"Über die algebraischen Strahlensysteme, in's Besondere über die der ersten und zweiten ordnung," Abh. d. Kon. Akad. d. Wiss. zu Berlin (1866), 1-120. Collected Papers, pp. 453-573.

On algebraic ray systems; in particular, on those of orders one and two.

E. E. KUMMER

[Read to the Academy of Science on 9 August 1866.]

Translated by D. H. Delphenich

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§ 1.

Definitions and general properties of algebraic ray systems.

The ray systems that will be regarded as algebraically determined in the sequel will be the ones whose general theory I have developed in the mathematical journal published by Borchardt in Bd. 57, page 189, *et seq*., namely, the ones that consist of a doubly-infinite family of straight lines, in such a way that the analytical representation of an arbitrary ray of the system will include two independent variables. Such a ray system shall be called *algebraic* when all of the equations that determine the rays are algebraic.

A finite number of rays go through each arbitrary point of space in any algebraic ray system; this number shall be called the *order* of the system. A ray system in which n rays go through any arbitrary point of space shall be called a *ray system of order n.*

The determination of the n rays of a ray system of order n that go through an arbitrarily-given point of the space depends upon an equation of degree n, which can never have more than n roots, except in the case where all of its coefficients are individually equal to zero, in which case, infinitely many rays will give satisfactory values. For an n^{th}-order ray system, one can never have more than n individual rays going through a point, but there can be points through which infinitely many rays of the system go, and which generally define a conical surface as a continuum. Those points, through which go, not n distinct rays, but infinitely many rays that define a conical surface, shall be called *singular points* of the ray system, and the cone that includes all of the rays that go through such a point shall be called the *ray cone* that belongs to this singular point. The case can also occur where not just a simply-infinite family of rays of the system go through certain points of space and define a ray cone, but a doubly-infinite family of rays; i.e., all of the straight lines that go through this point belong to the same ray system. The rays that go through such a point will, however, then define a complete system of rays for it, and indeed, a first-order ray system, since one ray of this system would go through each point of space, and each such first-order ray system would detach from the n^{th}-order system, such that one would have only one ray system of lower order instead of the n^{th}-order ray system, and in which such points would no longer occur.

If one draws an arbitrary plane through an algebraic ray system then a finite number of rays of the system will lie in it, in general; this number shall determine the *class* of the system. Namely, a ray system shall be said to be of *class k* when k rays of the system lie in any arbitrary plane. The determination of the k rays of a ray system of class k that lie in any chosen plane depends upon an equation of degree k, which can have no more than k roots, except when all of its coefficients are zero and any arbitrary values of the unknown quantities, and therefore infinitely many of them, will satisfy it. There can therefore also be planes in which infinitely many rays of the system lie, which, as a singly-infinite family of straight lines in the plane, will define the family of all tangents to a curve that lies in this plane. Such a plane, which includes a singly-infinite family of rays, shall be called a *singular plane* of the system, and the curve that is enveloped by it will be called a *planar ray curve.* A doubly-infinite family of lines that lie in a plane, which therefore encompasses all of the straight lines that lie in this plane, will yield a ray system in its own right, and indeed a ray system of order zero, since no ray will go through an arbitrary point of space, and of class one, since any arbitrary plane will cut out a ray that lies in the plane of this ray system, in such w way that the class of the system will be reduced by one unit.

In the theory of algebraic ray systems, it is of especial importance to distinguish the simple, irreducible ray systems from the composite, reducible ones that consist of two or more simple ray systems. Four quantities are necessary for the determination of an arbitrary straight line in space so, in the absence of a more precise determination, all lines in space will thus define a four-fold infinite system; should it be a two-fold infinite ray system then one would require two equations in order to determine the position of a straight line. However, two equations, which are necessary for the algebraic determination of a doubly-infinite ray system, do not ordinarily represent a pure, simple ray system, but one that is endowed with additional structures, which can be other ray systems, or also a ray cone or isolated rays. Here, one finds the same situation that one encounters in the theory of space curves, which can be represented by two equations – i.e., as the intersection of two surfaces – that are, in general, not pure, but endowed with additional structures, namely, with other curves or isolated points. The exclusion of the additional structures can be achieved for ray systems, just as it is for space curves, only by adding to the two required equations, yet another equation that depends upon them. A *simple*, or *irreducible*, ray system will be defined as one that is represented by nothing but algebraic equations, such that all of the rays that it contains will satisfy these equations. A *composite* – or *reducible* – ray system is, accordingly, one in which a part of the rays that comprise it – and indeed, a part that itself includes a two-fold infinite family of rays – defines a ray system that is definable by algebraic equations, in its own right. If two ray systems partially cover it in such a way that the rays that are common to both systems constitute a two-fold infinite family then they will not be irreducible. Therefore, if one desires to unite the one definite algebraic equation with the other one then one can represent the part that is common to them by these equations alone.

I shall choose x, y, z to be the quantities that determine each ray of the system, which essentially includes two independent variables, just as in the aforementioned treatment of the coordinates of the starting point of the ray, and the cosines of the angle that the ray defines with the three rectangular coordinate axes: viz., ξ, η, ζ. Since all of the algebraic equations that will be used in what follows to determine the ray system will be homogeneous relative to ξ, η, ζ, one can also think of there being quantities among them that are merely proportional to the three stated cosines, such that the equation $\xi^2 + \eta^2 + \zeta^2 = 1$ is superfluous. A definite starting point for all rays, as was assumed in the treatise cited, shall not be used in what follows. The lack of an equation in x, y, z that represents the starting point of all rays would make the ray system be triply-infinite if no other condition appeared in place of it; in order for it to be only two-fold infinite, it must fulfill the condition that when one chooses an arbitrary point of a given ray to be the starting point, the given ray will always be included among the n rays that go through this point. This condition can also be expressed as follows: All of the equations of the system, which can be always be represented as rational equations in the six quantities x, y, z, ξ, η, ζ, must still be equations of the same ray system when one replaces x, y, z with $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$, and for any arbitrary value of the quantity ρ; for any arbitrary value of the quantity ρ, the $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$ will then be the coordinates of an arbitrary point on the ray x, y, z, ξ, η, ζ, and for this arbitrary point of the ray, the equations of the ray system will then give precisely the same values for ξ, η, ζ that they give for the point x, y, z, such that every point of this ray can be assumed to be its starting point. By means of this condition, a single equation of a ray system will generally imply an entire series of other equations for the system along with it; if one converts x, y, z into $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$, and arranges the rational equation in the $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$ in powers of ρ then all of the terms that include the various powers of ρ must vanish individually. The new equations that arise in this way shall be called the *derived equations* of the given ones, and indeed the *first derived equation* will be the one that comes from setting the coefficient of ρ equal to zero in the equation that is ordered in powers of ρ, the *second derived equation* is the one that comes from setting the coefficient of ρ^2 equal to zero, etc. In each derived equation that follows, x, y, z will belong to a space where dimension is one less than before, while the dimension of the space that relates to ξ, η, ζ will be one unit higher than before in each successive derived equation. If the original equation is of degree m in x, y, z then it will imply m derived equations, in general, but they can also be fulfilled identically in certain cases, and thus either all of them or all of the ones that follow a certain one might not be present at all. The derived equations will be completely absent when the quantities x, y, z are present in the original equation only in the specific connection:

$$u = y\zeta - z\eta, \quad v = z\xi - x\zeta, \quad w = x\eta - y\xi,$$

such that these can represent one equation among the six quantities u, v, w, ξ, η, ζ.

The *focal surface* of an algebraic ray system order n and class k will be defined to be the geometric locus of all points of space for which two of the n rays that go through that point coincide. On the other hand, the focal surface can also be defined to be the surface that contacts all of the planes for which two of the k rays of the system that lie in them unite into a single ray. All rays of the system contact the focal surface twice, but conversely, not all of the straight lines that contact focal surfaces twice will belong to one and the same ray system. One can encounter the case, moreover, in which several completely different ray systems have one and the same focal surface, or – what amounts to the same thing – that the complete ray system that is defined by all of the doubly-contacting straight lines is a reducible one that consists of several different ray systems of lower orders and lower classes.

Any singular point of the ray system from which a ray cone emanates is likewise a singular point – namely, a *node* – of the focal surface. All of the rays of this cone, which, as rays of the system, contact the focal surface twice, will then have one of these two contact points in common at the center of the ray cone, which must then be a node since infinitely many of the tangents to the focal surface will emanate from it and contact that surface in yet a second point, and since each tangential plane of the ray cone will be a tangential plane of the focal surface at this point. The ray cone itself is the enveloping cone that lies in the focal surface at this node, or also a part of this enveloping cone, when it is reducible and consists of several cones of lower degree or even planes.

The focal surface of the algebraic ray system can also degenerate into curves, and indeed, either in such a way that just one sheet of the focal surface becomes a curve, or such that both sheets of the focal surface become curves. In place of the demand that each ray of the two sheets must contact the focal surface twice, one must, in turn, demand that it must go through the curve or through both curves that take the place of the focal surface. A curve through which all rays of a system go shall be called a *focal curve*. Any point of a focal curve is likewise a singular point of the ray system, since infinitely many rays emanate from it and define a ray cone. If both sheets of the focal surface degenerate into a focal curve then all rays of the system will go through these two curves. The two focal curves can, however, also coalesce into a single one; in that case, all of the rays of the system will cut this focal curve twice.

The system that is polar reciprocal to a ray system of order n and class k is a ray system of order k and class n; n rays that all go through a point in the first system will correspond to n lines in the polar system that lie in one and the same plane, and k rays that lie in a plane will correspond to k rays in the polar system that go through one and the same point. The focal surface of the polar reciprocal system will become the polar reciprocal surface of the focal surface of the given system, because the condition that a straight line should contact a surface twice will remain preserved in the polar reciprocal system.

For the simplest possible analytical representation of the ray system – namely, the one in which all ray systems of a specified order and class are exhausted – it is convenient to represent all of the ray systems that go into each other by collinear transformations of a single one, which can always be chosen in such a way that it includes 15 constants less than the most general system that subsumes all collinear ones. This simpler system, in turn, exhibits all of the essential properties of the entire group of systems that are collinear to it. The order and class of the ray system then remain unchanged under a collinear transformation, and also, all singular points and singular planes of the system remain essentially unchanged, since the ray cone that is associated with them, along with the planar ray curves, keep the same degree and the same singularities under such a transformation. The focal surfaces of the collinear systems are only collinear surfaces with the same degree and the same singularities. The transition from a certain ray system whose determining data are $x, y, z, \xi, \eta, \zeta$, to the most general collinear system with the determining data $x', y', z', \xi', \eta', \zeta'$, comes about when one substitutes the following values for $x, y, z, \xi, \eta, \zeta$:

$$x = \frac{p}{s}, \qquad y = \frac{q}{s}, \qquad z = \frac{r}{s},$$
$$\xi = sp' - ps', \qquad \eta = sq' - qs', \qquad \zeta = sr' - rs',$$

where:

$$\begin{aligned} p &= ax' + a_1 y' + a_2 z' + a_3, & p' &= a\xi' + a_1\eta' + a_2\zeta', \\ q &= bx' + b_1 y' + b_2 z' + b_3, & q' &= b\xi' + b_1\eta' + b_2\zeta', \\ r &= cx' + c_1 y' + c_2 z' + c_3, & r' &= c\xi' + c_1\eta' + c_2\zeta', \\ s &= dx' + d_1 y' + d_2 z' + d_3, & s' &= d\xi' + d_1\eta' + d_2\zeta'. \end{aligned}$$

§ 2.

First-order ray systems.

Since only one ray goes through each arbitrary point x, y, z of any first order ray system, the ratios of the three quantities ξ, η, ζ that determine the direction of this ray must be single-valued, algebraic, and therefore rational, functions of the three coordinates x, y, z of the starting point. One can then choose the two equations:

$$P\xi + Q\eta + R\zeta = 0, \qquad U\xi + V\eta + W\zeta = 0,$$

which are linear and homogeneous with respect to ξ, η, ζ, and in which P, Q, R, U, V, W are entire rational functions of x, y, z, to be the most general form of the two original equations of any first-order ray system. As a necessary, and likewise sufficient, condition for these two equations to, in fact, determine a first-order ray system, one can add that these two equations must be consistent with all of their derived equations; i.e., that all of these equations must yield the same values of the ratios $\xi : \eta : \zeta$ for arbitrary values of the x, y, z. The complete solution to the problem of finding all first-order ray systems, when regarded from a purely algebraic standpoint, consists of determining the six entire rational functions P, Q, R, U, V, W in all possible ways such that they satisfy the given condition. However, it seems simpler and more appropriate to find the solution to this problem in the following more geometric way:

The focal surface of any algebraic ray system is defined to be the geometric locus of all points of space through which two infinitely-close rays of the system go, but only one ray goes through a point for a first-order ray system, and when two of them go through it, infinitely many of them must go through it, as well, so it follows that any point of the focal surface must be a singular point of the system from which a ray cone emanates. It follows further from this that only focal curves can occur here, instead of the focal surface; if a ray cone emanated from each point of a surface then the ray system would necessarily be triply-infinite. Therefore:

I. *First-order ray systems have only focal curves, instead of focal surfaces.*

There are now two cases to distinguish: First, the one in which the first-order ray system has a single space curve that enters both sheets of the focal surface at once as its focal curve, which will be cut twice by every ray of the system, and secondly, the one in which two separate focal curves are present, each of which are cut once by all rays of the system.

If a single focal curve is present that is cut twice by all rays of the system then it must be assumed that it is irreducible. If it consists of several curves then one must consider only the ray systems that are associated with each of the individual irreducible curves. The rays of the system that go through any arbitrary point of space will be those straight lines that go through that point and cut the focal curve twice, so they will give the directions to the virtual double points of the focal curve when they are considered from that point. The orders of the complete ray systems that are associated with this focal curve then agree precisely with the number of virtual double points of this curve. Since the third-degree space curves always have one and only one virtual double point, it then follows that the ray systems that have a third-degree space curve for their focal curve and consist of all straight lines that cut it twice will be first-order systems. If one cuts such a system with an arbitrary plane then the focal curve will be cut in three points, and the three connecting lines for these three points will be the rays of the system that lie in this plane, and the system must then be of class three. Thus:

II. *The set of all straight lines that cut a third-degree space curve twice defines a ray system of order one and class three.*

In order to represent this general type of first-order ray system by equations, I set:

$$p = ax' + a_1 y' + a_2 z' + a_3\,, \qquad r = cx' + c_1 y' + c_2 z' + c_3\,,$$
$$q = bx' + b_1 y' + b_2 z' + b_3\,, \qquad s = dx' + d_1 y' + d_2 z' + d_3\,.$$

The three equations:

$$r^2 - qs = 0, \qquad sp - qr = 0, \qquad q^2 - pr = 0,$$

then represent, in turn, the most general equations for all third-degree space curves, and indeed only them, with no concomitant straight lines. Now let x, y, z be the coordinates of an arbitrary point in space, so for all values of ρ, $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$ will then be the coordinates of points on the straight line that goes through the point x, y, z in the direction that is determined by ξ, η, ζ. In order for this straight line to cut the curve twice, one must have that for two values of ρ, $x' = x + \rho\xi$, $y' = y + \rho\eta$, $z' = z + \rho\zeta$ will then be the three equations that are quadratic in ρ that one obtains when one substitutes these values x', y', z' into the three equations of the curve, so all three of them must have two roots. This condition gives the equations for the ray system as:

$$P\xi + Q\eta + R\zeta = 0, \qquad U\xi + V\eta + W\zeta = 0,$$

where:

$$\begin{aligned}
P &= a\ (r^2 - qs) + b\ (ps - qr) + c\ (q^2 - pr),\\
Q &= a_1(r^2 - qs) + b_1(ps - qr) + c_1(q^2 - pr),\\
R &= a_2(r^2 - qs) + b_2(ps - qr) + c_2(q^2 - pr),\\
U &= b\ (r^2 - qs) + c\ (ps - qr) + d\ (q^2 - pr),\\
V &= b_1(r^2 - qs) + c_1(ps - qr) + d_1(q^2 - pr),\\
W &= b_2(r^2 - qs) + c_2(ps - qr) + d_2(q^2 - pr).
\end{aligned}$$

Any two such equations for the ray system, which are quadratic in x, y, z, will have only one derived equation, since the second two derived equations will be fulfilled identically, and these two derived equations will be fulfilled by the original two equations themselves. The original two equations will be fulfilled identically for all of the points of the focal curve, and the two derived ones that agree with each other, which are of degree two in the ξ, η, ζ, will then give, in turn, each point of the second-degree ray cone that is associated with the focal curve.

The third-degree space curves are the only ones that have just one virtual double point; all space curves of higher degree will have more than one. Therefore, any complete ray system that consists of all straight lines that cut a space curve of higher degree twice must necessarily have an order that is higher than one. It is still not proved that the ray systems with a third-order focal curve are the only third-order systems that have a focal curve that enters both sheets of the focal surface at once. Therefore, one could possibly encounter the case in which the complete ray system with an irreducible focal curve of higher degree could be composed of several separate ray systems of lower order for which first-order ray systems could also occur. A precise examination of this problem is all the more indispensable since, as we will show later, the complete ray systems with an irreducible focal surface often, in fact, decompose into ray systems of lower order.

Thus, let an irreducible, n^{th}-degree, space curve be given as the focal curve of a complete ray system that consists of all straight lines that cut this curve twice. All rays that go through one and the same point of the focal curve will define a ray cone of degree $n - 1$ on which the entire focal curve lies; this ray cone will be an irreducible cone. If it decomposed into two or more cones of lower degrees then the irreducible focal curve, which must be cut by all of the rays of this cone, would have to lie partially on one cone and partially on the other one, which is impossible, since an irreducible space curve that lies partially on irreducible surface must lie upon it completely. Since this ray cone is irreducible, all of the straight lines that lie in it must be rays of one and the same irreducible system, and since the same thing is true for all of the cones that emanate from the continuously-following points of the focal curve, it will then follow that this entire family of ray cones must belong to one and the same irreducible ray system. All of the rays that lie in this family of ray cones will, however, completely exhaust all of the straight lines that cut the focal curve twice, and only in the case where the focal curve has real double points does one add all of the straight lines that go through such a double point to them, which will define a ray system of order one and class zero, in its own right. Thus:

III. *The set of all straight lines that cut an irreducible space curve twice always defines a single irreducible ray system, if one excludes the lines that go through a real double point of the focal curve and cut it at no other point.*

Since, furthermore, any space curve of degree higher than three will still always have more than one virtual double point, even when it has real double points, and since the number of virtual double points will determine the degree of the ray system that is associated with this curve, one can now infer with certainty:

IV. *Except for the ray systems with a third-degree focal curve, there are no other first-order ray systems that have an irreducible focal curve that belongs to both sheets of the focal surface at once.*

We shall thus investigate first-order ray systems that have two distinct focal curves, and whose rays all cut the one focal curve, as well as the other one. Either of the two focal curves, one of which will have degree m, while the other will have degree n, is to be regarded as an irreducible curve; if one of the two curves consisted of curves of lower degree then the ray system itself would decompose into several special ray systems. An n^{th}-degree ray cone emanates from an arbitrary point of the focal curve of degree m that goes through the n^{th}-degree focal curve and which is irreducible, since the n^{th}-degree curve that lies on it is an irreducible one. All of the rays that lie on such a cone will then belong to one and the same irreducible system. If one now lets the center of this cone move continuously along the m^{th}-degree curve then one will obtain a continuous family of n^{th}-degree ray cones whose rays must all belong to one and the same irreducible system. However, all rays of this family of ray cones will collectively subsume straight lines that intersect both focal curves at once, with the one exception of those straight lines that go through an intersection point of the two focal curves, when such a thing exists. Any arbitrary straight line that goes through the intersection point of both focal curves will fulfill the condition of cutting both focal curves, so it will belong to the complete ray system that has these two curves as its focal curves. The straight lines that go through an intersection point will, however, define a first-order ray system in their own right that may be separated from the complete ray system. If one now takes an arbitrary point of space and constructs the two cones of degrees m and n from it, each of which goes through one of the two focal curves, then these two cones will intersect in $m \cdot n$ straight lines that cut the two focal curves at once. The complete ray system will then be of order $m \cdot n$; however, if the two focal curves intersect at m points then one can remove n first-order ray systems from the complete ray system, and what remains will be an irreducible ray system of order $mn - \mu$. A first-order ray system with two different focal curves can then exist only under the condition that one has $mn - \mu = 1$; i.e., that the two focal curves have a number of intersection points that is smaller by one than the product of their degrees.

In order to now investigate whether, or under which conditions, two space curves of orders m and n, resp., can have $mn - 1$ intersection points without collapsing into a single curve, I draw one of the cone surfaces of degree $n - 1$ through the curve of degree n, such that its center lies on the curve itself. The m^{th}-degree curve, which, by assumption, cuts the n^{th}-degree curve in $mn - 1$ points, must then cut this cone of degree $n - 1$ in at least $mn - 1$ points. The number of intersection points of the m^{th}-degree curve with the cone of degree $n - 1$ will be, however, $m(n - 1)$, so one must then have $m(n - 1) \geq mn - 1$ if the m^{th}-degree curve is to not lie completely in the cone of degree $n - 1$. The latter is, however, impossible, since the same thing would then be true for each of the infinitely many cones of degree $n - 1$ that one can construct for the n^{th}-degree curve, so the m^{th}-degree curve must lie on each of these cones, and therefore coincide completely with the n^{th}-degree curve. The condition $m(n - 1) \geq mn - 1$ is, however, not fulfilled except when $m = 1$, and, as a result, $\mu = n - 1$. Since this condition for the existence of first-order ray systems with two distinct focal curves at once is sufficient, one will then have the theorem:

V. *The ray systems that have a straight line and an n^{th}-degree space curve that cuts it at $n - 1$ points for their focal curves are all ray systems of order one and class n, and there are no other first-order ray systems that have two distinct focal curves except for them.*

One recognizes that, in fact, two such focal curves will always yield a first-order ray system from the fact that the rays that go through an arbitrary point of space must lie in the plane that goes through the straight focal line, and intersect only at one point that is not the intersection point of the two curves. The fact that this system has class n follows from the fact that an arbitrary plane cuts the straight focal line in one and the other of n points, and that the n straight lines that go from an intersection point to the n intersection points with the n^{th}-degree focal curve will comprise the n rays of the system that lie in the plane.

As the simplest special case of this general type of first-order ray system, we can mention: The ray systems of order one and class one that have two straight, non-intersecting focal lines, and furthermore, the ray systems of order one and class two whose focal curves are a conic section and a straight line that does not lie in the same plane as that conic section but intersects it, etc.

In order to represent this type of first-order ray system by equations, I shall take the straight focal line to be the z-axis; the most general equations for all of the n^{th}-degree curves that cut the z-axis in $n - 1$ points are then:

$$\phi(x', y') + \phi_1(x', y') = 0, \qquad z' \psi_1(x', y') + \psi(x', y') = 0,$$

where $\phi, \phi_1, \psi, \psi_1$ are homogeneous functions of x' and y', of degrees $\mu + 1$, $\nu + 1$, ν, resp., while $\mu + \nu + 1 = n$. This n^{th}-degree curve has ν asymptotes, which are parallel to the z-axis, and which yield ν infinitely-distant intersection points of the curve with the z-

axis. In fact, one has $z' = \infty$ for $\psi_1(x', y') = 0$, and when the ν values of y' / x' that this equation yields are substituted into the equation $\phi(x', y') + \phi_1(x', y') = 0$ that will give ν associated values of x' and y' that are not infinite, in general. The first equation, which does not include z', and which then represents the projection of the curve onto the $x'y'$-plane, shows that this projection has a μ-fold point at the origin of the coordinates, so, along with the ν infinitely-distant intersection points, μ generally finitely-distant intersection points of the curve with the z-axis will be present. One obtains the equations of the ray system that has the z'-axis and this curve for its focal curves when one subjects the general straight line at the point x, y, z that goes in the direction ξ, η, ζ to the conditions that it cut the z'-axis, as well as the n^{th}-degree curve. The first condition immediately gives:

$$y\xi - z\eta = 0$$

as the one equation of the ray system. The second condition demands that if x', y', z' are replaced with the coordinates of any point of the straight line $x + \rho\xi, y + \rho\eta, z + \rho\zeta$ in the equations of the focal curve then they would both be satisfied for the same values of ρ. By means of the first equation of the ray system, one has $y + \rho\eta = y/x(x + \rho\xi)$, so if one were to set:

$$x' = x + \rho\xi, \qquad y' = (x + \rho\xi), \quad z' = z + \rho\zeta$$

then the two equations of the curve would give:

$$(x + \rho\xi)\ \phi(x, y) + x\phi_1(x, y) = 0$$
$$x(z + \rho\zeta)\ \psi_1(x, y) + (x + \rho\xi)\ \psi(x, y) = 0,$$

and the elimination of ρ from them would yield:

$$(z\ \phi(x, y)\ \psi_1(x, y) - \phi_1(x, y)\ \psi(x, y))\ \xi = \psi_1(x, y)\ (\phi(x, y) + \phi_1(x, y))\ \zeta$$

as the second equation of the ray system.

From the first-order ray systems, which were completely exhausted in the foregoing, one can likewise obtain all first-class ray systems when one forms the polar reciprocal system. Since a straight focal line becomes a straight focal line under it, but a curved focal line becomes a developable focal surface, it will follow that all ray systems of class one can have only straight lines for their focal lines and only developable surfaces for their focal surfaces.

The ray system of order one and class three that has a third-order focal curve will have a ray system of order three and class one for its polar reciprocal, and will have a developable surface of degree four for its focal surface that consists of all of the straight lines that contact this surface twice. An arbitrary plane cuts a fourth-degree curve with three cusps out of this focal surface, and one of them, in fact, has only a single double tangent, which gives the ray that lies in that plane. The lines of intersection of the three planes that make up the enveloping cone of this fourth-degree developable surface that emanate from an arbitrary point of space will be the three rays of the system that go through these arbitrary points and contact the focal surface twice.

The ray system of order one and class n that has a straight focal line and a focal curve of degree n that intersects it $n - 1$ times will have a ray system of order n and class one for its polar system, and will have a straight focal line and a developable focal surface of class n that will be contacted by the focal line at $n - 1$ points. We separate out $n - 1$ ray systems of order zero and class one from all the straight lines that cut the straight focal lines and contact the developable focal surface. All of the straight lines that go through an arbitrary point of the straight focal line and contact the developable focal surface will, in fact, lie on n planes, of which, the $n - 1$ of them that contact the developable surface at its $n - 1$ contact points with the straight focal line since all points of the straight focal line will remain themselves invariant will thus give the $n - 1$ particular ray systems of order zero and class one. If one cuts the system with an arbitrary plane then a curve of class n will be cut out of the focal surface, which will be contacted by n of the intersection points of this plane with the straight lines going to the straight focal line. However, these straight lines that cut the focal line and contact the focal surface will belong to $n - 1$ of the $n - 1$ special ray systems of order zero and class one, so only one of them will remain as the ray of the system of order n and class one that lies in this plane. The rays of the system that go through an arbitrary point of space must all lie in the plane that goes through the straight focal line. This plane cuts out a curve of class n from the focal surface, and the n tangents themselves that go through this arbitrary point will be the n rays of the n^{th}-order system that go through this point.

§ 3.

Second-order ray systems in general.

Since two rays go through each arbitrary point in a second-order algebraic ray system, the ratios $\xi : \eta : \zeta$ that determine the directions of the rays that go through the point x, y, z will be determined from the equations of the ray system as two-valued algebraic functions of x, y, z. Therefore, among the three quantities ξ, η, ζ, one of them must necessarily be a homogeneous linear one, and a homogeneous quadratic equation will come about, so one will have two equations of the form:

$$P\xi + Q\eta + R\zeta = 0, \tag{1}$$

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2D\eta\zeta + 2E\zeta\xi + 2F\xi\eta = 0, \tag{2}$$

in which $P, Q, R, A, B, C, D, E, F$ are entire rational functions of x, y, z. These two equations generally imply two sequences of derived equations that must be fulfilled, along with the two original ones, if this is to actually represent a ray system, and one obtains all possible second-order ray systems when one determines the nine quantities that appear as coefficients in these two equations as entire rational functions of x, y, z in all possible ways such that all of these derived equations will be fulfilled by the values of the ratios $\xi : \eta : \zeta$ that are given by the original two, and indeed, for all arbitrary values of x, y, z.

If one denotes the coordinates of any arbitrary point along a ray that goes through x, y, z in the direction ξ, η, ζ by x', y', z' then one will have:

$$x' - x : y' - y : z' - z = \xi : \eta : \zeta,$$

and one can then replace ξ, η, ζ with the proportional quantities $x' - x$, $y' - y$, $z' - z$ in the homogeneous equations (1) and (2), from which, the first equation will represent a plane that goes through the point x, y, z, and the second one will represent a second-degree cone whose center lies at x, y, z. The two rays of a second-order system that go through a point of space will be determined from the two equations (1) and (2) as the two intersection points of a plane and a second-degree cone whose center lies in this plane. Equation (1), as the equation of the plane that both of the rays that go through the point x, y, z lie in, can be varied in an infinitude of ways, since a second-degree cone will be completely determined by not just two given edges, but by five of them initially. In fact, one can also multiply the first equation by an arbitrary expression of the form $U\xi + V\eta + W\zeta$ and add the product to the second equation without changing the system of these two equations and without changing the fact that the second equation represents a second-degree cone that includes the same two rays.

