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On atmospheric ray refraction

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Up to now, the atmospheric refraction of rays has been treated almost exclusively only upon the basis of ratios of magnitudes that are coincidentally appropriate to our Earth for the practical use of astronomy and geodesy. On has, in turn, a series of very interesting phenomena that his theory features when they are considered from a more general, more mathematical viewpoint, which seem to have been completely unnoticed, up to now. The brief exposition of the phenomena that I would like to give here will perhaps also be of interest because even though it does not pertain to our Earth, it still must actually pertain to the larger heavenly bodies – e.g., Jupiter – even when the strengths of the atmosphere of such a heavenly body are significantly less that those of the Earth atmosphere.

1. The curvilinear path of a light ray in an inhomogeneous, transparent, simply-refracting medium whose absolute refraction exponent n is a continuous function of the rectangular coordinates x, y, z of position is determined by the following differential equations:

$$d\left(n\frac{dx}{ds}\right) = \frac{\partial n}{\partial x}ds\,,$$
$$d\left(n\frac{dy}{ds}\right) = \frac{\partial n}{\partial y}ds\,,$$
$$d\left(n\frac{dz}{ds}\right) = \frac{\partial n}{\partial z}ds\,,$$

in which ds refers to the differential arc length, and:

$$\frac{\partial n}{\partial x},\ \frac{\partial n}{\partial y},\ \frac{\partial n}{\partial z}$$

refer to the partial differential quotients of n. As is easy to show, one of these three equations will be a consequence of the other two, such that two of them will suffice for the complete determination of the path of the light rays.

If one compares these equations with the known differential equations by which one determines the equilibrium position of a flexible string that is acted upon by a given force then one will remark that they are completely identical with them if the refraction exponent n is the same function of x, y, z that the stress T is at each point of the string. One recognizes the basis for this identity almost immediately when one applies the principle of least action to one and the other equation. The refraction exponent n is known to be inversely proportional to the velocity of light at the point considered, so one will then have:

$$\frac{ds}{dt} = \frac{1}{n}, \qquad dt = n\, ds,$$

and since the motion of light from one point to another takes place in the shortest time:

$$\int n\, ds$$

must be a minimum. On the other hand, the equilibrium of the flexible string demands that the sum of all the stresses of the individual arc elements between any two given points must be a minimum. Thus

$$\int T\, ds$$

must be a minimum. One finds the above differential equations from these conditions from the rules of the calculus of variations, which must be the same for both problems when T is the same function of x, y, z as n is.

2. The general differential equations will now be applied to the case in which the refraction exponent n is any function of the distance from a fixed center, so the refraction exponent will have the same value for all points of the outer surface of a ball of arbitrary radius in the transparent medium, which is a case that approximates the atmosphere of a heavenly bodies in a regular state. Every curve that a light ray describes in such a medium will obviously lie completely within a plane that goes through the center; if it were chosen to be the x, y coordinate plane then one would have $z = 0$, and n would be a function of:

$$r = \sqrt{x^2 + y^2}\,.$$

The third differential equation will be fulfilled identically in this case, but the other two, which already sufficed for the complete solution of the problem, will give

$$y\, d\left(n\frac{dx}{ds}\right) - x\, d\left(n\frac{dy}{ds}\right) = 0$$

when one multiplies one of them by y and the other one by x and subtracts them. If one develops this equation and sets:

$$x\frac{dy}{ds} - y\frac{dx}{ds} = p$$

then one will obtain:

$$n\,dp + p\,dn = 0,$$

so

$$np = C$$

will be a first integral. If one introduces polar coordinates by setting:

