'_{mlk} = a'_{mlk} - 2\overset{R}{\varepsilon'}_{mlk} = a'_{mlk} - 2\varepsilon'_{mlk} - 2\overset{Q}{\varepsilon'}_{mlk} = g'_{mlk} - 2\overset{Q}{\varepsilon'}_{mlk}. \tag{70}$$

With this, we can put the relation (69) into the form:

$$\Gamma_{mlk} \equiv g'_{mlk} + h_{mlk} = A_{mlk} + 2\overset{Q}{\varepsilon'}_{mlk} \equiv B_{mlk}. \tag{71}$$

Here, g'_{mlk} represents the elastic deformation part and h_{mlk} represents the Cosserat-Nye curvature part of the deformed state that originates in the process of dislocation. It follows rigorously that the insertion of extra matter does not lead to stress-free structural curvatures. If one constructs the anti-symmetric part of eq. (71) then $\overset{Q}{\varepsilon'}_{mlk}$ follows naturally. The usual remaining equation:

$$\Gamma_{[ml]k} \equiv \alpha_{mlk} = A_{k\kappa}(\partial_m A_l^{\kappa} - \partial_l A_m^{\kappa})/2, \tag{72}$$

which was the expression of the fundamental geometric theorem in the reduced theory, can guarantee the connection on the body only up to a point, as far as the dislocations are concerned. The certainty of the connection in the presence of dislocations *and* extra matter is given by the complete eq. (71), which is therefore to be regarded as the expression of the fundamental geometric theorem in the general theory ([24]).

The terms g'_{mlk} and h_{mlk} appear in eq. (71) exactly as they do in the reduced theory (cf., eq. (33)). Nevertheless, there is an essential difference: the Riemann tensor of the connection Γ_{mlk} does not vanish, since the term $2\overset{Q}{\varepsilon'}_{mlk}$ gets added to A_{mlk}, and the connection Γ_{mlk} no longer has a form that is reducible to one with a vanishing Riemann tensor.

We can proceed in essentially two ways:

(a) We write eq. (71) in the form:

$$A_{mlk} = g'_{mlk} + h_{mlk} - 2\overset{Q}{\varepsilon'}_{mlk} \tag{73}$$

and form the Riemann tensor from both sides. The left-hand side gives:

$$A_{nmlk} = 2[\partial_m A_{mlk} - c^{pq} A_{nkq} A_{mlp}]_{[nm]} = 0, \tag{74}$$

which becomes:

$$[\partial_n(\Gamma_{mlk} - 2\overset{Q}{\varepsilon'}_{mlk}) - c^{pq}(\Gamma_{nkq} - 2\overset{Q}{\varepsilon'}_{nkq})(\Gamma_{mlp} - 2\overset{Q}{\varepsilon'}_{mlp})]_{[nm]} = 0, \tag{75}$$

in which $c^{pq} c_{qr} \equiv \delta^p_r$. If we set:

$$H_{mlk} = h_{mlk} - 2\overset{Q}{\varepsilon'}_{mlk} \tag{76}$$

then we can easily bring eq. (75) into a form that corresponds to eq. (27). It is:

$$\tfrac{1}{2}\{\varepsilon^{jnm}\varepsilon^{ilk}[\nabla_n(-2\varepsilon_{mlk} + H_{mlk}) - c^{pq}(-2\varepsilon_{nkq} + H_{nkq})(-2\varepsilon_{mlp} + H_{mlp})]\}_{(ij)} = 0. \tag{77}$$

The method for determining the internal stresses that was discussed in connection with eq. (27) can therefore be employed here in a practically unaltered way. However, along with the internal stresses, the elastic deformations, and thus the g'_{mlk}, and, from eq. (71), the lattice curvature and the connection A_{mlk} are known, so that the state of the body is completely determined from the dislocation density and the extra matter that $\overset{Q}{\varepsilon}_{kl}$.

(b) The second way is closer to the considerations of KONDO. We construct the Riemann tensor directly from eq. (71). The left-hand side reads:

$$\Gamma_{nmlk} \equiv 2[\partial_n \Gamma_{mlk} - g^{pq} \Gamma_{nkq} \Gamma_{mlp}]_{[nm]}. \tag{78}$$

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[24] It entails the relation $A_{k\kappa}A^{\kappa}_{l} = a_{kl} - 2\varepsilon_{kl} - 2\overset{Q}{\varepsilon}_{kl}$ that came from (72), and this represents a condition that ε_{kl} must satisfy in order to maintain the connection. Cf., footnote 13.

One observes the essential difference between A_{nmlk} and Γ_{nmlk}: It resides in the appearance of g^{pq}, instead of c^{pq}, and says that Γ_{nmlk} is associated with the metric of the intermediate state (κ'), whereas A_{nmlk} corresponds to the metric of the intermediate state (κ). Thus, if we now construct the Riemann tensor from the right-hand side of (71), then A_{mlk} also gives a non-vanishing contribution. It is:

$$B_{nmlk} \equiv 2[\partial_n B_{mlk} - g^{pq} B_{nkq} B_{mlp}]_{[nm]} \tag{79}$$

and after a brief reorganization, it becomes:

$$B_{nmlk} = \overset{Q}{(2\varepsilon)}_{nmlk} - 4[(c^{pq} - g^{pq}) A_{nkq} A_{mlp} - g^{pq}(\overset{Q}{\varepsilon}_{n'kq} A_{mlp} + A_{nkq} \overset{Q}{\varepsilon}_{n'kq})]_{[nm]}, \tag{80}$$

$$\overset{Q}{(2\varepsilon)}_{nmlk} \equiv 2[2\partial_n \overset{Q}{\varepsilon}_{m'lk} - 4\, g^{pq} \overset{Q}{\varepsilon}_{n'kq}\, \overset{Q}{\varepsilon}_{m'lp}]_{[nm]}. \tag{81}$$

If we multiply (78) and (80) by $\varepsilon^{jnm}\varepsilon^{ilk}/4$ then we obtain the *Einstein equation for continuum mechanics:*

$$\Gamma^{ij} = B^{ij}, \tag{82}$$

which, when symmetrized, is equivalent to eq. (77), and therefore it can also be regarded as the fundamental equations for the internal stresses with given dislocation densities and extra matter. We will explicitly write out eq. (82) in the next section.

The fact that the presently unknown quantities A_{mlk} enter into B^{ij} is relatively inconsequential when one employs, e.g., an iterative approach to the solution of the Einstein equations, as was proposed for the solution of eq. (27) [**19**]. Since the A_{mlk} are small of the same order as the $\overset{Q}{\varepsilon}_{kl}$, one can compute the linear terms, in which one has set $B^0_{nmlk} = \overset{Q}{(2\varepsilon)}_{nmlk}$. The linear computation gives, *inter alia*, A^0_{mlk}, which is the first approximation for A_{mlk}. For the second step, one next computes B^1_{nmlk} with the help of A^0_{mlk}, etc. Here, we can do without a detailed representation for the solution of the fundamental equation for the determination of the internal stresses, since such a solution was given by KRÖNER and SEEGER [**19**]; also, cf. § 10.

§ 9. Discussion

Up till now, we have not spoken of the bulk forces, whose considerations are trivial compared to the problem that we just discussed. One can first solve the problem stepwise for the case of vanishing bulk forces, and then finally consider the bulk forces by way of a supplementary purely elastic calculation. If we think of this possibility as being implicitly contained in the representation of § 8 then the theory of § 8 is the theory of twelve degrees of freedom. One can completely reconcile this theory with the theory of fifteen degrees of freedom by adding another term $\overset{Q}{h}_{mlk}$ to B_{mlk} in eq. (71), which describes the rotational matter that was suggested in § 7. One thus formally obtains the asymmetric Einstein equations.

One easily convinces oneself that the equations of the reduced theory are obtained from those of the general theory when one sets $\overset{Q}{\varepsilon}_{kl} = 0$ in them (from which, it follows that $c^{pq} = g^{pq}$) – i.e., by ignoring the extra matter. Next, one has $B_{nmlk} = 0$, so the connection Γ_{mlk} is integrable, which is consistent with eq. (71). Eq. (82) becomes identical to eqs. (77) and (27).

In regard to eq. (60), BILBY and SMITH [**24**] have characterized the vanishing of the Riemann tensor A_{nmlk} as a physical requirement that must satisfied if a lattice is to be uniquely defined at every point of a crystal.

This postulate also remains true in general. It is true for the *lattice connection* A_{mlk}, but not for the general connection Γ_{mlk}, for which perhaps the name *state connection* would be appropriate, since the state is uniquely determined by it, as compared to A_{mlk}. This state connection is the one that KONDO used. In the event that one should confuse one connection for the other, there would be no contradiction between the theory of KONDO and that of BILBY, BULLOUGH, and SMITH. In particular, there also exists a lattice connection the general theory, which characterizes the distortion of the lattice in an intuitive way with the help of the associated law of parallel translation. Unfortunately, this intuitive appeal of the state connection is further reduced since it no longer describes a parallel translation of *lattice* vectors. Thus, this connection is very useful, due to the fact that it is an additive combination of the elastic deformation components and structure curvatures, since one can thank this additivity for the possibility of separating the elastic deformation from the usual effect, from which the determination of the internal stresses can be realized.

The summands that get added to the term $(2\overset{Q}{\varepsilon})_{nmlk}$ in the expression (80) for the Riemann tensor B_{nmlk} (the corresponding term in the Einstein tensor B^{ij}, resp.) seems at the moment to be a "beauty flaw," since it depends on the lattice torsion A^{k}_{κ}. We had the same "flaw" in our previous definition (10) of the local dislocation density. Most likely, this independence vanishes when one formulates the Einstein equations in the coordinates of the initial state (k), since this is bound up with some difficulties, and, up till now, it has not been verified.