The first derived equation of (1), which one obtains when one replaces x, y, z with $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$, and sets the coefficients of ρ equal to zero in the resulting equation, becomes, when arranged in powers of ρ:

$$\frac{dP}{dx}\xi^2 + \frac{dP}{dy}\eta^2 + \frac{dP}{dz}\zeta^2 + \left(\frac{dQ}{dz} + \frac{dR}{dy}\right)\eta\zeta + \left(\frac{dR}{dx} + \frac{dP}{dz}\right)\zeta\xi + \left(\frac{dP}{dy} + \frac{dQ}{dx}\right)\xi\eta = 0; \tag{3}$$

when it is not just the identity $0 = 0$, this will then likewise represent a second-degree cone that has its center at the point x, y, z, and on which lie the rays that go through that point and which will be cut out by the plane that equation (1) represents. Equation (2) can then always be replaced by the first derived equation of (1), with the exception of the case in which equation (1) has no derived equation, at all. In that particular case, where the first derived equation of (1) vanishes identically, one will have the equations:

$$\frac{dP}{dx} = 0, \qquad \frac{dQ}{dy} = 0, \qquad \frac{dR}{dz} = 0,$$
$$\frac{dQ}{dz} + \frac{dR}{dy} = 0, \qquad \frac{dR}{dx} + \frac{dP}{dz} = 0, \qquad \frac{dP}{dy} + \frac{dQ}{dx} = 0,$$

which must be true for all arbitrary values of x, y, z. Another differentiation of these six equations with respect to x, y, and z will show that all of the second partial differential quotients of the three quantities P, Q, R must be equal to zero, so these three quantities can only be linear functions of x, y, z. Their complete determination gives:

$$\begin{aligned} P &= a_2 y - a_1 z - b, \\ Q &= a\ z - a_2 z - b_1, \\ R &= a_1 x - a\ y - b_2, \end{aligned} \tag{4}$$

where a, a_1, a_2, b, b_1, b_2 are arbitrary constants. Thus:

VI. *The second-order ray systems will be, in general, completely determined by a linear equation of the form:*

$$P\xi + Q\eta + R\zeta = 0,$$

and its derived equations, and only in the one special case where this linear equation has the form:

$$(a_2\, y - a_1\, z - b)\; \xi + (a\, z - a_2\, x - b_1)\; \eta + (a_1\, x - a\, y - b_2)\; \zeta = 0$$

must one add a second quadratic equation in ξ, η, ζ *that is independent of it in order to determine the ray system.*

The focal surface of a second-order ray system will then be determined in such a way that two infinitely-close rays of the system must go through each point of it. The plane (1) and the cone (2), whose intersection gives the two rays that go through the point x, y, z, must then contact each other if the point x, y, z lies on the focal surface. As is known, this condition will be expressed by the equation:

$$(5) \qquad \begin{vmatrix} A & F & E & P \\ F & B & D & Q \\ E & D & C & R \\ P & Q & R & 0 \end{vmatrix} = 0,$$

which is therefore the equation of the focal surface. Since one can also take the first derived equation of (1) in place of equation (2), with the exception of the special case that was given in Theorem VI, one can then also represent the equation of the focal surface in the following form:

$$(6) \qquad \begin{vmatrix} 2\dfrac{dP}{dx} & \dfrac{dP}{dy}+\dfrac{dQ}{dx} & \dfrac{dR}{dx}+\dfrac{dP}{dz} & P \\ \dfrac{dP}{dy}+\dfrac{dQ}{dx} & 2\dfrac{dQ}{dy} & \dfrac{dQ}{dz}+\dfrac{dR}{dy} & Q \\ \dfrac{dR}{dx}+\dfrac{dP}{dz} & \dfrac{dQ}{dz}+\dfrac{dR}{dy} & 2\dfrac{dR}{dz} & R \\ P & Q & R & 0 \end{vmatrix} = 0.$$

However, these equations do not generally represent the focal surface concisely, since they are ordinarily endowed with superfluous factors that give certain ancillary structures to the focal surface that one must be freed of, as will be shown in the sequel. These focal curves are also included in those cases where the second-order ray system has focal curves instead of focal surfaces, which are, in fact, double curves of the surface that is given by equations (5) or (6), since the fact that a ray goes through a double curve of a surface, as an intersection in two infinitely-close points of the surface, is to be deemed a point of contact, and therefore the condition that any ray of the system must contact the focal surface twice will also be fulfilled in such a way that it contacts it only once, and in addition, goes through a double curve of it, or in such a way that it cuts the double curve twice.

Since there are second-order ray systems that have real focal surfaces that do not degenerate into focal curves, the second-order ray systems subdivide into the following three distinct types:

1. Ray systems that have only focal curves.
2. Ray systems that have one focal curve and one focal surface.
3. Ray systems that have no focal curves, but only focal surfaces.

These distinct types will now be considered individually.

§ 4.

Second-order ray systems that have focal curves instead of focal surfaces.

If a second-order ray system has a single irreducible focal curve that is cut twice by all rays then two rays that emanate from an arbitrary point in space will necessarily lie in the directions of two apparent double points of the focal curve that is considered for that point. The focal curve must then be a space curve with two apparent double points, and it also cannot have more than two apparent double points, because otherwise more than two rays would emanate from each point of space, which, from Theorem III must belong to an irreducible system. The fourth-order space curves that arise from the complete intersection of two second-order surfaces are, however, known to be the only curves that have two – and no more than two – apparent double points. The one irreducible focal curve of a second-order ray system must then be necessarily one such space curve, and such a focal curve must also always belong to a second-order ray system. If one intersects such a system with an arbitrary plane then four points will be cut out of the focal curve, and the six straight lines that go through any two of these four points will be the six rays of the system that lie on this plane, so it will be of class six. Therefore:

VII: *The set of all straight lines that cut a space curve twice that is defined by the intersection of two second-degree surfaces defines a ray system of order two and class six, and there are no other second-order ray systems that have a single irreducible focal curve.*

If $\phi = 0$ and $\psi = 0$ are two second-degree surfaces whose intersection gives the focal curve then the ray of the system that emanates from the point x, y, z in the direction ξ, η, ζ must cut both surfaces at the same two points, so if one replaces x, y, z with $x + \rho\,\xi, y + \rho\,\eta, z + \rho\,\zeta$ at $\phi = 0$ and $\psi = 0$ then these two points must give equations in the same two values of ρ that are quadratic in ρ. The two condition equations that are necessary for this are two of the equations that determine the system. One of them, namely:

$$(1) \qquad \left(\phi\frac{d\psi}{dx} - \psi\frac{d\phi}{dx}\right)\xi + \left(\phi\frac{d\psi}{dy} - \psi\frac{d\phi}{dy}\right)\eta + \left(\phi\frac{d\psi}{dz} - \psi\frac{d\phi}{dz}\right)\zeta = 0,$$

however, succeeds in determining the ray system completely, because its first derived equation gives the other one for the determination of the necessary equation for any two rays that emanate from any point in space. Equation (1) also gives yet another derived equation that is of degree three and which, in turn, gives yet another third-degree cone on which the two rays that emanate from a point must lie; a third derived equation comes about because it is fulfilled identically. Equation (1), as well as its first derived equation, is fulfilled identically for all points x, y, z that lie on the focal curve $\phi = 0$, $\psi = 0$, so that would not yield a way of determining ξ, η, ζ – i.e., the directions of the rays that go through such a point – so only the second derived equation will remain as the equation of the third-degree cone of rays that emanate from any point of the focal surface.

The second-order ray systems that have two different focal curves will be obtained by the same method that was carried out completely in § 2 for the corresponding type of first-order ray system, so we can discuss it more briefly here. Here, just as in the previously-treated cases, all straight lines that cut the two irreducible focal curves of degrees m and n, with the exclusion of the ones that go through only the intersection points of these two curves, must belong to one and the same irreducible ray system of the two straight lines that go through the intersection point of the two focal curves, but define just as many first-order ray systems of class zero as the number of intersection points that are present. It then follows from this in the same way that these two curves of degrees m and n can be focal curves of a second-order ray system only when they intersect at $mn - 2$ points. The necessary condition for two irreducible space curves of orders m and n to intersect at $mn - 2$ points without coinciding in just one of them is, in turn, obtained in the same way as $m\,(n - 1) \geq mn - 2$ and $m\,(m - 1) \geq mn - 2$, and because this condition will be fulfilled only in the following two cases – firstly, when m and n are both equal to two, and secondly, when one of the two numbers is equal to one – it will then follow that:

VIII: *Second-order ray systems with two distinct focal curves can come about only when either both focal curves are conic sections that cut it in two points or when one of them is a straight line and the other one is an n^{th}-degree curve that cuts this straight line at $n - 2$ points.*

The fact that two conic sections that lie in different planes and intersect at two points will, in fact, give a second-order ray system as their focal curves follows from the fact that the two second-degree cones that go through these two conic sections at an arbitrary point of space will intersect in four straight lines, two of which will always go through the two intersection points of the conic section, and in turn, will belong to two special first-order systems such that the other two straight lines must belong to a second-order ray system. If one draws an arbitrary plane through such a ray system then each of these two second-degree focal curves will intersect at two points and the four straight lines that connect the two intersection points of the one focal curve with the two intersection points of the other one will be the four rays of the system that lie in this plane, so the system will then have class four. Therefore:

IX: *The set of all straight lines that go through two conic sections that intersect each other twice and lie in two distinct planes defines a ray system of order two and class four, with the exception of the ones that go through just the intersection points themselves.*

Ray systems of this kind can also be regarded as special cases of the ones that were given in Theorem VII that have a single focal curve of degree four. Namely, if one lets one of the two second-degree surfaces whose intersection is the fourth-degree focal curve go to a system of two planes then two conic sections that intersect at two points will enter in place of that curve. The class of the system will be reduced by two units, in such a way that the rays that lie in the planes of the two conic sections will define two ray systems of class one and order zero, which will drop out. One thus obtains the analytical representation of this kind of ray system immediately from that of present kind when one replaces ψ with pq, where p and q are two linear functions of x, y, z. The original linear equation in ξ, η, ζ, which determines the ray system completely along with its two derived ones, will then be:

$$P\,\xi + Q\,\eta + R\,\zeta = 0,$$

where

$$(2)\qquad \begin{aligned} P &= \phi p\frac{dq}{dx} + \phi q\frac{dp}{dx} - pq\frac{d\phi}{dx}, \\ Q &= \phi p\frac{dq}{dy} + \phi q\frac{dp}{dy} - pq\frac{d\phi}{dy}, \\ R &= \phi p\frac{dq}{dz} + \phi q\frac{dp}{dz} - pq\frac{d\phi}{dz}. \end{aligned}$$

One recognizes the fact that a straight focal line, along with an n^{th}-degree focal line that cuts it at $n - 2$ points, will always, in fact, yield a second-order ray system immediately from the fact that the plane that is drawn through an arbitrary point of space and the straight focal line will cut n rays out of the n^{th}-degree cone through the n^{th}-degree focal curve at the same point of space, $n - 2$ of which will consistently go through the $n - 2$ fixed intersection points of the two focal curves, and will, in turn, define $n - 2$ first-order ray systems, such that only two rays will remain, which will belong to a second-order ray system. This ray system will be of class n, so an arbitrary plane will cut the n^{th}-degree focal curve at n points, and the straight lines that go from these n points to the one intersection point of the plane with the straight focal line will be the n rays of the system that lie in that plane. Hence:

X: *The set all straight lines that go through a given straight line and an n^{th}-degree curve that cuts it at $n - 2$ points defines a ray system of order two and class n, with the exception of the ones that go through just the $n - 2$ intersection points.*

If one chooses the straight focal line to be the z-axis then one can express an n^{th}-degree curve that cuts it at $n - 2$ points in the most general way by means of the two following equations:

$$(3)\qquad f + f_1 + f_2 = 0, \qquad\qquad z\,f + g + g_1 = 0,$$

where ϕ, ϕ_1, ϕ_2, f, g, g_1 are complete and homogeneous functions of only x and y that have degrees μ, $\mu - 1$, ν, $\nu + 1$, ν, respectively. The first of these equations represents the projection of the curve onto the xy-plane, which is then a plane curve of degree μ with an $\mu - 2$-fold point at the coordinate origin, which then corresponds to $\mu - 2$ intersection points of the curve with the z-axis. One will have $z = \infty$ for the ν values of the y / x that satisfy the equation $f = 0$, from the second equation, and the first equation will give two values of x and y to each of these values of y / x, which are finite, in general. These values will give 2ν asymptotes to the curve that are parallel to the z-axis, so they will have, in turn, 2ν infinitely-distant intersection points with the straight focal lines, and the total number of all of these intersection points will therefore be $\mu + 2\nu - 2$. Since the curve itself is of degree $\mu + 2\nu$, it corresponds completely to the conditions that were posed.

One obtains the first equation of the ray system whose focal curves are that curve and the z-axis immediately when one requires that both of the rays that emanate from the arbitrary point x, y, z must cut the z-axis:

$$y\,\xi - x\,\eta = 0. \tag{4}$$

Since this equation of the system, which is linear in ξ, η, ζ, has no derived equations, a second equation of the system must be determined in some other way, which one finds when one replaces x, y, z with $x + \rho\,\xi$, $y + \rho\,\eta$, $z + \rho\,\zeta$ in the two equations of the focal curve and then eliminates ρ. If one then observes that from the first equation of the system one has:

$$y + \rho\,\eta = \frac{y}{x}(x + \rho\,\xi)$$

then one will obtain:

$$\begin{aligned}&(x + \rho\,\xi)^2\,\phi + x\,(x + \rho\,\xi)\,f_1 + x^2 f_2 = 0,\\ &x\,(z + \rho\,\zeta)\,f + (x + \rho\,\xi)\,g + x\,g_1 = 0,\end{aligned}$$

and the elimination of ρ will yield:

$$(x\,f\,\zeta - z\,f\,\xi - g_1\,\xi)^2\,f - (x\,f\,\zeta - z\,f\,\xi - g_1\,\xi)\,(x\,f\,\zeta + g\,\xi)\,\phi_1 + (x\,f\,\zeta + g\,\xi)\,\phi_2 = 0 \tag{5}$$

as the second equation of the ray system.

§ 5.

Second-order ray systems that have one focal curve and one focal surface.

If a ray system has one focal curve and one focal surface then all rays of the system must go through focal curve and likewise contact the focal surface. The focal curve, as well as the focal surface, are both assumed to be irreducible, because if one of them consisted of two separate components then the ray system itself would also have to consist of two separate components. For the examination of all second-order ray systems that belong to that type, it is preferable to distinguish the two main cases, viz., the one for which the focal curve lies on the focal surface and the one for which it does not.

I will first examine the case for which the focal curve does not lie on the focal surface.

If the focal surface has a degree that is higher than two then an arbitrary ray of the system that goes through the focal curve and contacts the focal curve once must also cut that curve at one or more points, in addition to the fact that it must contact it. If one now considers one of these intersection points to be the starting point of the rays of the system then since it is a point of the focal surface, two infinitely-close rays of the system will go through it in the direction of a tangent to the focal surface, and along with to the former ray, three rays of the same system will also go through that point, so the ray system cannot be of order two without that point being a singular point of it. However, the point considered cannot be singular for any ray of the system, since otherwise infinitely-many singular points would lie on the focal surface, which would yield a second focal curve of the system when connected continuously. The focal surface can therefore not have a degree that is higher than two; however, it cannot have a lower degree, either, since otherwise there could be no contact with the rays of the system. Therefore:

XI: *If a second-order ray system has one focal surface and one focal curve that does not lie on it then the focal surface must be a second-degree surface.*

Since the focal surface has degree two, all of the rays of the systems that emanate from an arbitrarily-chosen point of the focal curve will define a ray cone of degree two that is the enveloping cone of the focal surface that belongs to that point. If one temporarily excludes the case in which the focal surface is a conic surface of degree two – so this enveloping cone decomposes into two planes – then all of the rays of this enveloping cone will belong to the same irreducible system, and likewise also all rays of the continuous family of ray cones, which one obtains when one lets the starting point vary continuously. Therefore, all straight lines that go through the focal curve and contact the second-degree focal surface will be rays of one and the same irreducible system. If the focal curve of the system has degree n then all of the straight lines that emanate from an arbitrary point of space and simultaneously go through the curve and contact the focal surface will, firstly, lie on the n^{th}-degree cone that has that point for its center and goes through n^{th}-degree focal curve, and secondly, the rays of the irreducible system that emanate from that point and all $2n$ intersecting lines of these two cones will be on the second-degree cone at that point that lies on the focal surface. The ray system can then have order two only when $n = 1$, and thus, when the focal curve is a straight line. The fact that a second-degree focal surface and straight focal line that lies upon it actually give a second-order ray system, and the fact that it is also of class two follow quite simply from the fact that two tangents to a conic section can be drawn from an arbitrary point. Therefore:

XII: *The set of all of the straight lines that contact an arbitrary, non-conical, second-degree surface and go through a straight line that does not lie on it defines a ray system of order two and class two.*

If one chooses the straight focal line to be the z-axis and takes:

$$\phi = a\,x^2 + b\,y^2 + c\,z^2 + 2\,d\,y\,z + 2\,e\,z\,x + 2\,f\,x\,y + 2\,g\,x + 2\,h\,y + 2\,i\,z + k = 0$$

to be the equation of the focal surface then one will obtain the following two equations of the ray system by the same method as in the previously-treated cases:

$$y\,\xi - x\,\eta = 0,$$

$$(1)\qquad \left(\frac{d\phi}{dx}\xi + \frac{d\phi}{dy}\eta + \frac{d\phi}{dz}\zeta\right)^2 = 4f\,(a\,\xi^2 + b\,\eta^2 + c\,\zeta^2 + 2\,d\,\eta\,\zeta + 2\,e\,\zeta\,\xi + 2\,f\,\xi\,\eta),$$

which both have no derived equations, and therefore represent the ray system in its purest form. The two points at which the straight focal lines cuts the second-degree focal surface are two singular points of this ray system, from which plane pencils of rays emanate that lie in the tangential planes that contact the surface at those two points.

It now remains for us to examine the case that was previously excluded in which the focal surface is a second-degree cone. In this case, the ray cone that emanates from any arbitrary point of the n^{th}-degree focal curve consists of two plane pencils of rays that lie in the tangential planes that go through each of those two points, and these two plane pencils can either belong to one and the same ray system or also two different ones, since the complete ray system can decompose into two ray systems here, in such a way that one of these two pencils of rays belongs to one system and the other one belongs to the other system.

If the two pencils of rays belong to one and the same ray system, and if the n^{th}-degree focal curve do not go through the center of the conical focal surface then the ray system will necessarily have order $2n$, so the proof that was given for the case of non-conical, second-degree, focal surfaces remains completely valid in this case. In order for the ray system to be of order two, one must then have $n = 1$, and one obtains only a special case of the ray system that was presented in Theorem XII. However, if the n^{th}-degree focal curve goes through the center of the conical focal surface one or more times then the order of the system will be reduced by two units with each such passage, because two coincident rays of the system would then emanate from each point of space that would contact the conical focal surface at the center, and also cut the focal surface at the same point, which would, in themselves, define two coincident first-order ray systems that emanate from the center of the cone. However, if the focal curve goes through the cone $n - 1$ times (so it will be an $n - 1$-fold point of the focal curve) then the order of the ray system will be reduced by $2n - 2$ units, and it will be a second-order ray system. The focal curve must then necessarily be a plane curve, because only an n^{th}-degree planar curve can have an $n - 1$-fold point. The rays of such a system that emanate from an arbitrary point of space lie, firstly, in the two tangential planes to the conical focal surface that go through that point, and secondly, in the n^{th}-degree conic surface that goes through the focal curve, which has an $n - 1$-fold edge, due to the $n - 1$-fold point of the focal curve. Each of the two planes cuts out the $n - 1$-fold edges from the conic surface, along with one straight line. The twice-excised $n - 1$-fold edges of the cone give $2n - 2$ coincident straight lines that emanate from each point of space to the center of the focal surface, and thus, $2n - 2$ coincident ray systems of order one and class zero. The two remaining straight lines that are cut out of the cone by the two planes are the two rays that emanate from each point of space and belong to the second-order ray system that has this n^{th}-degree curve for its focal curve and the second-degree cone for its focal surface. An arbitrary plane cuts the focal curve at n points and the focal surface in a conic section, and two tangents to this conic section go through each of these n points, so $2n$ rays of the system will lie in a plane, and the system will be of class $2n$. Therefore, in the special case where the plane in which the n^{th}-degree focal curve lies is a tangential plane to the conical focal surface, one of the two plane pencils of rays that emanate from each point of the focal surface will remain in the plane of the curve itself for all points of the focal curve, and that plane will contain n coincident rays systems of order zero and class one, by whose omission the class of the systems would be reduced by n units. One then has the following theorem:

XIII: *The set of all straight lines that contact a second-degree cone and go through an n^{th}-degree plane curve that has an* $(n - 1)$*-fold point at the center of the cone defines a ray system of order two and class* $2n$*; however, in the special case in which the plane of the focal curve is a tangential plane of the conical focal surface, the ray system will only be of class* n*.*

If one chooses the center of the cone to be the coordinate origin and the plane of the focal curve to be the xy-plane then the focal surface will become:

$$\phi = a\,x^2 + b\,y^2 + c\,z^2 + 2\,dyz + 2\,czx + 2\,fxy,$$

and the focal curve will:

$$z = 0, \quad \psi(x, y) + \psi_1(x, y) = 0,$$

where $\psi(x, y)$ and $\psi_1(x, y)$ are entire, homogeneous functions of x and y, such that the former is of degree n and the latter is of degree $n - 1$. One then obtains the following two equations of the ray system by the method that was used already in the previous cases:

$$\text{(2)} \qquad \begin{gathered} \left(\frac{d\phi}{dx}\xi + \frac{d\phi}{dy}\eta + \frac{d\phi}{dz}\zeta\right)^2 = 4\phi\,(a\xi^2 + b\eta^2 + c\zeta^2 + 2\,d\eta\zeta + 2\,c\zeta\xi + 2\,f\xi\eta), \\ \psi(x\zeta - z\xi, y\zeta - z\eta) + \zeta\,\psi_1(x\zeta - z\xi, y\zeta - z\eta) = 0, \end{gathered}$$

which therefore still do not represent them, having been cleansed of the $2n - 2$ coincident ray systems of order one and class zero that emanate from the center of the cone. One can exhibit an equation of the form $P\xi + Q\eta + R\zeta = 0$ from these two equations that, together with its derived equations, represents the ray system purely and completely, and since the expression for the functions P, Q, R will then be very complicated, I would not like to develop them here.

Once one has ascertained completely the second-order ray system that comes about for a second-degree conical focal surface and an n^{th}-degree focal curve, when the two planar pencils of rays that emanate from both points of the focal curve belong to one and the same irreducible ray system, one must then examine the case in which these pencils of rays belong to two different ray systems that both have the same focal surface and focal curve. In this case, the two planes of the ray pencil – and thus, the two tangential planes of the conical focal surfaces that go through an arbitrary point x, y, z of the focal curve – can be expressed rationally in terms of the coordinates of that point. However, each of the two tangential planes to the cone $\phi = 0$ that are drawn through a point x, y, z contains only the one irrational quantity $\sqrt{\phi}$; should they be rational for each point of the focal curve then one would need to have $\sqrt{\phi} = M / N$ for all points of the focal curve, where M and N are entire, rational functions of x, y, z; the one equation of the focal curve must then be of the form $N^2\phi - M^2 = 0$. This equation interprets geometrically as the statement that the focal curve must lie on a surface that contacts the second-degree cone $\phi = 0$ in a curve without cutting it. The focal curve can then likewise cut that cone nowhere, but only contact it, and when it is of degree n, it will contact the cone n times, because at each contact point two of the $2n$ intersection points of the n^{th}-degree curve with the second-degree surface must unite into a contact point. The n^{th}-degree focal curve can also go through the center of the cone then, in which case, the number of actual contact points will be diminished since each passage of the curve through the center of the cone is to be counted as a contact, since two intersection points then combine into one. If the curve goes through the center of cone μ times then it will have only $n - \mu$ contact points; the focal curve will then lie on a conic surface of degree $n - \mu$ that has the same center as the second-degree cone of the focal surface, and which contacts it at $n - \mu$ straight lines. The complete ray system that consists of all the straight lines that cut the focal curve and contact the focal surface, which is of order $2n$, will be of order $2n - 2\mu$, when all of the rays that go through only the center are isolated, which by themselves define 2μ coincident rays systems of order one and class zero, and it will subsume only the two ray systems that each contain one of the two families of plane pencils of rays that emanate from the focal curve. If one of these two ray systems should now be of order two then two of the rays in it that emanate from an arbitrary point of space cannot lie in the same one of the two tangential planes to the conical focal surface that go through that point, but one of them must line in one tangential plane, while the other lies in the other tangential plane. If both of them lie in the same tangential plane then that plane must, as the plane of the two rays that go through the arbitrary point x, y, z of space, be rationally expressible in terms of x, y, z, which is not the case, since it necessarily contains the irrational quantity $\sqrt{\phi}$, which will not be rational for any point of space, but only for all points of the focal curve. An arbitrary tangential plane of the focal curve now cuts the n^{th}-degree focal curve at $n - \mu$ points, besides the μ points that coincide with the center, and a plane pencil of rays that lies in that tangential plane will belong to each of these $n - \mu$ points. However, only one of these $n - \mu$ pencils of rays can belong to the second-order ray system. If two or more of them belonged to that system then two or more rays of the system that lie in that plane would go through every point that lies in that tangential plane, which is impossible, since the two rays of the system that that emanate from a point will always lie in two different tangential planes that go through that point. If one now moves this one plane pencil of rays that shall belong to the second-order system along the entire focal curve, and with it, the tangential plane in which it lies, as well, then the tangential plane can never return to a position that it had previously occupied under all of this motion, because otherwise two ray pencils of the system would lie in that plane. The tangential plane can then go around the second-degree cone just once, and in the same sense. It follows further from this that any tangential plane can cut the second-degree conical focal surface to the focal curve at only two points. If more than two points were cut out then the plane pencil of rays that belongs to the second-order system and whose center traverses the entire focal curve, and in turn, must gradually come to all of the points that will be cut from a certain tangential plane to the focal curve, and with them, the tangential plane in which it lies, must either turn back or go around the focal surface several times. $n - m$ must then be necessarily equal to 2, so the focal curve must then lie on a second-degree cone that contacts the conical second-degree focal surface in two straight lines, so when its degree is equal to n, it must go through the center of the focal surface $n - 2$ times and contact it at two points. Since these conditions are not only necessary, but also sufficient, as one easily verifies, one has the following theorem:

XIV: *The set of all straight lines that contact a second-degree cone and likewise cut an* n^{th}*-degree curve that goes through the center of the cone* $n - 2$ *times and contacts the cone twice defines two distinct ray systems of order two and class n, with the exception of the straight lines that go through just the center.*

I will pass over the analytical representation of this kind of second-order ray system, since it offers no difficulties, although it is complicated.

All second-order rays systems that have one focal surface and one focal curve that does not lie on it are now exhausted with that, and the only case to be examined is then the one where the focal curve lies completely on the focal surface.

I assume that the focal curve that lies on the focal surface is an ν-fold curve on it, where the case $\nu = 1$ is not excluded, for which the focal curve is a simple curve that lies on the focal surface. An arbitrary ray of the system that goes through the ν-fold curve of the focal surface and contacts the focal surface, in addition, must cut it at some point when the focal surface has a degree higher than $\nu + 2$. However, because it is a point of the focal surface, two infinitely-close rays of the system must go through such an intersection point in the direction of a tangent, and also the ray that cuts the focal surface at this point, as well. The system cannot be of order two when the degree of the focal surface is higher than $\nu + 2$; the degree of the focal surface can also not be one lower, because otherwise no ray that went through the focal curve could contact it at another point. If the focal curve is a curved line then every straight line that goes through two points of it will cut out 2ν points from the focal surface, but since the degree of that surface is equal to $\nu + 2$, this can only happen for the values $\nu = 1$ or $\nu = 2$, while in the other cases the focal curve must be a straight n-fold line in the focal surface of degree $\nu + 2$. A curvilinear focal curve can thus exist on a focal surface only when it is a simple curve on a third-degree focal surface, or a double curve in a fourth-degree surface. The fact that these two special cases do not, however, give a second-order ray system will be show as follows.

