THE RELATIONSHIPS BETWEEN MAXWELL'S ETHER EQUATIONS AND THE THEORY OF FUNCTIONS

A CONTRIBUTION TO THE THEORIES OF RELATIVITY AND ELECTRONS

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CHAPTER 1

Properties of four-dimensional space under special transformations. Relationship to quaternions.

MINKOWSKI's vector analysis is based upon the linear-geometric structures of EUCLIDIAN space. The number of dimensions plays no privileged role in it at all, since the space-time continuum, with its four dimensions, defines a special case of the EUCLIDIAN space with a general dimension of n. However, it possesses precisely the properties of four-dimensional space in regard to the orthogonal transformations (i.e., rotations), which distinguishes it from all other spaces. Those properties, which are closely linked with HAMILTON's quaternions, allow, on the one hand, a fundamental common equation and unified picture of quaternion calculus for all of four-dimensional vector analysis, and on the other hand, make it possible to extend the latter by adapting the limits of field theory to the electromagnetic field in a natural way in the MINKOWSKIan context.

The scalar product of two vectors, like the vectorial one, can be introduced by the requirement that one should build a system of quadratic structures from the components of two vectors with the property that the new system is connected with the old one in a homogeneous manner under a rotation of the axis-cross. The scalar product defines one such system with its invariance, while the vectorial product with $\binom{n}{2}$ terms defines another. The possibilities are then exhausted, in general (except for the most general, but trivial, case in which all possible products whatsoever of two arbitrary components taken together will likewise yield a system that one desires). However, it is precisely for dimension $n = 4$ that yet another system – and in fact, a three-parameter one – can be constructed.

We write down the vectorial product of two four-vectors (X_1, Y_1, Z_1, T_1) and (X_2, Y_2, Z_2, T_2), with the usual notations, as well as the "dual" vector; both of them are six-vectors:

$$\left.\begin{aligned} \mathcal{F}_{yz} &= Y_1Z_2 - Y_2Z_1, \\ \mathcal{F}_{zx} &= Z_1X_2 - Z_2X_1, \\ \mathcal{F}_{xy} &= X_1Y_2 - X_2Y_1, \\ \mathcal{F}_{xt} &= X_1T_2 - X_2T_1, \\ \mathcal{F}_{yt} &= Y_1T_2 - Y_2T_1, \\ \mathcal{F}_{zt} &= Z_1T_2 - Z_2T_1, \end{aligned}\right\} \qquad \left.\begin{aligned} \mathcal{F}^{\#}_{yz} &= \mathcal{F}_{xt} = X_1T_2 - X_2T_1, \\ \mathcal{F}^{\#}_{zx} &= \mathcal{F}_{yt} = Y_1T_2 - Y_2T_1, \\ \mathcal{F}^{\#}_{xy} &= \mathcal{F}_{zt} = Z_1T_2 - Z_2T_1, \\ \mathcal{F}^{\#}_{xt} &= \mathcal{F}_{yz} = Y_1Z_2 - Y_2Z_1, \\ \mathcal{F}^{\#}_{yt} &= \mathcal{F}_{zx} = Z_1X_2 - Z_2X_1, \\ \mathcal{F}^{\#}_{zt} &= \mathcal{F}_{xy} = X_1Y_2 - X_2Y_1. \end{aligned}\right\} \tag{1.1}$$

Since the dual vector is covariant, along with the original one, the same thing will be true of the sum or difference of the two. Therefore, only three distinct quantities arise in both cases, namely:

$$\left.\begin{aligned} &\mathcal{F}_{xt} \pm \mathcal{F}_{zy}, \\ &\mathcal{F}_{yt} \pm \mathcal{F}_{zx}, \\ &\mathcal{F}_{zt} \pm \mathcal{F}_{xy}. \end{aligned}\right\} \tag{1.2}$$

(The two signs are understood to mean either/or.) This system of terms then likewise has the property of being connected homogeneously and linearly with the corresponding one in the transformed system. If we now imagine that the spatial part of a four-vector is either real or pure imaginary, and that the temporal part is correspondingly imaginary (real, resp.) then we will see that the last three quantities that were written down define complex numbers. However, a complex number is characterized by its real, as well as its imaginary, parts, such that when one allows complex numbers, the original six-vector can be replaced with those three quantities. If we add a fourth one in the form of the scalar product of the two vectors then we will obtain the following system, which includes both kinds of multiplication:

$$\left.\begin{aligned} &X_1T_2 \quad -X_2T_1 \quad \pm(Y_1Z_2 \quad -Y_2Z_1), \\ &Y_1T_2 \quad -Y_2T_1 \quad \pm(Z_1X_2 \quad -Z_2X_1) \\ &Z_1T_2 \quad -Z_2T_1 \quad \pm(X_1Y_2 \quad -X_2Y_1), \\ &X_1X_2 \quad -Y_2Y_1 \quad +Z_1Z_2 \quad +T_1T_2\,. \end{aligned}\right\} \tag{1.3}$$