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

then one will get:

$$\frac{nr^2\,d\varphi}{\sqrt{dr^2 + r^2 d\varphi^2}} = C,$$

so

$$d\varphi = \frac{Cdr}{r\sqrt{r^2n^2 - C^2}}\,.$$

If one now assumes that the transparent medium surrounds an opaque ball with a radius of R in the manner of an atmosphere then one will have to consider only the values of r that are greater than R. Therefore, if one sets $r = R + v$ and $R\varphi = u$ then u and v can be regarded as coordinates of the curve, and indeed, u – viz., the abscissa – will be an arc of the great circle on the sphere and v – viz., the ordinate – will be the altitude of the point of the curve considered above the outer surface of the ball that is erected at the endpoint of the abscissa. One will then have:

$$u + B = \int_G \frac{RC\,dv}{\sqrt{(R+v)\sqrt{(R+v)^2n^2 - C^2}}}\,.$$

For the determination of the two integration constants, it shall be assumed that the light ray starts from the point of the sphere whose coordinates are $u = 0$ and $v = 0$, and that its initial direction makes an angle of inclination i with the horizontal plane at that point. One will then have:

$$B = 0, \quad C = n_0 \cos i,$$

where n_0 refers to the value of n for $v = 0$. The equation of the curve of the light ray will then become:

$$u = \int_G \frac{R^2 n_0 \cos i\,dv}{\sqrt{(R+v)\sqrt{(R+v)^2n^2 - R^2n_0^2\cos^2 i}}}\,.$$

3. Now, there are two essentially different cases to distinguish, namely, the first case, in which the quantity under the square root sign, which shall be briefly denoted by:

$$V = (R + v)^2\, n^2 - R^2 \cos^2 i,$$

shall be equal to zero for no positive value of v, and otherwise positive, and the second case, in which that quantity shall be equal to zero for some positive value of v. The refraction exponent n, as a function of the altitude v, shall temporarily be left arbitrary and subject to only the condition that is must be a single-valued function of v that approaches a finite limit for $v = \infty$ that cannot be smaller than one, and furthermore, that n itself, as well as its first and second differential quotients, must not become infinitely large for any positive value of v.

If V is not equal to zero from $v = 0$ to $v = \infty$ then u will also become constantly larger with increasing v, but for $v = \infty$, u will take on a finite value, as is easy to show. If one denotes it by c then:

$$c = \int_0^\infty \frac{R^2 n_0 \cos i \, dv}{(R+v)\sqrt{V}}.$$

It follows from this that the curve of the light ray will always have a rectilinear asymptote, and that the angle that this asymptote makes with the vertical at the point $u = 0$, $v = 0$, when it is expressed as the arc for radius one, will be equal to:

$$\frac{c}{R} = \int_0^\infty \frac{R n_0 \cos i \, dv}{(R+v)\sqrt{V}};$$

Thus, the angle of refraction – which might be denoted by Θ – for objects whose distance is very large in comparison to the radius R, as the difference between the direction of the asymptote and the direction of the tangent to the initial point, will be:

$$\Theta = \frac{c}{2} - \frac{n}{2} + i;$$

this can also be easily put into the form:

$$\Theta = \int_0^\infty \frac{R \cos i}{R+v} \left(\frac{n}{\sqrt{V}} - \frac{1}{\sqrt{(R+v)^2 - R^2 \cos^2 i}} \right) dv.$$

4. I shall now consider the other case, in which V is equal to zero, for one, or also for several, positive values of v. Let $v = b$ be the smallest of these values, and let the value of the abscissa that belongs to it be $u = a$, so:

$$a = \int_0^b \frac{R^2 n_0 \cos i \, dv}{(R+v)\sqrt{V}}.$$

One now comes to the question of whether this well-defined integral – or a – has a finite or an infinitely-large value. From *Taylor's* theorem, one has:

$$V(v) = V(b) - (b - v)\, V'(b) + \frac{(b-v)^2}{2}\, V''(\varepsilon),$$

where ε is a quantity that lies between the limits of 0 and b. Now, since it was assumed above that the function n, along with its first two differential quotients, were not infinite the same thing will obviously be true, as well, for the function V and its first two differential quotients V' and V'' for all positive finite values of v. Now, when $V'(b)$ is not equal to zero, but V decreases from positive values to negative ones, so $V'(b)$ will have a negative value, where $V(b)$ is equal to zero, V will have the form:

$$V = (b - v)\, W,$$

where W will have a finite, non-zero value for $v = b$. One concludes from this by known rules that in this case – viz., where V' is not equal to zero for $v = b$ – the integral a will have a finite value. By contrast, if V', as well as V, are both equal to zero for $v = b$ then V will take the form:

$$V = (b - c)^2\, W_1,$$

where W_1 will not be infinite for $v = b$, from which, it would follow that the integral a would have an infinitely large value in that case.