A ray cone emanates from each point of the focal curve that envelops the focal curve. If three ray cones go through the same point then it will be a singular point of the second-order system, because three rays of the system that lie in three different ray cones must go through it. Now, if the ray cones that emanate from all points of the focal curve are of degree two or more than three of them will intersect in eight or more points, and if they have a common intersection curve then that curve must itself be a focal curve of the system since three rays of the system would go through each of its points, and since the system should have only one focal curve, that curve must be identical with the previous focal curve, and it must therefore also go through center of the ray cone, but this is possible only in the case where the centers of the three cones lie along a line, and that line will be a focal line, which is contrary to the assumption. The eight or more singular points of the ray system must likewise be nodes of the focal surface when just as many rays of the system emanate from every other point of the focal surface as the number of times that its tangential plane intersects the focal curve. However, a fourth-degree surface with a double curve of degree higher than one cannot have eight nodes, but at most four of them, when the double curve has degree two, and none of them when it is of degree three. The ray cone that emanates from any point of the focal curve cannot then be of degree two or higher in either of the present cases, but can be only a plane pencil of rays. An entire family of plane pencils of rays can, however, exist only when the focal surface is enveloped by all planes of that pencil of rays, and thus, when it is a developable surface. However, the only fourth-degree developable surface that has a curve double curve is the one whose edge of regression is of degree three, and if it were assumed to be the focal surface and its edge of regression were assumed to be the focal curve then that would not give a ray system, at all, because none of the straight lines that go through the edge of regression can contact the surface at a point that lies outside of that edge of regression. The third-degree surface, which must be a developable, can be only a conic surface, because other third-degree developable surfaces do not exist. A plane pencil of rays must emanate from each point of the focal curve that lies on this third-degree cone whose rays contact a certain straight line of the plane. The focal curve, which is curved, by assumption, must cut all straight lines of cone, and thus also the ones that will be met at all points of the rays of the one ray pencil; one of these rays must then also meet the point at which the focal curve cuts the straight line, but since it is a point of the focal curve, a second pencil of rays will go from it that does not include the one ray of the first ray pencil that goes through its center, because its plane cannot go through the center of the first one. One more ray of the system must then go through this second point of the focal curve, in addition to the plane pencil of rays, which is impossible.

Since these two special cases yield no second-order ray systems, all that remains is the general case in which a focal surface of degree n contains an $n - 2$-fold straight line as its focal line. This case always gives a second-order ray system. Since the rays of the system that emanate from an arbitrary point of space must cut the straight focal line, they will then lie in a plane that goes through the focal line, but that plane cuts just one conic section out of the surface, in addition to the $n - 2$-fold straight line, and it has only two tangents that go through the given point, which are the two rays of the system that go through it. If one intersects the system with an arbitrary line then a curve of degree n that has an $n - 2$-fold point will be cut out of the focal surface, and since the number of the tangents that go through this multiple point is $2n - 2$, the system then has class $2n - 2$. Therefore:

XV: *The set of all straight lines that go through an* $(n - 2)$*-fold straight line of a* n^{th}*-degree surface and contact that surface defines a ray system of order two and class* $2n - 2$.

If one takes the n – 2-fold straight line to be the z-axis then one can put the most general equation of that n^{th}-degree surface into the following form:

$$(3) \qquad \phi + 2\phi_1 + \phi_2 + 2z(\psi + \psi_1) + z^2\chi = 0,$$

where $\phi, \phi_1, \phi_2, \psi, \psi_1, \chi$ are entire homogeneous functions of x and y alone, and indeed ϕ, ψ, χ are of degree n – 2, ϕ_1 and ψ_1 are of degree n – 1, and ϕ_2 is of degree n. One then obtains the following two equations for the ray system from the method that was used in the previously-treated cases:

$$(4) \qquad y\xi - x\eta = 0, \quad U\,\xi^2 + 2\,V\xi\eta + W\,\eta^2 = 0,$$

where:

$$U = \phi_1^2 - \phi\,\phi_2 + 2z\,(\phi_1\,\psi_1 - \phi_2\,\psi) + z^2\,(\psi_1^2 - \phi_2\,\chi),$$
$$V = x\,(\psi\,\phi_1 - \psi_1\,\phi + \psi\,\phi_2 - \psi_1\,\phi_1 + z\,(\phi_1\chi - \psi\,\psi_1 + \phi_2\,\chi - \psi_1^2)),$$
$$W = x^2\,((\psi + \psi_1)^2 - (\phi + 2\phi_1 + \phi_2)\chi).$$

In the examination of the ray systems with one focal surface and a focal curve that lies on it, it was always assumed that the contact point of the focal surface with the individual rays of the systems were different from the intersection points of the rays with the focal curve, so it still remains for us to investigate those ray systems whose focal surfaces are contacted by all rays at the same point at which they cut the focal curve. Such a ray system consists of a family of plane pencils of rays that emanate from all points of the focal curve, and each of which lies in a tangential plane to the surface and consists of all of the tangents to it that go through the contact point. Since such a ray system is determined completely by the simply-infinite family of tangential planes that contact the focal surface along the focal curve, one can alter the focal surface in infinitely many ways without changing the ray system, when that one family of tangential planes thus remains unchanged. If one chooses the developable surface that is enveloped by this simply-infinite, continuous family of tangential planes in each case then the ray system that consists of the continuous family of plane pencils of rays that lie in these planes will necessarily be irreducible when this developable surface and the focal curve that lies on it are irreducible, so it must then be the complete second-order ray system itself that subsumes all rays of all of these ray pencils. It is associated, firstly, with the fact that only two planes of this family of enveloping planes of the developable surface will go through an arbitrary point of space, and then with the fact that a pencil of rays lies in each of these planes, so if more than two planes were to go through each arbitrary point of space then more than two rays of the system would also go through that point. Secondly, it is also requisite for this that no more than one ray pencil should lie in each plane of the family, so the focal curve that lies in the developable surface, along which, the centers of all ray pencils lie, should cut all straight lines of the developable surface only once. These two conditions are also sufficient for such a second-order ray system to actually exist. The condition for two enveloping planes of the developable focal surface to go through each point of space replies that this developable focal surface must necessarily be a second-degree cone. The condition that the focal curve that lies on this cone should cut each straight line of it only once will be fulfilled in the most general way by a focal curve that is cut out of that cone by an n^{th}-degree surface that has an $n - 1$-fold node at the center of the cone. Although such a focal curve will actually intersect any ray at n points, one does not count the $n - 1$ intersection points that coincide with the center of the cone, since the ray pencils that belong to it will unite into only ray systems of order one and class zero, which will then drop out. The focal curve then becomes a curve of degree $2n$ with a $2n - 2$-fold point at the center of the cone. An arbitrary plane cuts this curve at $2n$ points, and one ray of the intersecting plane lies in each of the $2n$ plane pencils of rays that emanate from that point, so the system will then have class $2n$. Therefore:

XVI: *The set all straight lines that contact a second-degree cone at all of the points of a curve that is cut out from an n^{th}-degree surface with an $(n - 1)$-fold point that lies at the center of the cone defines a ray system of order two and class $2n$, with the exception of straight lines that only go through the center of the cone.*

If one takes the center of the cone to be the coordinate origin then the equation of the n^{th}-degree surface that has an $n - 1$-fold node will have the form $\psi(x, y, z) + \psi_1(x, y, z) = 0$, where ψ and ψ_1 are entire, homogeneous functions of x, y, z, one of which has degree n, while the other of which has degree $n - 1$. Let the equation of the cone be:

$$\phi = ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy = 0.$$

If one now sets, to abbreviate:

$$\begin{aligned}
\phi' &= (ax + fy + cz)\xi + (fx + by + dz)\eta + (ex + dy + cz)\zeta, \\
\phi'' &= a\xi^2 + b\eta^2 + c\zeta^2 + 2d\eta\zeta + 2e\zeta\xi + 2f\xi\eta
\end{aligned}$$

then one will obtain the following two equations of this ray system:

$$\begin{gathered}
\phi'^2 - \phi\phi'' = 0, \\
\psi(x\phi' - \xi\phi, y\phi' - \eta\phi, z\phi' - \zeta\phi) + \phi\,\psi_1(x\phi' - \xi\phi, y\phi' - \eta\phi, z\phi' - \zeta\phi) = 0.
\end{gathered} \tag{5}$$

I will skip over the equations that one can define from these two equations, which are linear with respect to ξ, η, ζ, because they are complicated.

All ray systems of the kind that were presented in this paragraph must be exhausted by the method of investigation that was applied to the problem of establishing the ray systems with one focal curve and one focal surface, and there can be no ray systems of the stated kind that are not obtained as special cases, or also limiting cases, of them.

§ 6.

General properties of second-order ray systems that have focal surfaces, but no focal curves.

If a ray system has no focal curve then its focal surface will be contacted by each ray twice, and both contacts will then be generally proper contacts at those points of the surface that are endowed with only one well-defined tangential plane, and not merely intersections of the rays with the surface at double points or double curves on it. Since the focal surface is contacted by all rays of the system twice, it cannot have a degree that is lower than four, although for ray systems of order two it can also not have a degree that is higher than four. Now, since two infinitely-close rays of the second-order system emanate from each point of the focal surface in the direction of a tangent, two more infinitely-close rays would then emanate from such an intersection point of the former ray with the focal surface, in addition to that ray itself, and therefore at least three rays. Any such point must then be a singular point of the second-order ray system, and any ray of the system must go through a singular point of the system. Since this cannot happen for a ray system with no focal curve, it will then follow that:

XVII: *The focal surfaces of all second-order ray systems that have no focal curves are fourth-degree surfaces.*

I take this occasion to remark that the proof of the theorem assumes that the two contact points of any ray with the focal surface are two different points, in general. If these two contact points of any ray combine into one for all rays of the system then that would give a ray system whose rays each contact the focal surface at just one point, but in such a way that each ray goes through three infinitely-close points of the focal surface. The ray systems of this kind that can exist on third-degree focal surface can, however, never be of order two, because not just two, but three, infinitely-close rays will emanate from each point of the focal surface.

The complete system of all straight lines that contact a fourth-degree surface twice is a ray system of order twelve and class twenty-eight. As is known, 12 straight lines will go through an arbitrary point of space that will contact a fourth-degree surface twice – hence, 12 rays of the system – and an arbitrary plane will cut a fourth-degree curve out of the focal surface whose 28 double tangents will be the rays of the system that lie in that plane. If a fourth-degree surface is the focal surface of a second-order ray system then it must separate an autonomous second-order ray system from this complete ray system of order 12 and class 28, such that a ray system of order 10 still remains that can itself be further composed of ray systems of lower order. The 12 rays of the complete that emanate from an arbitrary point of space will be determined by an equation of degree 12 whose coefficients are rational functions of the coordinates x, y, z of the starting point. If the fourth-degree surface is to be the focal surface of a second-order ray system then this equation must be reducible and contain a second-degree factor whose coefficients are rational functions of x, y, z. Conversely, if this equation contains such a second-degree then the fourth-degree focal surface must belong to a second-order ray system. A complete examination of the conditions under which this twelfth-degree equation would contain a second-degree rational factor would then yield all second-degree ray systems that have no focal curves. It then seems simpler and more appropriate to apply another, more geometric, method to the complete investigation of them, which mainly amounts to just a discussion of the linear equation $P\xi + Q\eta + R\zeta = 0$, which must be true for all second-order ray systems.

Let the three entire, rational functions P, Q, R in the equation:

$$P\xi + Q\eta + R\zeta = 0 \tag{1}$$

be n^{th}-degree functions of the coordinates x, y, z, to which, the fourth coordinate t, which makes things homogeneous, will be added, such that P, Q, and R are entire, homogeneous functions of degree n of the four coordinates x, y, z, t, of which, it will also always be assumed that all three do not have a common factor. As was shown above, equation (1) must still be the equation of the same ray system when one simultaneously converts x into $x + \rho\xi$, y into $y + \rho\eta$, and z into $z + \rho\zeta$ for every arbitrary value of ρ. For the sake of brevity, let:

$$P(x + \rho\xi, y + \rho\eta, z + \rho\zeta) = P',$$
$$Q(x + \rho\xi, y + \rho\eta, z + \rho\zeta) = Q',$$
$$R(x + \rho\xi, y + \rho\eta, z + \rho\zeta) = R',$$

so one has the general equation:

$$P'\xi + Q'\eta + R'\zeta = 0, \tag{2}$$

which must be true for any value of ρ, and which likewise represents the equations (1), along with all of its derived equations.

The last of these derived equations, which one obtains when one develops equation (2) in powers of ρ and sets the coefficients of ρ'', which is the highest power of ρ, equal to zero, must be fulfilled identically for a ray systems that have no focal curves and can thus yield no determination of the quantities ξ, η, ζ. This latter equation, in fact, no longer includes x, y, z, and t, but only ξ, η, ζ in $n + 1$ dimensions, along with constants. Therefore, if one replaces ξ, η, ζ with $x' - x$, $y' - y$, $z' - z$ then it will represent a cone of degree $n + 1$, on which the two rays of the systems that go through the point x, y, z must lie, and which remain congruent and parallel to themselves for all points in space. All ray of the systems are thus parallel to the rays of an arbitrarily-chosen, but well-defined, one of these cones. If one cuts this well-defined cone with an infinitely-distant plane then all rays of the system can be regarded as going through this one infinitely-distant intersection curve, so it will be an infinitely-distant focal curve of the system. The last derived equation cannot exist for the ray systems that have no focal curve, but must be fulfilled identically. It can thus be represented as:

$$P(\xi, \eta, \zeta, 0)\ \xi + P(\xi, \eta, \zeta, 0)\ \xi + P(\xi, \eta, \zeta, 0)\ \xi = 0,$$

and since thismust be satisfied identically, one can also convert ξ, η, ζ into x, y, z, which will then yield:

$$Px + Qy + Rz = 0 \qquad \text{for } t = 0,$$

and one can also express the condition for the three functions P, Q, R in such a way that the S in the equation:

(3) $$Px + Qy + Rz + St = 0$$

must likewise be an entire, homogeneous function of degree n of x, y, z, t.

Now, let x, y, z, ξ, η, z be the determining data of an arbitrary line of the system, which will be regarded as a fixed line, so, for all values of the varying parameter λ:

(4) $$\zeta(x' - x) + \lambda\,\zeta(y' - y) - (\xi + \lambda\eta)(z' - z) = 0$$

will be the equation of a family of planes that go through the fixed line. In addition to this fixed ray that goes through the arbitrary point of the fixed line whose coordinates are $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$, there is a second ray of the system, and, as was shown above:

(5) $$P'(x' - x) + Q'(y' - y) + R'(z' - z) = 0$$

is the plane in which these two rays that go through the point $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$ will lie. Now, should this second ray lie in the plane (4) with the plane through the fixed line then the plane (3) itself must be the same as the plane (4), so the two equations must exist:

$$P'\lambda = Q', \qquad P'(\xi + \lambda\,\eta) = -R'\zeta,$$

one of which will already follow from the other, using equation (2). The equation:

(6) $$P'\lambda = Q'$$

is then the necessary and sufficient condition for the second ray that goes through the point $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$ to lie in the plane (4). Equation (6) is of degree n with respect to ρ, so it gives n values for ρ. The one fixed ray $x, y, z, \xi, \eta, \zeta$ will then be cut by n of the rays of the system that lie in the arbitrary plane that goes through it, such that precisely $n + 1$ rays will lie in that plane. One then has the following theorem:

XVIII: *If the three entire rational functions P, Q, R in the linear equation for a second-order ray system $P\xi + Q\eta + R\zeta = 0$ are of degree n then the ray system will be of class $n + 1$.*

If one considers the λ in equation (6) to be a function of ρ then λ will be a rational fractional function of ρ whose numerator and denominator are of degree n. If the quantity λ remains unchanged under an infinitely-small change in ρ – i.e., if $d\lambda / d\rho = 0$ – then two infinitely-close rays of the system will lie in a plane (4) that is determined by one such value of λ; this plane is therefore a tangential plane to the focal surface of the ray system. The condition $d\lambda / d\rho = 0$ gives:

(7) $$Q\frac{dP}{d\rho} - P\frac{dQ}{d\rho} = 0,$$

which is an equation of degree $2n - 2$ in ρ. $2n - 2$ tangential planes of the focal surface will then go through the fixed line that contacts that surface outside of the fixed ray itself. No other tangential planes can exist in addition to these $2n - 2$ tangential planes that go through the fixed ray and whose contact point does not lie in the fixed ray itself, so two infinitely-close rays of the system will lie in each tangential plane that must then cut the fixed ray that lies in the tangential plane at two infinitely-close points. The number of tangential planes to the focal planes that go through the fixed ray will then be precisely $2n - 2$. Now, the class of a surface is known to be determined by the number of its tangential planes that go through an arbitrary fixed straight line, so it is generally equal to the number of these tangential planes. However, if that fixed straight line contacted the surface once then the class would be two units larger than the number of tangential planes that go through the fixed straight line and whose contact point does not lie upon that fixed straight line itself, and if it contacted the surface twice then the class would be four units larger than that number. In the present case, the fixed ray is straight line that contacts the focal surface twice, and through which $2n - 2$ tangential planes to that surface go, so the focal surface will then have class $2n + 2$. Since, from Theorem XVIII, the ray system has class $n + 1$, it then follows:

XIX: *The class of the fourth-degree focal surface that belongs to a second-order ray system is always twice as large as the class of that ray system.*

The n values of ρ that are given by equation (6), when considered to be functions of λ, generally vary simultaneously with λ; i.e., the n intersection points of the rays that lie in the plane (4) with the one fixed ray will change position along that fixed line when this plane is rotated around it. However, it can also be the case that a certain number of the roots of equation (6) are completely independent of λ, so a certain number of these n rays will always intersect them in the same points when the plane (4) is rotated around the fixed ray. Such a family of rays that all go through the same fixed point on the fixed ray defines a ray cone whose center is a singular point of the system. The condition for equation (6) to have λ independent roots ρ is that one must have $P = 0$ and $Q' = 0$ for these values of r, and in turn, by means of equation (2), one must also have $R' = 0$. It then follows from this that:

XX: *If the three equations $P' = 0$, $Q' = 0$, $R' = 0$ are fulfilled simultaneously for a certain value of ρ then the ray $x, y, z, \xi, \eta, \zeta$ will go through a singular point of the ray system whose coordinates are $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$.*

The ray $x, y, z, \xi, \eta, \zeta$ that is assumed to be fixed can also be arranged so that the three equations:

$$P' = 0, Q' = 0, R' = 0$$

are fulfilled identically, not merely for a single well-defined value of ρ, but also for every any arbitrary value of ρ. Equation (6) is then fulfilled identically for all arbitrary values of the ρ and λ, thus, so one draws an arbitrary plane through this ray, any point of that ray will be an intersection point of it with another ray that lines in that plane. This is not possible, except when either that plane contains an entire family of rays of the system that cut the fixed ray at all points, or when this ray consists of two rays that coincide in such a way that any point of this ray can be regarded as the intersection point of the two coincident rays. In the first case, an entire ray cone must emanate from each point of that ray, and that ray must be a focal line of the ray system, but since the second-order ray systems that have focal curves were exhausted completely in the previous paragraphs, and are excluded here, all that remains is the other case in which that ray consists of two coincident rays of the system. Thus:

XXI: *Those rays for which the three equations $P' = 0$, $Q' = 0$, $R' = 0$ are fulfilled identically for each arbitrary value of ρ consist of two rays of the system that coincide. For that reason, they shall be called* **double rays**.

If one replaces $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$ with simply x, y, z then x, y, z will no longer mean just the coordinates of the starting point of the ray $x, y, z, \xi, \eta, \zeta$, but they will be the coordinates of any point on that straight line for any arbitrary value of ρ; one must accordingly replace P', Q', R' with P, Q, R. Theorem XXXI then yields that:

XXII: *If the three n^{th}-degree surfaces:*

$$P = 0, \quad Q = 0, \quad R = 0$$

include common straight lines then they will be double rays of the ray system, and conversely, every any double ray of the system will be a common straight line to these three surfaces.

If the three n^{th}-degree surfaces $P = 0$, $Q = 0$, $R = 0$ have any common point whatsoever – let it be a common intersection point of these three surfaces or suppose that it belongs to a common intersection curve of them – then that point must be either a singular point of the system from which a ray cone emanates or it must lie on a double ray. Namely, if any simple ray $x, y, z, \xi, \eta, \zeta$ of the system goes through that point then, by assumption, one will have $P' = 0$, $Q' = 0$, $R' = 0$ for the well-defined value $\rho = 0$ for such a simple ray, so from Theorem XX, x, y, z will be a singular point with a ray cone. However, when no simple ray of the system goes through that point, a double ray must necessarily go through it. There can then be no point of space at all through which no ray of the system goes in an algebraic ray system, so it must be true that either just as many rays go through each point of space as the order of the system would give, which can also be united with multiply-coincident rays, or infinitely many rays that define a ray cone must go through it. If follows further from this that the three surfaces can have no intersection curve that is common to all three, so any point of it must either be a singular point with a ray cone, so the intersection curve is a focal curve, which is a case that is excluded here, or a double ray must go through each point of that curve, and thus the three surfaces must contain an entire family of common straight lines that collectively define a ruled surface that is common to all three, and would give a common factor of the three functions P, Q, R, which is likewise excluded. One then has:

XXIII: *The three surfaces $P = 0$, $Q = 0$, $R = 0$ have no other common lines of intersection than the double rays of the system, and all common points of intersection of them that do not lie on these double rays are singular points of the ray systems from which ray cones emanate.*

One obtains the precise determination of the number of all double rays that are contained in a ray system of order two and class $n + 1$ in the following way. Let:

$$\alpha x' + \beta y' + \gamma z' + \delta t = 0 \tag{8}$$

be an arbitrary plane that is considered to be fixed and will be chosen in such a way that it contains no singular points of the ray system and none of the $n + 1$ rays of the system lie at infinity, nor do any of the $\frac{(n+1)n}{2}$ intersection points of any two of these $n + 1$ rays. Let x, y, z be the coordinates of any of these intersection points of two rays, so for these points the fixed plane (8) must be the same as the plane of the two rays that emanate from x, y, z, which would have the equation:

$$P\,(x' - x) + Q\,(y' - y) + R\,(z' - z) = 0, \tag{9}$$

as was shown above. The condition for these two planes to be identical gives the three equations:

$$\frac{P}{\alpha} = \frac{Q}{\beta} = \frac{R}{\gamma} \qquad \text{and} \qquad \alpha x + \beta y + \gamma z + \delta t = 0, \tag{10}$$

which must then be satisfied by the coordinates of all $\frac{(n+1)n}{2}$ intersection points of any two rays of the system that lie in the plane (8). In addition, the same equations are also satisfied by the coordinates of the intersection points of all double rays of the system with the plane (8). One will then have $P = 0$, $Q = 0$, $R = 0$, and $\alpha x + \beta y + \gamma z + \delta t = 0$. However, it can happen that no other points besides the aforementioned ones satisfy these three equations (10); if P, Q, and R are not all three equal to zero then the plane of the two rays of the system that go through the point x, y, z will be determined completely and will be identical with the plane (8), in such a way that this point will necessarily be an intersection point of two rays that lie in the plane. However, if one has $P = 0$, $Q = 0$, and $R = 0$ simultaneously for a point x, y, z then, from Theorem XXII, this point will be either a singular point of the system or a point on a double ray, and because, by assumption, the plane (8) goes through no singular point of the system, all of the points that satisfy the three equations (10) will necessarily be only intersection points of the plane (8) with the double rays of the system. The three equations (10) would yield precisely n^2 points that satisfy them, since two of them are of degree n and one of them has degree one, as long as a certain number of them do not necessarily lie at infinity. In order to ascertain these infinitely-distant points, I make use of equation (3):

$$P\,x + Q\,y + R\,z + S\,t = 0,$$

which must be satisfied by the three n^{th}-degree functions P, Q, R in such a way that S is likewise an entire function of degree n. One easily infers from this equation that P, Q, and R must be put into the following forms:

$$
\begin{aligned}
P &= y\,\phi_2 - z\,\phi_1 - t\,\psi, \\
Q &= z\,\phi\; - x\,\phi_2 - t\,\psi_1\,, \\
R &= x\,\phi_1\; - y\,\phi - t\,\psi_2\,,
\end{aligned}
\tag{11}
$$

where ϕ, ϕ_1, ϕ_2 are entire, rational, homogeneous functions of x, y, z that have degree n – 1, and ψ, ψ_1, ψ_2 are entire, rational, homogeneous functions of x, y, z, t that have the same degree. When combined with the equation $\alpha\,x + \beta\,y + \gamma\,z + \delta\,t = 0$, these expressions yield:

$$
\begin{aligned}
\beta P_2 - \gamma P_1 &= x\,(\alpha\phi + \beta\phi_1 + \gamma\phi_2) + t\,(\gamma\psi_1 - \beta\psi_2 + \delta\phi), \\
\gamma P - \alpha P_2 &= y\,(\alpha\phi + \beta\phi_1 + \gamma\phi_2) + t\,(\alpha\psi_2 - \gamma\psi + \delta\phi_1), \\
\alpha P_1 - \beta P &= z\,(\alpha\phi + \beta\phi_1 + \gamma\phi_2) + t\,(\beta\psi - \alpha\psi_1 + \delta\phi_2).
\end{aligned}
\tag{12}
$$

For all infinitely large values that satisfy the three equations (10), one then has:

$$t = 0, \quad \alpha\,x + \beta\,y + \gamma\,z = 0, \qquad \alpha\phi + \beta\phi_1 + \gamma\phi_2 = 0,$$

and because one of these equations has degree n – 1, while the other two have degree one, there are exactly n – 1 infinite values and, in turn, $n^2 - n + 1$ finite well-defined values of the coordinates x, y, z that satisfy the three equations (10). Since of the $n^2 - n + 1$ points thus determined, $\frac{n(n+1)}{2}$ of them will be the intersection points of any two rays that lie in the plane (8), what will remain are $n^2 - n + 1 - \frac{n(n+1)}{2} = \frac{(n-1)(n-2)}{2}$ points that will be the intersection points of the double rays of the system with the plane (8), and as a result, that will give the number of these double rays itself. Therefore:

XXIV: *Any ray system of order two and class n + 1 has precisely* $(n - 1)(n - 2)/2$ *double rays.*

The ray systems of class two and three then have no double rays at all, those of class four have one, those of class five have three, those of class six have six, etc.

If a double ray is cut by any other ray of the system then the two coincident rays of the double ray will go through that intersection point in addition to the other ray, and thus, at least three rays, from which, it follows that this point must be a singular point of the system with a ray cone. The double ray itself must belong to the rays of this cone, and must be a double edge of it, so two straight lines that are coincident at all points and not merely infinitely-close with a single intersection point can lie only on a double edge of the cone. Therefore:

XXV: *Any intersection point of a double ray with any other ray of a second-order system is a singular point of the ray system with a ray cone that has a double edge at the double ray.*

The fact that, conversely, any double edge of a ray cone is also a double ray of the system follows from the fact that a plane that is drawn through the double edge will cut out two completely coincident rays of the system.

A double ray, unlike a simple ray, will thus cut some other ray of a system at each of its points, because otherwise each of its points would have to be a singular point with a ray cone and the double ray would have to be a focal line of the system, and there are only isolated well-defined singular points on any double ray, moreover, through which all rays of the system that intersect them will go. All straight lines that cut a well-defined simple ray will define a ruled surface that must always decompose for a double ray into cone surfaces whose centers lie at the singular points of the double ray.

If one draws a plane through a fixed ray in the ruled surface whose generating straight lines are the rays of the system that cut that fixed ray then the intersection curve will consist of only the fixed line itself and generating straight lines of the surface that lie in that plane and which are the n rays of the system that lie in the plane and cut the fixed ray; however, the fixed ray itself will be cut three times, namely, once as the straight line through which all generating straight lines of the surface go, and two more times, as well, because the motion of the straight line will generate the ruled surface, so in its motion along the fixed ray it will go through that surface twice, namely, when its intersection point comes to one of the two points at which the fixed ray contacts the focal surface. The planes that are drawn through the fixed ray then cut this straight line out of the ruled surface as a triple one, and the n rays of the system that lie in the plane, as well, which cut the fixed ray, so:

XXVI: *The ruled surface that is defined by all of the rays of a system that intersect a fixed ray is a surface of degree* $n + 3$.

When the fixed ray that is cut from all generating straight lines of this $n + 3$-degree surface goes through a singular point of the ray system, the cone that belongs to this point will define a subset of that ruled surface.