If we now consider the vector (X_1, Y_1, Z_1, T_1) to be a quaternion – whose so-called "scalar" part is represented by the temporal part of the vector – and multiply it by the quaternion ($-X_2$, $-Y_2$, $-Z_2$, T_2) then we will get a quaternion whose components can also be represented as a sequence in just the manner that was written down for their product, and indeed when we choose the lower (i.e., negative) sign. By contrast, one chooses the positive sign when the sequence of those two quaternion factors is the opposite one. However, that product can no longer be referred to as simply a vector, since it is not covariant under an orthogonal transformation of its vector components. On the other hand, the old and new components of the product depend upon each other in a homogeneous, linear way (and that is certainly what is crucial in regard to the theory of relativity); except that the matrix of the transformation is different from the original matrix. We thus arrive at an extension of the original vector concept that admits a unified combination of four-vectors, six-vectors, and scalars. We assume that the number of components is four throughout and that these components should transform in a homogeneous, linear manner under arbitrary, orthogonal transformations, in which the coefficients can be different from those of the coordinate transformations. For the sake of brevity, let me refer to such a totality of four quantities as a *versor*, while the word *vector*, in the older sense of four-vector, shall still be used in the case of covariance. We are actually dealing with an extension of the purely-geometric concept of "line segment." The versor can also be defined by a line segment that does not, however, generally keep its direction under a rotation of the axis-cross, but will also experience a well-defined rotation. In addition, the components can also be complex.

We define quaternion multiplication to be the basic operation, instead of scalar and vectorial multiplication by themselves. We saw that in order to obtain a versor under that multiplication, the spatial part of a vector must be taken to have a negative sign. That

shall be called the *conjugate* of the vector (or quaternion), and denoted with an overbar, in analogy to complex numbers. Thus:

$$\mathcal{F} = (X, Y, Z, T), \qquad \bar{\mathcal{F}} = (-X, -Y, -Z, T). \tag{1.4}$$

Finally, the unit vectors along the individual axes shall be denoted by the symbols 1_x, 1_y, 1_z, and 1_t, so a vector will be represented in terms of them as follows:

$$\mathcal{F} = X\,1_x + Y\,1_y + Z\,1_z + T\,1_t\,. \tag{1.5}$$

The rules of calculation for the unit that falls along the time axis are identical with the ones for the ordinary unit, such that one can also set:

$$1_t = 1. \tag{1.6}$$

Moreover, as is known, the distributive, as well as the associative, law for multiplication is valid, while the rule for commutation finds its expression as follows ([1]):

$$\mathcal{F}\mathcal{G} = \overline{\bar{\mathcal{G}}\,\bar{\mathcal{F}}}\,. \tag{1.7}$$

The product $\mathcal{F}\bar{\mathcal{F}}$ – which is a pure temporal versor, and can also be regarded as simply a number – represents the square of the length of the vector. One can also derive division from it directly. The quotient of two vectors:

$$\mathcal{X} = \frac{\mathcal{F}}{\mathcal{G}}, \tag{1.8}$$

shall be determined from the equation:

$$\mathcal{X}\mathcal{G} = \mathcal{F}. \tag{1.9}$$

One will then have:

$$\mathcal{X}\mathcal{G}\bar{\mathcal{G}} = \mathcal{F}\bar{\mathcal{G}}\,, \tag{1.10}$$

so:

$$\mathcal{X} = \frac{\mathcal{F}\,\bar{\mathcal{G}}}{\mathcal{G}\,\bar{\mathcal{G}}}\,. \tag{1.11}$$

The division is then converted into a multiplication and a pure scalar division.

([1]) At this point, let us mention the remarkable fact that all rules of multiplication will also be true for quadratic matrices, in particular, orthogonal ones, where the conjugate of a matrix is understood to mean the matrix that arises when one switches the horizontal rows with the vertical ones.

CHAPTER 2

Characterizing a four-dimensional rotation by two quaternions.

A remarkable connection exists between quaternions and general, orthogonal transformations in four-dimensional space that makes it possible to determine an arbitrary rotation of the axis-cross in a simple and natural way. Six independent quantities are required for that determination, since the sixteen coefficients of the transformation matrix must satisfy the ten orthogonality conditions.

If we take a quaternion of length one:

$$p = (p_1, p_2, p_3, p_4), \tag{2.1}$$

and if we multiply it by the vector:

$$\mathcal{F} = (X, Y, Z, T). \tag{2.2}$$

The components of the product are:

$$\left.\begin{aligned} X' &= +p_4X - p_3Y + p_2Z + p_1T, \\ Y' &= +p_3X + p_4Y - p_1Z + p_2T, \\ Z' &= -p_2X + p_1Y + p_4Z + p_3T, \\ T' &= -p_1X - p_2Y - p_3Z + p_4T. \end{aligned}\right\} \tag{2.3}$$

If we regard that to be a transformation of the vector $\mathcal{F}$ into $\mathcal{F}'$ then we will see that we are dealing with a rotation, so the matrix of the transformation will be:

$$\mathcal{P} = \begin{vmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{vmatrix}; \tag{2.4}$$

it belongs to the so-called *anti-symmetric* matrices. The same thing will be true when the quaternion defines the second factor. The matrix, for which the symbol $\mathcal{Q}$ shall be used, shall then be:

$$\mathcal{Q} = \begin{vmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{vmatrix}. \tag{2.5}$$

