"Über die Strahlensysteme, deren Brennflächen Flächen vierten Grades mit sechzehn singulären Punkten sind," Monats. d. König. Preuss. Akad. Wiss. Berlin (1864), 495-499; *Collected Papers*, v. 2., ed. A. Weil, Springer, Berlin, 1975.

On the ray systems whose focal surfaces are surfaces of degree four with sixteen singular points.

By. E. E. Kummer

Translated by D. H. Delphenich

Hr. Kummer made the following communication on the ray systems whose focal surfaces are of fourth degree and have sixteen singular points.

In a treatise that was read on 18 April of this year on the surfaces of degree four with sixteen singular points, I proved that the totality of all their doubly-contacting tangents always consists of several separate ray systems, namely, four ray systems of order two and class two and a ray system of order four and class four. I further remarked that this latter ray system likewise decomposes into two separate ray systems of order two and class two for the Fresnel wave surface and the ones that are collinear to it. Since then, I have thoroughly carried out the investigation of whether that ray system of order four and class four is still decomposable for the general surfaces of degree four with 16 singular points and found that they always consist of two separate ray systems of order two and class two, such that each such surface will then be the focal surface of six special ray systems of order two and order class.

In order to prove this, I would like to give the complete development of these six ray systems for the surface of degree four with 16 singular points that is given in the form:

$$(ayz + bzx + c\,(1 + 2k)\,xy + dx + ey + fz)^2 - 4k\,(k + 1)\,xyp'q' = 0,$$

$$\text{(A)} \qquad \begin{aligned} p' &= cy + \frac{bz}{k+1} + \frac{d}{k}, \\ q' &= cx + \frac{az}{k} + \frac{e}{k+1} \end{aligned}$$

(see equation (4), page 251, of the Monatsberichte of this year). Let x, y, z be the coordinates of an arbitrary point in space, and let ξ, η, ζ be proportional to the cosines of the angles that the rays that go through x, y, z make with the coordinate axes, so any ray system of order two and class two will be given by two equations in the quantities x, y, z, ξ, η, ζ, one of which will be homogeneous and of degree two in ξ, η, ζ, and the other of which will be homogeneous and of degree one, and which must arranged in such a way that they will remain unchanged when one sets $x + \rho\xi$, $y + \rho\eta$, $z + \rho\zeta$, in place of x, y, z.

When represented in this way, one will have the following six ray systems that will have the surface above for their focal surfaces:

$$\text{I}\quad \left\{ \begin{aligned} & d\xi + az\eta - ay\zeta = 0, \\ & \left(acy + abz + \frac{ad}{k} + \frac{bek}{k+1} - cfk\right)\xi^2 + a^2 x\eta\zeta - (abx + af)\zeta\xi - (a^2 z + acx + ae)\xi\eta = 0. \end{aligned} \right.$$

$$\text{II}\quad \left\{ \begin{aligned} & bz\xi + e\eta - bx\zeta = 0, \\ & \left[abz - bcx + \frac{ad(k+1)}{k} - \frac{be}{k+1} - cf(k+1)\right]\eta^2 + b^2 y\zeta\xi \\ & \qquad - (aby + bf)\eta\zeta + (bcy - b^2 z - bd)\xi\eta = 0. \end{aligned} \right.$$

$$\text{III}\quad \left\{ \begin{aligned} & cy\xi - cx\eta + f\zeta = 0, \\ & \left(acy - bcx + \frac{ad}{k} - \frac{be}{k+1} - cfk\right)\zeta^2 + c^2 z\xi\eta + (bcz + cd)\xi\zeta - (acz + c^2 x + ce)\eta\zeta = 0. \end{aligned} \right.$$

$$\begin{matrix}\text{IV,} \\ \text{V,} \\ \text{VI}\end{matrix}\quad \left\{ \begin{aligned} & \left(cy + \frac{bz}{k+1} + \frac{d}{k}\right)\xi - \left(\frac{az}{k} + cx + \frac{e}{k+1}\right)\eta - \left(\frac{bx}{k+1} - \frac{ay}{k} + f\right) = 0, \\ & kx\eta\zeta - (k+1)y\zeta\xi + z\xi\eta = 0. \end{aligned} \right.$$

This last ray system represents three different ones, since k can have three different values that are given as roots of the cubic equation:

$$cfk^3 + \left(g - \frac{ad}{2} - \frac{be}{2} - \frac{3cf}{2}\right)k^2 + \left(g - \frac{3ad}{2} + \frac{be}{2} + \frac{cf}{2}\right)k - ad = 0.$$

k enters into the three ray systems I, II, and III only in such a way that they remain the same for all three values of k.