If a ray system has a ray cone of degree g and one draws though a plane through a ray of the system that does not belong to that ray cone and the center of the g^{th}-degree ray cone then it will contain one more ray in addition to g rays that cut from the ray cone, and thus, at least $g + 1$ rays, and since $n + 1$ rays of the system will lie in each plane, it then follows that:

XXVII: *A ray system of order two and class* $n + 1$ *can include no ray cone whose degree is higher than n.*

I will now consider the ray cone into which the ruled surface of degree $n + 3$ decomposes when the fixed ray in it is a double ray of the system. If one draws a plane through a double ray then $n - 1$ rays of the system will lie in it, along with double ray, and which will cut the double ray at only at singular points. If the number of singular points that are contained on a double ray is equal to h then the ruled surface that define all of these rays of the system that cut that double ray will consist of h ray cones, each of which has the double ray for a double edge. Let $g_1, g_2, \ldots, g_h$ then be the degree of these h ray cones, so $g_1 - 2$ rays will go through the first singular point in an arbitrary plane that goes through the double ray, $g_2 - 2$ rays will go through the second point, etc. The number of all of the rays that cut the double ray and lie in this plane will then be equal to $g + g_1 + g_2 + \ldots + g_h - 2h$, and because this number must be equal to $n - 1$, one will has be:

$$g_1 + g_2 + \ldots + g_h = n - 1 = 2h.$$

On the other hand, because these h cones collectively define only a special case of a ruled surface of degree $n + 3$ that consists of all rays of the system that cut a given ray, one has:

$$g_1 + g_2 + \ldots + g_h = n + 3.$$

The number h of the singular points that lie on a double ray must then be equal to two. This likewise implies that the two ray cones that belong to the two singular points of a double ray must have degree at least three; if one of them would then have degree less than three then, since the two have degree $n + 3$ collectively, the other one would need to have a degree higher than n, which is impossible, from Theorem XXVII. The condition for each of these cones to have the double ray for its double edge would not be sufficient to establish that, because a ray cone of degree two would also fulfill that condition if it consisted of two planes that intersected along the double line, and thus consisted of two ray pencils that laid in both planes and emanated from the singular points. One then has the theorem:

XXVIII: *Two singular points with ray cones of degree at least three lie on any double ray of a second-order system.*

Each of the ruled surfaces of degree $n + 3$ that is defined by a ray of the system that intersects an arbitrary fixed ray must always go through the all singular points of the ray system, and indeed, it must go through any singular points with a ray cone of degree g precisely g times in such a way that such a point must be a g-fold point of the surface. A ray cone of degree g will, in fact, be cut by the fixed line of the ruled surface of degree g at g points, and the g rays of the ray cone that go through these g points will likewise be g generating straight lines of the surface that go through the singular point. Two such ruled surfaces whose fixed guiding lines do not lie in the same plane will always have $n + 3$ rays of the system in common with each other, namely, those lines that go through the $n + 3$ intersection points of the fixed guiding rays of the one surface with the other one; three such ruled surfaces will generally have no rays of the system in common. Three lines of the system go through each common point of three such surfaces because a generating straight line that goes through this point – hence, a ray of the system – will lie in each of these surfaces (so with the exception of those cases in which two of these three lines are identical, so a common ray to two of these surfaces), will cut the third one, such that only two different rays of the system will go through the common point of three surfaces, each common point of these three surfaces must be a singular point of the ray system. If the three surfaces had a common intersection curve then it would have to be a focal curve of the ray system, because three different rays of the system must go through any arbitrary point.

The number of all intersection points of three $n + 3$-degree surfaces that have no common intersection curve is $(n + 3)^3$. The number of those intersection points that are not singular points of the ray system – i.e., ones at which only one common ray to two of these surfaces will cut the third one – is equal to $3(n + 3)^2$, because any two surfaces of degree $n + 3$ will have rays in common that cut the third surface of degree $n + 3$. If one now generally lets m_e denote the number of those singular points of the ray system from which ray cones of degree g emanate then one will have m_1 singular points with plane pencils of rays, through which each of the three surfaces goes just once, so each of them will contain only one of its intersection points. Each of the three surfaces goes through each of the m_2 singular points with ray cones of degree two twice, which gives 2^3 intersection points that lie in each of these singular points; therefore, $2^3 m_2$ intersection points of the three surfaces lie at these m_2 singular points. Generally, each singular point with a ray cone of degree g is a union of g^3 intersection points of the three surfaces, because each of them goes through it g times. On the other hand, the number of all intersection points of the three surfaces is then equal to:

$$3(n + 3)^2 + m_1 + 2^3\, m_2 + 3^3\, m_3 + \ldots,$$

which is a series that is established only up to the term $n^3\, m_n$, because ray cones of degree higher than n cannot exist. When both expressions for the number of intersection points are set equal to each other that will give:

XXIX: *If m_g generally denotes the number of all singular points of the ray system from which ray cones of degree g emanate then one will have:*

$$n(n + 3)^2 = m_1 + 2^3\, m_2 + 3^3\, m_3 + \ldots + n^3\, m_n\,.$$

I now consider the double curve of one such ruled surface of degree $n + 3$ that it must have, in addition to the triple straight line that lies at the fixed ray. The n generating lines that lie in an arbitrary plane that goes through fixed ray, along with the fixed ray, intersect it at $\frac{n(n-1)}{2}$ points, which are intersection points of that plane with the double curve. One adds to them the intersection points of the plane with the double curve that lie on the fixed ray itself, and their number is equal to $2\,(n - 1)$. As is known, any generating straight line of a ruled surface of degree $n + 3$ will be, in fact, cut by $n + 1$ other generating straight lines, and these intersection points will be points of the double curve. Of them, the two intersection points are to be counted with the two generating straight lines that lie at the fixed ray, so what remain are precisely $n - 1$ intersection curves of any generating straight line with the double curve. Each of the two generating straight lines that lie on the fixed ray thus contains $n - 1$ intersection points with the double curve, from which, it follows that the fixed ray will go through $2\,(n - 1)$ point of the double curve. The number of all points of the double curve that lie in the plane considered – thus, the degree of that curve – is therefore $\frac{n(n-1)}{2} + 2(n-1) = \frac{(n-1)(n+4)}{2}$.

I now add a second ruled surface of the same kind and consider the intersection points of the double curve of the first surface with the second surface, whose number must be equal to $\frac{(n-1)(n+4)(n+3)}{2}$, since the curve has degree $\frac{(n-1)(n+4)}{2}$ and the surface has degree $n + 3$. These intersection points are again singular points of the ray system, in general, since two rays that lie in the first surface and intersect in the double curve go through each of them, along with a ray that lies in the second surface, as well. Only those intersection points for which the ray that lies in the second surface is identical with one of the two that lie in the first surface (so only two distinct rays go through it) are non-singular points of the ray system. Since the second surface has $n + 3$ generating lines in common with the first one, and since each of them cuts the double curve in $n - 1$ points, the number of those intersection points of the double curve of the first surface with the second surface that are not singular points of the ray system is equal to $(n - 1)(n + 3)$, while all the remaining intersection points must distribute themselves on the m_1 singular points with plane pencils of rays, the m_2 singular points, with ray cones of degree two, and in general, on the m_g singular points with ray cones of degree g. Each of the two ruled surfaces goes through a singular point with a g^{th}-degree ray cone g times, so the double curve of the first surface must, in turn, go through that point $\frac{g(g-1)}{2}$ times, since any two passages of the surface will give a branch of the double curve that goes through that point. Since that point is likewise a g-fold point of the second surface, it will unite $\frac{g^2(g-1)}{2}$ intersection points of the double curve of the first surface with the second surface. The m_g points with ray cones of degree g then contain $\frac{g^2(g-1)}{2} m_g$ intersection points. If one now takes $g = 1, 2, 3, \ldots, n$ and adds the number that was found for those intersection points that do not exist at singular points of the system then one will get the number of all intersection points of the double curve of the first surface with the second surface as:

$$(n-1)(n+3) + 2m_2 + 9\,m_3 + 24\,m_4 + \ldots + \frac{n^2(n-1)}{2} m_n .$$

This number, when set equal to the one that was given above, will give the theorem:

XXX: *If m_g generally denotes the number of all singular points of the ray system from which ray cones of degree g emanate then one will have:*

$$\frac{(n-1)(n+2)(n+3)}{2} = 2m_2 + 9m_3 + 24\,m_4 + \ldots + \frac{n^2(n-1)}{2} m_n .$$

Many other theorems of the same kind can be developed in a similar way, although the two that were given are completely sufficient for the use that we would like to make of them in what follows for the exhibition of all second-order ray systems that have no focal curves. In regard to the possible exceptions to the validity of these two theorems, it must be remarked that the only possible exception is the special case in which two or more of the singular points of the ray system combine into one, although no exceptions will, in fact, exist for that case either when it is generally considered to be a limiting case, and when, in turn, the same kind of enumeration of points is applied to it as in the general case.

If one considers the relationship of the ray cone to the focal surface, for which it is always an enveloping cone, then it is obvious for the ray cones of degree two and higher that the center must be the node of the focal surface. Firstly, the center of any ray cone must be a point of the focal surface, since an intersection of infinitely-close rays of the system exists in it, and secondly, it is a contact point for infinitely-many tangents to the focal surface that do not lie in a plane. The case in which the ray cone has degree one – hence, it is a plane pencil of rays – demands special consideration, since in that case any point of the contact conic section of the singular tangential planes in which the ray pencil must lie can possibly be its center. As is known, six straight lines emanate from an arbitrary non-singular point of a fourth-degree surface that does not also lie in the contact conic section of the singular tangential plane of that surface with a singular tangential plane, each of which contact the surface at that and yet another point, and one of these six doubly-contacting straight lines must be the ray of the second-order ray system that goes that point of the focal surface and has fourth-degree surface for its focal surface. If one lets the point of the focal surface from which these six doubly-contacting straight lines emanate come infinitely close to the contact conic section of the singular tangential plane then each of these six lines will become one of the six nodes of the fourth-degree focal surface that lie in each singular tangential plane, while those of these six lines that is a ray of the second-order ray system go through a well-defined one of the six nodes, and since the same must be the case for all continuously following points of the contact conic section, a ray pencil will emanate from one of these six nodes that is the ray pencil of the second-order ray system that lies in this singular tangential plane whose center therefore also lies at a node of the focal surface. One then has the following two theorems:

XXXI: *The center of any ray cone is likewise a node of the fourth-degree focal surface.*

and

XXXII: *A planar ray pencil that emanates from a node lies in any singular tangential plane to the fourth-degree focal surface.*

In order to also examine the positions of the $\frac{(n-1)(n-2)}{2}$ double rays, which, from Theorem XXIV, each ray system of order two and class $n + 1$ must possess, I now consider two of these double rays, which shall not lie in the same plane. Let the two singular points, which must lie in one of these double rays, be a and b, while the two that lie in the other one are *c* and *d*. From Theorem XXVIII, a ray cone will emanate from the point *a* that will have degree at least three, but since each intersection point of a double ray is a singular point in it, this second double ray must include at least three singular points, if this ray cone does not perhaps have a second double edge, which goes through one of the two singular points of the second double edge, which goes through *c*, and which would be a third double ray that links the two points *a* and *c*. It will be likewise excluded that *b*, as well as yet a fourth double ray, must go through either *c* or *d*, and it follows further from this, from Theorem XXIV, that the ray system must have at least six double rays, and therefore must have class at least six. Moreover, since, as was shown above, the two ray cones that belong to the two singular points of the same double ray in a system of class $n + 1$ are always of degree $n + 3$, the two ray cones that belong to the points *a* and *b* must collectively have degree at least eight, so one of them must have degree at least four. In order for it to cut out no other singular points from singular ray *cd* than *c* and *d*, it must also necessarily have degree only four, and the two points *c* and *d* at which it can cut the double ray, and nowhere else, must be cut out by two double edges of it, which in turn must be two double rays of the system that go from this singular point to *c*, in one case, and *d*, in the other. Since the two ray cones at *a* and *b* have degrees at least eight, and one of them has degree four, the other one must have degree at least four, from which, it likewise follows that it also can have no higher degree, and that also two double rays can go from it to the two points *c* and *d*. Since, as was shown, each of the two ray cones at *a* and *b* must have degree precisely four, the ray system must be of class six; it then contains no other double rays besides the six double rays that were ascertained already, which define the six edges of a tetrahedron. Two double rays that do not lie in a plane can thus be present only in this system of order two and class six, while in all other second-order ray systems any two of double rays that are present must lie in the same plane, and thus intersect, which is not possible unless they all go through a single point. Therefore:

XXXIII: *All double rays in any second-order ray system must always intersect in one and the same point, with the exception of a single system of class six whose six double rays define the six edges of a tetrahedron.*

It is also easy now to determine the degree of all ray cones whose centers lie on double rays. For the special ray system whose six double rays define a tetrahedron, it was already shown that its singular points, through which three double rays go, belong to a ray cone of degree four. In all other ray systems in which all $\frac{(n-1)(n-2)}{2}$ double rays must go through a single point, one has, in addition to those of a singular point with $\frac{(n-1)(n-2)}{2}$ double rays, only ones which go through a double ray and ones that go through no double ray, and the latter have still not be an given any attention. A singular point lies on any double ray with that one double ray, along with a singular point through which all double rays will go. The ray cone of those singular points that contain only one double ray (which, as was just shown before, cannot have a degree that is less than three) must now have a degree that is precisely three. If it had a higher degree then, since it has only a double edge, it would cut each of the other double rays that are present in more than two points, and if it were the single double ray of the system then that system would have to be of class four, and the two ray cones that have this ray for their double edge would need to have degree six collectively, so each of them would have degree three, since neither of the two can have a degree that is less than three. Now, since one of the two singular points that lie on the same double ray is always of degree three, the other one must be of degree n, and both of them will collectively have degree $n + 3$. None of the singular points through which no double rays go can have a degree higher than two, since if one of them had a higher degree then it would cut the existing double ray in more than two points, since it can have no double edges, such that each such ray must contain more than two singular points, or when no double ray is present at all, so the ray system has class only two or three, and this ray cone, from Theorem XXVII cannot have a degree higher than two. I summarize all of these considerations on the degree of the ray cone in the following theorem:

XXXIV: *All of the singular points of a second-order ray system, through which go* $(g - 1)(g - 2)/2$ *double rays, have ray cones of degree g, and conversely:* $(g - 1)(g - 2)/2$ *double rays go through each center of any ray cone of degree g. The number of double rays that go through a singular point is always a trigonal number:* 0, 1, 3, 6, …

The class of a second-order ray system that has no focal curve cannot rise to any arbitrary level, as is already obvious from the facts that its focal surface has degree just four, and that no ray system on a fourth-degree surface can have a class that is higher than 28. It is now easily inferred from the theorem that was just proved that such second-order rays system can no longer exist from class eight on up. Namely, for class eight or higher, 15 or more double rays must be present, which must all go through one and the same singular point, and the ray cone that belongs to such a point must have a degree that is seven or higher. Each ray cone is, however, an enveloping cone of the focal surface that emanates from a node of that surface or a subset of that cone, when it is reducible, and that all-enveloping cone will have degree just six for any fourth-degree surface; ray cones of degree higher than six can therefore not exist. Thus:

XXXV: *There exists no second-order ray system that has no focal curves and whose class is higher than seven.*

The fact that second-order ray systems of class two, three, four, five, six, and seven that have no focal curves actually exist can be shown by a special examination of them, which I will now commence.

§ 7.

Ray systems of order two and class two that have no focal curves.

From Theorem XVIII, for ray systems of order two and class two, the three functions P, Q, R in the first equation that are linear in the ξ, η, ζ, will be of degree one in x, y, z for every ray system of second order, so $n = 1$. It will then follow from Theorem XXVII that ray systems of class two will have no other singular points than the ones that are associated with a planar ray pencil. The number of these singular points is obtained immediately from the equation in Theorem XXIX, which gives $m_1 = 16$ for $n = 1$. From Theorem XXXI, these 16 singular points with planar ray pencils must also be nodes of the focal surface and the 16 planar ray pencils must lie in 16 singular tangential planes of the focal surface. One then has the theorem (*):

XXXVI. *Ray systems of order two and class two have* 16 *singular points with planar ray pencils; their focal surfaces are fourth-degree surfaces with* 16 *nodes and* 16 *singular tangential planes.*

As was just shown for the case of $n = 1$ before us, the first (linear) equation of the ray system might have no derived equation, so it must have the form that was given in Theorem VI:

(1) $$(a_2\, y - a_1\, z - bt)\, \xi + (a\, z - a_2\, x - b_1\, t)\, \eta + (a_1\, x - a\, y - b_2\, t)\, \zeta = 0.$$

All that remains for us to do is then to find the second equation of this ray system, which one does not derive, so it must be of second degree in the ξ, η, ζ, and must therefore have the following form:

(2) $$A\xi^2 + B\eta^2 + C\zeta^2 + 2D\eta\zeta + 2E\zeta\xi + 2F\xi\eta = 0.$$

If one determines the quantities A, B, C, D, E, F as functions of x, y, z in such a way that this second equation likewise has no derived equation then one will obtain the following general expressions for them with no difficulty:

(3) $$\begin{aligned}
A &= c_2 y^2 - 2d\ yz + c_1 z^2 - 2 f_2 yt + 2g_1 zt + h\, t^2, \\
B &= c\ z^2 - 2d_1 zx + c_2 x^2 - 2 f\ zt + 2g_2 xt + h_1 t^2, \\
C &= c_1 x^2 - 2d_2 xy + c\ y^2 - 2 f_1 xt + 2g\ yt + h_2 t^2, \\
D &= -d\ x^2 + d_1 xy + d_2 zx - c\ yz + (e_2 - e_1) xt + f\ yt - g\ zt + i\ t^2, \\
E &= -d_1 y^2 + d_2 yz + d\ xy - c_1 zx + (e - e_2) yt + f_1 zt - g_1 xt + i_1 t^2, \\
F &= -d_2 z^2 + d\ zx + d_1 yz - c_2 xy + (e_1 - e) zt + f_2 xt - g_2 yt + i_2 t^2.
\end{aligned}$$

Since the equations (1) and (2) that this determines have no derived equations, and therefore no further restricting condition is present, these two equations will give ray systems of order two and class two for all arbitrary values of their constants; they also represent the most general ray system of order two and class two, as we will likewise show. If one sets:

(4) $$\begin{aligned}
a_2 y - a_1 z - b\ t &= r, \\
a\ z - a_2 x - b_1 t &= r_1, \\
a_1 x - a\ y - b_2 t &= r_2
\end{aligned}$$

for the sake of brevity then, as was shown above in § 3, the focal surface of this system will be expressed by the following equation:

* [D.H.D.: The focal surface is then a *Kummer surface*, in the modern terminology.]

(5)
$$\begin{vmatrix} A & F & E & r \\ F & B & D & r_1 \\ E & D & C & r_2 \\ r & r_1 & r_2 & 0 \end{vmatrix} = 0.$$

In this form, it is clearly of sixth degree, but it includes the factor t^2, which will drop out when the associated determinant is developed, such that all that will remain is an equation of degree four, as it must be.

Everything now comes down to finding the simplest form for the equations of the ray systems of order two and class two, and insofar as they are also to be regarded as the most general ones, one hopes to obtain all of the ray systems of order two and class two from collinear transformations of this one form. To that end, I consider the obviously very special case, in which all of the constants in equation (2) are zero, with the single exception of e, e_1, e_2, and I set $e_2 - e_1 = \delta$, $e - e_2 = \delta_1$, $e_1 - e = \delta_2$, such that one has $\delta + \delta_1 + \delta_2 = 0$; I will leave equation (1) unchanged. The two equations of this ray system are:

(6)
$$r\xi + r_1\eta + r_2\zeta = 0,$$
$$\delta x\eta\zeta + \delta_1 y\,\zeta\,\xi + \delta_2\,\zeta\,\xi\,\eta = 0,$$

where
$$\delta + \delta_1 + \delta_2 = 0.$$

The focal surface of this system is:

(7)
$$\begin{vmatrix} 0 & \delta_2 z & \delta_1 y & r \\ \delta_2 z & 0 & \delta x & r_1 \\ \delta_1 y & \delta x & 0 & r_2 \\ r & r_1 & r_2 & 0 \end{vmatrix} = 0,$$

or, when developed:

(8)
$$\delta^2 x^2 r^2 + \delta_1^2 y^2 r_1^2 + \delta_2^2 z^2 r_2^2 - 2\,\delta_1\,\delta_2\,y\,z\,r_1\,r_2 - 2\,\delta_{12}\delta z\,x\,r_2\,r - 2\,\delta\delta_1\,x\,y\,r\,r_1 = 0,$$

which can also be put into the simple irrational form:

(9)
$$\sqrt{\delta x r} + \sqrt{\delta_1 y r_1} + \sqrt{\delta_2 z r_2} = 0.$$

These equations represent the most general fourth-degree surface with 16 nodes, insofar as all other surfaces of this type will only be collinear transformations of the surface that is represented by each of these equations (7), (8), (9), as I have confirmed in a communication to the Monatsberichten der Akademie in the year 1864, page 246; the form that was chosen here agrees completely with the one that was given there, up to the constants that I have chosen somewhat differently here, in the interests of symmetry. It follows immediately from this that *equations* (6) *represent the most general ray system of order two and class two, insofar as all ray systems of this type are only collinear transformations of the equations that are included in that system.* Since the focal surfaces of all of these ray systems are collinear to the focal surface of the ray system (6), this system itself must also be collinear to the one that is defined by equations (6).

In order to ascertain the positions of the 16 singular points of the system and the 16 planar ray pencils that are associated with them, I present the equations of the 16 singular tangential planes and the coordinates of the 16 nodes of the focal surface in their entirety:

Singular tangential planes:

(10)

1.	$x = 0,$	9.	$\frac{\varepsilon_2 y}{b_2} - \frac{\varepsilon_1 y}{b_1} - \frac{\varepsilon t}{a} = 0,$
2.	$y = 0,$	10.	$\frac{\varepsilon z}{b} - \frac{\varepsilon_2 x}{b_2} - \frac{\varepsilon_1 t}{a_1} = 0,$
3.	$z = 0,$	11.	$\frac{\varepsilon_1 x}{b_1} - \frac{\varepsilon y}{b} - \frac{\varepsilon_2 t}{a_2} = 0,$
4.	$t = 0,$	12.	$\frac{\varepsilon x}{a} + \frac{\varepsilon_1 y}{a_1} + \frac{\varepsilon_2 z}{a_2} = 0,$
5.	$a_2\, y - a_1\, z - b\, t = 0,$	13.	$\frac{\varepsilon_2' y}{b_2} - \frac{\varepsilon_1' z}{b_1} - \frac{\varepsilon' t}{a} = 0,$
6.	$a\, z - a_2\, x - b_1\, t = 0,$	14.	$\frac{\varepsilon' z}{b} - \frac{\varepsilon_2' x}{b_2} - \frac{\varepsilon_1' t}{a} = 0,$
7.	$a_1\, x - a\, y - b_2\, t = 0,$	15.	$\frac{\varepsilon_1' y}{b_1} - \frac{\varepsilon' x}{b} - \frac{\varepsilon_2' t}{a_2} = 0,$
8.	$b\, x + b_1\, y + b_2\, z = 0,$	16.	$\frac{\varepsilon' x}{a} + \frac{\varepsilon_1' y}{a_1} + \frac{\varepsilon_2' z}{a_2} = 0.$

Nodes:

(11)

1.	$x = 0,$	$y = -\frac{b_2 t}{a},$	$z = \frac{b_1 t}{a},$
2.	$y = 0,$	$z = -\frac{bt}{a_1},$	$x = \frac{b_2 t}{a_1},$
3.	$z = 0,$	$x = -\frac{b_1 t}{a_2},$	$y = \frac{bt}{a_2},$
4.	$t = 0,$	$\frac{x}{a} = \frac{y}{a_1} = \frac{z}{a_2},$	
5.	$y = 0,$	$z = 0,$	$t = 0,$
6.	$z = 0,$	$x = 0,$	$t = 0,$
7.	$x = 0,$	$y = 0,$	$t = 0,$
8.	$x = 0,$	$y = 0,$	$t = 0,$

9.	$x = 0,$	$y = -\dfrac{\varepsilon_2' bt}{\varepsilon' a_2},$	$z = \dfrac{\varepsilon_1' bt}{\varepsilon' a_1},$
10.	$y = 0,$	$z = -\dfrac{\varepsilon' b_1 t}{\varepsilon_1' a},$	$x = \dfrac{\varepsilon_2' b_1 t}{\varepsilon_1' a_2},$
11.	$z = 0,$	$x = -\dfrac{\varepsilon_1' b_2 t}{\varepsilon_2' a_1},$	$y = \dfrac{\varepsilon' b_2 t}{\varepsilon_2' a},$
12.	$t = 0,$	$\dfrac{\varepsilon x}{\delta a} = \dfrac{\varepsilon_1 y}{\delta_1 a_1} = \dfrac{\varepsilon_2 z}{\delta_2 a_2},$	
13.	$x = 0,$	$y = -\dfrac{\varepsilon_2 bt}{\varepsilon a_2},$	$z = \dfrac{\varepsilon_1 bt}{\varepsilon a_1},$
14.	$y = 0,$	$z = -\dfrac{\varepsilon b_1 t}{\varepsilon_1 a},$	$x = \dfrac{\varepsilon_2 b_1 t}{\varepsilon_1 a_2},$
15.	$z = 0,$	$x = -\dfrac{\varepsilon_1 b_2 t}{\varepsilon_2 a_1},$	$y = \dfrac{\varepsilon b_2 t}{\varepsilon_2 a},$
16.	$t = 0,$	$\dfrac{\varepsilon' x}{\delta a} = \dfrac{\varepsilon_1' y}{\delta_1 a_1} = \dfrac{\varepsilon_2' z}{\delta_2 a_2},$	

where the quantities ε, ε_1, ε_2 – or their quotients, moreover – are determined by the equations:

$$\varepsilon + \varepsilon_1 + \varepsilon_2 = 0, \qquad \frac{\delta ab}{\varepsilon} + \frac{\delta_1 a_1 b_1}{\varepsilon_1} + \frac{\delta_2 a_2 b_2}{\varepsilon_2} = 0 \tag{12}$$

in a double-valued way, and ε, ε_1', ε_2' are the associated second values. One then has the quadratic equation:

$$\delta ab\varepsilon_1^2 + (\delta ab + \delta_1 a_1 b_1 - \delta_2 a_2 b_2)\, \varepsilon_1 \varepsilon + \delta_1 a_1 b_1 \varepsilon^2 = 0 \tag{13}$$

for the ratio $\varepsilon : \varepsilon_1$, and it follows from this that:

$$\varepsilon\varepsilon' : \varepsilon_1\varepsilon_1' : \varepsilon_2\varepsilon_2' = \delta ab : \delta_1 a_1 b_1 : \delta_2 a_2 b_2 . \tag{14}$$

If one denotes the nodes, and also the singular tangential planes, simply by the numbers at the top of the column then one can represent the six singular tangential planes that go through a node, and likewise the six nodes that lie in a singular tangential plane, by the following table:

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	I.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
(15)	II.	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
	III.	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
	IV.	8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
	V.	7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
	VI.	6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11

Here, the first column means: In the singular tangential plane 1, there lie nodes 1, 9, 13, 8, 7, 6, and vice versa: Through the node 1, there go the singular tangential planes 1, 9, 13, 8, 7, 6. All sixteen columns have the corresponding double meaning. The order of the points and planes is deliberately chosen in such a way that the reciprocity relationship that exists between them will become obvious in this way. The fact that each of the sixteen points always lies in the planar ray pencil that is associated with the ray system that is defined by (6) comes from the fact that there as just as many nodes as singular tangential planes.

From the fact that the six singular tangential planes that go through a node of the focal surface all have an equal status, and the fact that all six of them indeed have the same right to include a ray pencil in a system of order two and class two that that go through their common intersection point (since one of them includes such a ray pencil) one can infer that each fourth-degree surface with 16 nodes will likewise be the focal surface for six different ray systems of order two and class two at once. In fact, the same focal surfaces (7), (8), or (9) are associated with the following six distinct ray systems of order two and class two:

$$\text{I.}\qquad \begin{cases} (a_2 y - a_1 z - bt)\xi + (az - a_2 x - b_1 t)\eta + (a_1 x - ay - b_2 t)\zeta = 0, \\ \delta x\eta\zeta + \delta_1 y\zeta\xi + \delta_2 z\xi\eta = 0, \end{cases}$$

$$\text{II.}\qquad \begin{cases} \left(\dfrac{\varepsilon_2' y}{b_2} - \dfrac{\varepsilon_1' z}{b_1} - \dfrac{\varepsilon' t}{a}\right)\xi + \left(\dfrac{\varepsilon' z}{b} - \dfrac{\varepsilon_2' x}{b_2} - \dfrac{\varepsilon_1' t}{a_1}\right)\eta + \left(\dfrac{\varepsilon' x_1}{b_1} - \dfrac{\varepsilon' z}{b} - \dfrac{\varepsilon_2' t}{a_2}\right)\zeta = 0, \\ \varepsilon x\eta\zeta + \varepsilon_1 y\zeta\xi + \varepsilon_2 z\xi\eta = 0, \end{cases}$$

$$\text{III.}\qquad \begin{cases} \left(\dfrac{\varepsilon_2 y}{b_2} - \dfrac{\varepsilon_1 z}{b_1} - \dfrac{\varepsilon t}{a}\right)\xi + \left(\dfrac{\varepsilon z}{b} - \dfrac{\varepsilon_2 x}{b_2} - \dfrac{\varepsilon_1 t}{a_1}\right)\eta + \left(\dfrac{\varepsilon_1 x}{b_1} - \dfrac{\varepsilon z}{b} - \dfrac{\varepsilon_2 t}{a_2}\right)\zeta = 0, \\ \varepsilon' x\eta\zeta + \varepsilon_1' y\zeta\xi + \varepsilon_2' z\xi\eta = 0, \end{cases}$$

$$\text{IV.}\qquad \begin{cases} bt\xi + a_1 z\eta - ay\zeta = 0, \\ \left(\delta_2 a_2 y + \delta_1 a_1 z + (\delta_2 a_2 b_2 - \delta_1 a_1 b_1)\dfrac{t}{a}\right)\xi^2 - \delta ax\eta\zeta - (\delta_1 a_1 x + \delta_2 ay + \delta_1 b_1 t)\zeta\xi \\ \quad - (\delta_1 a_1 x + \delta_2 ay - \delta_1 b_1 t)\xi\eta = 0, \end{cases}$$

$$\text{V.}\quad\begin{cases} b_1 t\eta + a_1 x\zeta - a_1 z\xi = 0, \\ \left(\delta ay + \delta_2 a_2 x + (\delta ab - \delta_2 a_2 b_2)\dfrac{t}{a}\right)\eta^2 - \delta_1 a_1 y\zeta\xi - (\delta_2 a_2 y + \delta_1 az + \delta bt)\xi\eta \\ \quad - (\delta_2 a_1 x + \delta ay - \delta_2 b_2 t)\eta\zeta = 0, \end{cases}$$

$$\text{VI.}\quad\begin{cases} b_2 t\zeta + a_2 y\xi - a_2 x\eta = 0, \\ \left(\delta_1 a_1 x + \delta ay + (\delta_1 a_1 b_1 - \delta ab)\dfrac{t}{a}\right)\zeta^2 - \delta_2 a_2 z\xi\eta - (\delta az + \delta_1 a_2 x + \delta_1 b_1 t_1)\eta\zeta \\ \quad - (\delta a_2 y + \delta_1 a_1 z - \delta bt)\zeta\eta = 0, \end{cases}$$

where $\delta + \delta_1 + \delta_2 = 0$ and ε, ε_1, ε_2, ε', ε'_1, ε'_2 are determined by the equations that are given in (12).