For the sake of brevity, we would like to refer to these two types of transformations as $\mathcal{P}$-transformations ($\mathcal{Q}$-transformations, resp.) and the two matrices as $\mathcal{P}$-matrices ($\mathcal{Q}$-matrices, resp.). The $\mathcal{P}$-transformations and $\mathcal{Q}$-transformations by themselves each define subgroups of the general group of orthogonal transformations; that is, two successively-performed $\mathcal{P}$-transformations will again lead to a $\mathcal{P}$-transformation, and a corresponding statement will be true for the $\mathcal{Q}$-transformations; that follows from the associative law of multiplication. Namely, let:

$$\left.\begin{aligned} \mathcal{F} &= p_1 \mathcal{F}', \\ \mathcal{F}' &= p_2 \mathcal{F}'', \end{aligned}\right\} \tag{2.6}$$

so:

$$\mathcal{F} = (p_1\, p_2)\, \mathcal{F}''. \tag{2.7}$$

On the other hand, let:

$$\left.\begin{aligned} \mathcal{F} &= \mathcal{F}' q_1, \\ \mathcal{F}' &= \mathcal{F}'' q_2, \end{aligned}\right\} \tag{2.8}$$

so:

$$\mathcal{F} = \mathcal{F}''\, (q_2\, q_1). \tag{2.9}$$

A quaternion belongs to any $\mathcal{P}$-matrix or $\mathcal{Q}$-matrix. If we write it as an index then, from the equations that were just written down, we will have the rules:

$$\left.\begin{aligned} \mathcal{P}_{p_1} \mathcal{P}_{p_2} &= \mathcal{P}_{p_1 p_2}, \\ \mathcal{Q}_{p_1} \mathcal{Q}_{p_2} &= \mathcal{Q}_{p_1 p_2} \end{aligned}\right\} \tag{2.10}$$

for the product of two $\mathcal{P}$-matrices ($\mathcal{Q}$-matrices, resp.).

If we then perform a $\mathcal{Q}$-transformation after a $\mathcal{P}$-transformation then we will again obtain an orthogonal transformation, and indeed – as one can show – the most general one. The composition of the $\mathcal{P}$-group and the $\mathcal{Q}$-group then yields the group of general orthogonal transformations. Since the direction of the two quaternions p and q can be chosen arbitrarily, while its length must be unity, we will, in fact, require six terms. The sequence of the two transformations is, moreover, irrelevant, so the commutative law:

$$\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} \tag{2.11}$$

will be true here. The general, orthogonal transformation of the vector $\mathcal{F}$ into $\mathcal{F}'$ can thus be once more given by the simple equation:

$$\mathcal{F}' = p\,\mathcal{F}\,q\,. \tag{2.12}$$

Any orthogonal matrix can be represented as the product of a $\mathcal{P}$-matrix and a $\mathcal{Q}$-matrix, and indeed, the associated quaternions p and q will be determined uniquely (up to multiplication by – 1). One can refer to them as the *characteristics* of the transformation.

The resulting matrix will be represented very simply by the characteristics. We write the components of the product:

$$p\,1_x\,q\,, \qquad p\,1_y\,q\,, \qquad p\,1_z\,q\,, \qquad p\,q \tag{2.13}$$

underneath each other in each column in the existing sequence, so those sixteen components will yield the sixteen coefficients of the orthogonal matrix. For example, let the general, orthogonal matrix be denoted as follows:

	x	y	z	t
x'	α_{11}	α_{12}	α_{13}	α_{14}
y'	α_{21}	α_{22}	α_{23}	α_{24}
z'	α_{31}	α_{32}	α_{33}	α_{34}
t'	α_{41}	α_{42}	α_{43}	α_{44}

(2.14)

One will then have:

$$\alpha_{11}\,1_x + \alpha_{21}\,1_y + \alpha_{31}\,1_x + \alpha_{41} = p\,1_x\,q, \tag{2.15}$$

and so on. One can also proceed in such a way that one writes the corresponding components of the products:

$$\bar{p}\,1_x\,\bar{q}\,, \qquad \bar{p}\,1_y\,\bar{q}\,, \qquad \bar{p}\,1_z\,\bar{q}\,, \qquad \bar{p}\,\bar{q} \tag{2.16}$$

in a *row.*

Conversely, if the problem is to find the characteristics of a given matrix then we will proceed symmetrically as follows: Let the columns be considered to be the sequence of quaternions $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$. One will then have:

$$\left.\begin{aligned} p &= -\frac{1}{4\lambda}(\mathcal{U}_1 1_x + \mathcal{U}_2 1_y + \mathcal{U}_3 1_z - \mathcal{U}_4), \\ q &= -\frac{1}{4\mu}(1_x \mathcal{U}_1 + 1_y \mathcal{U}_2 + 1_z \mathcal{U}_3 - \mathcal{U}_4), \end{aligned}\right\} \tag{2.17}$$

in which λ and μ mean simply numbers. They will be determined (up to the factor ± 1) by the requirement that the length of p and q must be equal to unity, and that one must have:

$$\lambda\mu = (\alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_{44}), \tag{2.18}$$

in addition.

The fourth column of a purely-spatial transformation should be considered to be a quaternion:

$$\mathcal{U}_4 = 1. \tag{2.19}$$

One then has $pq = 1$, or:

$$q = \overline{p}\,. \tag{2.20}$$

The subgroup of purely-spatial transformations can then be defined by the fact that the two characteristics are conjugate to each other.