The notion of quasi-plastic deformation (distortion, resp.) was already used in the linear theory (cf. REIDER [45] and KRÖNER [46]). The reduced theory will be ruled by the equation:

$$\nabla\times\boldsymbol{\beta} = -\nabla\times\boldsymbol{\beta}^{P} \equiv \boldsymbol{\alpha}. \tag{83}$$

The dislocation density was defined as a measure of the perturbation caused by the connection during plastic deformation. It can play no role in the determination of the stresses, if the stress-free distortions $\boldsymbol{\beta}^{P}$ come about as a result of the process of dislocation or other influences, such as the insertion of matter, temperature fluctuations, electro- and magnetostriction, etc. One can therefore use a theory of dislocations to construct a theory of such quasi-distortions, and define a quasi-dislocation density ([25]):

[25] This definition of the quasi-dislocation density is certainly unrelated to the three degrees of freedom that are connected with Div $\boldsymbol{\alpha}$.

$$\overset{Q}{\boldsymbol{\alpha}} \equiv -\nabla\times\boldsymbol{\beta}^{Q}. \tag{84}$$

If one has both dislocations and quasi-dislocations (= general theory) at the same time, then one obtains the equation:

$$\nabla\times\boldsymbol{\beta} = \boldsymbol{\alpha} + \overset{Q}{\boldsymbol{\alpha}}. \tag{85}$$

In general, one must compute $\boldsymbol{\alpha}$, $\overset{Q}{\boldsymbol{\alpha}}$ from the physical givens in the problem. If one constructs the rotation of the right-hand side and then takes the symmetric part (symbol *S*), then what remains is:

$$\nabla\times\boldsymbol{\varepsilon}\times\nabla = \boldsymbol{\eta} + \overset{Q}{\boldsymbol{\eta}}, \qquad \boldsymbol{\eta} = (\boldsymbol{\alpha}\times\nabla)^{S}, \qquad \overset{Q}{\boldsymbol{\eta}} = (\overset{Q}{\boldsymbol{\alpha}}\times\nabla)^{S}. \tag{86}$$

One easily verifies that these extended compatibility equations are identical with the symmetric Einstein equation (82) when one linearizes them. In order to compare them, we write eq. (82) as:

$$\tfrac{1}{2}\{\varepsilon^{jnm}\varepsilon^{ilk}\,[\nabla_n(-2\varepsilon_{mlk} + h_{mlk}) - g^{pq}(-2\varepsilon_{nkq} + h_{nkq})(-2\varepsilon_{mlp} + h_{mlp})]\}_{(ij)} = B^{ij}. \tag{87}$$

If one linearizes the equations then one obtains (cf., (80, 81)):

$$\tfrac{1}{2}\{\varepsilon^{jnm}\varepsilon^{ilk}\nabla_n(-2\varepsilon_{mlk} + h_{mlk} - 2\overset{Q}{\varepsilon}_{mlk})\}_{(ij)} = 0. \tag{88}$$

Obviously, the first summand corresponds to the term $\nabla\times\ \boldsymbol{\varepsilon}\times\nabla$ in (86), the second summand corresponds to $\boldsymbol{\eta}$, and the third one to $\overset{Q}{\boldsymbol{\eta}}$.

If we add the quantity $-2\overset{Q}{\varepsilon}_{mlk}$, which characterizes the extra matter, to the quantity h_{mlk} that describes the dislocation in eq. (76), as if we have a resulting dislocation density, then we also effectively used the introduced of the quasi-dislocation in the nonlinear theory. Usually, in the nonlinear theory, coupling terms arise between quantities with different meanings, e.g., products of h_{mlk} with $\overset{Q}{\varepsilon}_{mlk}$, etc. If one ignores these typical nonlinear phenomena then the physical result of the nonlinear equations is the same as that of the linear equations, and one can clarify the most fundamental questions about the linear equations. One of the objectives of this endeavor is to understand from this how the knowledge of the linear theory already involves, to a considerable degree, the knowledge of the fundamental concepts of not only the reduced, but also the general nonlinear continuum theory of dislocations and internal stresses.

We conclude this discussion with a remark on the restriction to metric bodies that we also employed in the general theory. We described the deformed state through the connection Γ_{mlk} and found the following form for it:

$$\Gamma_{mlk} = g'_{mlk} + h_{mlk}. \tag{89}$$

In this, g'_{mlk} represents the (in general, incompatible) elastic deformations and h_{mlk} represents the (in general, incompatible) Cosserat-Nye structure curvatures, which collectively contribute fifteen degrees of freedom. From differential geometry, it is known that the general *metric* connection has precisely the form (89). The most general metric continuum is thus the Cosserat continuum, which is incompatible with deformations and curvatures.

In the preceding statement, the word "metric" means: The distance between two arbitrary points of the body is well defined. This demand is satisfied by, e.g., a body that is composed of nothing but fibers, or a body that is continuously endowed with rips, as one might obtain by perhaps rolling a metal piece, when one does not make the holes too big. Such a body is no longer a continuum in the usual sense. This assumption was what led us to choose the (provisional) terminology of "general theory."

III. The Integration of the Fundamental Equations

In this chapter, we treat the integration of the Einstein equations. The most important device for that is the stress tensor function χ. The principal meaning of the tensor field of stress functions in the continuum theory of dislocations and internal stresses has already been discussed in numerous references [**8**, **19**]. SCHAEFER [**47**] originated a physical meaning for the stress functions that seems particularly adapted to the differential geometric aspects of the theory: The stress functions represent the reactions (in the sense of LAGRANGE) to the constraint condition that the body must remain in Euclidian space.

In [**19**], it was shown that the nonlinear problem of summing the internal stresses in a series can be iteratively replaced with the treatment of linear summation problems. It is clear that one can also treat the second boundary-value problem ([26]) (the combined problem – given internal stresses and boundary forces – resp.) in a similar linear iterative way. Thus, one first solves the linear problem, so that the tensor field of stress functions χ^0 satisfies the boundary conditions, but not the nonlinear differential equations (it is the Einstein equation, in which the elastic deformations have been replaced by the stress functions with the help of the material law). One must therefore add a second tensor field to χ^0 that satisfies the conditions for a free boundary and provides for the satisfaction of the differential equation. We will explain this in somewhat more detail later.

Now, the linear three-dimensional boundary-value problem is already very difficult, so one can presently shy away from more of such problems, which one will require for the iteration. Insofar as the partial problems to be solved are all of the same type, since they always involve the same boundary, the intermediate results of the first computation can be further used in the solution of the following problem. Thus, if one would like to carry out the computation, say, with the help of Green functions then one would arrive at the same function as the complete iteration would provide. If one uses the method of series development, which has such practical importance, then most of the work is taken up with the computation of the matrix elements; however, one can then use them for all of the iterations. In conclusion, the modern methods of computation are particularly well suited to the demands of iterative processes. The amount of time that is required for a program to carry one out depends only upon the number of steps; the computation time of the machine itself increases linearly with this number. From this standpoint, the treatment of nonlinear problems does not therefore seem to be much more complicated than the treatement of linear problems.

In the following sections, we present generalities on the stress function Ansatz and a brief representation of iterative processes. In § 11 the second boundary-value problem of the linear theory will be treated.

§ 10. The stress function Ansatz

The equilibrium condition for the stresses is written in terms of the coordinates of the deformed state:

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[26] We restrict ourselves throughout to the particularly important second boundary-value problem (given boundary forces), which also includes the case of the free boundary.

$$\nabla_i \sigma^{ij} = 0. \tag{1}$$

They will be satisfied identically by the Ansatz:

$$\sigma^{ij} = -\varepsilon^{ilk} \nabla_l \varphi_{k.}{}^{j}, \qquad \varphi_{k.}{}^{j} = \varepsilon^{ilk} \nabla_n \chi_{km}, \qquad (?) \tag{2}$$

or also:

$$\sigma^{ij} = -\varepsilon^{jnm} \varepsilon^{ilk} \nabla_n \nabla_l \chi_{mk}, \tag{3}$$

as Beltrami [**48**] first showed. Obviously, we have the following equations for the first order stress function tensor $\varphi_{k.}{}^{j}$:

$$\nabla_j \varphi_{k.}{}^{j} = 0, \qquad \varphi_{k.}{}^{j} = 0. \qquad (?) \tag{4}$$

Since condition (1) implies that the symmetric tensor σ^{ij} has three degrees of freedom, one can impose three conditions on the second order symmetric stress tensor χ_{ij} without restricting to the manifold of stress states that are derived from χ_{ij}. The conditions are certainly not arbitrary. We are interested only in "admissible" conditions, which are the ones that imply no restriction of the stresses. For example, the conditions $\chi_{xx} = \chi_{yy} = \chi_{zz} = 0$ are admissible (MORERA [**49**]), as are $\chi_{xy} = \chi_{yz} = \chi_{zx} = 0$ (MAXWELL [**50**]), $\chi_{xz} = \chi_{yz} = \chi_{zz} = 0$, $\chi_{xy} = \chi_{yy} = \chi_{zz} = 0$, $\chi_{xx} = \chi_{xz} = \chi_{zz} = 0$ (BLOCH [**51**]). There are no other admissible combinations with three vanishing Cartesian components of χ_{ij} [**51**]. If one computes the stress function tensor in non-Cartesian components then the question of admissibility can get very difficult. We shall return to this notion.

The stress functions satisfy certain differential equations that one obtains when one addresses the (extended) compatibility equations:

$$-\varepsilon^{jnm} \varepsilon^{ilk} \nabla_n \nabla_l \varepsilon_{mk} = \eta^{ij}, \tag{5}$$

with the aid of the material law. (From § 8, these equations are indeed the linearized Einstein equations.) Instead of eq. (5), one takes advantage of the (extended) Beltrami form ([**8**], pp. 55)([27]):

$$\Delta' \sigma^{ij} + \frac{1}{1+\nu} (\nabla^i \nabla^j - a^{ij} \Delta) \sigma_k^k = 2G \eta^{ij}, \qquad \Delta' \equiv \nabla^l \nabla_l. \tag{6}$$

These equations are true only when (1) is valid.

It has now been known for a few years that the aforementioned secondary conditions for the stress functions can be formulated in such a way that one is led to problems in bipotential theory or potential theory. Thus, there are essentially two different possibilities, one of which leads to a solution of the summation problem, while the other seems to offer certain advantages in the treatment of boundary value problems. Both Ansätze are in contrast to the aforementioned covariant ones, which clarifies their results.

a) In order to describe the Ansatz of KRÖNER-MARGUERRE [**36**, **52**], we introduce the abbreviations:

[27] Observe: The statement $\nabla^l \nabla_l = \partial^l \partial_l$ is only true for Cartesian coordinate systems, and it is only by the use of scalar quantities that Δ' is equal to the Laplace operator that one usually denotes by Δ.

$$\chi'_{ij} \equiv \left(\chi_{ij} - \frac{\nu}{1+2\nu}\chi^k_k a_{ij}\right)/2G, \qquad \eta'^{\,ij} \equiv 2G\left(\eta^{ij} + \frac{\nu}{1-\nu}\eta^k_k a^{ij}\right) \tag{7}$$

with the inverses:

$$\chi_{ij} \equiv \left(\chi'_{ij} + \frac{\nu}{1-2\nu}\chi'^{\,k}_k a_{ij}\right)/2G, \qquad \eta^{ij} \equiv \left(\eta'^{\,ij} - \frac{\nu}{1+2\nu}\eta^k_k a^{ij}\right)/2G. \tag{8}$$

If we impose the following secondary condition (in analogy with the known Lorentz convention in electrodynamics):

$$\nabla_\iota \chi'^{\,ij} = 0 \tag{9}$$

(from which, it also follows that $\nabla_\iota \varphi^{\,ij} = 0$) then by substituting this into (3) the Beltrami equations become equivalent to:

$$\Delta'\Delta'\chi'^{ij} = \eta^{ij}, \tag{10}$$

or:

$$\Delta'\Delta'\chi^{ij} = \eta'^{ij}, \tag{11}$$

resp. In an infinite medium the general solution of (10) and (9) is:

$$\chi'^{ij}(\mathfrak{r}) = -\frac{1}{8\pi}\iiint \eta^{ij}(\mathbf{r}')\,|\,\mathbf{r}-\mathbf{r}'\,|\,dV'; \tag{12}$$

hence, it is likewise the solution of the summation problem for internal stresses with a given source density η^{ij}. For a finitely extended medium the secondary condition (9) is not automatically satisfied, so it needs more consideration [**8**].