One obtains the remaining five ray systems that are associated with the same focal surface from the one that was given first by applying a suitable collinear transformation, under which the equation of the focal surface will be converted into an equation with the same analytical form that differs from the given one only by the values of the constants, such that a goes over to a', a_1 into a'_1, a_2 into a'_2, etc., although the two equations of the ray system will be essentially different. There is also an entire cycle of collinear transformations of the equation of the focal surface into itself, under which the values of the constants a, a_1, a_2, b, b_1, b_2, δ, δ_1, δ_2 will also remain unchanged, although that is precisely why the ray systems will also remain unchanged, such that one of these ray systems cannot be used for the purpose of deriving the remaining five of them. In order to derive system IV from system I, I make the following linear substitution:

$$\begin{aligned} x' &= a_2 y - a_1 z - bt, & x &= -\frac{b_1 y'}{b} - \frac{b_2 z'}{b} + \frac{t'}{b}, \\ y' &= y, & y &= y', \\ z' &= z, & z &= z', \\ t' &= bx + b_1 y + b_2 z, & t &= -\frac{x'}{b} + \frac{a_2 y'}{b} - \frac{a_1 z'}{b}; \end{aligned}$$

it follows from this that:

$$\begin{aligned} az - a_2 x - b_1 t &= \frac{hz'}{b} + \frac{b_2 x'}{b} - \frac{a_2 t'}{b}, \\ a_1 x - a y - b_2 t &= \frac{b_2 x'}{b} - \frac{hy'}{b} + \frac{a_1 t'}{b}, \end{aligned}$$

in which we have set:

$$ab + a_1 b_1 + a_2 b_2 = h.$$

By this substitution, the equation of the focal surface is converted into an equation of the same form, with the altered constants.

$$a' = \frac{h}{b}, \qquad a_1' = \frac{b_2}{b}, \qquad a_2' = -\frac{b_1}{b},$$
$$b' = -\frac{1}{b}, \qquad b_1' = \frac{a_2}{b}, \qquad b_2' = -\frac{a_1}{b},$$

while δ, δ_1, δ_2 remain unchanged. From the formulas that were given in § 1 for the collinear transformation of the ray system, one has:

$$\xi = (b'\,x' + b_1'\,y' + b_2'\,z')\,(a_2'\,\eta' + a_1'\,\zeta') - (a_2'\,y + a_1'\,z - b'\,t)\,(b'\xi' + b_1'\,\eta' + b_2'\,\zeta'),$$
$$\eta = (b'\,x' + b_1'\,y' + b_2'\,z')\,\eta' - y'\,(b'\xi' + b_1'\,\eta' + b_2'\,\zeta'),$$
$$\xi = (b'\,x' + b_1'\,y' + b_2'\,z')\,\zeta' - z'\,(b'\xi' + b_1'\,\eta' + b_2'\,\zeta').$$

If one now substitutes the values of $x, y, z, t, \xi, \eta, \zeta$ in the two equations for ray system I then one will obtain, after carrying out the calculations:

$$b'\,t'\,\xi' + a'\,z'\,\eta' - a'\,y'\,\zeta' = 0,$$

$$\left(\delta_2 a_2' y' + \delta_1 a_1' z' + (\delta_2 a_2' b_2' - \delta_1 a_1' b_1')\frac{t'}{a'}\right)\xi'^2 - \delta a' x' \eta' \zeta' - (\delta_1 a_1' x' + \delta_2 a' y' + \delta_2 b_2' t)\zeta' \xi'$$
$$- (\delta_1' a' z' + \delta_2 a_2' x' - \delta_1 b_1' t')\xi' \eta' = 0,$$

as the two equations of a ray equation whose focal surface has the form of equation (9), with the constants a', b', etc. Since these equations agree completely with those of ray system IV, it will then follow that ray system IV will have the same focal surface (9) as ray system I. It then follows immediately from this that systems V and VI will have the same focal surface; they will then arise from IV by permuting the symbols $x, y, z, a, a_1, a_2, b, b_1, b_2$, while the focal surface remains unchanged. Ray systems II and III can be derived from I in the same way by linear transformations; one obtains them, however, much more simply when one remarks that the equation of the focal surface can also be put into the following form:

$$\sqrt{\varepsilon' x\left(\frac{\varepsilon_2 y}{b_2} - \frac{\varepsilon_1 z}{b_1} - \frac{\varepsilon t}{a}\right)} + \sqrt{\varepsilon_1' y\left(\frac{\varepsilon z}{b} - \frac{\varepsilon_2 x}{b_2} - \frac{\varepsilon_1 t}{a_1}\right)} + \sqrt{\varepsilon_2' z\left(\frac{\varepsilon_1 x}{b_1} - \frac{\varepsilon y}{b} - \frac{\varepsilon_2 t}{a_2}\right)} = 0.$$

If one also carries out the same change of constants by which equation (9) goes into this form on ray system I then one will obtain ray system III, and, if one permutes the roots of the quadratic equation by which ε, ε_1, ε_2 are given such that they go over to ε', ε_1', ε_2' then one will obtain ray system II from this.

The planar ray pencils that are associated with each of these six ray systems of order two and class two will be determined completely by the table (15) that was given above; that table is then arranged such that the numbers above the line mean the 16 singular points and the numbers in the rows I, II, III, IV, V, VI give the planes in which the planar ray pencil that is associated with the point lies for each of the six ray systems.

The complete system of all straight lines that contact a fourth-degree surface with 16 nodes twice includes, in addition to these six ray systems, 16 more ray systems of order zero and class one, each of which consists of all of the straight lines that lie in a singular tangential plane, since they are always straight lines that contact the surface twice, so the system is, in fact, of order 12 and class 28, as must be the case for any fourth-degree surface. One then has the theorem:

XXXVII. *Any fourth-degree surface with* 16 *nodes is the focal surface of six distinct ray systems of order two and class two and of* 16 *distinct ray systems of order zero and class one.*

As noteworthy special cases of this general ray system of order two and class two, I would like to mention here the case in which the fourth-degree focal surface with 16 nodes becomes a surface with one double line and the one in which it becomes a surface with two double lines.

If one sets $b_2 = 0$ then one will obtain the focal surface with a double line from $x = 0$, $y = 0$; the eight nodes 1, 2, 7, 8, 9, 10, 15, 16 will fall on these double lines whenever one joins any two of them – namely, 1 and 10, 2 and 9, 7 and 16, 8 and 15 – to a point; the eight singular tangential planes with the same names will go through the double line when one likewise joins any two of the corresponding points with the same names to a plane and makes them coincide. All that will then remain are eight special nodes that do not coincide and do not lie in the double lines, and eight singular tangential planes that do not coincide and do not go through the double line. Four of the six ray systems of order two and class two will remain – namely, I, II, IV, V – which consist of the ones that have no focal curve, although the two ray systems III and IV give only those ray systems of order two and class two that have the double line for their focal curve. Each of the four ray systems that have no focal lines will still have 16 singular points with 16 planar ray pencils when the two that coincide are counted twice everywhere. With a different type of enumeration, the theorem that was given in § 6 on the number of singular points for second-order ray systems would no longer be correct in such special or limiting cases, which was already stated expressly at the time.

If one specializes further by setting $a_2 = 0$, in addition to $b_2 = 0$, then one will obtain the fourth-degree focal surface with two non-intersecting double lines $x = 0$, $y = 0$ and $z = 0$, $t = 0$. It will then become a fourth-degree ruled surface, since it is known that two non-intersecting double lines can occur only in a fourth-degree ruled surface. Eight nodes will then fall on each of the two double lines when one combines them into a single one, and likewise eight singular tangential planes will go through each of the two double lines, two of which will coincide. The four ray systems I, II, IV, V will also still remain in this case as the ones that have no focal curve, while III and VI will drop away.

§ 8.

Ray systems of order two and class three that have no focal curves.

From Theorem XVIII, the three functions P, Q, R in the first linear equation of a ray system of order two and class three will be of degree two. If one then sets $n = 2$ in the two equations of Theorems XXIX and XXX then that will give:

$$50 = m_1 + 8m_2 \qquad \text{and} \qquad 10 = 2m_2 ,$$

so

$$m_1 = 10 \qquad \text{and} \qquad m_2 = 5.$$

The ray systems of this class then have 15 singular points, in all, 10 of which have plane pencils of rays and 5 of which have second-degree ray cones, and since the singular points of the system are likewise nodes and the planes of the ray pencil are singular tangential planes of the focal surface, one will have the following theorem:

XXXVIII: *The ray systems of order two and class three have* 15 *singular points, and, in fact,* 10 *of them have planar ray pencils and* 5 *of them have second-degree ray cones; their focal surfaces are fourth-degree surfaces with* 15 *nodes and* 10 *singular tangential planes.*

If one is given the first equation of a ray system of this class:

$$P\,\xi + Q\,\eta + R\,\zeta = 0 \tag{1}$$

then the second one will likewise be given as the first derived equation of that equation, but, as was shown above for the n^{th} derived equation, in general, the second derived equation must vanish identically, and here, where other derived equations do not exist, that condition will be the sufficient condition for the first equation, together with its one derived equation, to in fact give a ray system of order two and class two, which must likewise be the most general of that class. If one sets P, Q, R equal to entire, rational functions of degree two in x, y, z¸ t of a general form then the condition that the second derived equation must vanish identically will immediately give ten simple linear equations in the 3 times 10 constants of these second-degree functions, which will yield the following most general form of them:

$$\begin{aligned} P &= -f_1\,y^2 - e_2\,z^2 + d\,y\,z + e\,z\,x + f\,x\,y + g\,x\,t + h\,y\,t + i\,z\,t + k\,t^2, \\ Q &= -d_2\,z^2 - f\,x^2 + d_1\,y\,z + e_1\,z\,x + f_1\,x\,y + g_1\,x\,t + h_1\,y\,t + i_1\,z\,t + k_1\,t^2, \\ R &= -e\,x^2 - d_1\,y^2 + d_2\,y\,z + e_2\,z\,x + f_2\,x\,y + g_2\,x\,t + h_2\,y\,t + i_2\,z\,t + k_2\,t^2, \end{aligned} \tag{2}$$

with the one condition equation:

$$d + e_1 + f_2 = 0. \tag{3}$$

If one sets the first equation in the form:

(4) $$A\,\xi^2 + B\,\eta^2 + C\,\zeta^2 + 2\,D\,\eta\,\zeta + 2\,E\,\zeta\,\xi + 2\,F\,\xi\,\eta = 0$$
then one will get:

$$\begin{aligned} A &= 2\,(f\,y + e\,z + g\,t), & D &= -\,d\,x - d_1\,y - d_2\,z + (i_1 + h_2)\,t, \\ B &= 2\,(d_1\,z + f_1\,x + h_1\,t), & E &= -\,e\,x - e_1\,y - e_2\,z + (g_2 + i)\,t, \\ C &= 2\,(e_1\,x + d_2\,y + i_1\,t), & F &= -\,f\,x - f_1\,y - f_2\,z + (h + g_1)\,t. \end{aligned}$$

As was shown in general in § 3, the focal surface of this most general ray system of class three that is represented by both of equations (1) and (4) will be given by the following determinant:

(5) $$\begin{vmatrix} A & F & E & P \\ F & B & D & Q \\ E & D & C & R \\ P & Q & R & 0 \end{vmatrix} = 0,$$

which will obviously be of degree six, since P, Q, R are of degree two and A, B, C, D, E, F are of degree one, but it includes the factor t^2, which drops out in such a way that the focal surface will be of degree four, as it must be. The fact that the surface that is determined by this equation has, in fact, 15 nodes and ten singular planes is difficult to see in this most general form, so the simplest form of the ray systems shall also be presented here, which is likewise also the most general one, insofar as all ray systems of this class will be only collinear conversions of it.

To that end, I take the most general form of P, Q, R:

$$\begin{array}{llll} d = \delta, & h = a_2, & i = -\,a_1, & k = -\,b, \\ e_1 = \delta, & i_1 = a, & g_1 = -\,a_2, & k_1 = -\,b_1, \\ f_2 = \delta, & g_2 = a_1, & h_2 = -\,a, & k_2 = -\,b_2, \end{array}$$

while I take all other coefficients equal to zero, which makes:

(6) $$\begin{aligned} P &= \delta\,y\,z + r\,t, \\ Q &= \delta_1\,z\,x + r_1\,t, \\ R &= \delta_2\,x\,y + r_2\,t, \end{aligned}$$

in which r, r_1, r_2 denote the same quantities as in the previous paragraphs, and in which:

$$\delta + \delta_1 + \delta_2 = 0.$$

The focal surface of this ray system is:

$$(7)\qquad \begin{vmatrix} 0 & -\delta_2 z & -\delta_1 y & \delta yz + rt \\ -\delta_2 z & 0 & -\delta x & \delta_1 zx + r_1 t \\ -\delta_1 y & -\delta x & 0 & \delta_2 xy + r_2 t \\ \delta yz + rt & \delta_1 zx + r_1 t & \delta_2 xy + r_2 t & 0 \end{vmatrix} = 0,$$

which can be put into the following simpler form, from which, the factor of t^2 has been omitted:

$$(8)\qquad \begin{vmatrix} 0 & -\delta_2 z & -\delta_1 y & r \\ -\delta_2 z & 0 & -\delta x & r_1 \\ -\delta_1 y & -\delta x & 0 & r_2 \\ r & r_1 & r_2 & 2r_3 \end{vmatrix} = 0,$$

in which, for the sake of brevity, we have set:

$$b\,x + b_1\,y + b_2\,z = r_3\,.$$

The complete development of this determinant gives:

$$(9)\qquad \begin{aligned} \delta^2 x^2 r^2 + \delta_1^2 y^2 r_1^2 + \delta_2^2 z^2 r_2^2 - 2\delta_1\delta_2\, yz\, r_1\, r_2 - 2\delta_1\delta\, yzr_1\, r_2 - 2\delta\delta_1\, xyrr_1 \\ - 4\delta\delta_1\delta_2\, xyzr_3 = 0. \end{aligned}$$

This equation, which differs from equation (8) in the previous paragraph only by its last term, which is added to it, gives the most general form of the equation of all fourth-degree surfaces with 15 modes, insofar as all of these surfaces are only collinear conversions of the one in this form. The complete proof of this assertion will be achieved with no difficulty from the same method by which I developed the most general form of all fourth-degree surfaces with 16 nodes in the Monatsberichten of 1864, page 249. I will outline the actual proof here, which goes further into the present purpose of the investigation of ray systems. It follows from it that *all ray systems of order two and class three are only collinear conversion of those of the ray systems whose three determining functions P*, *Q*, *R are given by equations* (6).

The ten singular tangential planes to the focal surface have the following equations:

(10)

1. $x = 0$,
2. $y = 0$,
3. $z = 0$,

4 and 7. $\quad (\rho - \delta_1\, a_1\, b_1)\,\dfrac{y}{b_2} + (\rho - \delta ab - \delta_1\, a_1\, b_1)\,\dfrac{y}{b_1} + \delta b\, t = 0,$

5 and 8. $\quad (\rho - \delta_2\, a_2\, b_2)\,\dfrac{y}{b} + (\rho - \delta_1\, a_1\, b_1 - \delta_2\, a_2\, b_2)\,\dfrac{y}{b_2} + \delta_1\, b_1\, t_1 = 0,$

6 and 9. $(\rho - \delta_2\, a_2\, b_2)\,\dfrac{y}{b} + (\rho - \delta_1\, a_1\, b_1 - \delta_2\, a_2\, b_2)\,\dfrac{y}{b_2} + \delta_1\, b_1\, t_1 = 0,$

10. $b\,x + b_1\,y + b_2\,z = 0,$

in which ρ is determined in a double-valued way by the quadratic equation:

$$
\begin{aligned}
&\rho^2 - (\delta ab + \delta_1\, a_1\, b_1 + \delta_2\, a_2\, b_2)\,\rho + \delta_1\, a_1\, b_1 \delta_2\, a_2\, b_2 + \delta_2\, a_2\, b_2\, \delta\, a\, b \\
&\qquad + \delta\, a\, b\, \delta_1\, a_1\, b_1 - \delta\,\delta_1\delta_2\, b\, b_1\, b_2 = 0,
\end{aligned}
\tag{11}
$$

and in which one takes one of the values of ρ to be 4, 5, 6, for the singular tangential planes, but equal to 7, 8, 9 for the other.

The 15 nodes of the focal surface are determined most simply by the four singular tangential planes that go through each of them, which are given by the following table:

(12)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	1	2	3	1	2	3	1	2	1	1	4	7
4	5	6	4	5	6	5	4	4	2	3	3	2	5	8
7	8	9	8	7	7	6	6	5	3	4	5	6	6	9
10	10	10	9	9	8	7	8	9	10	7	8	9	10	10

Here, the numbers above the line denote the nodes and the ones beneath them denote the four numbers of the singular tangential planes that go through each node. Each of the 15 nodes belongs to a second-degree cone that envelops the focal surface, in addition to the four singular tangential planes, which will be denoted by the same number as the node. 9 nodes lie on each of the 15 enveloping second-degree cones, including the one that lies at the center; 9 of these cones go through each node, moreover. The nine nodes that lie in a cone and the nine cones that go through a node will be given systematically by the following table:

(13)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	2	1	1	2	1	1	4	2	1	1	4	7
2	2	2	3	3	2	3	3	2	5	3	3	2	5	8
3	3	3	4	4	4	4	5	6	6	4	5	6	6	9
5	4	4	5	5	5	7	7	7	7	7	8	9	10	10
6	6	5	6	6	6	8	8	8	8	11	11	11	11	11
8	7	7	7	8	9	9	9	9	9	12	12	12	12	12
9	9	8	11	10	10	10	10	10	10	13	13	13	13	13
12	11	11	12	12	13	11	12	14	14	14	14	14	14	14
13	13	12	14	14	14	15	15	15	15	15	15	15	15	15

If a number above the line is assumed to refer to a cone then the numbers beneath it will give the nine nodes that lie in it, and conversely, if the number above the line is assumed to refer to a node, the numbers beneath it will give the nine cones that go through that node.

The ray system of class three that is given by equation (6) includes the five ray cones with the same numbers at the singular points 11, 12, 13, 14, 15, but the plane pencils of rays at the points 1 to 10, whose planes are denoted by the same numbers in the same sequence. As the table shows, the five ray cones 11, 12, 13, 14, 15 lie in such a way that the center of each of them lies on the other four cones. The necessity of that condition for each ray system of class three also follows from the fact that if any two of the five ray cones do not lie in such a way that they reciprocally contain their centers then an arbitrarily-given plane that goes through those two centers would be cut out of the each of the two distinct rays of the system – hence, four, in all – such that it could not be of class three.

From a closer consideration of the table in (13), one sees that there are precisely six couplings of five of the 15 enveloping cones that fulfill the condition that the center lies on one of the other four, namely, the couplings: (11, 12, 13, 14, 15), (4, 5, 6, 10, 14), (7, 8, 9, 10, 15), (2, 3, 4, 7, 11), (1, 3, 5, 8, 12), and (1, 2, 6, 9, 13). One can conclude from this that the same focal surface will belong to six different rays systems of order two and class three whose ray cones are these six couplings. In fact, the following six ray systems of order two and class three have the same focal surface (9):

$$\text{I.}\qquad \begin{cases} P = \delta yz + (a_2 y - a_1 z - bt)t, \\ Q = \delta_1 zx + (ay - a_2 x - b_1 t)t, \\ R = \delta_2 xy + (a_1 x - ay - b_2 t)t, \end{cases}$$

$$\text{II and III.}\qquad \begin{cases} P = s(\delta b(\rho - \delta ab)(\rho' - \delta ab)x + \delta_1 b_1(\rho' - \delta ab)(\rho' - \delta_1 a_1 b_1)y \\ \qquad\qquad + \delta_2 b_2(\rho - \delta ab)(\rho - \delta_2 a_2 b_2)y), \\ Q = s_1(\delta_1 b_1(\rho - \delta_1 a_1 b_1)(\rho' - \delta_1 a_1 b_1)y + \delta_2 b_2(\rho' - \delta_1 a_1 b_1)(\rho' - \delta_2 a_2 b_2)z \\ \qquad\qquad + \delta b(\rho - \delta_1 a_1 b_1)(\rho - \delta ab)x), \\ R = s_2(\delta_2 b_2(\rho - \delta_2 a_2 b_2)(\rho' - \delta_2 a_2 b_2)z + \delta b(\rho' - \delta_2 a_2 b_2)(\rho' - \delta ab)x \\ \qquad\qquad + \delta_1 b_1(\rho - \delta_2 a_2 b_2)(\rho - \delta_1 a_1 b_1)y), \end{cases}$$

where

$$s = (\rho - \delta_1 a_1 b_1)\frac{y}{bz} + (\rho - \delta ab - \delta_1 a_1 b_1)\frac{y}{b_1} + \delta b\, t,$$

$$s_1 = (\rho - \delta_2 a_2 b_2)\frac{y}{b} + (\rho - \delta_1 a_1 b_1 - \delta_2 a_2 b_2)\frac{x}{b_2} + \delta_1 b_1 t_1,$$

$$s_2 = (\rho - \delta ab)\frac{y}{b_1} + (\rho - \delta_2 a_2 b_2 - \delta ab)\frac{y}{b} + \delta_2 b_2 t_2,$$

and where ρ and ρ' are the two roots of the quadratic equation (11):

$$\text{IV.}\qquad \begin{cases} P = \delta yz + (a_2 y - a_1 z - bt)t, \\ Q = \delta_2 zx - z\left((\delta_1 b_1 - a_2 a)\dfrac{y}{b} - (\delta_2 b_2 - aa_1)\dfrac{z}{b} + at \right) \\ R = \delta_1 xy + y\left((\delta_1 b_1 - a_2 a)\dfrac{y}{b} - (\delta_2 b_2 - aa_1)\dfrac{z}{b} + at \right) \end{cases}$$

$$\text{V.}\qquad \begin{cases} P = \delta_2 yz + z\left((\delta_2 b_2 - aa_1)\dfrac{y}{b_1} - (\delta b - a_1 a_2)\dfrac{x}{b_1} + a_1 t \right), \\ Q = \delta_1 zx - (az - a_2 x - b_1 t)t, \\ R = \delta xy - x\left((\delta_2 b_2 - aa_1)\dfrac{z}{b_1} - (\delta b - a_1 a_1)\dfrac{x}{b_1} + a_1 t \right) \end{cases}$$

$$\text{VI.}\qquad \begin{cases} P = \delta_1 yz - z\left((\delta b - a_1 a_2)\dfrac{y}{b_2} - (\delta_1 b_1 - a_2 a)\dfrac{x}{b_2} + a_2 t \right), \\ Q = \delta zx + x\left((\delta b - a_1 a_2)\dfrac{z}{b_2} - (\delta_1 b_1 - a_2 a)\dfrac{x}{b_2} + a_2 t \right), \\ R = \delta_2 xy + (a_1 x - ay - b_2 t)t. \end{cases}$$

The five remaining ray systems can be derived from the first one by the same method through collinear conversions, as in the corresponding case of the previous paragraph, and it can also be verified without difficulty (but not, at the same time, without a certain long-windedness) that they all have the same focal surface by defining and developing the equation of the focal surface for each of them.

The 10 plane pencils of rays and five ray cones that belong to these six ray systems will be given by the following table:

(14)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I.	1	2	3	4	5	6	7	8	9	10	(11)	(12)	(13)	(14)	(15)
II.	7	8	9	(4)	(5)	(6)	1	2	3	(10)	4	5	6	(14)	10
III.	4	5	6	1	2	3	(7)	(8)	(9)	(10)	7	8	9	10	(15)
IV.	10	(2)	(3)	(4)	9	8	(7)	6	5	1	(11)	3	2	4	7
V.	(1)	10	(3)	9	(5)	7	6	(8)	4	2	3	(12)	1	5	8
VI.	(1)	(2)	10	8	7	(6)	5	4	(9)	3	2	1	(13)	6	9

where the second-degree ray cones that are distinguished by parentheses are to be excluded. Since all two-fold contacting straight lines of the focal surface, along with these six rays systems, define 10 ray systems of order zero and class one that lie in the 10 singular tangential planes, one will have the following theorem:

XXXIX: *Any fourth-degree surface with* 15 *nodes and ten singular tangential planes is the focal surface of six distinct ray systems of order two and class three and of* 10 *distinct ray systems of order zero and class one.*

As a special case of one of them in which some of the 15 singular points combine into one, I point out the case in which:

$$\delta b_1\, b_2 + a\,(ab + a_1\, b_1 + a_2\, b_2) = 0,$$

for which one will have:

$$\rho = \delta_1\, a_1\, b_1 - \delta_2\, ab, \qquad \rho' = \delta_2\, a_2\, b_2 - \delta_1\, ab.$$

In this case, the three singular points 1, 4, 15 coalesce into one, which will be a uniplanar node for the focal surface whose osculating cone consists of two coincident planes. The three nodes 1, 4, 15 that belong to the enveloping cone of degree two decompose into two planes that are identical with two of the existing singular tangential planes, so they give six singular tangential planes that go through uniplanar node; the remaining 12 nodes each carry four singular tangential planes and its enveloping second-degree cone. The ray systems that belongs to a focal surface with 13 nodes, one of which is a uniplanar one, consist of six different ray systems of order two and class three, with the difference that each of them will carry only four second-degree ray cones, since the fifth one will decompose into two plane pencils of rays that emanate from the uniplanar node.

Another remarkable special case of ray systems of class three that one gets from the general equations that were presented for them, not immediately, but after a collinear conversion, is the one for which four times three nodes combine into four uniplanar nodes, and three of them consist of ordinary nodes. The general equation for the fourth-degree surfaces that have four uniplanar and three ordinary nodes is:

$$(yzyz + zx + xy + xt + yt + zt)^2 - 4xyzt = 0,$$

so the four uniplanar nodes are:

1.	$y = 0,$	$z = 0,$	$t = 0,$
2.	$z = 0,$	$x = 0,$	$t = 0,$
3.	$x = 0,$	$y = 0,$	$t = 0,$
4.	$x = 0,$	$y = 0,$	$t = 0,$

and the three ordinary nodes are:

5.	$x = +\,t,$	$z = -\,t,$	$t = -\,t,$
6.	$x = -\,t,$	$z = +\,t,$	$t = -\,t,$
7.	$x = -\,t,$	$z = -\,t,$	$t = +\,t.$

The ten singular tangential planes to the planes are:

1.	$x = 0,$	5.	$y + z = 0,$	8.	$x + t + 0,$
2.	$y = 0,$	6.	$z + x = 0,$	9.	$y + t + 0,$
3.	$z = 0,$	7.	$x + y = 0,$	10.	$z + t + 0,$
4.	$t = 0.$				

For each of the four uniplanar nodes, the enveloping cone that emanates from it consists of six of the ten singular tangential planes, and for each of the three ordinary nodes, it will consist of four singular tangential planes and a second-degree cone.

The six different ray systems of order two and class three that have this surface for their common focal surface are determined by the equations:

I. $z\,(y + t)\,\xi + t\,(z + x)\,\eta - y\,(x + t)\,\zeta = 0,$

II. $y\,(z + t)\,\xi - t\,(z + x)\,\eta + t\,(x + y)\,\zeta = 0,$

III. $-z\,(y + t)\,\xi + x\,(z + t)\,\eta + t\,(x + y)\,\zeta = 0,$

IV. $t\,(y + z)\,\xi + z\,(z + t)\,\eta - x\,(y + t)\,\zeta = 0,$

V. $t\,(y + z)\,\xi - x\,(z + t)\,\eta + y\,(x + t)\,\zeta = 0,$

VI. $-z\,(y + t)\,\xi + t\,(z + x)\,\eta + x\,(y + t)\,\zeta = 0,$

and their first derived equations. Two plane pencils of rays emanate from each of the four singular points 1, 2, 3, 4 in each of these six ray systems, but only one plane pencil of rays will emanate from two of the singular points 5, 6, 7, and a second-degree ray cone will emanate from the third one. If one considers these ray systems to be limiting cases of the general ray system of order two and class three that have ten singular points with plane pencils of rays and five with second-degree ray cones then it will be those cases in which four of the second-degree ray cones decompose into two plane pencils of rays that the two points that combine with them into a point will coincide, while of the three remaining singular points, one of them will carry a second-order ray cone and the other two will carry plane pencils of rays.