In the previous chapter, we introduced versors and saw that the product $\mathcal{F}\overline{\mathcal{G}}$ of the vectors $\mathcal{F}$ and $\mathcal{G}$ represents such a versor. We would now like to determine the matrix of its transformation, in addition. Let the orthogonal transformation be given by:

$$\left.\begin{aligned} \mathcal{F}' &= p\,\mathcal{F}\,q, \\ \mathcal{G}' &= p\,\mathcal{G}\,q. \end{aligned}\right\} \tag{2.21}$$

From the rules of multiplication, one will have:

$$\overline{\mathcal{G}}' = \overline{q}\,\overline{\mathcal{G}}\,\overline{p}\,, \tag{2.22}$$

and thus:

$$\mathcal{F}'\overline{\mathcal{G}}' = p\,\mathcal{F}\,\overline{\mathcal{G}}\,\overline{p}\,. \tag{2.23}$$

The matrix of the transformation will be represented by the product:

$$\mathcal{P}_p\,\mathcal{Q}_{\overline{p}}\,, \tag{2.24}$$

but also by:

$$\mathcal{P}_p\,\mathcal{Q}_q\,\mathcal{Q}_{\overline{q}}\,\mathcal{Q}_{\overline{p}} = (\mathcal{P}_p\,\mathcal{Q}_q)(\mathcal{Q}_{\overline{p}\,\overline{q}})\,. \tag{2.25}$$

When written out thoroughly, and taking into account the fact that $\overline{p}\,\overline{q}$ represents the fourth row of the matrix, one will have:

$$\left|\begin{array}{cccc||cccc} \alpha_{11} & \cdots & \cdots & \alpha_{14} & \alpha_{44} & \alpha_{43} & -\alpha_{42} & \alpha_{41} \\ \cdots & \cdots & \cdots & \cdots & -\alpha_{43} & \alpha_{44} & \alpha_{41} & \alpha_{42} \\ \cdots & \cdots & \cdots & \cdots & \alpha_{42} & -\alpha_{41} & \alpha_{44} & \alpha_{43} \\ \alpha_{41} & \cdots & \cdots & \alpha_{44} & -\alpha_{41} & -\alpha_{42} & -\alpha_{43} & \alpha_{44} \end{array}\right|. \tag{2.26}$$

The first matrix is the vector matrix itself. The transformation of the product $\mathcal{F}\overline{\mathcal{G}}$ then produces two orthogonal transformations as a result. The first of them is the transformation of the factors $\mathcal{F}$ and $\mathcal{G}$, and the second one is a well-defined $\mathcal{Q}$-transformation. Thus, if the versor $\mathcal{F}\overline{\mathcal{G}}$ is pictured as a line segment then that line

segment will experience a rotation that is given by the second matrix under a rotation of the coordinate system. This leaves out only the case of a purely-spatial transformation (for which, the second matrix will be equal to unity), and then the versor will go to an ordinary vector; things behave similarly for the product $\bar{\mathcal{F}}\mathcal{G}$.

In physical applications, the four-dimensional orthogonal transformation comes under consideration in the form of the LORENTZ transformation, for which the spatial coordinates are coordinates, while the time coordinate is imaginary. The coefficients of the transformation are accordingly part real and part pure imaginary. However, the two characteristics p and q will then be complex quantities. Here, we need a notation, namely, in order to be able to express the conjugate complex value of the complex quaternion:

$$p = p' + ip''. \tag{2.27}$$

Since the symbol $\overline{()}$ has already been assigned a different meaning, here, the similar symbol $()^*$ shall serve our purpose:

$$p^* = p' - ip''. \tag{2.28}$$

Now let the LORENTZ transformation be expressed by the equation:

$$\mathcal{F}' = p\,\mathcal{F}\,q. \tag{2.29}$$

If we set $-i$ in place of i everywhere then we will have:

$$\mathcal{F}'^* = p^*\,\mathcal{F}\,q^*. \tag{2.30}$$

If the vector has the property that its spatial part is real, while its temporal part is imaginary, then it will also maintain that property under the transformation (in the case of the LORENTZ transformation). However, it follows from this that the same relation must exist between $\mathcal{F}'^*$ and $\mathcal{F}^*$ that exists between $\bar{\mathcal{F}}'$ and $\bar{\mathcal{F}}$. Meanwhile, one has:

$$\bar{\mathcal{F}}' = \bar{q}\,\bar{\mathcal{F}}\,\bar{p}, \tag{2.31}$$

and a comparison of this with the previous equation will imply that:

$$\bar{q} = p^* \tag{2.32}$$

and

$$\bar{p} = q^*. \tag{2.33}$$

These two formulas are identical to each other.

We then see that in the case of the LORENTZ transformation, the conjugate of the one characteristic is equal to the complex conjugate value of the other real one, so the characteristics are invariant only under purely-spatial rotations, so one will have $\bar{q} = p$, and therefore also:

$$p^* = p. \tag{2.34}$$

Remark: At the conclusion of this chapter, which contains the formal basis for the following developments, along with the previous one, I would like to briefly mention that quaternion multiplication also seems to define a satisfactory basis for tensors, when one regards then as a quadratic matrix in which the transformation will take a vector to a vector, in turn. Namely, one arrives at a tensor matrix from the two vectors $\mathcal{F}$ and $\mathcal{G}$ when one writes the components of the four products:

$$-\mathcal{F}\,1_x\,\mathcal{G}, \qquad -\mathcal{F}\,1_y\,\mathcal{G}, \qquad -\mathcal{F}\,1_z\,\mathcal{G}, \qquad -\mathcal{F}\,\mathcal{G}, \tag{2.35}$$

underneath each other in a column in a manner that is similar to what we did for the construction of the orthogonal matrix from characteristics.

CHAPTER 3

Quaternion functions.