First of all, let V' be non-zero for $v = b$, so a finite value of the abscissa $u = a$ will belong to the value $v = b$ of the ordinate. Once v has reached the value b, starting from zero, it cannot become larger, since otherwise $\sqrt{V}$ would become imaginary; however, the curve of the light ray cannot break suddenly at the point $u = a$, $v = b$, since the light ray must remain in the continuously-transparent medium, so it must once more become smaller from there to v – hence, dv will be negative – and at the same time, the root $\sqrt{V}$ must take on a negative sign that can come about without disrupting continuity, since that square root will go through the value zero for $v = b$. The curve of the light ray will thus attain its maximum altitude at the point $u = a$, $v = b$, and then it will once more approach the sphere from there, and indeed in such a way that the increasing part of the curve will be completely symmetric to the decreasing part, and it will again return to the outer surface of the ball, and the distance from the starting point to the point where it again meets the sphere will be equal to $2a$, when measured as the arc length of a great circle of the sphere.

Here, one enters into the same situation as in the known phenomenon of air reflection, namely, when a light ray comes tangentially at an infinitely small angle to a layer of air that is relatively too thin for it to break through, it will suffer a kind of total reflection and turn back from there into the thicker layer of air.

Secondly, if V', as well as V, is equal to zero for $v = b$, so a becomes infinitely large, then the ordinate v will approach the limit $v = b$ when the abscissa u goes to infinity. The curve of the light ray will thus go around the sphere infinitely many times and asymptotically approaches a circle whose radius is equal to $R + b$, or whose height above the sphere is equal to b.

5. The various values of the inclination angle i shall now be brought under consideration. If the quantity $(R + v)^2\, n^2$ attains its smallest absolute value, which is at the same time smaller than $R^2 n_0^2$, for $v = b$, and one determines the acute angle $i = I$ by means of the equation $V = 0$ for $v = b$, namely:

$$\cos I = \frac{(R+\beta)\,n(\beta)}{R\,n_0},$$

then V can be equal to zero only when the inclination angle i lies between the limits of 0 and I, and for each of the limiting values of i there will also be actually one or more positive values of v for which $V = 0$; let the smallest of them be $v = b$. It will then follow from this that:

All light rays that depart from the sphere with an angle $i > I$ will go to an infinite distance from it and have rectilinear asymptotes; however, the ones that leave the sphere at an angle that is smaller than I will attain only a certain maximum altitude $v = b$ that is smaller than β, and then will generally return to the outer surface of the ball, but in the special case for which not only $V = 0$ for $v = b$, but also $V' = 0$, the light ray will approach a circle whose altitude above the sphere is equal to b.

This latter case of the circular asymptote of the light ray always comes about when $i = I$, since $v = \beta$ for that value of the angle of inclination and V will attain its minimum, whose value is equal to zero, so one will have both $V = 0$ and $V = 0$ for $v = \beta$. One can also enter in that case even for other values of i that between the limits 0 and I, namely, when the quantity V becomes equal to zero for certain other smaller values of v – e.g., for $v = \beta'$. The light ray will then also have a circular asymptote at the altitude β' for the value of the inclination angle i that is determined by the equation:

$$\cos i = \frac{(R+\beta')\,n(\beta')}{R\,n_0}.$$

6. In order to apply the results that were found for the phenomena of atmospheric ray refraction to other heavenly bodies, I will assume that they are spherical and consider the temperature of the atmosphere to be constant at the various altitudes, which are assumptions that suffice entirely for the present purpose, in which one does not arrive at the most precise numerical results that are possible, but only to the qualitative character of the phenomena. It is known that the density of the atmosphere, as a function of the altitude v above the outer surface of the heavenly body whose radius is R, when it is determined from the law of gravity and *Mariotte's* law, has the expression:

$$e^{-\frac{Rv}{\lambda(R+v)}},$$

when the density at the outer surface is set to unity, and λ equals the altitude of an air column of constant density one which exerts the same pressure on the outer surface of the heavenly body as its atmosphere actually does. Thus, from a known physical law:

$$n^2 = 1 + k\, e^{-\frac{Rv}{\lambda(R+v)}},$$

where k is the absolute refracting power of the air at the outer surface of the heavenly body, or $\sqrt{1+k}$ is its absolute refraction exponent. If one now sets:

$$\frac{Rv}{\lambda(R+v)} = w,$$

for the sake of brevity, then one will get the equation:

$$n = \int_0 \frac{R^2\sqrt{1+k}\cos i\, dv}{(R+v)\sqrt{V}}$$

for the curve of the light ray, where:

$$V = (R + v)^2 (1 + k\, e^{-w}) - R^2 (1 + k) \cos^2 i,$$

from which, one will obtain the following values for the first and second differential quotients of V:

$$V' = 2 (R + v) (1 + k\, e^{-w}) - \frac{R^2 k}{\lambda} e^{-w},$$

$$V'' = 2 + k\, e^{-w} \left(1 - \frac{R^2}{\lambda(R+v)}\right)^2 + k\, e^{-w}.$$

Since the second differential quotient V'' is always positive, as the expression itself shows, the first V' will be a function that increases along with v, so V' can be zero only for a single value of v, when it goes from positive to negative, and if this value of v that gives $V' = 0$ should be positive then V' will still be negative for $v = 0$, and conversely, if V' is negative for $v = 0$ then the equation $V' = 0$ will have a positive root, which shall be denoted by b. The condition that V' should be negative for $v = 0$ gives:

$$2R (1 + k) - \frac{R^2 k}{\lambda} < 0,$$

or

$$R > \frac{2\lambda(1+k)}{k}.$$

For those heavenly bodies for which this condition is not fulfilled, but:

$$R < \frac{2\lambda(1+k)}{k},$$

V' will be positive for all positive values of v, so V will constantly increase, and as a result, it will never be equal to zero for any value of the inclination angle i. Therefore, for these heavenly bodies, all rays that emanate from a point of the outer surface will go to infinity, and no other point of the outer surface can be seen from any point of that surface, strictly speaking. This is the case for our Earth, for which, in fact, under the assumption of a constant temperature of 0 degrees:

$$k = 0.000589, \quad \lambda = 7974\ m, \quad R = 6366198\ m,$$

so

$$\frac{2\lambda(1+k)}{k} = 27092000\ m,$$

which is a value that is larger than the radius R of the Earth.

7. The refraction phenomena of the heavenly bodies for which:

$$R > \frac{2\lambda(1+k)}{k}$$

shall now be examined more closely. For them, there will be a single positive value $v = \beta$ for which $V' = 0$, which can be calculated easily from the equations:

$$e^w + \frac{k}{2} w - \frac{Rk}{2\lambda} + k = 0, \qquad v = \frac{R\lambda w}{R - \lambda w}.$$

If one now determines the acute angle I, which is the value of i that satisfies the equation $V = 0$ for $v = \beta$, so:

$$\cos I = \frac{(R+\beta)\sqrt{1 + ke^{-\frac{R\beta}{\lambda(R+\beta)}}}}{R\sqrt{1+k}},$$

then for all values of the inclination angle i that lie between 0 and I, the equation:

$$V = 0$$

will have two real, positive roots, the smaller of which, $v = b$, will be the maximum altitude to which the light ray will rise above the heavenly body, and one will have:

$$\cos i = \frac{(R+b)\sqrt{1+ke^{-\frac{Rb}{\lambda(R+b)}}}}{R\sqrt{1+k}}.$$

The arc length $2a$ from the starting point at which the light ray again meets the heavenly body is:

$$2a = \int_0^b \frac{2R^2\sqrt{1+k}\cos i\,dv}{(R+v)\sqrt{V}}.$$

While i increases continuously from 0 to I, b will increase from 0 to β, and $2a$ will increase from zero to infinity. If one lets i_1, i_2, i_3, … denote the values of i that belong to the values of $2a$ that equal $R\pi$, $2R\pi$, $3R\pi$, …, resp., then these inclination angles will define an increasing sequence whose infinitely-distant term will be equal to I. If one now thinks of an arbitrary point on the outer surface of such a heavenly body as an observer then he must be able to survey the entire outer surface of the heavenly body from that point. It must appear to be a concave shell to him whose boundary – viz., the apparent horizon – is elevated above the true horizon by the angle I. In this shell, the entire outer surface of the heavenly body must be visible from an angle of zero up to i_1, and indeed the closer objects on it must appear to have their natural proportions right up to the antipodes, but the more distant ones, which lie closer to the antipodes, must increasingly flatten and, at the same time, grow narrower. The points that lie diametrically opposite to the observer must appear to be a complete circle that is extended under the inclination angle i_1 by which this first image completes the entire outer surface. In the annular interval between the angles i_1 and i_2, a second complete image of the entire outer surface must be visible, in which the observer must see himself in the boundary that is determined by the angle i_2, and indeed, from behind, and distorted into a complete circle. A third image of the entire outer surface will lie between the angles i_2 and i_3, a fourth between i_3 and i_4, and so forth, to infinity, and this infinite sequence of images, which soon become extraordinarily narrow, will conclude with the apparent horizon for the angle I.

If one now considers the value of i that is greater than I and for which V can therefore no longer equal zero then:

$$\frac{c}{R} = \int_0^\infty \frac{R\sqrt{1+k}\cos i\,dv}{(R+v)\sqrt{V}}$$

will be the angle that the asymptote of the light ray that goes to infinity will define with the vertical line at the point $u = 0$, $v = 0$. When i decreases continually from the value $\pi / 2$ to the value I here, the angle c / R will increase continually from zero to infinity. If one now denotes the values of i that give $c / R = \pi$, 2π, 3π, … by i', i'', i''', …, respectively, then they will define a decreasing sequence of quantities that will have the value I for their limiting value. An observer at an arbitrary point of the outer surface of the heavenly body, which can be chosen to be the coordinate origin $u = 0$, $v = 0$, can view the entire starlit sky down to the nadir from the zenith to the apparent zenith distance $\pi / 2 - i'$. A second complete, but very narrow, image of the entire starlit sky must appear between the angles i' and i'' (and thus, between the zenith distances $\pi / 2 - i'$ and $\pi / 2 - i''$), a third, even narrower, image must appear between the angles i'' and i''', and so forth to infinity. This infinite sequence of images, which become ever narrower quite rapidly, concludes with the angle I and the apparent horizon.

If one does not merely consider those light rays that emanate from a point on the outer surface of the heavenly body, or – what amounts to the same thing – meet such a point, but locates the observer at an arbitrarily distant point from the heavenly body – e.g., on any other heavenly body – then the same remarkable situation will occur. Namely, the entire outer surface – viz., the front and rear halves – of the heavenly body will also be visible from such a standpoint: first, in a disc-shaped principal image, and then an infinite sequence of these discs surrounding annular images. Furthermore, the entire starlit sky in the atmosphere of such a heavenly body will also be visible in infinitely-many images.

8. In order to now examine which heavenly bodies might fulfill the condition:

$$R > \frac{2\lambda(1+k)}{k},$$

upon which, these remarkable refraction phenomena will depend, one must make only *one* special assumption about the strength of the atmosphere when one takes the absolute refracting power of that surrounding air to be equal to that of Earth. One can measure the strength of the Earth atmosphere as being the one that would exist over the Earth at an altitude of 7974 meters when it has constant density equal to unity. The strength of the atmosphere of another heavenly body should now be measured in a similar way by the altitude h at which the surrounding air would have when it has the same density equal to one everywhere; i.e., the density that air has at the outer surface of the Earth.