§ 9.

Rays systems of order two and class four that have no focal curves.

From Theorem XXIV, the ray systems of class four for which the degree n of the three functions P, Q, R is equal to 3 will have a double ray. From Theorem XXXIV, the two singular points of the systems that lie at this double ray will have third-degree ray cones for which the double ray is a double edge, and since no other third-degree ray cones are present besides those two, one will have $m_3 = 2$. It one now sets $n = 3$ in the two equations of Theorems XXIX and XXX then one will get:

$$108 = m_1 + 8m_2 + 27\,m_3\,, \qquad 30 = 2m_2 + 9\,m_3\,,$$

so:

$$m_1 = 6, \qquad m_2 = 6, \qquad m_3 = 6,$$

and one will thus have the following theorem:

XL: *The ray systems of order two and class four have a double ray and* 14 *singular points and, in fact,* 6 *of them have planar ray pencils,* 6 *of them have second-degree ray cones, and* 2 *of them have third-degree ray cones; their focal surfaces are fourth-degree surfaces with* 14 *nodes and* 6 *singular tangential planes.*

The analytical representation of these ray systems rests upon the determination of three functions P, Q, R in the equation:

$$P\,\xi + Q\,\eta + R\,\zeta = 0, \tag{1}$$

so that equation, along with its derived equations, will determine the ray system completely. If one chooses the one double ray to be the z-axis, then from Theorem XXII, the three third-degree functions P, Q, R must be equal to zero for $x = 0$, $y = 0$, so they will have the form:

$$\begin{aligned} P &= x\phi + y\,\phi_1 + x\,y\,p, \\ Q &= x\phi' + y\,\phi_1' + x\,y\,p', \\ R &= x\phi'' + y\,\phi_1'' + x\,y\,p'', \end{aligned} \tag{2}$$

in which ϕ, ϕ', ϕ'' are second-degree functions that do not include y, so they will be homogeneous functions of degree two in x, z, t, and ϕ_1, ϕ_1', ϕ_1'' are homogeneous functions of degree two in y, z, t, but p, p', p'' are linear functions of x, y, z, t. If one now introduces the condition that the third derived equation must vanish identically – or what amounts to the same thing – that $Px + Qy + Rz$ must be of degree only three relative to x, y, z, then one will obtain:

$$\begin{aligned}
\phi_1 &= A_1\,y^2 + B_1\,yz + C_1\,z^2 + D_1\,y\,t + E_1\,z\,t + F_1\,t^2, \\
\phi_1' &= \quad + B_1'yz + C_1'z^2 + D_1'y\,t + E_1'z\,t + F_1'\,t^2, \\
\phi_1'' &= -A_1\,y^2 - C_1''yz \quad + D_1''y\,t + E_1''z\,t + F_1''\,t^2, \\
\\
\phi &= \quad + B\,x\,z + C\,z^2 + D\,x\,t + E\,z\,t + F\,t^2, \\
\phi' &= A'x^2 + B'x\,z + C'z^2 + D'x\,t + E'z\,t + F't^2, \\
\phi'' &= -B\,x^2 - C''x\,z \quad + D''x\,t + E''z\,t + F''t^2, \\
\\
p &= -A'x + H\,y + I\,z + Kt, \\
p' &= -H\,x - A_1\,y + I'z + K't, \\
p'' &= -(B' + I)\,x - (B_1 + I)\,y - (C' + B_1)\,z + K''t.
\end{aligned} \tag{3}$$

Now, let the coefficients that enter into these expressions be further determined in such a way that the first equation of the ray system and the two derived equations concur with each other, such that one of these three equations is a consequence of the other two. One obtains the derived equations from using the rule that was given above, when one replaces x, y, z with $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$, resp., in the original equation, so that equations must then be true for any arbitrary value of ρ. In the present case, it is now preferable to determine the two derived equations in such a way that one gives ρ two well-defined values, and indeed, $\rho = -x / \xi$, in the one case, and $\rho = -y / \xi$, in the other; the two equations that are obtained will then be completely equivalent with the ones that were found by the usual method of development in powers of ρ.

For the value $-x / \xi$, one will have:

$$x + \rho\xi = 0, \qquad y + \rho\eta = -\frac{\omega}{\xi}, \qquad z + \rho\zeta = +\frac{\upsilon}{\xi},$$

in which we have set $y\,\zeta - x\,\zeta = u$, $z\xi - x\zeta = \upsilon$, $x\eta - y\xi = \omega$, to abbreviate. Now, since $y + \rho\eta$ drops out, the equation $P\,\xi + Q\,\eta + R\,\zeta = 0$ will give, using the equation $u\,\xi + v\,\eta + w\,\zeta = 0$:

$$(4)\quad \begin{aligned} &C_1\upsilon^2 + A_1\omega^2 - B_1\upsilon\omega - C_1'uv + B_1'u\omega + (D\xi + D_1'\eta + D_1''\zeta)\,\omega t \\ &+ (E_1\xi + E_1'\eta + E_1''\zeta)\,\upsilon t + (F_1\xi + F_1'\eta + F_1''\zeta)\,\xi t^2 = 0. \end{aligned}$$

For the other value $\rho = -y / \xi$, one obtains in the same way:

$$(5)\quad \begin{aligned} &C_1u^2 + A'\omega^2 - B\upsilon\omega - Cuv - B'u\omega + (D\xi + D'\eta + D''\zeta)\,\omega t \\ &- (E\xi + E'\eta + E''\zeta)\,u\,t + (F\xi + F'\eta + F''\zeta)\,\eta\,t^2 = 0. \end{aligned}$$

These two equations, which enter in place of the two derived equations, must now be identical, when one consults the original equation (1). Since both of them are of degree two in ξ, η, ζ, and likewise of degree two in the quantities x, y, z, which contain only u, υ, ω, and since the original equation is of degree three in x, y, z, one can give just one equation for an equation that couples one of these two equations with the original one and has a degree higher than two, and which cannot then be identical to the other equation. It then follows from this that the two equations (4 and 5) must be identical, in their own right. Since the six quantities $u, \upsilon, \omega, \xi, \eta, \zeta$ couple to each other by just the one equation $\xi\,u + \eta\,\upsilon + \zeta\,\omega = 0$, but are otherwise completely independent, the identity of both equations must be true term-by-term when the terms $-E\,\xi\,u\,t$ is replaced with the two terms $+E\,\eta\,\upsilon\,t + E\,\zeta\,\omega t$ in the last one. Comparing the individual terms then gives:

$$(6)\quad \begin{matrix} C_1 = 0, & E_1 = 0, & E_1'' = 0, & F_1 = 0, & F_1'' = 0, \\ C' = 0, & E' = 0, & E'' = 0, & F' = 0, & F'' = 0, \end{matrix}$$

so both equations then have the form:

$$(7)\qquad \omega(\alpha\,u + \alpha_1 u + \alpha_2\omega + \beta\xi\,t + \beta_1\eta\,t + \beta_2\zeta\,t) + (\delta_2\upsilon\eta - \delta_2\omega\zeta)\,t - \gamma u\upsilon + \varepsilon\eta\xi t^2 = 0.$$

One then has:

$$
(8)\qquad
\begin{aligned}
&A_1 = \kappa\,\alpha_2, && B_1 = \kappa\,\alpha_1, && B_1' = \kappa\,\alpha, && C_1' = \kappa\,\gamma, && F_1' = \kappa\,\varepsilon,\\
&D_1 = -\,\kappa\beta, && D_1' = -\,\kappa\beta_1, && D_1'' = -\kappa\,(\beta_2 + \delta_1), && E_1' = \kappa\delta_1,\\
&A' = \lambda\alpha_2, && B = \lambda\alpha_1, && B' = -\,\lambda\alpha, && C = -\,\lambda\gamma, && F = \lambda\varepsilon,\\
&D = \lambda\beta, && D' = \lambda\beta_1, && D'' = \lambda(\beta_2 - \delta_1 - \delta_2), && E = \kappa\delta_2,
\end{aligned}
$$

in which κ and λ are two arbitrary quantities. If one sets:

$$H = a_2, \qquad I = -\,a_1, \quad I' = +\,a, \qquad K = -\,b, \qquad K' = -\,b_1, \qquad K'' = -\,b_2$$

then one will obtain, after employing the values of all the following expressions for the three functions P, Q, R:

$$
(9)\qquad
\begin{aligned}
P &= xyr + (ky^2 - \lambda x^2)\,s + (\gamma z^2 + \delta_2\,z\,t + \varepsilon\,t^2)\,\lambda x,\\
Q &= xyr_1 + (ky^2 - \lambda x^2)\,s_1 + (\gamma z^2 + \delta_2\,z\,t + \varepsilon\,t^2)\,ky,\\
R &= xyr_2 + (ky^2 - \lambda x^2)\,s_2 + \lambda x^2\,(\gamma z - (\delta_2 + \delta_1)\,t) + ky^2\,(\gamma z - \delta_1 t),
\end{aligned}
$$

where:

$$
\begin{aligned}
r &= a_2\,y - a_1\,z - b\,t, & s &= \alpha_2\,y - \alpha_1\,z - \beta\,t,\\
r_1 &= a\,z - a_2\,x - b_1\,t, & s_1 &= \alpha\,z - \alpha_2\,z - \beta_1\,t,\\
r_2 &= a_1\,x - a\,y - b_2\,t, & s_2 &= \alpha_1\,x - \alpha\,y - \beta_2\,t.
\end{aligned}
$$

Once the most general ray systems of order two and class four has been found, one again comes to the problem of finding the simplest ray system of the same kind, which can still amount to the most general one, insofar as all other ones are only collinear transformations of this simplest one. To that end, I set $\alpha = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, $\beta = 0$, $\beta_1 = 0$, $\beta_2 = 0$, $\gamma = 0$, $\varepsilon = 0$, and $\delta_1 + \delta_2 = -\,\delta$, so one gets:

$$
(10)\qquad
\begin{aligned}
P &= x\,y\,r + \lambda\,\delta_2\,x\,z\,t,\\
Q &= x\,y\,r_1 + \kappa\,\delta_2\,y\,z\,t,\\
R &= x\,y\,r_2 + \lambda\,\delta\,x^2\,t + \lambda\,\delta_1\,y^2\,t,
\end{aligned}
$$

and equation (7) gives the second equation of the ray system:

$$(11)\qquad \delta_2\,\upsilon\,\eta - \delta_1\,\omega\,\zeta = 0,$$

or when developed:

$$(12)\qquad \delta x\,\eta\zeta + \delta_1 y\,\zeta\,\xi + \delta_2\,z\,\xi\eta = 0.$$

The focal surface of this system will then be:

$$
(13)\qquad
\begin{vmatrix}
0 & \delta_2 z & \delta_1 y & P\\
\delta_2 z & 0 & \delta x & Q\\
\delta_1 y & \delta x & 0 & R\\
P & Q & R & 0
\end{vmatrix} = 0,
$$

which includes the superfluous factor of $x^2\,y^2$ in this form, but which drops out when one develops this determinant. The equation of the focal surface then becomes:

$$(14)\qquad (\delta xr + \delta_1 yr_1 - \delta_2\, z\, r_2)^2 - 4\delta\delta_1\,(yr + \lambda\delta_2\, zt)(xr_1 + \kappa\delta_2\, zt) = 0,$$

or

$$(15)\qquad \begin{aligned}&\delta^2\, x^2\, r^2 + \delta_1^2 y^2 r_1^2 - \delta_2^2 z^2 r_2^2 - 2\delta_1\delta_2\, y\, z\, r_1\, r_2 - 2\delta_2\delta\, z\, x\, r_2\, r - 2\delta\delta_1\, x\, y\, r\, r_1\\ &\qquad - 4\,\delta\,\delta_1\delta_2\,(k\, y\, r + \lambda\, x\, r_1)\, z\, t - 4\,\delta\,\delta_1\,\delta_2^2\, k\,\lambda\, z^2\, t^2 = 0.\end{aligned}$$

In fact, this equation represents a fourth-degree surface with 14 nodes and 6 singular tangential planes, and indeed the most general surface of this kind, insofar as all other ones are just collinear conversions of it. In the same sense, the ray system that is given simply by equations (10) is also the most general ray system of order two and class four.

The six singular tangential planes of this surface are:

$$(16)\qquad \begin{aligned}
&1. && z = 0,\\
&2. && p = \delta a_2\,\rho\, x + \delta_1\, a_2\, y - (\delta\, a\rho + \delta_1\, a_1)\, z = 0,\\
&3. && p' = \delta a_2\,\rho' x + \delta_1\, a_2\, y - (\delta\, a\rho' + \delta_1\, a_1)\, z = 0,\\
&4. && t = 0,\\
&5. && q' = a_2\,\rho' x - a_2\, y + (b + b_1\rho')\, t = 0,\\
&6. && q = a_2\,\rho\, x - a_2\, y + (b + b_1\rho)\, t = 0,
\end{aligned}$$

in which ρ and ρ' are the two roots of the quadratic equation:

$$(17)\qquad \delta(ab_1 - \delta_2\, a_2\, k)\,\rho^2 + (\delta ab + \delta_1\, a_1\, b_1 - \delta_2\, a_2\, b_2)\,\rho + \delta_1\,(a_1 b - \delta_2\, a_2\,\lambda) = 0.$$

By means of these expressions for the six singular tangential planes, one can also put the equations can into the following form:

$$(18)\qquad \sqrt{pq'} + \sqrt{p'q} + \sqrt{mzt} = 0,$$

where:

$$m = \delta(ab_1 - \delta_2\, a_2\, k)\,(\rho - \rho')^2.$$

The 14 nodes of the surface are:

1.	$p = 0,$	$p' = 0,$	$z = 0,$
2.	$q' = 0,$	$q = 0,$	$z = 0,$
3.	$p = 0,$	$q = 0,$	$z = 0,$
4.	$q' = 0,$	$p' = 0,$	$z = 0,$
5.	$q' = 0,$	$q = 0,$	$z = 0,$
6.	$p = 0,$	$p' = 0,$	$t = 0,$
7.	$q' = 0,$	$p' = 0,$	$t = 0,$
8.	$p = 0,$	$q = 0,$	$t = 0,$

9 and 10.	$p = 0$,	$p' = 0$,	$p'q - mzt = 0$,
11 and 12.	$q' = 0$,	$p' = 0$,	$pq' - mzt = 0$,
13 and 14.	$p = 0$,	$q = 0$,	$pq' - mzt = 0$.

The first eight nodes are such that three singular tangential planes go through each of them; in addition, an enveloping cone of degree three with a double edge will emanate from each of these eight points. Only two singular tangential planes go through each of the remaining six nodes, and two enveloping cones of degree two will emanate from each of them, as well.

The eight enveloping cones of degree three, which emanate from the first eight nodes, are arranged pair-wise such that the double points of any two of them will coincide, which are then the third-degree cones that emanate from the points 1 and 5, 2 and 6, 3 and 7, 4 and 8. The ray system of degree two and class four that was presented above has the third-degree cone that emanates from the two points 1 and 5 as its ray cone and the common double edges to them as the double ray. In addition, it has a second-degree ray cone that comes from each of the six pairs of second-degree cones that emanate from each of the six pairs of second-degree cones that emanate from the six nodes 9, 10, 11, 12, 13, 14. Finally, it has six plane pencils of rays that emanate from the six nodes 2, 3, 4, 6, 7, 8, which lie in the singular tangential planes, z, q, q', t, p', p, respectively. Since each ray system of order two and class four must have two third-degree ray cones with a common double edge as double ray, and since the focal surface has just four such pairs enveloping cones of degree three with common double edges, it then follows that no more than four such ray systems can lie on one and the same focal surface. The fact that any such fourth-degree surface is, in fact, the focal surface of four such ray systems follows simply from the commutability of the six singular tangential planes, which will leave the surface itself unchanged, although its nodes will be permuted. If one permutes q' with p and q with p' then the nodes 1 and 5 will go to 2 and 6, and one will obtain a second ray system of order two and class four that has the connecting line for the nodes 2 and 6 for its double ray; one likewise obtains the third ray system of that kind with the double ray that goes through the nodes 3 and 7 by permuting p' and q, and the fourth one, whose double ray goes through the nodes 4 and 8, by permuting p and q'. Therefore:

XLI: *Any fourth-degree surface with* 14 *nodes and* 6 *singular tangential planes is the focal surface of four distinct ray systems of order two and class four.*

The complete system of all straight lines that contact such a fourth-degree surface twice consists, in addition to the aforementioned four ray systems of order two and class four, of an irreducible ray system of order four and class six, and the six rays systems of order zero and class one that are defined by all of the straight lines that lie in the six singular tangential planes. I will skip over the analytical representation of the other three ray systems of order two and class four that lie on the same focal surface (15), since the expressions are too complicated.

§ 10.

Ray systems of order two and class five that have no focal curves.

The degree of the three functions P, Q, R is $n = 4$ for ray systems of class five. From Theorem XXIV, they have three double rays that, from Theorem XXXIII, go through the same point. The singular point of the ray systems at which the three double rays intersect has, from Theorem XXXIV, a fourth-degree ray cone for which the three double rays are double edges, and the three singular points that lie on the three double rays each have a third-degree ray cone with the singular ray as its double ray; one then has $m_4 = 1$, $m_3 = 3$. If one now sets $m_4 = 1$, $m_3 = 3$, $n = 4$ in the two equations of Theorems XXIX and XXX then that will give:

$$51 = m_1 + s\, m_2\,, \qquad 12 = 2m_2\,,$$

so $m_1 = 3$, $m_2 = 6$, $m_3 = 3$, $m_4 = 1$, and one will then have the following theorem:

XLII: *The ray systems of order two and class five have three double rays that go through one and the same point and* 13 *singular points, and, in fact, three of them have planar ray pencils, six have second-degree ray cones, three have third-degree ray cones, and one of them has a fourth-degree ray cone; their focal surfaces are fourth-degree surfaces with* 13 *nodes and three singular tangential planes.*

The analytic representation of this ray system will found by a method that is similar to the one that was applied to ray systems of class four. If one chooses the three double rays that go through a point to be the three coordinate axes and the planes that go through any two of them to be the x, y, z coordinate planes, and observes that the three double rays must be the three common straight lines to the three surfaces $P = 0$, $Q = 0$, $R = 0$, then one will obtain the following forms for these three fourth-degree functions:

$$(1)\qquad \begin{aligned} P &= y\,z\,\phi\; + z\,x\,\phi_1\; + x\,y\,\phi_2\; + x\,y\,z\,p, \\ Q &= y\,z\,\phi' + z\,x\phi_1' + x\,y\phi_2' + x\,y\,z\,p', \\ R &= y\,z\,\phi'' + z\,x\phi_1'' + x\,y\phi_2'' + x\,y\,z\,p'', \end{aligned}$$

where ϕ, ϕ', ϕ'' are homogeneous functions of degree two in y, z, t, ϕ_1, ϕ_1', ϕ_1'' are homogeneous functions of degree two in z, x, t, and ϕ_2, ϕ_2', ϕ_2'' are homogeneous functions of degree two in x, y, t, but p, p', p'' are linear functions of x, y, z, t. If one now introduces the necessary condition that $Px + Qy + Rz$ must be of only fourth degree in x, y, z then one will obtain the following formulas for the nine second-degree functions ϕ, ϕ', ϕ'', etc.:

$$\begin{aligned} \phi\; &= \quad Ay^2 + B\,yz + Cz^2 + D\,\,yt + E\,\,zt + F\,\,t^2, \\ \phi' &= \qquad\quad + B'yz + C'z^2 + D'yt + E'zt + F't^2, \\ \phi'' &= -B'y^2 + C'yz \qquad + D''yt + E''zt + F''t^2, \\ \phi_1 &= -B_1''y^2 + C''yz \qquad + D_1\,zt + E_1\,xt + F_1\,t^2, \end{aligned}$$

$$(2)\qquad \begin{aligned} \phi_1' &= A_1' z^2 + B_1' zx + C_1' x^2 + D_1' zt + E_1' xt + F_1' t^2, \\ \phi_1'' &= \quad + B_1'' zx + C_1'' x^2 + D_1'' zt + E_1'' xt + F_1'' t^2, \\ \phi_2 &= \quad + B_2\, xy + C_2\, y^2 + D_2\, xt + E_2\, yt + F_2\, t^2, \\ \phi_2' &= - B_2\, x^2 - C_2\, xy \quad + D_2' xt + E_2' yt + F_2' t^2, \\ \phi_2'' &= A_2'' x^2 + B_2'' xy + C_2'' y^2 + D_2'' xt + E_2'' yt + F_2'' t^2 . \end{aligned}$$

If one now replaces x with $x + \rho\xi$, y with $y + \rho\eta$, and z with $z + \rho\zeta$ in the first equation of the ray system:

$$P\xi + Q\eta + R\zeta = 0,$$

and successively gives the arbitrary quantity ρ the three values $\rho = -x / \xi$, $\rho = -y / \eta$, $\rho = -z / \zeta$, then after dropping the superfluous factors one will obtain the following three equations, which are only of degree two in ξ, η, ζ, and also in x, y, z:

$$(4)\qquad \begin{aligned} &C\upsilon^2 + A\omega^2 - B\upsilon\omega + B'\omega u - C'u\upsilon + (E\upsilon - D\omega + Ft\xi)\,\xi t \\ &\quad + (E'\upsilon - D'\omega + F't\xi)\,\eta t + (E''\upsilon - D''\omega + F''t\xi)\,\zeta t = 0, \end{aligned}$$

$$(5)\qquad \begin{aligned} &C_1'\omega^2 + A_1' u^2 - B_1' u\upsilon + B_1'' u\upsilon - C_1''\upsilon\omega + (E_1'\omega - D_1' u + F_1' t\eta)\,\eta t \\ &\quad + (E_1''\omega - D_1'' u + F_1'' t\eta)\,\zeta t + (E_1\omega - D_1 u + F_1 t\zeta)\,\xi t = 0, \end{aligned}$$

$$(6)\qquad \begin{aligned} &C_2'' u^2 + A_2''\upsilon^2 - B_2'' u\upsilon + B_2\,\upsilon\omega - C_2\,\omega u + (E_2'' u - D_2''\upsilon + F_2'' t\zeta)\,\zeta t \\ &\quad + (E_2\, u - D_2\,\upsilon + F_2\, t\zeta)\,\xi t + (E_2' u - D_2'\upsilon + F_2' t\xi)\,\eta t = 0. \end{aligned}$$

These three equations, which enter in place of the three derived equations, must now be identical with each other, and when one replaces $\omega\zeta$ with $- u\xi - \upsilon\eta$ by means of the equation $u\xi + u\eta + w\zeta = 0$, they must be identical term-by-term. If one then compares the terms that do not appear in all three equations then one will get:

$$(7)\qquad \begin{array}{llllll} C = 0, & A = 0, & E = 0, & D = 0, & E'' = 0, & D' = 0, \\ C_1' = 0, & A_1' = 0, & E_1' = 0, & D_1' = 0, & E_1 = 0, & D_1'' = 0, \\ C_2'' = 0, & A_2'' = 0, & E_2'' = 0, & D_2'' = 0, & E_2' = 0, & D_2 = 0, \\ & & & & & \\ & & F = 0, & F' = 0, & F' = 0, & \\ & & F_1' = 0, & F_1'' = 0, & F_1 = 0, & \\ & & F_2' = 0, & F_2' = 0, & F_2' = 0, & \end{array}$$

such that these equations will take on the form:

$$(8)\qquad \alpha\,\upsilon\,\omega + \beta\,\omega u + \gamma u\,\upsilon + \delta_1\, u\,\zeta\, t - \delta\,\upsilon\,\eta\, t = 0.$$

Now, in order for all of these three forms to be identical one must further have the equations:

$$(9)\qquad \begin{array}{lllll} B = -\kappa\alpha, & B' = \kappa\beta, & C' = -\kappa\gamma, & D'' = \kappa\delta_1, & E' = \kappa\delta_2, \\ B_1' = -\lambda\beta, & B_1'' = \lambda\gamma, & C_1'' = -\lambda\alpha, & D_1 = \lambda\delta_2, & E_1'' = \lambda\delta, \\ B_2'' = -\mu\gamma, & B_1 = \mu\alpha, & C_2 = -\mu\beta, & D_2' = \mu\delta, & E_2 = \mu\delta_1, \end{array}$$

where κ, λ, μ are arbitrary quantities and we have set $\delta_2 = -\delta - \delta_1$, so $\delta + \delta_1 + \delta_2 = 0$. If one substitutes these values for the coefficients into the nine functions that are denoted by ϕ then one will obtain the following expressions for P, Q, R:

$$(10)\qquad \begin{aligned} P &= \alpha(-\kappa y^2 z^2 + \lambda z^2 x^2 + \mu x^2 y^2) - \beta\mu y^2 x - \gamma\lambda z^2 x \\ &\qquad + \lambda\,\delta_2 z^2 x\,t + \mu\,\delta_1 x\,y^2 t + xyzp, \\ Q &= \beta(\kappa y^2 z^2 - \lambda z^2 x^2 + \mu x^2 y^2) - \gamma\kappa y^3 y - \alpha\mu z^3 y \\ &\qquad + \mu\,\delta x^2 y\,t + \kappa\,\delta_2\,y\,z^2\,t + xyzp', \\ R &= \gamma(\kappa y^2 z^2 + \lambda z^2 x^2 - \mu x^2 y^2) - \alpha\lambda x^3 z - \beta\kappa z^3 z \\ &\qquad + \kappa\,\delta_1 y^2 z\,t + \lambda\,\delta z\,x^2\,t + xyzp'', \end{aligned}$$

where the three linear expressions p, p', p'', according to the condition that $Px + Qy + Rz$ must be of degree only four in x, y, z, will be determined as follows:

$$(11)\qquad \begin{aligned} p &= (\gamma\mu + a_2)\,y - a_1\,z - b\,t, \\ p' &= (\alpha\kappa + a)\,z - a_2\,x - b_1\,t, \\ p'' &= (\beta\lambda + a_1)\,x - a\,y - b_2\,t. \end{aligned}$$

Now, since this again only comes down to finding a ray system of order two and class four from which all ray systems of that kind can be generated by collinear conversions, one can set $\alpha = 0$, $\beta = 0$, $\gamma = 0$ in the ones that were given here with no loss of generality; one then obtains the simpler ray system:

$$P\xi + Q\eta + R\zeta = 0,$$

$$(12)\qquad \begin{aligned} P &= xyzr + \lambda\,\delta_2\,z^2\,xt + \mu\,\delta_1\,x\,y^2\,t, \\ Q &= xyzr_1 + \mu\,\delta x^2\,yt + \kappa\,\delta_2\,y\,z^2\,t, \\ R &= xyzr_2 + \kappa\,\delta_1\,y^2\,xt + \lambda\,\delta z\,x^2\,t, \end{aligned}$$

where r, r_1, r_2 are the same linear expressions as in the previous paragraph, namely:

$$r = a_2\,y - a_1\,z - bt, \qquad r_1 = a\,z - a_2\,x - b_1\,t, \qquad r_2 = a_1\,x - a\,y - b_2\,t.$$

As the second equation of this ray system, one obtains from equation (8):

$$(13)\qquad \delta_1\,u\,\xi - \delta\,\upsilon\,\eta = 0,$$

or

$$\delta x \eta \zeta + \delta_1 y \zeta \xi + \delta_2 z \xi \eta = 0. \tag{14}$$

The focal surface of this system is then:

$$\begin{vmatrix} 0 & \delta_2 z & \delta_1 y & P \\ \delta_2 z & 0 & \delta x & Q \\ \delta_1 y & \delta x & 0 & R \\ P & Q & R & 0 \end{vmatrix} = 0. \tag{15}$$

In this form, it again contains the superfluous factor of $x^2\, y^2\, z^2$, which will drop out under the complete development of this determinant, with which, the equation of the focal will take on the following form:

$$\begin{aligned} &\delta x^2 r^2 + \delta_1 y^2 r_1^2 + \delta_2 z^2 r_2^2 - 2\delta_1\delta_2\, y\, z\, r_1\, r_2 - 2\, \delta_2\, \delta z\, x\, r_2\, r - 2\, \delta\delta_1\, xyrr_1 \\ &\quad - 4\, \delta\, \delta_1 \delta_2\, (\kappa y\, z\, r + \lambda\, z\, x\, r_1 + \mu\, x\, y\, r_2)\, t - 4\, \delta\, \delta_1 \delta_2\, (\delta\lambda\mu\, x^2 + \delta_1 \mu\, \kappa y^2 \\ &\qquad + \delta_2\, \kappa\lambda\, z^2)\, t^2 = 0. \end{aligned} \tag{16}$$

This equation, in fact, represents a fourth-degree surface with thirteen nodes and three singular tangential planes, and indeed it is the most general one of that kind, insofar as all other ones will only be collinear conversions of it.