The so-called HAMILTONian operator:

$$\nabla = \frac{\partial}{\partial x}1_x + \frac{\partial}{\partial y}1_y + \frac{\partial}{\partial z}1_z + \frac{\partial}{\partial t} \tag{3.1}$$

is equivalent to a vector, as far as its transformation is concerned. If we then perform the multiplication:

$$\bar{\mathcal{F}}\nabla = \frac{\partial\bar{\mathcal{F}}}{\partial x}1_x + \frac{\partial\bar{\mathcal{F}}}{\partial y}1_y + \frac{\partial\bar{\mathcal{F}}}{\partial z}1_z + \frac{\partial\bar{\mathcal{F}}}{\partial t}, \tag{3.2}$$

in which $\mathcal{F}$ shall mean a vector, then we will obtain a versor. If the components of $\mathcal{F}$ are denoted by *X*, *Y*, *Z*, *T* then the components of the versor will be:

$$\left.\begin{aligned} &\frac{\partial T}{\partial x} + \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} - \frac{\partial X}{\partial t}, \\ &\frac{\partial T}{\partial y} + \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} - \frac{\partial Y}{\partial t}, \\ &\frac{\partial T}{\partial z} + \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} - \frac{\partial Z}{\partial t}, \\ &\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} + \frac{\partial T}{\partial t}. \end{aligned}\right\} \tag{3.3}$$

If we set these expressions equal to zero then the system of equations thus-obtained will still preserve the properties of versors, and as a result, for any arbitrary rotation of the coordinate system, as well. These partial differential equations define certain functions of the four variables *x*, *y*, *z*, *t* – which we will assume to all be real, here – so we would like to give them the name of *quaternion functions*, and indeed consider the totality of the four values *X*, *Y*, *Z*, *T* to be *one* function. Namely, the same symbolic relationship exists between it and the quaternions that exists between complex functions and complex numbers. Symbolically, the CAUCHY-RIEMANN differential equations are included in the equation:

$$(u + iv)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = 0. \tag{3.4}$$

However, the analogy goes far beyond the formal level. In many regards, the equations:

$$\bar{\mathcal{F}}\nabla = 0, \tag{3.5}$$

or also:

$$\bar{\nabla}\mathcal{F} = 0, \tag{3.6}$$

can be regarded as the representatives of the basis for a theory of functions in four-dimensional space. General, many classical properties of complex functions are lost, but many of the fundamental ones remain preserved, or can be carried over in a corresponding way. Above all, one has the connection with the potential (which is four-dimensional here). Namely:

$$\nabla(\bar{\nabla}\mathcal{F}) = (\nabla\bar{\nabla})\mathcal{F} = \frac{\partial^2\mathcal{F}}{\partial x^2} + \frac{\partial^2\mathcal{F}}{\partial y^2} + \frac{\partial^2\mathcal{F}}{\partial z^2} + \frac{\partial^2\mathcal{F}}{\partial t^2} = 0; \tag{3.7}$$

that is, all four components of a quaternion function are potentials (which satisfy the LAPLACE equation). Conversely, any quaternion function can also be reduced to four potential functions. Namely, let Φ be expressed in the following way:

$$\Phi = \nabla\bar{\Phi}\,. \tag{3.8}$$

Φ will then be a quaternion function when:

$$\bar{\nabla}\nabla\bar{\Phi} = 0, \tag{3.9}$$

so the components of Φ will be potentials, and therefore the solution of the fundamental equations will be reduced to the solution of the LAPLACE equation (*vector potential*). In the following, the expression "the potential" shall mean the four-function Φ, from which the quaternion function can be derived in the manner that was just mentioned.

The fundamental classical integral theorem of CAUCHY, which allows the determination of complex functions from the boundary values, finds its complete analogue. Its proof is entirely similar to the one in function theory. We apply GAUSS's integral theorem to the equation:

$$\bar{\mathcal{F}}\nabla = \frac{\partial\bar{\mathcal{F}}}{\partial x}1_x + \frac{\partial\bar{\mathcal{F}}}{\partial y}1_Y + \frac{\partial\bar{\mathcal{F}}}{\partial z}1_z + \frac{\partial\bar{\mathcal{F}}}{\partial t} = 0, \tag{3.10}$$

and write it in the form:

$$\int \bar{\mathcal{F}}\, n\, df = 0, \tag{3.11}$$

in which *n* means the surface normal that points outward, which is taken to be a vector of unit length, and *df* is the surface element, and the integration shall be extended over an arbitrary closed surface (in four-dimensional space), which still envelops nothing but regular points. Furthermore, let:

$$\nabla\bar{\mathcal{G}} = 0 \tag{3.12}$$

for another function $\mathcal{G}$, so:

$$1_x \frac{\partial \bar{\mathcal{G}}}{\partial x} + 1_y \frac{\partial \bar{\mathcal{G}}}{\partial y} + 1_z \frac{\partial \bar{\mathcal{G}}}{\partial z} + \frac{\partial \bar{\mathcal{G}}}{\partial t} = 0. \tag{3.13}$$