b) The Ansatz of SCHAEFER is appropriate to the treatment of boundary value problems, i.e., to the solution of the homogeneous problem that goes with (6), which is the usual form of the Beltrami equations:

$$\Delta'\sigma^{ij} + \frac{1}{1+\nu}\nabla^i\nabla^j\sigma^k_k = 0. \tag{13}$$

One splits the stress function tensor χ^{jj} into a spherical tensor $(\Omega + \Theta^k_k/3)\,a^i$ and a small harmonic deviator $\Theta^{\,ij} - \Theta^k_k a^{ij}/3$ thusly:

$$\chi^{jj} = \Theta^{\,ij} + \Omega a^{ij}. \tag{14}$$

If one introduces this while applying (3) to (13), then one obtains:

$$\Delta'\Theta^{\,ij} = 0, \qquad \Delta\Omega = \frac{1}{1-\nu}\nabla^i\nabla^j\Theta^{\,ij}, \tag{15}$$

from which, it follows that:

$$\Delta\Delta\Omega = 0. \tag{16}$$

Since Ω only contributes to the anti-symmetric part of $\varphi^{\,ij}$, we also have:

$$\Delta' \varphi^{(ij)} = 0. \tag{17}$$

The right-hand side of eq. (15) submits to an elementary integration, and a possible form for the general solution is, e.g.:

$$\Omega = \frac{\frac{1}{2}}{1-\nu} x_i \nabla_j \Theta^{ij} + \nu, \tag{18}$$

such that Ω can be expressed in terms of the Θ^{ij} and another harmonic function v. Since the three-dimensional boundary value problem is triply harmonic, but the Θ^{ij} represent six harmonic functions, over four more harmonic functions will be further available. For example, it is admissible to set $\Theta_{xy} = \Theta_{xy} = \Theta_{xy} = \nu = 0$, so one then has three harmonic functions remaining, and the Ansatz takes on the MAXWELL form [**36**]. The degrees of freedom that come from the excess harmonic functions can therefore be adapted to some problem in question as much as possible. This seems to offer a certain advantage over the method of PAPKOVITCH-NEUBER [**37**, **38**], which includes only *one* excess harmonic function. As SCHAEFER [**47**] has shown, his stress functions are closely related to those of PAPKOVITCH-NEUBER; these authors use the functions Ω, $\nabla_i \Theta^{ij}$. The latter are not stress functions – in contrast to the Θ^{ij} – since the equilibrium condition cannot be *identically* satisfied by using them.

From the generally quite simple integration of the right-hand side of eq. (15) one needs to consider no further supplementary conditions. Schaefer's Ansatz then has a certain advantage over that of KRÖNER and MARGUERRE for the treatment of boundary value problems, which we will discuss in the following section.

Now, we shall briefly carry out the iteration process for the case of a body that is acted on by boundary forces and likewise includes internal stresses, whose sources will be regarded as the dislocations α^{kl} and the extra matter B^{ij}. We assume that the elasticity law takes the form:

$$\varepsilon_{kl} = s_{ijkl}\, \sigma^{ij} + s_{ghijkl}\, \sigma^{ij}\sigma^{gh} + \ldots \tag{18a}$$

so we specialize our attention to an isotropic medium.

If we substitute the stress functions into the Einstein equations (II, 87), with the help of the law (18a), then we obtain, as in [**19**]:

$$\Delta'\Delta'\chi^{ij} = \eta_0'^{ij} + P'^{ij} + Q'^{ij}. \tag{18b}$$

Here, the primed quantities in the right-hand side are connected with the unprimed quantities in the same way that η'^{ij} relates to η^{ij} in eq. (7). Let the P^i and the Q^{ij} be defined as in [**19**]. (They include the χ^{ij}, the Q^{ij}, and the α^{kl}, moreover, in a nonlinear form.) In η_0^{ij}, we have now included the matter tensor B^{ij}, which was not considered in [**19**]. Eq. (18b) takes exactly the same form as eq. (44) in [**19**], and the method of determining the stress functions that was presented here can be largely applied to it. We therefore set $P'^{ij} = Q'^{ij} = 0$ for the iteration, and determine the linear approximation to the stress function tensor χ_0^{ij}, in which we look for the particular integral of the equation:

$$\Delta'\Delta \overset{S_{ij}}{\chi_0} = \eta_0'^{ij}, \tag{18c}$$

and add it to a solution to the homogeneous equation:

$$\Delta'\Delta \overset{R_{ij}}{\chi_0} = 0, \tag{18d}$$

such that $\chi_0^{ij} = \overset{S_{ij}}{\chi_0} + \overset{R_{ij}}{\chi_0}$ satisfies the boundary conditions. The method for determining $\overset{R_{ij}}{\chi_0}$ follows in the next section.

Next, we can compute $P_0'^{ij}$ and $Q_0'^{ij}$ from χ_0^{ij} [**19**], and we come to the next iteration, which means looking for the particular integral to the equation:

$$\Delta'\Delta \overset{S_{ij}}{\chi_1} = P_0'^{ij} + Q_0'^{ij}, \tag{18e}$$

and $\overset{R_{ij}}{\chi_0}$ supplements a function χ_1^{ij} that satisfies the boundary conditions. $\chi_0^{ij} + \chi_1^{ij}$ is then the quadratic approximation to the stress function tensor. The next iteration involves:

$$\Delta'\Delta \overset{S_{ij}}{\chi_1} = P_0'^{ij} + Q_0'^{ij}, \tag{18f}$$

etc. Since the required elastic constants of order three and higher have not yet been experimentally determined, one must meanwhile satisfy oneself with the quadratic approximation. In the event that perhaps $\overset{Q}{\varepsilon}_{kl}$ is given instead of B^{ij}, one then encounters minor complications that, from the remarks at the end of § 8, can be regarded as already having been resolved (*).

§ 11. The treatment of the three-dimensional second boundary value problems with the help of the stress functions

The result of the treatment of three-dimensional boundary value problems with the help of Schaefer's stress function Ansatz depends largely on a skillful utilization of the freedom that comes from the excess of harmonic functions Θ^{ij}. At this point, we shall not go into the admissibility problem that is linked with that, which can become quite difficult when one uses curvilinear components for Θ^{ij}. One can possibly find clues in [**51**] or [**36**].

We write the boundary conditions as:

$$\mathfrak{n} \bullet \boldsymbol{\sigma} = -\mathfrak{n} \bullet (\nabla \times \boldsymbol{\varphi}) = -(\mathfrak{n} \times \nabla) \bullet \boldsymbol{\varphi} = (\mathfrak{n} \times \nabla) \bullet [\mathfrak{n} \times (\mathfrak{n} \times \boldsymbol{\varphi})] = \mathfrak{A}, \tag{19}$$

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* *Editor's note:* Meanwhile, it has been proved that it is sufficient to consider the boundary conditions just once, namely, at the last iteration.

in which $\mathfrak{A}$ is the surface density of the boundary forces, and $\mathfrak{n}$ is the external unit normal vector. These conditions are satisfied when one sets:

$$\mathfrak{n}\times(\mathfrak{n}\times\boldsymbol{\varphi}) = \mathfrak{n}\times\nabla\mathfrak{h} + \mathfrak{n}\times(\mathfrak{n}\times\nabla)\mathfrak{a} \tag{20}$$

only in the event that:

$$(\mathfrak{n}\times\nabla)\bullet(\mathfrak{n}\times\nabla)\mathfrak{h} = \mathfrak{A} \tag{21}$$

is true. From Schaefer [**35**], one can obtain the solution of these equations as the solution of an equilibrium problem in the boundary layer ("scale"). The required expenditure of work up to this point is trivial in comparison to the complete calculation. We therefore assume that $\mathfrak{h}$ is given.

As one sees, if $\mathfrak{a}$ is derived by substituting (20) into (19) then it can be chosen in a completely arbitrary way. The appearance of $\mathfrak{a}$ is understandable when one realizes that one can add an arbitrary field of the form $\nabla\mathfrak{a}$ to the $\boldsymbol{\varphi}$ without altering the stresses. From this, we conclude that the freedom in the choice of $\mathfrak{a}$ is closely coupled with the freedom that we mentioned in connection with the Θ^{ij}; $\mathfrak{a}$ is largely determined by giving definite values to the Θ^{ij}, and conversely.

In the sequel, we would like to restrict ourselves to bodies whose boundary is composed exclusively of (arbitrarily many) pieces of Cartesian coordinate surfaces x, y, z, = const. This is no restriction in reality since one can approximate any body in this way arbitrarily closely. If we make the required assignment $\Theta_{xy} = \Theta_{yz} = \Theta_{zx} = 0$, i.e., if we compute with Maxwell functions then, as one easily verifies, the boundary conditions can be written in the following way, in which one explicitly takes (20) into account:

x = const.:

$$\begin{aligned} &\partial_{zz}\chi_{yy} + \partial_{zz}\chi_{yy} = -A_x && \text{(a)}\\ &\partial_x\chi_{yy} = \partial_z H_z + \partial_y\textstyle\int dz\,(\partial_y H_z) && \text{(b)}\\ &\partial_x\chi_{zz} = \partial_y H_y + \partial_x\textstyle\int dy\,(\partial_z H_y) && \text{(c)} \end{aligned} \tag{22′}$$

y = const.:

$$\begin{aligned} &\partial_{xx}\chi_{zz} + \partial_{zz}\chi_{xx} = -A_y && \text{(a)}\\ &\partial_y\chi_{zz} = \partial_x H_x + \partial_z\textstyle\int dx\,(\partial_z H_x) && \text{(b)}\\ &\partial_y\chi_{xx} = \partial_x H_z + \partial_x\textstyle\int dz\,(\partial_x H_z) && \text{(c)} \end{aligned} \tag{22″}$$

z = const.:

$$\begin{aligned} &\partial_{yy}\chi_{xx} + \partial_{xx}\chi_{yy} = -A_z && \text{(a)}\\ &\partial_z\chi_{zz} = \partial_y H_y + \partial_x\textstyle\int dy\,(\partial_x H_y) && \text{(b)}\\ &\partial_z\chi_{yy} = \partial_x H_x + \partial_y\textstyle\int dx\,(\partial_y H_x) && \text{(c)} \end{aligned} \tag{22‴}$$

(The integrals give the contribution from $\mathfrak{a}$.) Thus, the H_i ($i = x, y, z$) follow uniquely from equations of the type:

$$(\partial_{xx} + \partial_{yy})H_i = A_i \qquad \text{for} \qquad x = \text{const.} \tag{23}$$

etc. Up till now, no one has succeeded in getting from this point to the standard problem of potential or bipotential theory, i.e., to link up with the boundary conditions. (The

differential equations for the Cartesian Θ^{ij} are indeed coupled.) Such a coupling occurs in the special case of a body that is bounded by two infinite parallel planes z = const., as we shall now show, since the boundary conditions for x = const. and y = const. are not valid here.