The three singular tangential planes are:

$$\begin{aligned} &t = 0, \\ &p = \delta\, a_2\, \rho\, x + \delta_1\, a_2\, y - (\delta a \rho + \delta_1\, a_1)\, z = 0, \\ &q = \delta\, a_2\, \rho' x + \delta_1\, a_2\, y - (\delta a \rho' + \delta_1\, a_1)\, z = 0, \end{aligned} \tag{17}$$

where ρ and ρ' are the two roots of the quadratic equation:

$$\begin{aligned} &\delta\,(ab_1 - \delta_2\, a_2\, \kappa - \frac{\delta a^2 \mu}{a_2})\, \rho^2 + (\delta\, ab - \delta_1\, a_1\, b_1 - \delta_2\, a_2\, b_2 - \frac{2\mu\, \delta\, \delta_1 a a_1}{a_2})\, \rho \\ &\qquad + \delta_1\, (a_1\, b - \delta_2\, a_2\, \lambda - \frac{\delta_1 a_1^2 \mu}{a_2}) = 0. \end{aligned} \tag{18}$$

The 13 nodes are: First of all, the following three:

1. $x = 0, \qquad y = 0, \qquad z = 0,$
2. $t = 0, \qquad y = \rho x, \qquad z = \dfrac{-\delta_2 a_2 \rho x}{\delta a \rho + \delta_1 a_1},$
3. $t = 0, \qquad y = \rho' x, \qquad z = \dfrac{-\delta_2 a_2 \rho' x}{\delta a \rho' + \delta_1 a_1},$

through which, any two singular tangential planes will go, and for which the sixth-degree enveloping cone will consist of a fourth-degree cone with three double edges and two planes.

Secondly, one has the node:

4. $t = 0, \qquad \frac{x}{a} = \frac{y}{b} = \frac{z}{c},$

through which, all three singular tangential planes will go, and which will belong to a third-degree enveloping cone with no double edge. Thirdly, the surface has 9 nodes, through each of which only one of the three singular tangential planes will go, and for which the completely enveloping sixth-degree cone will consist of a third-degree cone with a double edge, a second-degree cone, and a plane. Three of these nine nodes lie in the singular tangential plane $t = 0$, three of them, in $p = 0$, and three of them, in $q = 0$; the three that lie in $t = 0$ are:

5. $t = 0, \qquad y = 0, \qquad z = 0,$
6. $t = 0, \qquad z = 0, \qquad x = 0,$
7. $t = 0, \qquad x = 0, \qquad y = 0.$

The three nodes 8, 9, 10 that lie in the plane $p = 0$, as well as the three nodes 11, 12, 13 that lie in $q = 0$, depend upon a third-degree equation whose coefficients include the roots ρ or ρ' of the quadratic equation (18).

The ray system that is exhibited by (12) has one singular point with a fourth-degree ray cone and three double edges in node 1, further, the three singular points with third-degree ray cones, and a double edge at the points 5, 6, 7, then six singular points with second-degree ray cones at the nodes 8, 9, 10, 11, 12, 13, and the three singular points with plane pencils of rays at the nodes 2, 3, 4.

Since each ray system of order two and class five includes a fourth-degree ray cone with three double edges, but the fourth-degree surfaces with 13 nodes and three singular tangential planes have three nodes, from which, fourth-degree enveloping cones with three double edges emanate, it will then follow that such a surface cannot belong to more than three ray systems of that kind. Furthermore, since the node 1 goes to 2 under permutation of the two singular tangential planes p' and t and to 3 under permutation of p and t, it will then follows that, in fact, in addition to the ray system of order two and class five that was exhibited above, two other ones belong to that focal surface. Hence:

XLIII: *Any fourth-degree surface with thirteen nodes and three singular tangential planes is the focal surface of three different ray systems of order two and class five.*

The complete system of all straight lines that contact such a surface twice consists of a ray system of order six and class ten and three ray systems of order zero and class one, in addition to these three ray systems of order two and class four.

§ 11.

Ray systems of order two and class six that have no focal curves, of the first type.

As was proved in Theorem XXXIII of § 6, there are two different kinds of ray systems of order two and class six, one of which has six double rays that define the edges of a tetrahedron, and will be referred to as the first type. Only four singular points lie on the six double edges here, through each of which, three of the double rays go, so from Theorem XXXIV, fourth-degree ray cones with three double edges will belong to them, and one will then have $m_5 = 0$, $m_4 = 4$, $m_3 = 0$. If one substitutes these values, along with $n = 5$, into the equations of Theorems XXIX and XXX then one will obtain:

$$64 = m_1 + s\, m_2, \qquad 16 = 2m_2\,,$$

so $m_1 = 0$, $m_2 = 8$. One then has the following theorem:

XLIV: *The ray systems of order two and class four of the first type have six double rays, any three of which go through one and the same point, and furthermore, they have twelve singular points, eight of which have second-degree ray cones and four of which have fourth-degree ray cones with three double edges; their focal surfaces are fourth-degree surfaces with no singular tangential planes.*

We now have to determine the three fifth-degree functions P, Q, R in the first equation of the ray system:

(1) $$P\xi + Q\eta + R\zeta = 0,$$

which, as we showed above, must then satisfy the equation:

(2) $$Px + Qy + Rz + St = 0,$$

in which S is a fourth entire function of degree five. To that end, I choose the four faces of the tetrahedron, which have the six double rays for edges, to be the four coordinates x, y, z, s, where s should not be represent the infinitely-distant plane, which was denoted by t above, but a homogeneous, linear function of x, y, z, t:

(3) $$s = \alpha x + \beta y + \gamma z + t.$$

If one correspondingly sets:

(4) $$\sigma = \alpha\xi + \beta\eta + \gamma\zeta$$

then one can represent equations (1) and (2):

(5) $$(P - \alpha S)\,\xi + (Q - \beta S)\,\eta + (R - \gamma S)\,\zeta + S\sigma = 0,$$

and

(6) $$(P - \alpha S)\,x + (Q - \beta S)\,y + (R - \gamma S)\,z + Ss = 0.$$

As was shown above, the three surfaces $P = 0$, $Q = 0$, $R = 0$ must now include the six double rays as common straight lines, and equation (2) shows that the surface $S = 0$ must also go through the same six double rays, so one will also have $P - \alpha S = 0$, $Q - \beta S = 0$, and $R - \gamma S = 0$. It follows from this that these functions must have the following forms:

$$\begin{aligned} P - \alpha S &= yzs\phi + zsx\phi_1 + xyz\phi_3 + xyzsp, \\ Q - \beta S &= yzs\phi' + zsx\phi_1' + xyz\phi_3' + xyzsp', \\ R - \gamma S &= yzs\phi'' + zsx\phi_1'' + xyz\phi_3'' + xyzsp'', \\ S &= yzs\phi''' + zsx\phi_1''' + xyz\phi_3''' + xyzsp'''. \end{aligned} \tag{7}$$

If these expressions are substituted and one takes $x = 0$, $y = 0$, $z = 0$, $s = 0$ in succession then equation (2) will now show that one must have identically:

$$\begin{aligned} &\phi' y + \phi'' z + \phi''' s = 0, \\ &\phi_1'' y + \phi_1''' s + \phi_1 x = 0, \\ &\phi_2''' s + \phi_2 x + \phi_1' y = 0, \\ &\phi_3 x + \phi_3' y + \phi_3'' s = 0. \end{aligned} \tag{8}$$

One obtains the following expressions for the sixteen functions ϕ from this:

$$\begin{aligned} \phi &= Ay^2 + Bz^2 + Cs^2 + Dzs + Esy + Fyz, \\ \phi' &= -F'z^2 - E'''s^2 + D'zs + E'sy + F'yz, \\ \phi'' &= -F'y^2 - D'''s^2 + D''zs + E''sy + F''yz, \\ \phi''' &= -E'y^2 - D''z^2 + D'''zs + E'''sy + F'''yz, \end{aligned} \tag{9}$$

where $D' + E'' + F''' = 0$.

$$\begin{aligned} \phi_1' &= A_1'z^2 + B_1's^2 + C_1'x^2 + D_1'sx + E_1'xz + F_1'zs, \\ \phi_1'' &= -F_1'''s^2 - E_1x^2 + D_1''sx + E_1''xz + F_1''zs, \\ \phi_1''' &= -F_1''z^2 - D_1x^2 + D_1'''sx + E_1'''xz + F_1'''zs, \\ \phi_1 &= -E_1'''y^2 - D_1'''s^2 + D_1'''sx + E_1xz + F_1zs, \end{aligned} \tag{10}$$

where $D_1'' + E_1''' + F_1 = 0$.

$$\begin{aligned} \phi_2'' &= A_2''s^2 + B_2''x^2 + C_2''y^2 + D_2''xy + E_2''ys + F_2''sx, \\ \phi_2''' &= -F_2x^2 - E_2'y^2 + D_2''xy + E_2'''ys + F_2'''sx, \\ \phi_2 &= -F_2'''s^2 - D_2'x^2 + D_2xy + E_2ys + F_2sx, \\ \phi_2' &= -E_2'''s^2 - D_2x^2 + D_2'xy + E_2'ys + F_2'sx, \end{aligned} \tag{11}$$

where $D_2''' + E_2 + F_2' = 0$.

$$(12)\qquad \begin{aligned}
\phi_3''' &= A_3'''x^2 + B_3'''y^2 + C_3'''z^2 + D_3'''yz + E_3'''zx + F_3''xy,\\
\phi_3 &= \qquad\qquad -F_3'y^2 - E_3''z^2 + D_3\,yz + E_3\,zx + F_3\,xy,\\
\phi_3' &= -F_2\,x^2 \qquad\qquad - D_3''z^2 + D_2'\,yz + E_2'\,zx + F_3'\,xy,\\
\phi_3'' &= -E_3\,x^2 - D_3'y^2 \qquad\qquad + D_3'''yz + E_3'''zx + F_3'''xy,
\end{aligned}$$

where $D_3 + E_3' + F_3'' = 0$.

If one now replaces x with $x + \rho\xi$, y with $y + \rho\eta$, z with $z + \rho\zeta$ in equation (5), with which, s goes to $s + \rho\sigma$, and gives the arbitrary quantity ρ the special value $\rho = -x / \xi$ then if one sets, as above:

$$u = y\zeta - z\eta, \quad \upsilon = z\xi - x\zeta, \quad \omega = x\eta - y\xi,$$

and in addition:

$$u' = s\xi - x\sigma, \qquad \upsilon' = s\eta - y\sigma, \qquad \omega' = s\zeta - z\sigma,$$

one will obtain

$$(13)\qquad \begin{aligned}
&(A\omega^2 + B\upsilon^2 + Cu'^2 + D\upsilon u' - Eu'\omega - F\omega\upsilon)\xi\\
&+(-F''\upsilon^2 - E'''u'^2 + D'\upsilon u' - E'u'\omega - F'\omega\upsilon)\eta\\
&+(-F'\omega^2 - D'''u'^2 + D''\upsilon u' - E''u'\omega - F''\omega\upsilon)\zeta\\
&+(E\omega^2 - D''\upsilon^2 + D'''\upsilon u' - E'''u'\omega - F'''\omega\upsilon)\sigma = 0.
\end{aligned}$$

By means of the equations:

$$(14)\qquad \begin{gathered}
u\xi + \upsilon\eta + \omega\zeta = 0, \qquad uu' + \upsilon\upsilon' + \omega\omega' = 0.\\
\upsilon'\zeta - \omega'\eta + u\sigma = 0, \quad \omega'\xi - u'\zeta + \upsilon\sigma = 0, \quad u'\eta - \upsilon'\xi + \omega\sigma = 0,
\end{gathered}$$

and the equation $D' + E'' + F''' = 0$, equation (13) can be converted in such a way that x drops out as a common factor, and one will have:

$$(15)\qquad \begin{aligned}
&A\omega^2 + B\upsilon^2 - F\upsilon\omega + F'\omega u + F''u\upsilon + D\upsilon u' - E\omega u' - E'\omega\upsilon'\\
&\quad + D''\upsilon\omega' + Cu'^2 - E'''u'\upsilon' - D'''u'\omega' + E''uu' - F'''\upsilon\upsilon' = 0.
\end{aligned}$$

In the same way, for the special value $\rho = -y / \eta$, one obtains the equation:

$$(16)\qquad \begin{aligned}
&A_1'u^2 + B_1'\upsilon'^2 + F_1'u\upsilon' + F_1''u\omega' - F_1'''\upsilon'\omega' + D_1'\,\omega\upsilon' - E_1'\,u\omega'\\
&\quad - E_1''u\upsilon + D_1'''\upsilon'u' + C_1'\,\omega^2 + E_1'\,\upsilon\omega - D_1\omega u' + E_1'''uu' - D_1'''\upsilon\upsilon' = 0,
\end{aligned}$$

and for the special values $\rho = -z / \zeta$:

$$(17)\qquad \begin{aligned}
&A_2''\omega'^2 + B_2''\upsilon^2 + F_2''\upsilon\omega' + F_2'''u'\omega' + F_2\,\upsilon'u' + D_2''\,u\upsilon - E_2''\,u\omega'\\
&\quad + E_2'''\upsilon'\omega' - D_2\,\upsilon\omega + C_2''u^2 - E_2'\,\upsilon'u - D_1'\omega u + E_2\,uu' - F_2'\,\upsilon\upsilon' = 0.
\end{aligned}$$

Finally, with the special value $\rho = -s/\sigma$ one obtains:

$$
\begin{aligned}
(18) \qquad & A_3'''u'^2 + B_3'''\upsilon'^2 + C_3'''\omega'^3 - D_3'''\upsilon'\omega' + E_3'''\omega u' + F_3'''u'\upsilon' \\
& + D_3''u\omega' - E_3'\upsilon'\omega' - E_3\upsilon u' - F_3\omega u' + F_3'\omega\upsilon' + E_3'uu' - D_3\upsilon\upsilon' = 0.
\end{aligned}
$$

These four equations, which take the place of the first four derived equations, and which are of degree two relative to ξ, η, ζ, and likewise relative to x, y, z, must now be identical to each other, on the same basis as the corresponding equations in the previous two paragraphs, and since the six quantities u, υ, ω, u', υ', ω' are coupled only by the one equation $uu' + \upsilon\upsilon' + \omega\omega' = 0$, but are otherwise independent, they must coincide with each other term-by-term. Now, there is no term besides the two terms that include uu' and $\upsilon\upsilon'$ that appears in all four equations at once, so each of the remaining terms must have a zero coefficient in at least one of these equations, moreover. Therefore, all of these terms must have a zero coefficient in all of these equations – i.e., besides the twelve coefficients D', E'', F''', D_1''', E_1''', F_1, D_2''', E_2, F_2', D_3, E_3', F_3'' all of the remaining coefficients of the 16 functions ϕ must be zero. Since each of these four equations has the form $\delta_1 uu' - \delta\upsilon\upsilon' = 0$, one gets the following values for the twelve coefficients that are not zero, when one takes $\delta + \delta_1 + \delta_2 = 0$:

$$
(19) \qquad
\begin{array}{llll}
D' = \delta_2\kappa; & D'' = \delta\lambda, & D_3''' = \delta_2\,\mu, & D_3 = \delta\,\nu, \\
E'' = \delta_1\kappa; & E_1''' = \delta_1\lambda, & E_2 = \delta_1\,\mu, & E_3' = \delta_1\,\nu, \\
F''' = \delta\kappa; & F_1 = \delta_2\lambda, & F_2' = \delta\,\mu, & F_3'' = \delta_2\,\nu,
\end{array}
$$

so

$$
(20) \qquad
\begin{array}{llll}
\phi = 0, & \phi_1' = 0, & \phi_2'' = 0, & \phi_3''' = 0, \\
\phi' = \delta_2\kappa zs, & \phi_1'' = \delta\lambda sx, & \phi_2''' = \delta_2\mu xy, & \phi_3 = \delta\nu yz, \\
\phi'' = \delta_1\kappa sy, & \phi_1''' = \delta_1\lambda xz, & \phi_2 = \delta_1\mu ys, & \phi_3' = \delta_1\nu zx, \\
\phi''' = \delta\kappa yz, & \phi_1 = \delta_2\lambda zs, & \phi_2' = \delta\mu sx, & \phi_3'' = \delta_2\nu xy,
\end{array}
$$

and thus:

$$
(21) \qquad
\begin{aligned}
P - \alpha S &= x(\delta_2\lambda z^2s^2 + \delta_1\mu y^2s^2 + \delta\nu y^2z^2 + yzsp), \\
Q - \beta S &= y(\delta_2\kappa z^2s^2 + \delta\mu s^2x^2 + \delta_1\nu z^2x^2 + xzsp'), \\
R - \gamma S &= z(\delta_1\kappa s^2y^2 + \delta\lambda s^2x^2 + \delta_2\nu x^2y^2 + xysp''), \\
S &= s(\delta\kappa y^2z^2 + \delta_1\lambda x^2z^2 + \delta_2\mu x^2y^2 + xyzp''').
\end{aligned}
$$

I now take the fourth coordinate plane $s = 0$ to be the infinitely-distant plane $t = 0$, with which, one will get $\alpha = 0$, $\beta = 0$, $\gamma = 0$, by means of the equation $s = \alpha x + \beta y + \gamma z + t$, and then determine the linear expressions p, p', p'', p''' from the equation $Px + Qy + Rz + St = 0$ as:

$$
(22) \qquad p = a_2y - a_1z - bt, \qquad p' = az - a_2x - b_1t, \qquad p'' = a_1x - ay - b_2\,t,
$$

$$
p''' = bx + b_1y + b_1z\,.
$$

If one then lets r, r_1, r_2, r_3 denote the same things as above then one will get the following analytical representation of this ray system of class six:

$$P\xi + Q\eta + R\zeta = 0,$$

$$\begin{aligned} P &= x\,(\delta_2\,\lambda\,z^2\,t^2 + \delta_1\,\mu\,y^2\,t^2 + \delta\,\nu\,y^2\,z^2 + y\,z\,t\,r), \\ Q &= y\,(\delta_2\,\kappa z^2\,t^2 + \delta\,\mu\,x^2\,t^2 + \delta_1\,\nu\,z^2\,x^2 + x\,z\,t\,r_1), \\ R &= z\,(\delta_2\,\kappa y^2\,t^2 + \delta\,\lambda\,x^2\,t^2 + \delta_2\,\nu\,x^2\,y^2 + x\,y\,t\,r_2), \end{aligned} \tag{23}$$

and the equation $\delta_1 uu' - \delta\upsilon\upsilon' = 0$ will give:

$$\delta x\eta\zeta + \delta_1 y\zeta\xi + \delta_1\,z\xi\eta = 0 \tag{24}$$

as a second equation for this ray system. Once again, this representation is the most general one, insofar as all ray systems of this kind can be obtained by collinear conversions of the one presented here. The focal surface is:

$$\begin{vmatrix} 0 & \delta_2 z & \delta_1 y & P \\ \delta_2 z & 0 & \delta x & Q \\ \delta_1 y & \delta x & 0 & R \\ P & Q & R & 0 \end{vmatrix} = 0, \tag{25}$$

which then contains the superfluous factor $x^2\,y^2\,z^2\,t^2$; when it has been removed, the equation will take the form:

$$\begin{aligned} \delta^2\,x^2\,r^2 + \delta_2^2 z^2 r_1^2 + \delta_1^2 z^2 r_2^2 - 2\,\delta_1\,\delta_2\,y\,z\,r_1\,r_2 - 2\,\delta_2\,d\,z\,x\,r_1\,r - 2\,\delta\delta_1\,xy\,rr_1 \\ - 4\delta\delta_1\delta_2\,(\kappa y\,z\,t\,r + \lambda\,z\,x\,t\,r_1 + \mu\,x\,y\,t\,r_2 + \nu\,x\,y\,z\,r_3) \\ - 4\delta\delta_1\delta_2\,(\delta\lambda\mu\,x^2\,t^2 + \delta_1\mu\kappa y^2\,t^2 + \delta_2\,\kappa\lambda\,z^2\,t^2 + \delta\kappa\nu\,z^2\,x^2 + \delta_2\,\mu\nu\,x^2\,y^2) = 0. \end{aligned} \tag{26}$$

This equation, in fact, represents the most general form for the fourth-degree surface with twelve nodes that has no singular tangential planes. The first four nodes are:

1.	$x = 0,$	$y = 0,$	$z = 0,$
2.	$y = 0,$	$z = 0,$	$t = 0,$
3.	$z = 0,$	$x = 0,$	$t = 0,$
4.	$x = 0,$	$y = 0,$	$t = 0,$

while the remaining eight nodes depend upon an eight-degree equation that one obtains by elimination from the equations $P = 0$, $Q = 0$, $R = 0$. The enveloping sixth-degree cone, which emanates from a node, decomposes for one of these twelve nodes into a fourth-degree cone with three double edges and a second-degree cone. Any three of the four enveloping second-degree cones that emanate from the points 1, 2, 3, 4, intersect in the remaining eight nodes, which can also be represented as the eight intersection points of three second-degree surfaces. The enveloping fourth-degree cone that emanates from these first four nodes is so arranged that the three double edges of one of these emanating cones go through the other three nodes, such that these double edges are collectively the edges of the tetrahedron that has these four nodes for vertices. If one considers the fourth-degree enveloping cone that emanates from one of the remaining eight nodes – which I choose to be node 5 – then its three double edges will go through three nodes of the surface that are not the nodes 1, 2, 3, 4. I denote these three nodes by 6, 7, 8. These four nodes 5, 6, 7, 8 then have the same property as 1, 2, 3, 4 – namely, that they define the vertices of a tetrahedron whose six edges are the double edges of the four enveloping fourth-degree cones that emanate from these points. The same thing is also true for the remaining four nodes 9, 10, 11, 12.

The ray system that was presented has the four points 1, 2, 3, 4 as singular points, from which emanate the four fourth-degree ray cones with those three double edges, although the points 5, 6, 7, 8, 9, 10, 11, 12 are the 8 singular points from which second-degree ray cones. Since the four nodes 5, 6, 7, 8, and likewise also the four nodes 9, 10, 11, 12 have precisely the same relationship to each other and the remaining nodes as 1, 2, 3, 4, it will then follow that this focal surface contains three different ray systems of order two and class six. Then:

XLV: *Any fourth-degree surface with twelve nodes and no singular tangential planes is the focal surface of three distinct ray systems of order two and class six whose six double rays are the edges of a tetrahedron.*

This focal surface is associated with a ray system of order six and class ten, along with these three second-order ray systems.

As a noteworthy special case of this ray system, I point out the one for which one has $a = 0$, $a_1 = 0$, $a_2 = 0$, $b = 0$, $b_1 = 0$, $b_2 = 0$, so $r = 0$, $r_1 = 0$, $r_2 = 0$. Its focal surface:

$$\delta\lambda\mu\, x^2\, t^2 + \delta_1\mu\kappa y^2\, t^2 + \delta_2\lambda\kappa z^2\, t^2 + \delta\kappa\nu\, y^2\, z^2 + \delta_1\mu\nu\, z^2\, x^2 + \delta_2\mu\nu\, x^2\, y^2 = 0,$$

is the reciprocal figure to the center of curvature surface of a three-axis ellipsoid, and the three ray systems of order two and class six that this focal surface is associated with have three ray systems of order six and class two for their reciprocal polar ray systems, each of which is the system of all normals to an ellipsoid.

The second-order ray systems without focal curves that were treated up to now can be considered to be the special case of the ray system of order two and class six that was given by (23). If one sets $\nu = 0$ then one will obtain the ray system of order two and class five that was presented in § 10, in which one drops the common factor of t from the three functions P, Q, R, which will reduce the class by one unit. If one sets $\nu = 0$ and $\mu = 0$ then the two factors t and z will drop out, and one will obtain the ray system of order two and class four that was given § 9. If one sets $\nu = 0$, $\mu = 0$, $\lambda = 0$ then, since t, z, y will drop out, one will obtain the first of the ray systems of order two and class three that was given in § 8. Finally, when one sets $\nu = 0$, $\mu = 0$, $\lambda = 0$, and $\kappa = 0$, one also will obtain the first of the ray systems of order two and class two that was presented in § 7, and likewise its focal surface, although one then accepts $\delta x\eta\zeta + \delta_1 y\zeta\xi + \delta_2\, z\xi\eta = 0$ as the second equation of the ray system, which is remarkably the same for all of these ray systems.

§ 12.

Ray systems of order two and class six that have no focal curves, of the second type.

As an example of a ray system of class six, I point out the ones whose six double rays all go through and one and the same point. From Theorem XXXIV, that point will be a singular point of the system with a ray cone of degree five that has six double edges in which the six double rays lie. In addition, at each of the six double rays, there is another singular point with a third-degree ray cone that has the double ray for its double edge. One then has $m_5 = 1$, $m_4 = 0$, $m_3 = 6$, and since $n = 5$, one gets:

$$33 = m_1 + 8m_2, \qquad 8 = 2m_2$$

from the two equations in Theorems XXIX and XXX, so $m_1 = 1$, $m_2 = 4$. One then has the theorem:

XLVI: *The ray systems of order two and class four of the second type have six double rays that go through one and the same point, and furthermore, they have twelve singular points, one of which has a planar ray pencil, four of which have second-degree ray cones, six have third-degree ray cones with just one double edge, and one of them has a fifth-degree ray cone with six double edges. The focal surfaces of these systems are fourth-degree surfaces with twelve nodes and one singular tangential plane.*

The analytic representation of these ray systems will be found by the following method. Let, as above:

$$u = y\zeta - z\eta, \qquad \upsilon = z\,\xi - z\,\zeta, \quad \omega = x\eta - y\,\xi;$$

I then take the first equation of a ray system to be an equation of the following form:

$$(1) \qquad at u^2 + bt\upsilon^2 + 2p\,\upsilon\omega + 2q\,\omega u + 2ru\,\upsilon = 0,$$

where

$$p = d_1\,y + d_2\,z + d_3\,t, \quad q = e_2\,z + e\,x + e_3\,t, \quad r = f\,x + f_1\,y + f_3\,t.$$

This equation has only one derived equation, namely:

$$(2) \qquad (d_1\eta + d_2\zeta)\,\upsilon\omega + (e_2\zeta + e\xi)\,\omega u + (f\xi + f_1\eta)\,u\upsilon = 0.$$

The two equations (1) and (2) thus determine a ray system completely. Now, although the first equation, when developed, is of degree two relative to ξ, η, ζ, and the second equation is of degree three, this ray system will still be only one of second order. In order to show this, I put equation (1) into the form:

$$xu\,(e\omega + f\upsilon) + y\upsilon\,(f_1\,u + d_1\,\omega) + z\omega\,(d_2\,\upsilon + e_2\,u) + t\cdot M = 0,$$

where, to abbreviate, we have set:

$$2M = au^2 + b\upsilon^2 + 2d_3\,\upsilon\omega + 2e_3\,\omega\upsilon + 2f_3\,u\,\upsilon.$$

Equation (2), as the first derived equation of this one, then becomes:

$$\xi u\,(e\,\omega + f\,\upsilon) + \eta\upsilon\,(f_1u + d_1\,\omega) + \zeta\omega\,(d_3\;\upsilon + e_2\;u) = 0,$$

and from these two one obtains:

$$(3)\qquad \begin{aligned} \upsilon\omega\,(f_1 - e_2)\;u - d_2\,\upsilon - d_1\,\omega) &= \xi Mt,\\ \omega u\;(d_2 - f)\;\upsilon - e\,\omega - e_1u) &= \eta Mt,\\ u\,\upsilon\,((e - f_1)\,\omega - f_1u - f\upsilon) &= \zeta Mt. \end{aligned}$$

The quotients of any two quantities ξ, η, ζ are thus rational fractional functions of u, υ, ω, and when ω is eliminated by means of the equation $xu + y\upsilon + z\,\omega = 0$, they will be rational function of the one quantity $u\,/\,\upsilon$. If one also eliminates ω from equation (1) then one will obtain:

$$(4)\qquad (2qx - atz)\;u^2 + 2(px + qy - rz)\;u\,\upsilon + (2py - btz)\;\upsilon^2 = 0,$$

so the quantity $u\,/\,\upsilon$ will be two-valued, and in turn, the quotients of ξ, η, ζ will also be two-valued, so the ray system will have order two.

The ray system that is given by equations (1) and (2) must, in turn, also have an equation of the form $P\xi + Q\eta + R\zeta = 0$, and this also can, in fact, be derived from the given equations. I shall skip the derivation of this equation here, because it is an immediate special case of the results that were developed in the following paragraphs for the rays systems of order two and class seven.