These six ray rays will be represented symmetrically for the form of the focal surface:

$$\text{(B)}\qquad \phi^2 = 16\,Kxyz,$$

where

$$\phi = x^2 + y^2 + z^2 + 1 + 2a(yz + x) + 2b(zx + y) + 2c(xy + z)$$

$$K = a^2 + b^2 + c^2 - 2abc - 1$$

(cf., equation (10) in the cited paper), in such a way that it will suffice to write down a single one of them, namely:

$$\mathfrak{A}\xi + \mathfrak{B}\eta + \mathfrak{C}\zeta = 0,$$

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2D\eta\zeta + 2E\zeta\xi + 2F\xi\eta = 0,$$

where

$$\begin{aligned}
A &= -\,2y, \\
B &= 2\,(a - \sqrt{a^2-1}\,)\,z, \\
C &= -\,2y, \\
D &= 2[b - c\,(a - \sqrt{a^2-1}\,)]\,x - (a - \sqrt{a^2-1}\,)\,y + z, \\
E &= -\,2by, \\
F &= x + 2c\,(a - \sqrt{a^2-1}\,)\,z + a - \sqrt{a^2-1}\,, \\
\mathfrak{A} &= -\,(a + \sqrt{a^2-1}\,)\,y - z, \\
\mathfrak{B} &= (a + \sqrt{a^2-1}\,)\,x + 1, \\
\mathfrak{C} &= x + a + \sqrt{a^2-1}\,.
\end{aligned}$$

One will get the remaining five ray systems from this one when one simultaneously permutes the x, y, z; ξ, η, ζ; a, b, c; A, B, C; D, E, F; $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and gives the three square roots $\sqrt{a^2-1}$, $\sqrt{b^2-1}$, $\sqrt{c^2-1}$ both their values.

The focal surface of this ray system can be represented by the following determinant:

$$\begin{vmatrix} A & F & E & \mathfrak{A} \\ F & B & D & \mathfrak{B} \\ E & D & C & \mathfrak{C} \\ \mathfrak{A} & \mathfrak{B} & \mathfrak{C} & 0 \end{vmatrix} = 0,$$

which will agree with the given surface completely when it is properly developed.

Since the complete ray system of all doubly-contacting lines of a fourth-degree surface with 16 singular points consists of six second-degree ray systems, and any two rays can coalesce into a single one when the point from which they emanate lies on the focal surface itself, it will follow that one can associate each point of the surface with six other ones in such a way that their coordinates are determined rationally from those of the given point and that each of the six associated points will have one and the same tangent with the given one. If one takes x, y, z to be the given point of the surface and lets x', y', z' denote the coordinates of the corresponding point of the reciprocal polar surface (which is taken relative to the sphere $x^2 + y^2 + z^2 = 1$), and one further denotes the coordinates of the point that corresponds to x, y, z relative to the first ray system by x_1, y_1, z_1 then one will have:

$$x_1 = \frac{(a+\sqrt{a^2-1})y' - z'}{(a+\sqrt{a^2-1})z' - y'},$$

$$y_1 = \frac{-(a+\sqrt{a^2-1})\,x'-1}{(a+\sqrt{a^2-1})z'-y'},$$

$$z_1 = \frac{x'+a+\sqrt{a^2-1}}{(a+\sqrt{a^2-1})z'-y'}.$$

One will likewise get the remaining five from this by a suitable choice of symbols and signs for the square roots. One recognizes from this that the reciprocal polar figure is collinear to the original one, and that there are six different collinear conversions of the reciprocal polar surface, which are arranged such that the point of the reciprocal polar surface that belongs to a point of the given surface goes to one of the six corresponding points of the given surface.

The equation of the reciprocal polar surface for the surface:

$$\phi^2 = 16Kxyz$$

can be represented in the following form:

$$\Phi^2 = 16\,Kpqrs,$$

where

$$\Phi = p^2 + q^2 + r^2 + s^2 + 2a\,(qr + ps) + 2b\,(rp + qs) + 2c\,(pq + rs),$$

$$\begin{aligned}
p &= (a + \sqrt{a^2-1}\,)\,x' + 1,\\
q &= z' - (a + \sqrt{a^2-1}\,)\,y',\\
r &= y' - (a + \sqrt{a^2-1}\,)\,z',\\
s &= -x' - a - (a + \sqrt{a^2-1}\,),
\end{aligned}$$

from which one will obtain five other analogous expressions by permuting the symbols and the signs of the square roots.