It results from these two equations for $\mathcal{F}$ and $\mathcal{G}$ that:

$$\frac{\partial}{\partial x}(\bar{\mathcal{F}} 1_x \bar{\mathcal{G}}) + \frac{\partial}{\partial y}(\bar{\mathcal{F}} 1_y \bar{\mathcal{G}}) + \frac{\partial}{\partial z}(\bar{\mathcal{F}} 1_z \bar{\mathcal{G}}) + \frac{\partial}{\partial t}(\bar{\mathcal{F}}\,\bar{\mathcal{G}}) = 0, \tag{3.14}$$

or, with an application of GAUSS's theorem:

$$\int \bar{\mathcal{F}}\, n \bar{\mathcal{G}}\, df = 0. \tag{3.15}$$

In this, the interior of the surface must be regular with respect to $\mathcal{F}$, as well as $\mathcal{G}$. We will now choose $\mathcal{G}$ to be the function:

$$\mathcal{G} = \nabla \frac{1}{R^2}, \tag{3.16}$$

in which:

$$R^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 + (t - \vartheta)^2. \tag{3.17}$$

It has a single singular point, namely, the point:

$$x = \xi, \quad y = \eta, \quad z = \zeta, \quad t = \vartheta. \tag{3.18}$$

We remove it from a ball that envelops the integration domain, and ultimately arrive at the equation:

$$\bar{\mathcal{F}}(\xi, \eta, \zeta, \vartheta) = -\frac{1}{4\pi^2} \int \bar{\mathcal{F}}(x, y, z, t)\, n \bar{\nabla} \frac{1}{R^2} df \tag{3.19}$$

by an argument that is similar to the one for complex functions, in which the integration on the right is taken over surface that surrounds the point (ξ, η, ζ, ϑ) and is everywhere regular in its interior. This equation determines the values of the function in the interior of a regular spatial domain from its boundary values.

The analogy to CAUCHY's theorem breaks down especially when the latter is written in a corresponding form. In place of the present form:

$$f(\xi + i\,\eta) = \frac{1}{2\pi i} \int \frac{f(x + iy)}{(x - \xi) + i(y - \eta)}\, d(x + i\,y) \tag{3.20}$$

one can then also set:

$$f(\xi + i\,\eta) = \frac{1}{4\pi} \int f(x + iy)\, n \bar{\nabla} (\log R^2)\, ds, \tag{3.21}$$

in which *ds* means the line element, while *n* and *R* are the corresponding quantities that are also in the formula for $\mathcal{F}$, and:

$$\bar{\nabla} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}. \tag{3.22}$$

In four-dimensions, the logarithmic potential $\log R^2$ corresponds to the potential $1 / R^2$.

Results for quaternion functions that relate to series developments in increasing and decreasing powers of the distance ("Laurent series") can be derived from the integral theory that are similar to the ones that are true for complex functions, except that the commutative law is not fulfilled for multiplication here. Since that discussion has a purely mathematical interest for the most part, it shall be passed over here.

CHAPTER 4

Maxwell's equations.

If the components of the function $\mathcal{F}$ are complex functions and the fourth variable is imaginary:

$$t = i\tau, \tag{4.1}$$

in which t means time (in a system of units for which the speed of light $c = 1$), then the defining equations of the quaternion functions will go to MAXWELL's equations, and particularly in the special case for which we set $T = 0$. There then exists an extremely intrinsic connection between the fundamental equations of electromagnetism and the quaternion functions that were introduced in the previous chapter. The electric and magnetic field strengths do not define a six-vector in that picture, but a complex versor of the form:

$$\mathcal{F} = \mathcal{H} + i\mathcal{E}, \tag{4.2}$$

if $\mathcal{E}$ means the electric field strength, and $\mathcal{H}$ means the magnetic one. When written in the present three-dimensional symbolism, the following relations will, in fact, exist between the two field strengths:

$$\left.\begin{aligned} \frac{\partial \mathcal{H}}{\partial t} + \operatorname{curl} \mathcal{E} = 0, \\ \frac{\partial \mathcal{E}}{\partial t} - \operatorname{curl} \mathcal{H} = 0, \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} \operatorname{div} \mathcal{E} = 0, \\ \operatorname{div} \mathcal{H} = 0, \end{aligned}\right\} \tag{4.4}$$

so:

$$\left.\begin{aligned} \frac{\partial (\mathcal{H} + i\mathcal{E})}{\partial t} + \operatorname{curl} (\mathcal{H} + i\mathcal{E}) = 0, \\ \operatorname{div} (\mathcal{H} + i\mathcal{E}) = 0, \end{aligned}\right\} \tag{4.5}$$

or:

$$\left.\begin{aligned} \frac{\partial \mathcal{F}}{\partial t} - \operatorname{curl} \mathcal{F} = 0, \\ \operatorname{div} \mathcal{F} = 0. \end{aligned}\right\} \tag{4.6}$$

The equation:

$$\overline{\nabla} \mathcal{F} = 0 \tag{4.7}$$

combines both equations of the system for $T = 0$. Meanwhile, we would like to mostly liberate ourselves from that restriction, since it would contradict the whole spirit of the

investigation as an unfounded assumption, when considered from the standpoint that has been chosen here.