Next, one computes the harmonic function $\chi_{xx} - \chi_{yy}$ from its normal derivative on the boundary:

$$\partial_z(\chi_{xx} - \chi_{yy}) = \partial_y H_y - \partial_x H_x + \partial_x \int dy\, (\partial_x H_y) - \partial_y \int dx\, (\partial_y H_x). \qquad (24)$$

One can pose this problem, as one chooses, as a first or second boundary value problem in potential theory. If one chooses the first then one computes $\partial_z(\chi_{xx} - \chi_{yy})$ in the volume and therefore also has:

$$\chi_{xx} - \chi_{yy} = \int dz\, [\partial_z(\chi_{xx} - \chi_{yy})] + f(x, y), \qquad (25)$$

in which we have set $f(x, y) = 0$ if the stresses are to vanish at infinite.

With the knowledge of $\chi_{xx} - \chi_{yy}$, if we integrate the boundary condition (a):

$$(\partial_{xx} + \partial_{yy})\, \chi_{xx} = -A_z + \partial_{xx}(\chi_{xx} - \chi_{yy}) \qquad (26)$$

then we are in a position to compute the functions χ_{xx} on the boundary. Since the normal derivatives of χ_{xx} on the boundary are given (b), the value of χ_{xx} over the entire volume follows from solving a standard problem of bipotential theory. Thus, one also knows χ_{yy}.

One ultimately obtains the stress function χ_{zz} in an elementary way from the second condition (15), which one can put into the form:

$$\Delta\chi_{zz} = \frac{-1}{1-\nu}[\partial_{yy}(\Delta\chi_{xx} - \Delta\chi_{yy}) + \partial_{zz}(\Delta\chi_{xx} - \Delta\chi_{zz})] \qquad (27)$$

since $\Delta\Omega = \Delta\chi_{zz}$. From this, it follows that:

$$\chi_{zz} = \chi_{xx} + \int dz \int dz\, [\partial_{yy}(\Delta\chi_{xx} - \Delta\chi_{yy}) + (1 - \nu)\, \Delta\chi_{xx}] + zg(x, y) + h(x, y). \qquad (28)$$

Since we must set $g(x, y)$ and $h(x, y)$ equal to zero, for the same reasons as we gave above for $f(x, y)$, we also have ascertained χ_{zz}. It satisfies all of the differential equations and boundary conditions.

A specialization of the body that was treated here is the elastic half-space $z \leq 0$. If one sets $\chi_{zz} = zu + v$ with $\Delta u = \Delta v = 0$ then, instead of the bipotential problem, one must solve two standard problems of potential theory. The possibility of the problem of the elastic half-space giving way to three standard problems in potential theory has been know for some time [**53**]. Through the exclusive use of stress functions, this was first realized by SCHAEFER [**35**] in abbreviated form, although, in contrast to our treatment, he set χ_{yz}, χ_{zx}, and $\chi_{xx} - \chi_{yy}$ equal to zero.

The fact that the three-dimensional boundary-value problem for arbitrary boundary reduces to the standard problems in potential or bipotential theory (or even to a "tripotential theory") was very advantageous from the standpoint of modern computing, since these standard problems can naturally be programmed much better as problems

with coupled boundary conditions. Indeed it is very questionable whether such a reduction to a standard problem is possible in completely generality; indeed, it would be a great achievement if one could realize this simplification for at least one important type of body. Corresponding investigations were also useful for some special force distributions, e.g., substantially simplifying the boundary conditions (22), when one sets $\partial_i A_j - \partial_j A_i = 0$.

In the cases in which the reduction to standard problems is not valid, one must mostly cling to the methods of series developments of the Θ^{ij} in terms of harmonic functions. One then suitably replaces the χ_{ij} with the Θ^{ij} in the boundary conditions (22), in which perhaps the solution (18) for Ω is satisfactory. For a curved boundary one must adapt to the given conditions as best as one can.

We now see that there is advantage to the stress function method when computing the internal stresses on an infinite body with a free boundary. As we remarked in § 10, by solving the summation problem, one next obtains stress functions $\overset{R}{\chi}_{ij}$, e.g., as Maxwell functions [**8**]; these no longer satisfy the boundary conditions. Now, let $\overset{R}{\chi}_{ij}$ be the stress functions that one obtains by solving the boundary value problem and which one must add to $\overset{S}{\chi}_{ij}$ in order to obtain the resulting stress functions χ_{ij} that satisfy the free boundary conditions. If one replaces $\chi_{ij} = \overset{R}{\chi}_{ij} + \overset{S}{\chi}_{ij}$ in these conditions then one almost immediately obtains the boundary conditions for the determination of the $\overset{R}{\chi}_{ij}$. One can thus connect the boundary value problem directly with the summation problem without having to first ascertain the stresses that are associated with the $\overset{S}{\chi}_{ij}$, which are quite uninteresting. One is thus spared much elementary computation, and the integrations (23) also fall out. It is obviously very impractical to go from the solution of the summation problem to the displacements for the purpose of treating boundary-value problems.

Finally, let it be particularly emphasized that the stress function Ansatz, in contrast to the method of displacement fields, has its simplest formulation in terms of the coordinates of the final state. For many nonlinear problems this is undoubtedly a great asset; e.g., in any case where the external influences are given as functions of the final coordinates. Naturally, other problems will arise that naturally suggest a treatment in the initial coordinates. However, this type of problem seems to be in the minority.

IV. Para- and Diaelasticity ([28])

The appearance of para- and diaelasticity is very closely linked with the extra matter that was treated in the second chapter. In § 7, we remarked that the extra matter that we had introduced could be described as an elastic dipole density. The permanent or induced elastic dipole (- force couple tensor) is, however, the central notion in the theory of para- and diaelasticity (§ 12).

Strictly speaking, this is already included in the general theory. For that reason, we now endeavor to elaborate on a theory of para- and diaelasticity, since we have special applications in mind that already are largely suggested by that title: We regard the paraelastic (diaelastic, resp.) continuum as the analog of the paramagnetic (diamagnetic, resp.) body. One the one hand, we have that in such a theory, in contrast to the theory of chapter II, problems with singular internal stress sources are interesting above all others.

Because of the particular aspects of such problems the theory of para- and diaelasticity possesses a certain stigma in the eyes of continuum mechanics. Unfortunately, we must restrict ourselves completely to the linear approximation in this chapter.

§ 12. The elastic dipole ([29])

The most important notion in the theory of dia- and paraelasticity is that of the elastic dipole. It was introduced by BOUSSINESQ in a different connection, and is usually referred to as a force couple today. For our purposes, we recommend the former name. It allows one to see the analogy with electromagnetic phenomena much better, and it also plays an important role in the intended applications of force effects to elastic singularities: It is very comfortable and familiar to speak of forces that originate in a dipole.

We now give the *microscopic* definition of a (one-axis) dipole in an elastic continuum. For this, we associate two individual forces $\pm \mathfrak{f}$ of equal magnitude and opposite directions with two different points whose separation distance is described by the vector $\mathfrak{l}$ (Fig. 3). These forces shall now be moved together along the linea that connects them and their magnitude increases in such a way that the dyadic product $\mathfrak{l}\mathfrak{f}$ remains finite. One then defines the elastic dipole as the limiting value:

$$\boldsymbol{P} \equiv \lim \mathfrak{l}\mathfrak{f}. \qquad (1)$$

Figure 3. On the definition of the general one-axis elastic dipole

In contrast to the electric and magnetic dipole, the elastic dipole is a tensor of rank two, which corresponds to the fact that the elastic fields (stress and deformation) also have a tensor character.

[28] I would like to express my heartfelt thanks to Herrn Dr. J.D. ESHELBY, who has made many suggestions about the theory of para- and diaelasticity, for his correspondence on this topic.

[29] For this section, cf., the presentation in LOVE's Mathematical Theory of Elasticity [**54**].

(We will see that in the elastic case a dipole density represents a stress field in the same way that a magnetic dipole density represents a magnetic field.) As a simple dyadic product, the dipole in (1) is a special case of the more general (three-axis) dipole tensor. One must write this as a sum of up to three dyadic products:

$$\boldsymbol{P} \equiv \lim(\mathfrak{l}'\mathfrak{f}' + \mathfrak{l}''\mathfrak{f}'' + \mathfrak{l}'''\mathfrak{f}'''). \tag{2}$$

In this we mean, e.g., P_{xx} represents two forces located on the x-axis, say, separated by dx and pointing in the + and – x-direction, which corresponds to a force couple *without* moment (Fig. 4a), while P_{xy} represents two forces on the x-axis that are dx apart from each other and point in the + and – y-direction, which corresponds to a force couple with a torque in the z-direction.

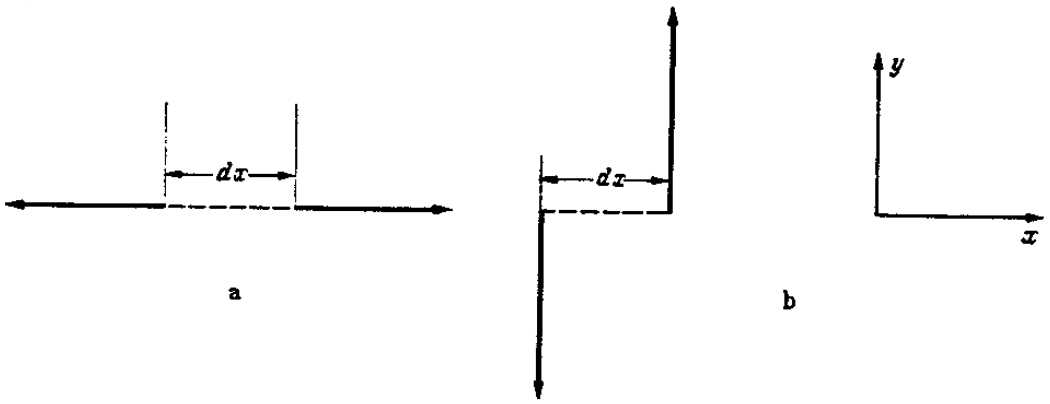

Figure. 4a and 4b. One-axis elastic dipole without (a) and with (b) a torque.

If one has a dipole with $P_{xy} = P_{yx}$ then the mutual moments are annulled, so the symmetric force couple tensor is therefore moment-free. It can be transformed to the principal axis so it then has a form as in Fig. 5.

If one contents oneself with macroscopic observations then for sufficiently small $\mathfrak{l}$ the passage to the limit in (2) is no longer essential. One can then – e.g., in Fig. 5 – enumerate all six forces by indices and obtain, when one denotes the position vectors of their points of application by $\mathfrak{x}^{(i)}$:

$$\boldsymbol{P} = \sum_{i=1}^{6} \mathfrak{x}^{(i)}\mathfrak{f}^{(i)} \tag{3}$$

as the macroscopically observed dipole.