One obtains the focal surface of this ray system immediately from equation (4) by the condition that the two values of $u\,/\,\upsilon$ must be equal to each other when x, y, z is a point of the focal surface, namely:

$$(5)\qquad (px + qy - rz)^2 - (2qx - atz)(2py - btz) = 0,$$

which can also be put into the following form:

$$(6)\qquad (px - qy)^2 - z\,(2prx + 2\;qry - 2apyt - 2bqxt\;r^2\,z - ab\;t^2\;z) = 0.$$

It then follows from this that the plane $z = 0$ is a singular tangential plane of the focal surface that contacts it at the conic section $z = 0$, $px - qy = 0$. The six nodes of the surface that lie in these singular tangential planes are determined by the three equations:

$$z = 0,\qquad px - qy = 0,\qquad prx + qry \text{-} apyt - bqxt = 0$$

so they are:

1. $z = 0,\qquad x = 0,\qquad y = 0,$
2. $z = 0,\qquad p = 0,\qquad q = 0,$
3. $z = 0,\qquad x = 0,\qquad q = 0,$
4. $z = 0,\qquad y = 0,\qquad p = 0.$

The two remaining nodes 5 and 6 that lie in $z = 0$ will be determined by a quadratic equation. From the form of equation (5), one further sees that the eight intersection points of the three second-degree surfaces:

$$px + qy - rz = 0, \qquad 2qx - atz = 0, \qquad 2py - btz = 0$$

are nodes of the focal surface, and since only two of these eight nodes – viz., 1 and 2 – lie in the plane $z = 0$, one thus obtains the six nodes that shall be denoted by 7, 8, 9, 10, 11, 12. The focal surface thus has 12 nodes, and one can easily convince oneself that it also has no other nodes beyond these 12. The second-order ray system that is given by the two equations (1) and (2) will then have a fourth-degree surface with 12 nodes and one singular tangential plane for its focal surface. If one examines the sixth-degree enveloping cone that emanates from the nodes then one will find that for each of the two nodes 1 and 2, this enveloping cone will consist of a fifth-degree cone with six double edges and a plane, and furthermore, for each of the four nodes 3, 4, 5, 6, it will consist of a third-degree cone with no double edge, a second-degree cone, and a plane, while for each of the six nodes 7, 8, 9, 10, 11, 12, it will consist of two third-degree cones, each of which has a double edge.

Equation (4) is identical with equation (1), with the exclusion of the case where $z = 0$, in which it is tacitly assumed that it can then be considered to be the first equation of the ray system. Since this equation is fulfilled identically for each of the six nodes 7, 8, 9, 10, 11, 12, only equation (2) for the ray system will be true for these points, which represents a third-degree cone with a double edge that is given by the equations $\xi / x = \eta / y = \zeta / z$, which must then be a ray cone of the system that emanates from the point considered. As the equations themselves show, the six double edges of the third-degree ray cones that emanate from the points 7, 8, 9, 10, 11, 12 all go through the point $x = 0$, $y = 0$, $z = 0$, so every double edge of a ray cone is a double ray of the system, moreover. The ray system that is given by the equations (1) and (2) is then a second-order ray system with six double rays that go through the same point, so it is then the desired ray system of order two and class six of the second kind. The fact that it also represents the most general ray system of that kind follows from the fact that its focal surface is the most general fourth-degree surface with 12 nodes and one singular tangential plane when arbitrary, linear functions of the coordinates are taken in place of the x, y, z, t. The one singular point of the ray system with the fifth-degree ray cone and six double edges is node 1, the six singular points with third-degree ray cones with double edges are the points 7, 8, 9, 10, 11, 12, the four singular points with second-degree ray cones are the nodes 3, 4, 5, 6, and the one plane pencil of rays of emanates from the singular point 2. Since the focal surface is likewise an enveloping cone of degree five with six edges that emanates from the node 2, which go through the six nodes 7, 8, 9, 10, 11, 12, and since yet a second enveloping third-degree cone emanates from each of these six nodes with a double edge that goes through the node 2, one will see that the focal surface belongs to a second ray system of the same kind that one can derive from the given one when one switches x and p, and likewise y and q, with which, the node 1 will go to the mode 2. Thus:

XLVII: *Any fourth-degree surface with* 12 *nodes and one singular tangential plane is the focal surface of two distinct ray systems of order two and class six whose six double rays go through a point.*

§ 13.

Ray systems of order two and class seven that have no focal curves.

As was shown above, the ray systems of class seven have ten double rays that go through the same point, and at that singular point they have a sixth-degree ray cone with ten double edges on which the double rays lie; one then has $m_5 = 1$ for them. In addition, a singular point with a third-degree ray cone that has this double ray as a double edge lies on each of ten double rays; one then has $m_3 = 10$. One also has $m_5 = 0$, $m_6 = 0$, since singular points can lie on only the double rays for ray cones of degree higher than two, and since the 11 singular points that lie on them have only ten third-degree ray cones and one of degree six. If one then sets $m_6 = 1$, $m_5 = 0$, $m_4 = 0$, $m_3 = 10$, along with $n = 6$, in the equations then, since the systems has class seven one obtains:

$$0 = m_1 + s\,m_2\,, \qquad\qquad 0 = 2m_2\,,$$

so $m_1 = 0$, $m_2 = 0$. It then follows from this that:

XLVIII: *The ray systems of order two and class seven have ten double rays that go through one and the same point, and furthermore, they have eleven singular points, one of which has a sixth-degree ray cone and ten double edges and ten of which have third degree ray cones with only one double edge. The focal surfaces of these systems are fourth-degree surfaces with eleven nodes, one of which must have an enveloping sixth-degree cone with ten double edges emanating from it.*

If one takes the first equation of a ray system to be the equation:

$$\text{(1)} \qquad atu^2 + bt\upsilon^2 + ct\omega^2 + 2p\,\upsilon\omega + 2q\,\omega u + 2ru\,\upsilon = 0,$$

where u, υ, ω, p, q, r have the same meaning as in the previous paragraphs, then it will have only one derived equation:

$$\text{(2)} \qquad (d_1\eta + d_2\zeta)\ \upsilon\omega + (e_2\zeta + e\xi)\ \omega u + (f\xi + f_1\eta)\ u\,\upsilon = 0,$$

and the two equations (1) and (2) will determine a ray system completely, and we shall prove that this is the desired ray system of order two and class seven, and indeed the most general one, insofar as the one considers all collinear conversions of it to be likewise included in that form. If one puts equation (1) into the form:

$$xu\,(e\,\omega + f\,\upsilon) + y\,\upsilon\,(f_1\,u + d_1\,\omega) + z\,\omega\,(d_2\,\upsilon + e_2\,u) + tM = 0$$

in the same way as what was done in the previous paragraphs, where

$$2M = au^2 + b\upsilon^2 + c\omega^2 + 2d_3\,\upsilon\omega + 2e_3\,\omega u + 2f_3\,u\upsilon,$$

and puts equation (2) into the form:

$$\xi u\,(e\omega + f\upsilon) + \eta\upsilon\,(f_1\,u + d_1\,\omega) + \zeta\omega\,(d_2\,\upsilon + e_2\,u) = 0$$

then one will obtain the same expressions for ξ, η, ζ in terms of x, y, z from this:

$$(3)\qquad \begin{aligned} \upsilon\omega\,((f_1 - e_2)\,u - d_2\,\upsilon + d_1\,\omega) &= \xi M\,t, \\ \omega u\,((d_2 - f)\,\upsilon - e\,\omega + e_2\,u) &= \eta M\,t, \\ uw\,((e - d_1)\,\omega - f_1\,u + f\,\upsilon) &= \zeta M\,t. \end{aligned}$$

If one now eliminates the quantity ω from equation (1) by using the equation $ux + \upsilon y + \omega z = 0$ then one will get:

$$(4)\qquad (atz^2 - 2qzx + ctx^2)u^2 + 2(ctxy - pxz - qyz + rz^2)\,u\upsilon + (btz^2 - 2\,pzy + cty^2)\,\upsilon^2 = 0.$$

If one eliminates the quantity ω from the expressions for ξ, η, ζ that are given by (3) using the same equations then the quotients of any two of the quantities ξ, η, ζ will be rational functions of u / υ, and because, from equation (4), u / υ is two-valued the quotients of any two of the quantities ξ, η, ζ will be two-valued functions of x, y, z, t, so the ray system will have order two.

Since the two values of u / υ that are given by the quadratic equation (4) must be equal to each other for each point of the focal surface, one will get:

$$(5)\qquad (ctxy - pxz - qyz + rz^2)^2 - (ctx^2 - 2qxz + atz^2) + (cty^2 - 2pyz + btz^2) = 0$$

as the equation of the focal surface of the second-order ray system that given by equations (1) and (2). This equation then contains the common factor z^2, and when the equation is freed of that factor, it will take the following form:

$$(6)\qquad \begin{aligned} x^2(p^2 - bct^2) + y^2(q^2 - cat^2) + z^2(r^2 - abt^2) + 2\,yz\,(atp - qr) \\ +2zx\,(btq - rp) + 2\,xy\,(ctr - pq) = 0, \end{aligned}$$

which can also be represented by the following symmetric determinant:

$$(7)\qquad \begin{vmatrix} at & r & q & x \\ r & bt & p & y \\ q & p & ct & z \\ x & y & z & 0 \end{vmatrix} = 0.$$

If one arranges for equation (1) to be an equation that has degree two with respect to ξ, η, ζ in the form:

(8) $$A\xi^2 + B\eta^2 + C\zeta^2 + 2\,D\eta\zeta + 2\,E\,\zeta\xi + 2\,F\,\xi\eta = 0$$

then one will get:

(9) $$\begin{aligned} A &= btz^2 - 2pzy + cty^2, \\ B &= ctx^2 - 2qxz + atz^2, \\ C &= aty^2 - 2ryx + btx^2, \\ D &= -px^2 + qxy + rxz - atyz, \\ E &= -qy^2 + ryz + pyx - btzx, \\ F &= -rz^2 + pzx + rzy - ctxy, \end{aligned}$$

and these six coefficients will be coupled by the following equations:

(10) $$\begin{aligned} Ax + Fy + Ez &= 0, &\qquad -Ax^2 + By^2 + Cz^2 + 2Dyz &= 0, \\ Fx + By + Dz &= 0, &\qquad +Ax^2 - By^2 + Cz^2 + 2Ezx &= 0, \\ Ex + Dy + Cz &= 0, &\qquad +Ax^2 + By^2 + Cz^2 + 2Fxy &= 0, \end{aligned}$$

and one additionally will get:

(11) $$\begin{aligned} &D^2 - BC = x^2\phi, \quad E^2 - CA = y^2\phi, \quad F^2 - AB = z^2\phi, \\ &AD - EF = yz\phi, \quad BE - FD = zx\phi, \quad CF - DE = xy\phi, \end{aligned}$$

where $\phi = 0$ is the equation for the focal surface.

One sees immediately that all six quantities A, B, C, D, E, F are equal to zero for the four points:

1. $x = 0,\quad y = 0,\quad z = 0,$
2. $y = 0,\quad z = 0,\quad t = 0,$
3. $z = 0,\quad x = 0,\quad t = 0,$
4. $x = 0,\quad y = 0,\quad t = 0.$

Moreover, equations (10) show that D, E, F must necessarily equal zero when A, B, and C are equal to zero without x, y, or z being zero. If one now eliminates the two quantities t and z from the three equations $A = 0$, $B = 0$, $C = 0$ then one will obtain an equation of degree seven for y / x, and z / x and t / x will be expressed rationally in terms of y / x. In addition to the stated four points, there are also seven points that do not lie in any of the four coordinate planes $x = 0$, $y = 0$, $z = 0$, $t = 0$, for which, the six quantities A, B, C, D, E, F will be simultaneously equal to zero. The first equation of the ray system will be fulfilled identically or these eleven points – which, as equations (11) shows, are likewise eleven nodes of the focal surface – without yielding a determination of the direction of the rays that emanate from them. These points will be, in turn, singular points of the ray system from which ray systems emanate that are determined by the second equation of the ray system. For the first point $x = 0$, $y = 0$, $z = 0$, the second equation will be fulfilled in addition to the first one, such that the ray cone that belongs to that point will remain undetermined, but for each of the ten remaining singular points the second equation will give a third-degree ray cone with a double edge that is determined by the equations $\xi / x = \eta / y = \zeta / z$, and in turn, will always go through the origin of the coordinates. The ray system will then have ten third-degree ray cones with one double edge, and thus ten double rays, so it will necessarily be the desired ray system of order two and class seven, and the point $x = 0$, $y = 0$, $z = 0$, through which the ten double rays go, will be the singular point that has the sixth-degree ray cone, which has ten double edges.

Of the eleven nodes of the focal surface, only one of then – viz., $x = 0$, $y = 0$, $z = 0$ – has the property that a sixth-degree enveloping cone with ten double edges emanates from it, while the sixth-degree enveloping cones that emanate from the remaining ten nodes will decompose into two third-degree cones, one of which has a double edge, but not the other one. It then follows from this that, besides this a ray system of order two and class seven, the same focal surface can belong to no others of the same kind, and no other second-order ray systems at all.

§ 14.

Representation of a ray system of order two and class seven by an equation that is linear in ξ, η, ζ, and special cases of these systems.

From the two equations (1) and (2) that were found in the previous paragraph for the ray systems of order two and class seven, the three functions P, Q, R in the linear equation:

$$P\xi + Q\eta + R\zeta = 0, \tag{1}$$

which each second-order ray system must satisfy, will be determined in the following way: If one first eliminates u, then υ, then ω from equation (1) by means of the equation $ux + \upsilon y + \omega z = 0$ then one will obtains the three equations:

$$\begin{aligned} C\upsilon^2 - 2D\upsilon\omega + B\omega^2 &= 0, \\ A\omega^2 - 2\,E\omega u + Cu^2 &= 0, \\ Bu^2 - 2\,Fu\upsilon + A\upsilon^2 &= 0, \end{aligned} \tag{2}$$

where A, B, C, E, F have the same meanings as in the previous paragraph. That will give the following values for the quotients of any two of the quantities u, υ, ω:

$$\begin{aligned} &\frac{\upsilon}{\omega} = \frac{D + x\sqrt{\phi}}{C}, \qquad \frac{\omega}{u} = \frac{E + y\sqrt{\phi}}{A}, \qquad \frac{u}{\upsilon} = \frac{F + z\sqrt{\phi}}{B}, \\ &\frac{\omega}{\upsilon} = \frac{D - x\sqrt{\phi}}{B}, \qquad \frac{u}{\omega} = \frac{E - y\sqrt{\phi}}{C}, \qquad \frac{\upsilon}{u} = \frac{F - z\sqrt{\phi}}{A}. \end{aligned} \tag{3}$$

If one now substitutes the values for ξ, η, ζ that were found in (3) in the previous paragraph in the equation $P\xi + Q\eta + R\zeta = 0$ then, after dropping the common factors, one will get:

$$(4)\quad P\left((f_1 - e_2) - d_2\frac{\upsilon}{u} + d_1\frac{\omega}{u}\right) + Q\left((d_2 - f) - e\frac{\omega}{\upsilon} + e_1\frac{u}{\upsilon}\right) + R\left((e - d_1) - f_1\frac{u}{\omega} + f\frac{\upsilon}{\omega}\right) = 0,$$

and when the values that are found for u / u, w / u, w / u, etc., are substituted in this:

$$(5)\quad \begin{aligned} &\frac{P}{A}\Big((f_1 - e_2)A - d_2(F - z\sqrt{\phi}) + d_1(E + y\sqrt{\phi})\Big) \\ &+\frac{Q}{A}\Big((d_2 - f)B - e(D - x\sqrt{\phi}) + e_2(F + z\sqrt{\phi})\Big) \\ &+\frac{R}{A}\Big((e - d_1)C - f_1(E - y\sqrt{\phi}) + f(D + x\sqrt{\phi})\Big) = 0. \end{aligned}$$

Since the same equation is also true when one gives $\sqrt{\phi}$ the opposite sign, one will get the following two equations:

$$(6)\quad \begin{aligned} &\frac{P}{A}\big((f_1 - e_2)A - d_2F + d_1E\big) + \frac{Q}{B}\big((d_2 - f)B - eD + e_2F\big) + \frac{R}{C}\big((e - f_1)C - f_1E + fD\big) = 0, \\ &\frac{P}{A}(d_1y - d_2z) + \frac{Q}{B}(d_2 - f) + \frac{R}{C}(fx + f_1y) = 0, \end{aligned}$$

and since P, Q, R are entire functions of x, y, z, t with no common factors, one will thus obtain the following values for them:

$$(7)\quad \begin{aligned} P &= A\big\{(fx + f_1y)\big((d_2 - f)B - eD + e_2F\big) - (e_2z + ex)\big((e - d_1)C - f_1E + fD\big)\big\}, \\ Q &= B\big\{(d_1y + d_2z)\big((e - d_1)C - f_1E + fD\big) - (fx + f_1y)\big((f_1 - e_2)A - d_2F + d_1E\big)\big\}, \\ R &= C\big\{(e_2z + ex)\big((f_1 - e_2)A - d_2F + d_1E\big) - (d_1y + d_2z)\big((d_2 - f)B - eD - e_2F\big)\big\}. \end{aligned}$$

These expressions for P, Q, R all have the common factor t, and when it is dropped, one will obtain the following representation for the:

<u>Ray systems of order two and class seven</u>

$$P\xi + Q\eta + R\zeta = 0,$$

$$P = AK, \qquad Q = BC, \qquad R = CM,$$

$$A = btz^2 - 2pzy + cty^2,$$
$$B = ctx^2 - 2qxz + atz^2,$$
$$C = aty^2 - 2ryx + btx^2,$$

$$K = q_0\,(ay - 2f_3\,x)((d_1 - e)\,y + fz) + q_0\,bx\,((d_1 - e)\,x - f_1\,z)$$
$$+ r_0\,(az - 2e_3\,x)((d_2 - f)\,z + ey) + r_0\,cx\,((d_2 - f)\,x - e_2\,y) + 2q_0\,r_0\,d_3\,x,$$

(8)
$$L = r_0\,(by - 2d_3\,y)((e_2 - f_1)\,z + d_1\,x) + r_0\,cy\,((e_2 - f_1)\,y - d_2\,x)$$
$$+ p_0\,(bx - 2f_3\,y)((e - d_1)\,x + f_1\,z) + p_0\,ay\,((e - d_1)\,y - f\,z) + 2r_0\,p_0\,e_3\,y,$$

$$M = p_0\,(cx - 2e_3\,z)((f - d_2)\,x + e_2\,y) + p_0\,az\,((f - d_2)\,z - e\,y)$$
$$+ q_0\,(cy - 2d_3\,z)((f_2 - e_2)\,y + d_2\,x) + q_0\,bz\,((f_1 - e_2)\,z - d_1\,x) + 2p_0\,q_0\,f_3\,z,$$

in which p_0, q_0, r_0 refer to the values of p, q, r when $t = 0$, namely:

$$p_0 = d_1\,y + d_2\,z, \qquad q_0 = e_2\,z + e\,x, \qquad r_0 = f\,x + f_1\,y.$$

The three functions P, Q, R, by which the ray systems of order two and class seven are determined completely, when combined with the derived equations, are functions of degree seven, as they must be, since the degree is always one unit less than the class of the ray system. The ten double rays are common straight lines to the three surfaces $P = 0$, $Q = 0$, $R = 0$, and indeed the three coordinate axes belong to the these ten double rays; the seven remaining ones are obtained as the ones that are common to the three third-degree cones $K = 0$, $L = 0$, $M = 0$, which are straight lines that are determined by a seventh-degree equation.

If one takes $c = 0$ then the common factor of z will drop out of the three functions P, Q, R, and one will obtain the general representation for the:

<u>Ray systems of order two and class six of the second kind</u>

$$P\xi + Q\eta + R\zeta = 0,$$

where

$$P = (btz - zpy)\,K, \qquad Q = (atz - 2qx)\,C, \qquad R = CM,$$

$$K = q_0\,(ay - 2f_3\,x)\,((d_1 - e)\,y + f\,z) + q_0\,bx\,((d_1 - e)\,x - f_1\,z)$$
$$+ r_0\,(az - 2e_3\,x)\,((d_2 - f)\,z + e\,y) + 2\,q_0\,r_0\,d_3\,x,$$

(9)
$$L = p_0\,(bx - 2f_3\,y)\,((e - d_1)\,x + f_1\,z) + p_0\,ay\,((e - d_1)\,y - f\,z)$$
$$+ r_0\,(bz - 2d_3\,y)\,((e_2 - f_1)\,z + d_1\,x) + 2\,p_0\,r_0\,e_3\,y,$$

$$M = -\,2\,e_3\,p_0\,((f_1 - d_2)\,x + e_2\,y) + p_0\,a\,((f - d_2)\,z - e\,y)$$
$$-\,2\,d_3\,q_0\,((f_1 - e_2)\,y + d_2\,x) + q_0\,b\,((f_1 - e_2)\,z + d_1\,x) + 2\,q_0\,r_0\,d_3\,x\,.$$

Of the six double rays that go through the coordinate origin, which must be six of the straight lines that are common to the three surfaces $P = 0$, $Q = 0$, $R = 0$, one of them will lies along the z-axis, while the other five will be the straight lines that are common to the two third-degree cones $K = 0$, $L = 0$, and the second-degree $M = 0$, which will be determined by an equation of degree five.

If one sets $c = 0$ and $b = 0$ then the common factors z and y will drop out of the three functions P, Q, R, and one will obtain the:

<u>Ray systems of order two and class five</u>

$$P\xi + Q\eta + R\zeta = 0,$$

$$P = -2p\big(q_0\,(ay - 2f_3\,x)((d_1 - e)\,y + fz) + r_0\,(az - 2e_3\,x)((d_2 - f)\,z + cy$$
$$+ 2d_3\,q_0\,r_0\,x\big),$$

(10) $$Q = (atz - 2qx)\begin{pmatrix} -2d_3r_0\left((e_2 - f_1)z + d_1x\right) + 2e_3r_0\,p_0 \\ -2f_3p_0\left((e - d_1)x + f_1z\right) + ap_0\left((e - d_2)y - fz\right) \end{pmatrix},$$

$$R = (aty - 2rx)\begin{pmatrix} -2d_3q_0\left((f_1 - e_2)y + d_2x\right) - 2e_3\,p_0\left((f - d_2)x + e_2y\right) \\ -2f_3p_0q_0 + ap_0\left((f - d_2)z - ey\right) \end{pmatrix}.$$

The focal surface of this system is:

(11) $$\begin{vmatrix} at & r & q & x \\ r & 0 & p & y \\ q & p & 0 & z \\ x & y & z & 0 \end{vmatrix} = 0,$$

or, when developed:

(12) $$x^2\,p^2 + y^2\,q^2 + z^2\,r^2 - 2yzqr - 2zxrp - 2xypq + 2ayzpt = 0.$$

This representation for the ray systems of class five has a completely different form fromthe one that was given in § 10. It is likewise the most general of all of them, and for that reason, one can take both representations to each other by collinear conversions. Likewise, the focal surface of this system will represent the most general fourth-degree surface with 13 nodes and three singular tangential planes, and one can convert the one form of the equation into the other one by collinear conversions. The three singular tangential planes for this equation of the surface are simply $y = 0$, $z = 0$, and $p = 0$.

If one sets $a = 0$, in addition to $c = 0$, $b = 0$, then the two factors of z and y drop out of the three functions, along with the common factor of x, and since the degrees of these functions will be reduced by three units, the class will also be reduced by three units, and one will obtain the:

Ray systems of order two and class four

$$P\xi + Q\eta + R\zeta = 0,$$

$$P = p\,\big(f_3\, q_0\, ((d_1 - e)\, y + f\, z) + e_3\, r_0\, ((d_2 - f)\, z + ey) - d_3\, q_0\, r_0\big),$$

$$Q = q\,\big(d_3\, r_0\, ((e_2 - f_1)\, z + d_1\, x) + f_3\, p_0\, ((e - d_1)\, x + f_1\, y) - e_3\, r_0\, p_0\big), \tag{13}$$

$$R = r\,\big(e_3\, p_0\, ((f - d_2)\, x + e_2\, y) + d_3\, q_0\, ((f_1 - e_2)\, y + d_2\, x) - f_3\, p_0\, q_0\big).$$

The focal surface for this is:

$$x^2\, p^2 + y^2\, q^2 + z^2\, r^2 - 2yzqr - 2zxrp - 2xypq = 0, \tag{14}$$

or, in irrational form:

$$\sqrt{xp} + \sqrt{yq} + \sqrt{zr} = 0. \tag{15}$$

This form, which is different from the form that was found in § 9 for ray systems of class four, likewise includes the most general ray system of this kind, and the one form can be considered to be a collinear conversion of the other form. The equation of the focal surface in this form has the advantage that the six singular tangential planes $x = 0$, $y = 0$, $z = 0$, $p = 0$, $q = 0$, $r = 0$ become immediately apparent.

In order to obtain the ray systems of class third from this by further specialization, I introduce the following condition equation, with the current constants:

$$\frac{d_3}{k} + \frac{e_3}{l} + \frac{f_3}{m} = 0, \tag{16}$$

where I have set:

$$\begin{aligned} k &= d_1\, d_2 - d_1\, f - d_2\, e, \\ l &= e_2\, e - e_2\, d_1 - e\, f_1, \\ m &= f\, f_1 - f\, e_2 - f_1\, d_2, \end{aligned} \tag{17}$$

which are quantities that satisfy the following equations:

$$\begin{aligned} k\,(e_2 - f_1) + l\, d_2 - m\, d_1 &= 0, \\ l\,(e_2 - f_1) + m\, e - k\, e_2 &= 0, \\ m(d_1 - e) + k\, f_1 - l\, f &= 0, \end{aligned} \tag{18}$$

and if I further set:

$$d_3 = k\delta, \qquad e_3 = l\,\delta_1, \qquad f_3 = m\,\delta_2$$

then the equation:

$$\delta + \delta_1 + \delta_2 = 0 \tag{19}$$

will exist between the quantities δ, δ_1, δ_2. By means of this condition equation, the three quantities P, Q, R of the previous cases have the common factor:

$$kx + ly + mz = 0,$$

and when this is omitted, one will obtain the:

<u>Ray systems of order two and class three</u>

$$(20)\qquad \begin{aligned} &P\xi + Q\eta + R\zeta = 0, \\ \\ &P = p\,(\delta_2 f\ q_0 + \delta_1\, e\, r_0), \\ &Q = q\,(\delta\, d_1\, r_0 + \delta_2\, f_1\, p_0), \\ &R = r\,(\delta_1\, e_2\, p_0 + \delta\, d_2\, q_0), \\ \\ &p = d_1\, y + d_2\, z + dkt, \\ &q = e_2\, z + e\, x + \delta_1\, lt, \\ &r = f\, x + f_1\, y + \delta_2\, m\, t, \end{aligned}$$

whose focal surface has the equation:

$$(21)\qquad \sqrt{xp} + \sqrt{yq} + \sqrt{zr} = 0$$

for the special values of the linear expressions p, q, r that were given here, so this equation represents the most general fourth-degree surface with 15 nodes and ten singular tangential planes.

Finally, one also obtains the ray systems of class two from the ones given here for class four when one establishes the conditions:

$$(22)\qquad \begin{aligned} k &= d_1\, d_2 - d_1\, f + d_2\, e = 0, \\ l &= e_2\, e - e_2\, d_1 - e\, f_1 = 0, \\ m &= f\, f_1 - f\, e_2 - f_1\, d_2 = 0, \end{aligned}$$

which are essentially just two equations, because as equations (18) show, if two of them are true then the third one will be fulfilled automatically. If one sets:

$$(23)\qquad \begin{aligned} &d_1 = \delta a_2, \qquad e_2 = \delta_1 a, \qquad f = \delta_2 a_1, \\ &d_2 = -\delta a_1, \qquad e = -\delta_1 a_2, \qquad f_1 = -\delta_2 a, \end{aligned}$$

in order to fulfill these conditions in a symmetrical manner, and one sets:

$$(24)\qquad d_3 = -\,\delta b, \qquad e_3 = -\,\delta_1\, b_1\,, \qquad f_3 = -\,\delta_2\, b_2\,,$$

in addition, then one will get:

$$\delta + \delta_1 + \delta_2 = 0,$$

$$(25)\qquad \begin{aligned} p &= \delta(a_2\, y - a_1\, z - b\, t), \\ q &= \delta_1(az - a_1\, x - b_1\, t), \end{aligned}$$

$$r = \delta_2(a_1\, x - a\, y - b_2\, t),$$

or when one makes use of the notation for these linear equations that was applied above, one will get $p = \delta r$, $q = \delta_1\, r_1$, $r = \delta_2\, r_2$. The factors that are enclosed in parentheses in the expressions for P, Q, R in (13) will be equal to each other when one substitutes these values, and if one drops them, such that only $P = r$, $Q = r_1$, $R = r_2$ remain then:

(26) $$r\xi + r_1\,\eta + r_2\,\zeta = 0$$

will emerge as the first equation of the ray system of class two, which is the same as the one that was presented in § 7. One can likewise find the second equation for the ray system of class two, which is quadratic with respect to *x*, *h*, *z*, as a special case of the equations that were given for the ray systems of class seven. Namely, if one puts $c = 0$, $b = 0$, $a = 0$, $p = \delta r$, $q = \delta_1\, r_1$, $r = \delta_2\, r_2$ in equation (1) of § 13 then that will give:

(27) $$\delta r\, vw + \delta_1\, r_1\, wu + \delta_2\, r_2\, uv = 0,$$

and from the coupling of this equation with the first equation:

(28) $$r\,\xi + r_1\,\eta + r_2\,\xi = 0,$$

one will obtain:

$$\delta x\,\eta\zeta + \delta_1\, y\,\zeta\,\xi + \delta_2\, z\,\xi\eta = 0,$$

which is the second equation of the first ray system of class two that belongs to six focal surfaces.

The ray systems of class seven (which are the ones with the highest class that have no focal curves at all for the ray systems of second order) thus subsume all ray systems of lower classes as special cases, with the exception of those ray systems of class six whose six double rays define the edges of a tetrahedron.

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