At this point, we would like to touch upon the question of the transformation of the function $\mathcal{F}$ when we go from a given coordinate system to an equivalent one. We saw that the fundamental equations remain invariant when $\mathcal{F}$ is considered to be a vector. However, the electric and magnetic field strengths collectively define a six-vector in the MINKOWSKI picture, which obeys completely different transformation formulas. It is the nature of things that the transformation of the versor $\mathcal{F}$ is not at all determined uniquely. It is then easy to recognize that along with $\mathcal{F}$:

$$\mathcal{F}' = \mathcal{F}\, q_0 \tag{4.8}$$

is also a quaternion function, in the event that q_0 means a constant quaternion whose length we choose to be $| q_0 | = 1$, in order to eliminate a mere similarity transformation in advance. However, that equation means an orthogonal transformation of the function $\mathcal{F}$, and in fact, a $\mathcal{Q}$-transformation. We then see that a three-dimensional manifold of transformations of the components of a quaternion function that leave the defining equations untouched already exists in one and the same coordinate system. If we now perform a rotation of the axis-cross with the characteristics p and q then the transformation of the versor $\mathcal{F}$ will generally again be given by the formula:

$$\mathcal{F}' = p\, \mathcal{F}\, q\, q_0\,; \tag{4.9}$$

i.e., an arbitrary $\mathcal{Q}$-transformation with the characteristic q_0 can be performed after the vector transformation. In particular, if we choose:

$$q_0 = \bar{q}\, \bar{p}\,, \tag{4.10}$$

so the formula of the transformation will be:

$$\mathcal{F}' = p\, \mathcal{F}\, \bar{p}\,, \tag{4.11}$$

then we will be dealing with the usual transformation of the electromagnetic field. The first case that was mentioned – viz., in which $\mathcal{F}$ is a vector ($q_0 = 1$) to begin with – is present in a gravitational field. Here, the field strengths can be derived from a scalar gradient (viz., the temporal component of the general potential Φ) as its gradient. The potential remains invariant under a rotation, so the field strengths transform as a vector.

The transformation of the potential Φ will generally be given by the formula:

$$\Phi' = \bar{q}_0\, \bar{q}\ \Phi\, q. \tag{4.12}$$

If we replace q_0 with its value for an electromagnetic field then the vector transformation will result for Φ. Thus, while the field strength is a vector for a gravitational field, the potential (viz., "vector potential") will be a vector for the electromagnetic field. In all cases, field strengths, as well as potentials, represent versors.

CHAPTER 5

The electron as a function-theoretic singularity.

The fundamental equations of MAXWELL's theory, in their original sublime simplicity and lack of ambiguity, shall be valid only for the pure ether, while for matter, they will be extended with new, foreign terms. That process is in stark contrast to the actual mathematical spirit of those equations. We are dealing with partial differential equations, and what that means, as well as a wealth of possibilities that already lie hidden beneath the simplest types, can be given by a classical example in function theory, which is a magnificent edifice that is built upon only the soil of the CAUCHY-RIEMANN equations. Whether quaternion functions can also be the basis for a correspondingly complex discipline is debatable, but certainly the theory of functions shows us the path, the method, and the general viewpoint for the treatment of partial differential equations to begin with. Therefore, nothing can be said of an extension by terms that are foreign and do not belong to functions. There are probably places where the equations would lose their validity, although they are not arbitrary, but are determined by the nature of the function itself. At such places, one can no longer speak of equations at all, since the differential quotients would lose their meaning. In contradiction to the regular points of the function, one cares to refer to them as its *singular points*. They are characteristic insofar as they make it possible to determine the function in question from their position and their behavior in their immediate neighborhood in a natural way. By such a function-theoretic interpretation – which already lies close to the remarkable affinity between the functions that we speak of here with complex functions that we touched upon in the previous chapter, moreover –the problem of matter – especially, its atomic structure – will take on an exceptionally harmonic solution. *Matter represents the singular loci of the functions that are determined by the differential equations that are valid in the ether.*

The characteristic role of singularities that was mentioned just now finds its physical sense in the fact that all effects have their starting point in matter. The basic paradox of the theory of the electron – viz., how a structure can be held together by nothing but expansive forces – will become pointless here, while the discontinuity of matter seems to be a self-explanatory consequence, since the singular points will naturally define a discrete set in regular space. A field theory that is consequently developed (and one wishes that it should be constructed in the spirit of partial differential equations) would then present no contradiction to atomism, but would even lead to it directly.

However, the fundamental equations define a linear and homogeneous system. As a result, arbitrary particular solutions can be superimposed with each other in an entirely arbitrary manner. The positions of the individual singularities, like the functions that are associated with them as particular solutions, are completely independent of each other. In reality, we find both continual interaction, on the one hand, and strict determinacy, on the other. The fundamental equations cannot arrive at an explanation for nature then, but they must be extended by a new principle. The next chapter would like to make a contribution towards the solution of that question.

CHAPTER 6

Hamilton's principle.

We would now like to assume that the fourth variable is pure imaginary and should be connected with time t by the equation:

$$t = i\tau. \tag{6.1}$$

The basic particular function of the potential equation will then be the following function:

$$\Phi = \frac{f(\tau')\sqrt{1-v^2}}{r(1+\dot{r})}, \qquad v^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 . \tag{6.2}$$

Here, we are dealing with a singularity that represents a line (viz., a *world line*) when it is mapped into (x, y, z, t) space. The spatial coordinates of its points ξ, η, ζ can be arbitrary functions of time (except that no superluminal velocities can occur), v is the velocity, and τ' is the retarded time:

$$\tau' = \tau - r, \tag{6.3}$$

in which r means the distance from the starting point (x, y, z) to (ξ, η, ζ), but taken at the time point τ'. Dots represent differentiation with respect to τ', in any case.

New particular solutions can be defined by differentiating this solution with respect to the coordinates of the starting point (x, y, z, τ), and then define new ones by the same process, in turn, etc., and upon combining all of these particular solutions, we will ultimately obtain an infinite series in increasing powers of the distance, which will define the general solution of the potential equation for the singularity in question. This series (which corresponds to the LAURENT development in the theory of functions, here) contains an infinite series of arbitrary functions of time as its coefficients.