The following generalization now becomes important: Let a small closed surface f be defined inside a continuum. Suppose that a surface force density $\mathfrak{A}(\mathfrak{x})$ with $\int d\mathfrak{A} = 0$ is defined on it. At a great distance, this acts like a total dipole:

$$P^{ij} = \int_f x^i dA^j \,. \tag{4}$$

We must point out here that the definition (4) of the dipole $\boldsymbol{P}$ is valid only in a continuum that is elastically homogeneous in the neighborhood of the dipole. In particular, the

elastic constants inside of *f* must be the same as they are outside of it. If one alters the internal elastic constants, perhaps by the application of a force A, then one likewise alters the displacement field outside of *f*, and also by volumes that are just as small as the domain bounded by *f*. In order to arrive at a meaningful definition one must, however, demand that the field of the dipole, which vanishes at infinity in the exterior of *f*, is uniquely determined outside of *f* by being given P^{ij} and the elastic constants ([30]).

For the intended applications, the situation in the neighborhood of the dipole is mostly very complicated. Nevertheless, one can make definite statements about the dipole when one perhaps considers the elastic displacement field at a great distance from it that we discussed before.

As one knows, an isolated force P^i at the origin of an infinitely extended elastically homogeneous medium produces a displacement field:

$$s_j(\mathfrak{x}) = S_{ij}(\mathfrak{x})\, P^i \qquad (5)$$

with:

$$S_{ij}(\mathfrak{x}) = F_{ij}(\vartheta, \varphi) / r. \qquad (6)$$

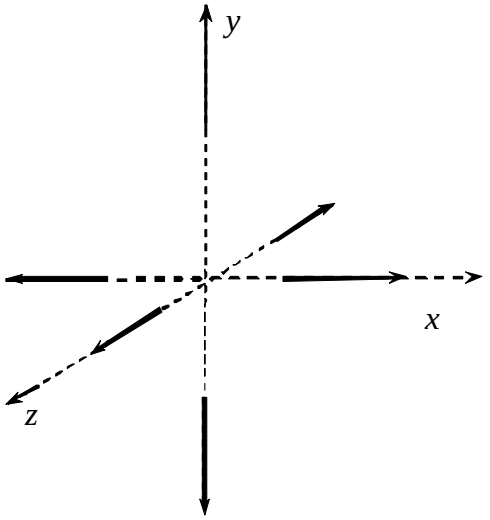

Figure 5. The three-axis symmetric elastic dipole principal-axis representation.

(r, ϑ, φ are spherical coordinates.) Like S_{ij}, $F_{ij}(\vartheta, \varphi)$ is a tensor of rank two. In an isotropic medium, S_{ij} has the form (λ, μ = Lamé constants):

$$S_{ij} = \frac{1}{8\pi\mu}\left(-\frac{\lambda+\mu}{\lambda+2\mu}\,\nabla_i\nabla_j + a_{ij}\Delta\right) r. \qquad (7)$$

The displacement field of an elastic dipole is then:

$$s_k = P^{ij}\,\nabla_i\, S_{jk}\,; \qquad (8)$$

it vanishes at infinity like $1/r^2$. Eq. (8) shows the possibility of defining an elastic dipole in terms of the displacement field that it evokes at a great distance away from it. We call the definition of a dipole by means of eq. (8) the *macroscopic* definition. It is also experimentally measurable since one generally does not know the forces that enter into an application of the definition (2) or (4), but the displacement field (8) is accessible to measurement (more precisely: the deformation field that is associated with the displacement (8)).

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[30] At close range, the elastic field that is associated with 𝔄 can also be regarded as a quadrupole, octupole, etc. An elastic quadrupole is defined, in analogy with (1), by lim ⌊***P***. The generalization of the theory to such multipoles encounters no special difficulties. We will therefore ignore it, since, up till now, no applications for multipoles have been found.

We shall now briefly derive the most important formulas for the interaction of a dipole with an elastic deformation field ε_{kl} that comes from other sources.

Let a homogeneous, not particularly isotropic, continuum K be of the sort that is acted on by boundary forces so that it exhibits a homogeneous deformation. Suppose that there is a dipole of the type (4) at an arbitrary location, and the stress field σ^{ij} is produced by only this dipole. The interaction energy associated with the field is then:

$$\int_K \sigma'^{ij} \varepsilon_{ij} \, dV = \varepsilon_{ij} \int_K \sigma'^{ij} \, dV \,. \tag{9}$$

An application of Gauss's theorem for the domains inside of f and outside of it gives:

$$\int_K \sigma'^{ij} \, dV = \int_f x^i \sigma'_{(i)}{}^{jk} df_k - \int_f x^i \sigma'_{(a)}{}^{jk} df_k + \int_F x^i \sigma'_{(i)}{}^{jk} dF_k \,, \tag{10}$$

because no bulk forces are present ($\nabla_i \sigma'^{ij} = 0$). By ($i$) and ($a$) we intend that one takes the values immediately inside and outside of f, resp. The third integral vanishes on the boundary F of the body since the conditions for a free boundary are associated with only the σ'^{ij}. The other two integrals give $-\int x^i \, dA^j$, if $\sigma'_{(i)}{}^{jk} - \sigma'_{(a)}{}^{jk} df_k = -dA^j$, and are indeed the conditions for the stresses on the boundary surface f. This gives the interaction energy between a dipole and a deformation field, which is the potential energy of a dipole in a deformation field ε_{ij}, as follows:

$$E = -P^{ij} \varepsilon_{ij}. \tag{11}$$

This formula is also valid for variable deformation fields, since it can depend only on the deformations at the position of a dipole when one brings the dipole into a pre-existing field.

In the *elementary* derivation of the formula (11), which was given here for the first time, we regarded the force A as an external force. In the following sections, we will establish that eq. (11) is also true when the A^i are internal forces; one is then dealing with a torque stress dipole, so to speak.

Intuitively speaking, E is also the energy that one gains when one takes the dipole from a place where the deformation is null to a place where the deformation is ε_{ij}. If one has an inhomogeneous field ε_{ij} then eq. (11) says that one gains $dE = -P^{ij} \, d\varepsilon_{ij}$ when one displaces the dipole through dx^i, and the deformations at the two places differ by $d\varepsilon_{ij}$. Therefore, in an inhomogeneous deformation field the force:

$$K_i = P^{jk} \, \nabla_j \, \varepsilon_{ki} \tag{12}$$

acts on an elastic dipole. Furthermore, from eq. (11), the formula for the torque that is exerted on a dipole in a homogeneous deformation field easily follows:

$$L^k = 2\varepsilon^{jkl} P_l{}^j \varepsilon_{ki} \,. \tag{13}$$

In our treatment of the problem we shall not speak of force effect that the dipole exerts on the surface itself. This force effect comes about due to the fact that for a given dipole

strength the internal energy of a dipole, hence, the expression $\frac{1}{2}\int \sigma'^{ij}\varepsilon'_{ij}\,dV$, depends on the position of the dipole in the body. From ESHELBY [**35**], one can understand this force as also being the effect of "imaginary forces" that one ascribes to the fact that the displacement field of the dipole that is given by the expression (8) does not satisfy the boundary conditions. Naturally, the variable proper energy of the dipole has nothing to do with the energy described in eq. (11). On the contrary, in formula (12), one must add the "imaginary field" to ε_{ki} when one wishes to consider the "force the dipole exerts on itself."

We make the following remark concerning the validity of eqs. (11) through (13): Since the elasticity law did not enter into the derivation of this formula, it is also valid for arbitrary anisotropies in the elastic constants. From this, it seems possible that it is also valid in the realm of nonlinear theory, although this has not been verified up till now.

Eqs. (11) through (13) all refer to symmetric dipoles. Not much can be said at this time about the behavior of anti-symmetric dipoles, and for this reason, one is referred to [**46**].

§ 13. The paraelastic continuum

An elastic continuum is called paraelastic when it contains a (microscopically) regular or irregular arrangement of permanent elastic dipoles that can be rotated; i.e., they can be rotated about one (elastic) field direction.

From now on, we assume that the individual dipoles are distributed so densely that one cannot macroscopically distinguish them, so that they can also be described by a dipole density that is everywhere independent of position as long as the body is regarded macroscopically. These assumptions simplify the deeper investigations with no loss of generality.

The mobility of the dipole shall not be considered at the moment. We can produce a single dipole in a continuum perhaps by the following operations:

One cuts out a small region B from the volume V and deforms it homogeneously and without resulting stresses (perhaps plastically, with or without a change of volume) by way of $\overset{P}{\varepsilon}_{kl}$. With the help of the surface forces $-A^i$ one makes this deformation go away in a purely elastic way ($\varepsilon_{kl} = -\overset{P}{\varepsilon}_{kl}$), such that the total deformation of B becomes:

$$\overset{P}{\varepsilon}_{kl} + \varepsilon_{kl} = 0. \tag{14}$$

At this point, one puts B back in the hole and lets it merge with its neighborhood. The total deformation of the entire medium is then null. It shall now be deformed so that an external surface force density A^j acts on the separation surface. Since the elastic constants are the same everywhere in the medium, the associated displacement field is, at least at large distances, the same as that of an elastic dipole:

$$P^{ij} = \int_f x^i dA^j\,. \tag{15}$$

The body now exists in a state of pure internal stress since the external forces, which maintain the elastic deformations of the internal region ($\varepsilon_{kl} = -\overset{P}{\varepsilon}_{kl}$), are now compensated for by the forces A^j.

As one knows, the permanent dipole is very simply expressed by the "imprinted" deformation $\overset{P}{\varepsilon}_{kl}$ when one once more returns to the state of eq. (14). With the help of Gauss's law ($\nabla_k \sigma^{jk} = 0$, $\nabla_k x^i = \delta^i_k$) one then has:

$$\int_f x^i dA^j = -\int_f x^i \sigma^{jk} df_k = -\int_B \sigma^{ij} dV = \int_B \overset{P}{\sigma}{}^{ij} dV\,, \tag{16}$$

hence:

$$P^{ij} = \int_B \overset{P}{\sigma}{}^{ij} dV\,, \tag{17}$$

when one has:

$$\sigma^{ij} = c^{ijkl}\,\varepsilon_{kl}, \qquad \overset{P}{\sigma}{}^{ij} = c^{ijkl}\,\overset{P}{\varepsilon}_{kl}\,. \tag{18}$$

Eq. (17) shows that one can also consider the dipole P^{ij} to be the total dipole that corresponds to a dipole density:

$$\overset{P}{\sigma}{}^{ij} = dP^{ij}/dV \tag{19}$$

in the volume V. Due to the relationship between $\overset{P}{\sigma}{}^{ij}$ and $\overset{P}{\varepsilon}_{kl}$ it seems reasonable to regard $\overset{P}{\sigma}{}^{ij}$ as an "imprinted stress"; cf., also RIEDER [**45**].

By substituting (18) into (17) one obtains the important result:

If one squeezes into an arbitrarily shaped hole (volume V) in an elastic continuum (elastic moduli c^{ijkl}) a piece of the same material, whose form differs from that of the hole by a homogeneous deformation, then one produces internal stresses that are the stresses of a permanent elastic dipole:

$$P^{ij} = c^{ijkl}\,\overset{P}{\varepsilon}_{kl}\,V \tag{20}$$

at a large distance. If one exchanges the hole and its contents around an inhomogeneous deformation $\overset{P}{\varepsilon}_{kl}$ then instead of $\overset{P}{\varepsilon}_{kl}$ showing up in eq. (20), one obviously sees the mean value $\int_B \overset{P}{\varepsilon}_{kl}\,dV\,/\,V$ of $\overset{P}{\varepsilon}_{kl}$.