If one lets R_1, k_1, λ_1 denote the values that these three quantities have for our Earth, under the assumption of a constant temperature of zero degrees, such that:

$$R_1 = 6366198\ m, \qquad \lambda_1 = 7974\ m, \qquad k_1 = 0.000589,$$

and one takes the mass of the Earth to be equal m_1 and the value of gravity at its outer surface to be equal g_1, while R, λ, k, m, g mean the corresponding quantities for another heavenly body, then one will have:

$$\frac{g}{g_1} = \frac{mR_1^2}{m_1R^2}.$$

The pressure of the h meter-high air column of unit density relates to the pressure of the λ_1 meter-high air column on the Earth like hg to $\lambda_1\, g_1$, so the density of the lowest layer of air to the other heavenly body, which shall be denoted by δ, relates to the unit density like hg to $\lambda_1\, g_1$, in any case, and one will have:

$$\delta = \frac{hg}{\lambda_1 g_1}.$$

Since $n^2 - 1 = k_1$ for air of unit density, moreover, $n^2 - 1 = k_1\delta$ for air of density δ, so $k = k_1\delta$, or:

$$k = \frac{hgk_1}{\lambda_1 g_1}.$$

The air column that has height h for unit density will have height h / δ for density δ, so one will have:

$$\lambda = \frac{h}{\delta} = \frac{\lambda_1 g_1}{g}.$$

The condition:

$$R > \frac{2\lambda(1+k)}{k}$$

will then give:

$$\frac{Rhgk_1}{\lambda_1 g_1} > \frac{2\lambda_1 g_1}{g}\left(1 + \frac{hgk_1}{\lambda_1 g_1}\right),$$

or

$$h > \frac{2\lambda_1^2 g_1^2}{Rg^2 k_1 - 2\lambda_1 g_1 hk_1},$$

so when the strength of the atmosphere that is to be measured by h satisfies this condition, the remarkable phenomena that were explained above will occur.

9. I take Jupiter as a special case, whose radius is roughly 10.86 times as big as the Earth radius, and whose mass is about 338 times as big as that of the Earth. One will thus have:

$$\frac{R}{R_1} = 10.86, \qquad \frac{m}{m_1} = 338, \qquad R = 691356000$$

for Jupiter, and as a result:

$$\frac{g}{g_1} = 2.866, \qquad l = 2782, \qquad k = \frac{h \cdot 0.001688}{7974}.$$

One will then find that the quantity condition above for the strength of the atmosphere that Jupiter must have in order for this kind of phenomena to take place on it is:

$$h > 389,$$

and since 389 is somewhat smaller than the twentieth part of 7974, it will then follow that it would already be sufficient if the atmosphere of Jupiter were also only as strong as the twentieth part of the Earth atmosphere.

However, one makes the assumption about the strength of the Jupiter atmosphere that seems to be the most reasonable one – namely, that the mass of the air on Jupiter relates to that of the Earth as the total mass of Jupiter to that of Earth – so one will have:

$$4R^2\pi h : 4R_1^2\pi\lambda_1 = 338 : 1,$$

from which, one will get that h = 22852; that would then make:

$$k = 0.00484, \quad \lambda = 2782.$$

One the finds the value of β = 11394 m from the equation $V' = 0$, and from that, the value of $I = 3^{\circ}\ 48'$. With that assumption, the outer surface of Jupiter will appear to be a concave shell whose boundary, as the apparent horizon, is elevated by $3^{\circ}\ 48'$ above the true horizon.

I further remark that the expression "visible" that was used in the foregoing is to be taken only in its geometric sense, namely, that, in fact, the light rays in the given directions would arrive at the eye under the assumptions that were made. Namely, if one brings under consideration the weakening or the complete disappearance that the light rays suffer in an atmosphere that is not absolutely transparent – e.g., as they do for our Earth – then only a few of them will actually remain in the physiological and physical sense of the given phenomena. Only a certain part of the principal, first image of the Jupiter outer surface that is directed by the higher or lower degree of transparency will actually be indistinguishable from the second, third, and following images. Likewise, of the entire starlit sky, not once will the first image be clearly distinguished completely, and even the Sun, which never rises or sets on Jupiter, due to the refraction of its rays, since its image can never sink below the apparent horizon, will also be the first image if it goes too far below the true horizon, which must then appear to be flattened into a very narrow ellipse that is hardly recognizable to the eye. In the vicinity of the apparent horizon, only a blue strip will appear in place of the infinitely many images of the entire sky and the entire outer surface of Jupiter.

Berlin, in July 1860.