One can therefore produce a paraelastic continuum in which one has removed very many holes in a homogeneous stress-free continuum and filled them in with material of the same type.

We would now like to establish that the formulas that were derived in the last section (11) to (13) are also valid for the "internal" dipoles (internal stress dipoles) that were discussed in this section. It is now clear that the state of the continuum outside of f is the same in either case when one chooses the same surface f, the forces A^j, the field ε_{kl}, and the position x^i of the dipole. From this, it follows that the boundary forces that act on the continuum that was imagined in either case produces the same displacement when one moves the dipole from x^i to $x^i + dx^i$. Under both operations the boundary forces perform the same work, which is equal to the variation of the elastic energy content of the body;

indeed, the same assumptions are also true in the case of the "external" dipole since its binding forces A^j collectively do no work since $\int_f dA^j = 0$. From this, it follows that the dipole itself experiences the same forces through the boundary forces (distant effect) or through the deformation field (near effect), resp., i.e., eq. (12) is also valid for an internal dipole. From this, one easily comes to eq. (11) when one takes into account the work that was done by the force K_k when it displaced a point with null deformation to a point with a deformation ε_{kl}. One obtains:

$$\int_{\varepsilon_{ij}=0}^{\varepsilon_{ij}} K_k dx^k = \int P^{ij} \nabla_k \varepsilon_{ij} dx^k = P^{ij} \int \nabla_k \varepsilon_{ij} dx^k = P^{ij} \varepsilon_{ij}, \tag{21}$$

and the result does not depend upon the path. Thus, one can think of eq. (11) as the potential energy of a dipole in a deformation field for the case of an internal dipole, as well.

Now, let there be a difference between the states in the cases of the external and internal dipoles. Inside of *f*, one has, e.g., a dilatation for the external dipole when the direction of A^j points outward; obviously, the opposite is true for the internal dipole. Thus, we assert that the integral (9) gives zero for the internal dipole, which clashes with the known theorem of COLONETTI [**56**] that the interaction energy (defined by eq. (9)) between internal and external stresses vanishes. Therefore, the potential energy of the internal dipole is not defined as it was in § 12, but we must include the work done by external forces (which also leads to the same result for external dipoles). For these problems, one should confer the careful analysis of ESHELBY [**55**], and particularly [**44**], pp. 95, et seq., and also [**8**], § 19.

The method of ESHELBY for treating the forces that originate in elastic singularities as well as the interaction energy is more general than ours since not only dipoles are considered. The same is true for the investigations of REIDER [**57**], which can be regarded as an extension of Eshelby's work. Compared to that case, the derivation that we gave in § 12 has, we believe, the advantage of relative simplicity.

We must further mention that eqs. (11, 12) can also be obtained in a third way. Due to the previously observed equivalence of the elastic dipole with an infinitesimal dislocation loop, one effortlessly obtains, e.g., (12) from the Peach-Koehler formula ([**55**], [**8**], pp. 86) for the force on a line element due to a dislocation through a stress field by integrating around the loop. We would like to at least write down the Peach-Koehler formula, which represents the analog of the Lorentz force in electrodynamics, and is fundamental to dislocation theory: In a stress field $\boldsymbol{\sigma}$ the line element $d\mathfrak{L}$ with Burgers vector $\mathfrak{b}$ experiences the force:

$$d\mathfrak{K} = d\mathfrak{L} \times \boldsymbol{\sigma} \bullet \mathfrak{b} \,. \tag{22}$$

We now come to the issue of the mobility of dipoles. Let the bodies I and II be two cylindrical paraelastic continua with just as many holes (perhaps spherical ones) of equal sizes. Let all of the holes in I be characterized by $\overset{P}{\varepsilon}_{xx} = a > 0$, and in II let there be just as many holes characterized by $\overset{P}{\varepsilon}_{xx} = a$, $\overset{P}{\varepsilon}_{yy} = a$, $\overset{P}{\varepsilon}_{zz} = a$. If both bodies were of the same

length before the introduction of the holes then afterward body I will be longer than body II. (One thinks of the production of paraelasticity as being perhaps the result of introducing oriented ellipsoids into a spherical hole in a ball. Under the partial relaxation that follows the introduction the ellipsoids that were oriented in the *z*-direction will be lengthened in the *z*-direction, along with the neighborhood of the ellipsoids, to some extent. Corresponding statements are true for ellipsoids that were oriented in the *x* and *y*-directions.)

If one now imposes an external stress σ^{ij} on the samples – perhaps a homogeneous traction σ_{xx} in the *x*-direction – then one will measure the same final deformations for both of them, i.e., the same elastic moduli. Then, as we pointed out, there are no integral interactions between internal and external stresses in the approximation of linear elasticity (COLONETTI's theorem).

The results are thus altered when one now gives the dipoles the freedom to rotate. (Perhaps one temporarily thinks of the connection between dipoles and their neighborhood as non-existent and rotates the dipole through maybe 90^o.) For example, one might rotate all of the $\overset{P}{\varepsilon}_{xx}$ and $\overset{P}{\varepsilon}_{yy}$ dipoles in body II into the *z*-direction, such that it has the same length as body I; it is thus plastically (or, if you wish, quasi-plastically) lengthened. The rotation of the dipole is thus nothing but a plastic deformation. If this rotation is the result of applied external traction forces then they can do work, and they therefore impose the constraint on the dipole that rotation be in the direction of the field.

In most of the applications only discrete orientations are possible for the dipoles, and the number of "flipped" dipoles depends linearly on the applied stress. For our example, we can thus assume that partial fracture that is proportional to the external stress takes the defect $\overset{P}{\varepsilon}_{xx}, \overset{P}{\varepsilon}_{yy}$ to the state $\overset{P}{\varepsilon}_{zz}$ under the action of the stress σ_{zz}. The macroscopically observed elastic moduli are thus seen to be degraded. Instead of the usual Hookean law:

$$\sigma^{ij} = c^{ijkl}\, \varepsilon_{kl}\,, \tag{23}$$

one obtains the law:

$$\sigma^{ij} = (c^{ijkl} + r_P^{ijkl})\, \varepsilon_{kl}\,. \tag{24}$$

ε_{kl} is the macroscopically observed deformation and r_P^{ijkl} is the paraelastic susceptibility of the sample.

The fact that the concept of a paraelastic continuum is physically meaningful rests on the fact that many real bodies behave like paraelastic ones. We will give examples of this in § 15.

§ 14. The diaelastic continuum

An elastic continuum will be called diaelastic when a (microscopically) regular or irregular distribution of elastic dipoles is induced in it by the action of an elastic field. One also speaks of the elastic polarization of the medium.

We make the same simplifying assumptions about the arrangement of induced dipoles as we did in the case of permanent dipoles in a paraelastic continuum. The medium then appears to be macroscopically homogeneous again.

A simple example shows the essential details. If one acts on a continuum with a small spherical hole in it with a hydrostatic pressure then this hole shrinks in the manner that an application of Hooke's law to the magnitude of the pressure would imply. Very simple elasticity-theoretic computations in this case show that the elastic field of the medium is combined with the homogeneous field that one obtains in the absence of the hole, and the field of an elastic dipole that one imagines to be at the midpoint of the hole, and indeed, in this particularly symmetrical case, a so-called center of compression (i.e., the dipole tensor is a spherical tensor here, cf. LOVE [**54**]). One says that an elastic dipole is induced in the hole by the action of the stress.

If the body has no hole in it, but it does have a small spherical defect with varying elastic constants (an "inhomogeneity") then the same statements are true for the rest of the body as before. The induced dipole strength is then generally something else, and indeed it has the same sign as in the case of the hole when the defect is softer, or on the contrary, harder than the rest of the body. A homogeneous stress prevails in the defect.

One obtains a sensible result when a body with spherical inhomogeneities is subjected not to a hydrostatic pressure, but to an arbitrary external force that would produce a constant stress $\overset{A}{\sigma}{}^{ij}$ in the absence of inhomogeneities (NIESEL [**59**], ESHELBY [**60**]). The stress field will also be homogeneous inside the defect ($\overset{I}{\sigma}{}^{ij}$), and in the exterior of the defect the stress field will be combined with stress field $\overset{A}{\sigma}{}^{ij}$ of an elastic dipole P_{ind}^{ij} that one imagines to be at the midpoint of the ball.

In order to ascertain $\overset{I}{\sigma}{}^{ij}$ and P_{ind}^{ij} a complicated boundary-value problem must now be solved. If $\overset{P}{\varepsilon}{}^{ij}$ is an imprinted (stress-free) deformation (§ 13), which must have a defect with the same elastic constants as the matrix, then if it is to appear to be a permanent dipole of precisely the same type and strength as P_{ind}^{ij} then one obtains, by way of example, when the defect and everywhere else are elastically isotropic (cf. ESHELBY [**60**], pp. 389-390):

$$\text{Spur}(\overset{P}{\varepsilon}{}^{ij}) = A\ \text{Spur}(\overset{A}{\varepsilon}{}^{ij}), \qquad \text{Dev}(\overset{P}{\varepsilon}{}^{ij}) = B\ \text{Dev}(\overset{A}{\varepsilon}{}^{ij}), \tag{25}$$

$$\text{Spur}(\overset{I}{\varepsilon}{}^{ij}) = (A\alpha + 1)\ \text{Spur}(\overset{A}{\varepsilon}{}^{ij}), \qquad \text{Dev}(\overset{I}{\varepsilon}{}^{ij}) = (B\beta + 1)\ \text{Dev}(\overset{A}{\varepsilon}{}^{ij}), \tag{26}$$

with:

$$A = \frac{K_I - K}{(K - K_I)\alpha - K}, \qquad B = \frac{G_I - G}{(G - G_I)\beta - G}, \tag{27}$$

$$\alpha = \frac{K}{K + 4G} = \frac{1}{3}\frac{1+\nu}{1-\nu}, \qquad \beta = \frac{2}{5}\frac{K + 6G}{K + 4G} = \frac{2}{15}\frac{4 - 5\nu}{1 - \nu}. \tag{28}$$

Dev stands for "deviator," K_I, G_I, and K, G are the compressive and shear moduli in the defect and matrix, resp., ν is the Poisson number of the matrix, and finally $\overset{I}{\varepsilon}{}^{ij}$ is the elastic deformation in the defect, hence $\overset{I}{\sigma}{}^{ij} = c^{ijkl}\ \overset{I}{\varepsilon}{}^{ij}$. For α and β we have $\frac{1}{3} \leq \alpha \leq 1$, $\frac{6}{15} \leq \beta \leq \frac{8}{15}$.

One easily verifies that $\overset{I}{\varepsilon}{}^{ij}$ and $\overset{A}{\varepsilon}{}^{ij}$ have the same sign whereas the sign of $\overset{P}{\varepsilon}{}^{ij}$ (hence, the induced dipole) is equal to the latter signs only when the defect is weaker than the rest

of the body. Furthermore, if $\overset{I}{\varepsilon}{}^{ij}$ is greater or less than $\overset{A}{\varepsilon}{}^{ij}$ depending on whether the defect is softer or harder, resp. The converse is true for the associated stresses.

All of these statements, in particular the ones about the homogeneity of the state in the defect, are also qualitatively valid for ellipsoidal defects, and also for anisotropic elastic constants in one or the other sub-bodies. Quantitatively, one then deals with substantially more complicated formulas instead of eqs. (25) and (26).

One can now produce a diaelastic continuum in which one has removed very many small cavities from a homogeneous stress-free continuum, and then filled them with matter with other elastic properties in a stress-free way, or simply left them empty. The diaelasticity brings about a reduction of the effective (or macroscopic) moduli of the sample in the event that the defects are softer than the rest of the body; in any other case, one obtains an increase in the moduli. The reduction effect comes about, e.g., as follows: We assume there are macroscopically homogeneous stresses σ^{ij} in the sample. σ^{ij} is then independent of the elastic properties of the sample and is given only by the external forces (by traction tests in the *z*-direction one has, e.g., σ_{zz} = const., with all other stress components vanishing). As we have already established, the mean deformation of the (weak) defect is greater than that of the rest of the body, but the opposite is true for the mean stress. Consequently, the mean stresses of the rest of the body must naturally be greater than the stresses σ^{ij}. The surplus consists of precisely the stresses that correspond to the polarization of the medium, which are the stresses $\sigma^{ij}_{ind} = c^{ijkl}\,\overset{P}{\varepsilon}_{kl}$ (cf. § 13, perhaps eq. (20)). Thanks to this induced stress, the mean deformation of the rest of the body is greater than the macroscopic deformation of the body with defects, and the mean deformation of the defect is even greater. Hooke's law:

$$\sigma^{ij} = c^{ijkl}\,\varepsilon_{kl}\,, \qquad (28)(?)$$

which is valid for a continuum without defects is no longer valid for the macroscopic deformation ε_{kl}, but one has the law:

$$\sigma^{ij} = (c^{ijkl} + r_D^{ijkl})\,\varepsilon_{kl}\,, \qquad (29)$$

with r_D^{ijkl} as the diaelastic susceptibility.

In the case of hard defects r_D^{ijkl} is obviously positive eq. (29).

Again, the physical reality of the concept of a diaelastic medium lies in the fact that many real bodies behave diaelastically.

In the former Gedanken experiments, we have either *jammed* certain cavity stress-free elastic continua with matter with the *same* elastic properties, or *filled* it with matter with *other* elastic properties in a *stress-free* way. The continua are then purely paraelastic (diaelastic, resp.). In general, one can *jam* matter with *other* elastic properties into these cavities. When the defects are mobile, this gives rise to a paraelastic reduction of moduli, and, at the same time, to a diaelastic reduction (increase, resp.) of moduli depending on whether the defects are softer or harder than the rest of the body.

§ 15. The physical reality of para- and diaelasticity

We shall now discuss some examples of the phenomena of para- and diaelasticity. The solid matter that one finds in Nature generally has a crystalline structure. An ideal crystal shows neither para- nor diaelasticity. However, if one brings – say – interstitial atoms into an ideal crystal then the forces between the adjacent atoms will be altered; something will generally be forced between them. An elastic deformation in the neighborhood of the interstitial atoms is produced that falls off at large distances like $1/r^3$. One can therefore macroscopically regard the interstitial atoms as an elastic dipole.

In general, the interstitial atoms do not push on all sides with the same strength. For example, a carbon atom in iron represents an elastic dipole with tetragonal symmetry. The components of this dipole may be determined experimentally, and one obtains ([**8**], pp. 153):

$$P_{xx} = 11.2 \text{ [eV]}, \quad P_{yy} = P_{zz} = 4.6 \text{ [eV]}, \quad P_{xy} = P_{yz} = P_{zx} = 0, \tag{30}$$

when one denotes the cubic axes of the iron crystal by x, y, z. Whereas, by eq. (12), a spherically symmetric dipole experiences a force only through the hydrostatic part of the deformation, the tetragonal dipole (30) responds with, e.g., a pure shear deformation $\varepsilon_{xx} - \varepsilon_{yy}$ in addition. For example, the deformation field of screw dislocation is a pure shear. The description of the carbon atoms in iron as (spherically symmetric) centers of compression that was attempted in previous approximations thus led to no interaction energy between screw dislocations and the carbon atoms in iron. This was qualitatively corrected by COCHARD, SCHÖCK, and WIEDERSICH [**61**] by considering the tetragonality of the distortion produced by a carbon atom. From this, it emerged that that the aforementioned interaction energy had the same order of magnitude as that of a carbon atom with a step dislocation, which produces a deformation field with strong hydrostatic components. This is an example of a type of result that always reappears.

For many material properties of commercial iron the role of the carbon atoms varies quite widely. The elastic behavior that was largely described by eqs. (11) to (13) helps to clarify many observed macroscopic properties of iron. There have already been some important results; e.g., on the meaning of the known stretching limit effects (COTTRELL [**12**], SCHÖCK, and SEEGER [**62**]), and the Snoek effect ([**63**], [**64**], cf. also [**8**], § 31). For the former, the motion of the carbon atoms in the deformation field of the dislocation of the iron is decisive, since it can lead to a blockage of the flow of the body's responsible dislocations. For the latter, it is the the flipping of the dipole into the field direction by the action of a traction stress that can generally give rise to a strong mechanical damping of the body. The Snoek effect is precisely the verification of paraelastic flipping behavior that was described in § 13.

We will now satisfy ourselves with these examples of the effects of interstitial atoms. The carbon atom in iron is a typical example for the numerous other possible combinations whose mechanical behavior is likewise largely governed by eqs. (11) through (13).

Some other important pointlike lattice defects are foreign atom substitutions and cavities. These exist at the regular lattice sites and likewise the force behavior in their neighborhood. In a simple cubic lattice they have lattice symmetry and can therefore be described as spherical dipoles. However, from eq. (13), a spherical dipole has no torque

so these lattice defects can give rise to no paraelasticity. This happens in very few symmetric lattices and all of the other ones are altered when perhaps two such foreign atoms or a pair of cavities appear together. Such pairs have – also in a cubic lattice – at most tetragonal symmetry and can thus lead to effects that are similar to the ones that we previously discussed in the example of interstitial atoms (cf., e.g., ZENER [**64**]).

All of the lattice defects that we discussed up till now are also diaelastic since they represent regions in which the elastic constants have been altered ("inhomogeneities"). Meanwhile, this diaelasticity can generally be proved only when there is no paraelasticity, as well, since permanent dipole moments are generally stronger than the induced ones. For example, two spherical dipoles in an isotropic medium exert no force on each other, because, from eq. (8), the deformation field of a compression center has no hydrostatic part. In this case, the interaction between two such defects ignores the reciprocal elastic polarization of the lattice defects (CRUSSARD [**65**], TELTOW [**66**], ESHELBY [**67**]). The diaelastic effect due to foreign atom substitution and cavities is therefore probably important in face-centered cubic lattices.

There are numerous other macroscopic pointlike lattice defects that lead us to suspect that they would produce elastic behavior that is similar to the aforementioned. Among them are such interesting defects as the *F*-centers in alkali halogenides (JACOBS [**68**]). Also, the mechanism for mechanical relaxation in iron that was discussed by BASS [**69**] ultimately rested on similar effects. We cannot go into the multiplicity of phenomena any further here, so we refer the reader to the soon-to-appear survey article of ESHELBY [**70**].

Another class of applications for the theory of para- and diaelasticity is that of bodies with microscopically large defects that have different properties from the surrounding matter. For the treatement of such bodies the results of NIESEL [**59**], and especially ESHELBY [**60**], for spherical and ellipsoidal defects have shown to be particularly useful. For these applications, we also refer to the article of ESHELBY [**70**] and mention only two more results here.

The elastic constants of macroscopically isotropic polycrystals may be computed exactly from the constants of single crystals by the fact that the polarizability of a crystallite that is composed of polycrystals vanishes in the mean [**71**].

b) The opposing obstruction that the crystallite presents to the plastic deformation of polycrystals, which comes about in a favorable orientation for the crystallite to begin flowing, is then the least favorable orientation, and it plays an essential role in less than 1% of all deformations. The method by which such a computation is accomplished follows from the remark that an initially flowing crystallite creates elastic dipoles, which corresponds to plastic deformation $\overset{P}{\varepsilon}_{kl}$ inside of a basic substance that is only elastically deformed (cf. eq. (20)). The internal stresses that are thus produced impede the favorable orientation and demand the unfavorable orientation in the crystallite to such a degree that all crystallites can flow steadily in less than 1% of all deformations in practice. The more qualitative theory of this behavior that was developed by GREENOUGH [**72**] therefore plays its greatest role for very small deformations (KRÖNER and DEBATIN, unpublished).

Here, we can give only a small sample of the multiplicity of paraelastic and diaelastic phenomena, but we still hope that this brief representation suffices to give an impression of the uses and possible applications of the theory of para- and diaelasticity.

V. Summary and Outlook

§ 16. Dislocations as elementary sources of internal stresses

In the previous treatment of the continuum theory of dislocations and internal stresses [**8**] the notion that "the dislocation is the elementary source of internal stresses" occupied a central position. This idea laid the groundwork for the following argument:

In a linear theory of internal stresses the stress tensor field satisfies the divergence condition:

$$\text{Div } \boldsymbol{\sigma} = 0, \tag{1}$$

and the material law has the usual form for the linear elasticity theory:

$$\sigma^{ij} = c^{ijkl}\, \varepsilon_{kl}. \tag{2}$$

Any symmetric tensor field $\boldsymbol{\sigma}$ that has the form Rot $\boldsymbol{\varphi}$ satisfies relation (1). Thus, in an infinitely extended medium there are also stresses when no external forces act on it, namely, the internal stresses. Obviously eqs. (1) and (2) are still insufficient to determine the state of the medium. We are missing the condition that we are dealing with a continuum that is connected with the body in question. At this point, we will make no presentation of the circumstances surrounding states with internal stresses. We assume that the body, which was initially in an ideal state, was subjected to such operations or processes that it ultimately remained in an altered state, namely, a state of internal stresses. Which type of operation or process we are talking about is unimportant; the possibilities are numerous. Therefore, since we are constructing a continuum theory we demand that:

a) The body, as a continuum, is connected initially and finally.

b) One can identify the body in the initial state with the body in the final state. We have not especially insisted on this condition up till now, since it is self-explanatory, so to speak. For the moment, it is therefore good to discuss it to some degree.

We would like to understand condition b) in the following manner: If we direct our attention to an arbitrary mass element of the body in the initial state (perhaps we color it), then this element will also be found again in the final state, although it will be altered. We thus exclude the possibility that the mass element might disintegrate into small pieces that are no longer connected. Furthermore, mass elements that are close in the initial state shall also be close in the final state, and finally, we will not allow the entire mass element to vanish or for new ones to appear. On the other hand, the introduction or removal of matter *from* the mass element is permitted. These are the demands that one must reasonably place on a continuum theory of solid bodies. They may be summarized mathematically in the form:

$$\text{Rot } \boldsymbol{\beta}^{G} = 0, \tag{3}$$

in which $\boldsymbol{\beta}^{G}$ is the total distortion tensor in the linear theory, and is given by the sum of the total deformation $\overset{G}{\boldsymbol{\varepsilon}}$ and the total rotation $\overset{G}{\boldsymbol{\omega}}$.

One can now remove a mass element from the final state that has an elastic deformation of $-\boldsymbol{\varepsilon}$ from the relaxation and thus the rotation $\boldsymbol{\omega}$ that the structure (the lattice) sustained compared to the initial state. We call the sum of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ the elastic distortion tensor $\boldsymbol{\beta}$ (in the linear theory), and, in place of eq. (3), we write:

$$\text{Rot}\,\boldsymbol{\beta} = -\,\text{Rot}\,(\boldsymbol{\beta}^G - \boldsymbol{\beta}) \equiv \boldsymbol{\gamma}. \tag{4}$$

It so happens that one can evaluate the right-hand side of this equation when one has sufficient precision in one's knowledge of the operations and processes that take the body from its initial state to its final one. We thus add eq. (4) to eqs. (1) and (2) as the third law. The system of equations thus obtained is then, as one can easily show, sufficient for the determination of the internal stress state from the "sources" $\boldsymbol{\gamma}$. The associated computation gives the incompatibility tensor $(\boldsymbol{\gamma}\times\nabla)^S$ and the stress functions, and is always the same as the computation that may have also led to $\boldsymbol{\gamma}$ as a result. If this were, e.g., a plastic distortion then $\boldsymbol{\beta}^G - \boldsymbol{\beta}$ would be the plastic distortion tensor $\boldsymbol{\beta}^P$, and $\boldsymbol{\gamma}$ would be identical with the crystallographic dislocation density $\boldsymbol{\alpha}$. If one treats an insertion of extra matter or some other quasi-plastic distortion then $\boldsymbol{\beta}^G - \boldsymbol{\beta}$ is the quasi-plastic distortion tensor $\boldsymbol{\beta}^Q$ and $\boldsymbol{\gamma}$ is the quasi-plastic dislocation density $\overset{Q}{\boldsymbol{\alpha}}$ ([31]).

We are now close to regarding $\boldsymbol{\gamma}$ as a generalized dislocation density, as well as regarding it as the origin of internal stresses. From this standpoint, the following statement is therefore valid: The dislocation is the elementary source of internal stresses. In this picture, it appears that, e.g., the elastic dipole is equivalent to an infinitesimal dislocation loop (the complete dipole tensor is equivalent to three dislocation loops), as was shown in [**8**].

We emphasize that this is only one possible standpoint, which has its analog in the theory of the magnetic fields of stationary currents in the Amperian equivalence of magnetic dipoles and infinitesimal current loops ([32]).

Another standpoint that one can take is to understand that the dislocation density only relates to the part of $\boldsymbol{\gamma}$ that describes a plastic distortion: These are then the crystallographic dislocations that are used in crystallography, whose Burgers vectors must principally be a lattice vector. From this standpoint, one then has other sources of internal stresses, namely, as long as one remains in mechanics (therefore temperature fluctuations and magnetic effects, etc,. and their associated stresses may be disregarded in the computation of the internal stresses), one has the pointlike lattice defects, which are elementary elastic dipoles. This is the standpoint that the differential geometric theory comes to next, in which we clearly distinguish between the extra matter and the dislocations. Thus, it was remarked in § 9 that the first standpoint also has its justification in the general theory.

Finally, it may be easily shown that there is a decomposition of the density that leads to no internal stresses, but only structure curvatures. One obtains internal stresses only when the incompatibility tensor $(\boldsymbol{\gamma}\times\nabla)^S$ does not vanish. For this reason, one can, by the

[31] In this section, we shall not go into the possibility that dislocations might end inside of the body, which is still problematic at present.

[32] This standpoint was also taken by DEHLINGER [**73**] in his abstracted presentation on the special needs of metallurgy, which especially suggests the possibility of applications.

same right as one did with $\boldsymbol{\gamma}$, declare that $(\boldsymbol{\gamma}\times\nabla)^S$ is the source of elementary internal stresses. Since this standpoint also has its justification in the general theory – here, the vanishing of $(\boldsymbol{\gamma}\times\nabla)^S$ means the vanishing of the Einstein tensor that is associated with the Christoffel symbols g'_{mlk} – we have shown that the basic concepts of the linear theory are abundantly manifest in the general theory. Which of the three standpoints one chooses in a special case depends upon the physical details of the problem. For example, the theory of para- and diaelasticity obviously belongs to the second standpoint.

§ 17. The unsolved problems

In this section we shall direct our attention to some problems in the general theory whose solutions remain a disturbing void to date.

For the moment, we remain in context of the stationary theory, and we begin with geometry. It is completely justified for one to say that in these spaces the reduced theory (nine degrees of freedom) is practically complete. Fundamental unsolved problems do not remain.

Something less far-reaching is our knowledge of the three degrees of freedom that belong to the (macroscopic) extra matter. Above all, the problem remains that was mentioned in § 9 of the formulation of the Einstein equations in the final state, for which one hopes to find solutions that might describe the connection between the deformation $\overset{Q}{\varepsilon}_{kl}$ that describes extra matter and the matter tensor B^{ij}, without the intermediary of introducing any distortion tensors (as we did in (80)), perhaps in the form:

$$B^{ij} = \tfrac{1}{4}\,\varepsilon^{jnm}\,\varepsilon^{ipq}\,(2\overset{Q}{\varepsilon})_{nmlq}. \tag{5}$$

The problem of the last three degrees of freedom is essentially unsolved. The appearance of dislocations that end in the interior poses great difficulties in the analysis. Since the applications have not yet suggested such dislocations up till now the requirement does seem very urgent, but from the standpoint of fundamental principles, it is naturally desirable.

On the other hand, there is actually an ongoing investigation of the problems connected with the Cosserat torque stresses, particularly the question of the material law and the question of solving the field equations. The possibility of understanding the internal stresses due to dislocations continuum mechanically in this way – not only macroscopically, but also microscopically – is very attractive since at this point in time very little can be said about the possible effects.

Certainly, the greatest deficiency in the foregoing situation is the absence of a dynamical theory, which, above all, should treat the motion of dislocations and foreign atoms (extra matter). The importance of lattice defects in all phenomena that pertain to solid bodies shows from the outset that this is certainly a rich domain of applications for such a theory.

The dynamical theory must go beyond the present state of electrodynamics; it is undoubtedly impossible to arrive at such a theory that expresses the dislocations (velocities, resp.) as a vector field, i.e., reduces to three functional degrees of freedom.

Such a dynamical theory can make interesting effects come to light. It was FRANK [**74**] ([33]) who first succeeded in computing the motion of an isolated dislocation, which then gave the sound velocity as the limiting velocity that the dislocation can attain at most, as with a particle with non-vanishing rest mass and the velocity of light. With increasing velocity a contraction of the stress field that can be described by formulas that are completely analogous to the ones that one finds in the case of electrons that are accelerating to higher velocities. There is also a sort of special relativity theory, in which the velocity of light is replaced by the velocity of sound. Since one generally has various sound velocities the multiplicity of phenomena is very large.

On the other hand, by the irradiation of samples in reactors it is now possible to bring light atoms with supersonic velocities into the body, which generally gives rise to a "sonic Čerenkov effect." For "slow particles" the associated Mach waves fall into the hypersonic regime. The most recent progress in this regime gives hope that in the foreseeable future it might be possible to experimentally examine (e.g., to count them) these waves and therefore the particles produced. The class of problems that is presently so important, that of radiation damage in solid bodies, now places an increasing number of possible applications at our disposal, not only for the stationary theory, but also for the yet-to-be defined dynamical theory.

Finally, we mention another large and urgent unsolved problem, namely, the ultimate specification of the connection between the continuum theory of dislocations and internal stresses and phenomenological plasticity theory (the unification of domains, resp.). Only through the selfsame application of this theory can we solve the problem that was stated in the Introduction under (1), which one could call the basic problem of continuum mechanics. The specification of the connection might then follow perhaps along the lines defined by BILBY, GARDNER, and STROH [**25**], to whose work we refer. KONDO [**21**] also pursues the same objective in a somewhat different way.

§ 18. Relationships with general relativity theory

Every expert on general relativity theory who studies the general continuum theory of dislocations and internal stresses will recognize the great similarity between the two theories, which will considerably enhance his understanding of the latter one. The general continuum theory has very much to thank general relativity for: Through the emergence of this theory the development of higher differential geometry was given considerable impetus, and it took on the present elegant form that it presents, as well as representing complicated relationships in the simplest ways.

In comparison to general relativity theory (its extension, resp.) the general continuum theory is free of any speculation. It now remains only to apply established laws to its derivation. This shows us that this theory can be regarded as a consistent theory of physical reality in which the notions of connection, Einstein tensor, etc., still work.

It has been know for some time that internal stresses have something to with Riemannian geometry, but previously no one did anything with this knowledge. It is noteworthy that we now know that Riemannian geometry is much too narrow in scope for continuum mechanics. The number of functional degrees of freedom must be raised from

[33] SAÉNZ has a given a summary presentation [**75**].

nine to fifteen: The general metric connection Γ_{mlk} appears in place of the Christoffel symbols. All of the states of the continuum that are allowed by Γ_{mlk} also occur in Nature.

The questions that emerge now are: Is there any physically sensible basis that allows us to subject the Universe to such a drastic reduction to a Riemannian geometry? Is there a basis for assuming that the Universe can be described by a connection that is less general than the most general metric connection? There is no such basis in continuum mechanics, and it might be quite difficult for us to find one for the Universe.

We have already regarded the continuum in its deformed state from the non-Euclidian standpoint, in which we joined with BILBY and his co-workers in defining the connection by the use of a non-Euclidian law of parallel translation. In the Riemann-Cartan continuum thus obtained, e.g., an isolated interstitial atom moves like a star does in the Universe. This has certainly not been verified, as of yet. However, there is certainly no doubt that just as the latter follow geodesic lines so do the interstitial atoms, as long as they are subjected to no forces. The forces that originate in the stresses on the interstitial atoms cannot therefore be computed, and they appear to be transformed away (in all probability) through the introduction of the new law of parallel translation, just as the gravitational forces are transformed away by the introduction of Riemannian parallelism. (Here, we shall ignore the details that the lattice structure stipulates for the real continuum in the case of interstitial matter.) This picture also shows the physical similarity of the two theories very incisively.

We shall not go into any speculation here, since we have many more such observations, because we believe that an ongoing investigation of the connections between general relativity and the general continuum theory of dislocations and internal stresses can be of considerable benefit to both theories.

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