An electron can then be regarded as a structure with infinitely many degrees of freedom. The more precisely that the values of the function must be known, the more terms that there must be in the series, and also the more degrees of freedom that must be considered. Meanwhile, at a suitable distance, the influence of the higher terms in comparison to the first one will always be smaller, and ultimately we will come to a distance at which we can already truncate the series at its first term. That case (when one is not actually dealing with the internal structure of atoms or molecules) can probably be physically realized for the macroscopic physical applications. We then come to the usual (physical) solution of the potential equation.

That now raises the question of how one can determine the coefficients of this series, as well as the spatial course of the singularity, which in nature are indeed determined uniquely by external causes that are no less than arbitrary and at each instant (at least, the inorganic kind). Questions of that kind are answered classically in physics by posing variational principles, and in especially HAMILTON's principle, which seems to take on a universal significance. We would now like to attempt to give it a formulation that corresponds to our approach.

HAMILTON's principle states that for given initial and final states of the system, the so-called action integral:

$$W = \int_{\tau_1}^{\tau_2} (T - \Psi)\, d\tau \tag{6.4}$$

will be the smallest possible for the events that actually occur. In this, T is the total kinetic energy, and Ψ is the potential energy of the system. For the electromagnetic field, the kinetic energy will take the form of the electric energy, and the potential will take the form of the magnetic energy. In order to get the total energy of the field, one must integrate over all of electromagnetic space, since the energy is, in fact spread over space. That implies a three-dimensional integration over the spatial coordinates, and then an integration over time is added to HAMILTON's principle. We then see directly how things look in the eyes of the principle of relativity. *We will have to perform an integration over all of four-dimensional space* and must look for the boundary values of that integral. The integrand in that process shall depend upon merely the values of the basic electromagnetic function – or *field strength* $\mathcal{F}$ – at the point in question and remain invariant under a LORENTZ transformation, as a pure scalar quantity. It will then follow in advance from this that we must necessarily direct our attention to the absolute value of $\mathcal{F}$ (its square:

$$\mathcal{F}\bar{\mathcal{F}} = X^2 + Y^2 + Z^2 + T^2, \tag{6.5}$$

resp.) as the only invariant of a vector. The same thing will also be true when the field strengths do not obey the vector transformation, but are subject to a $\mathcal{Q}$-transformation. One will then have:

$$\mathcal{F}\, q_0\ \overline{\mathcal{F}\, q_0} = \mathcal{F}\bar{\mathcal{F}}\,. \tag{6.6}$$

One can then already establish the form of the action integral from mere dimensional considerations, along with the principle of relativity, namely, as:

$$\int \mathcal{F}\bar{\mathcal{F}}\ dx\, dy\, dz\, d\tau. \tag{6.7}$$

However, this expression harmonizes wonderfully with the form of HAMILTON's principle that was written down to begin with. Namely, one has:

$$\mathcal{F}\bar{\mathcal{F}} = (\mathcal{H} + i\,\mathcal{E})\ (\bar{\mathcal{H}} + i\bar{\mathcal{E}}) = \mathcal{H}\bar{\mathcal{H}} - \mathcal{E}\bar{\mathcal{E}} + i(\mathcal{E}\bar{\mathcal{H}} + \mathcal{H}\bar{\mathcal{E}})\,. \tag{6.8}$$

The real part is thus equal to the square of the magnetic field strength minus the square of the electric field strength. If we first perform the integration over the spatial coordinates at constant time then we will, in fact, obtain the difference of magnetic and electric energies that are contained in the total field (a constant factor that depends upon the system of units that are used does not enter into this), in the sense of MAXWELL's theory and corresponding to the usual three-dimensional picture. For such a division of the variables, however, the four-dimensional world would take the form of a circular

cylinder with infinitely-large dimensions, which contradicts the spirit of the principle of relativity. The given initial and final positions of the world would be the lower and upper bases of that cylinder, resp. We must look for a boundary for the universe that is admissible to the theory of relativity, and the solution to that problem will be unavoidably necessary for us when that boundary lies at infinity. [The demand that one must "integrate over all space" must then be accomplished by a corresponding passage to the limit ([1]).]

Above all, we shall address the prescribed initial and final state of the world, between which the variation is carried out. It would be very unlikely that each world-line would be prescribed to have any well-defined point as its starting point and any other as its endpoint, and that these limit points would be dispersed throughout the universe in any sort of way. I believe that the idea suggests itself with an almost-intuitive certainty that a *single* point should be assumed to be the starting point of the universe, and a single point should be assumed to the endpoint. That is, a single point should define the starting point of all world-lines, and likewise there should be a single point where all world-lines must conclude. We are dealing with only a single function $\mathcal{F}$, whose singularities make up only a part of it, while they first produce the entire function only in their totality. That unity will now be achieved because of the fact that they all possess a common starting point and endpoint, and thus, to some extent, define a single closed line.

With that assumption, the question of the boundary of the universe can also be answered uniquely. Namely, since no superluminal velocities can enter in, all world-lines must necessarily remain inside of the ball that expands away from the starting point with the speed of light, just as it must remain inside of a ball that contracts from the endpoint with the speed of light (in order for it to be able to arrive at the latter). In the four-dimensional picture then, the "possible" world would then fall inside the interior of the cone:

$$\tau = \tau_1 + \sqrt{x^2 + y^2 + z^2}\,, \tag{6.9}$$

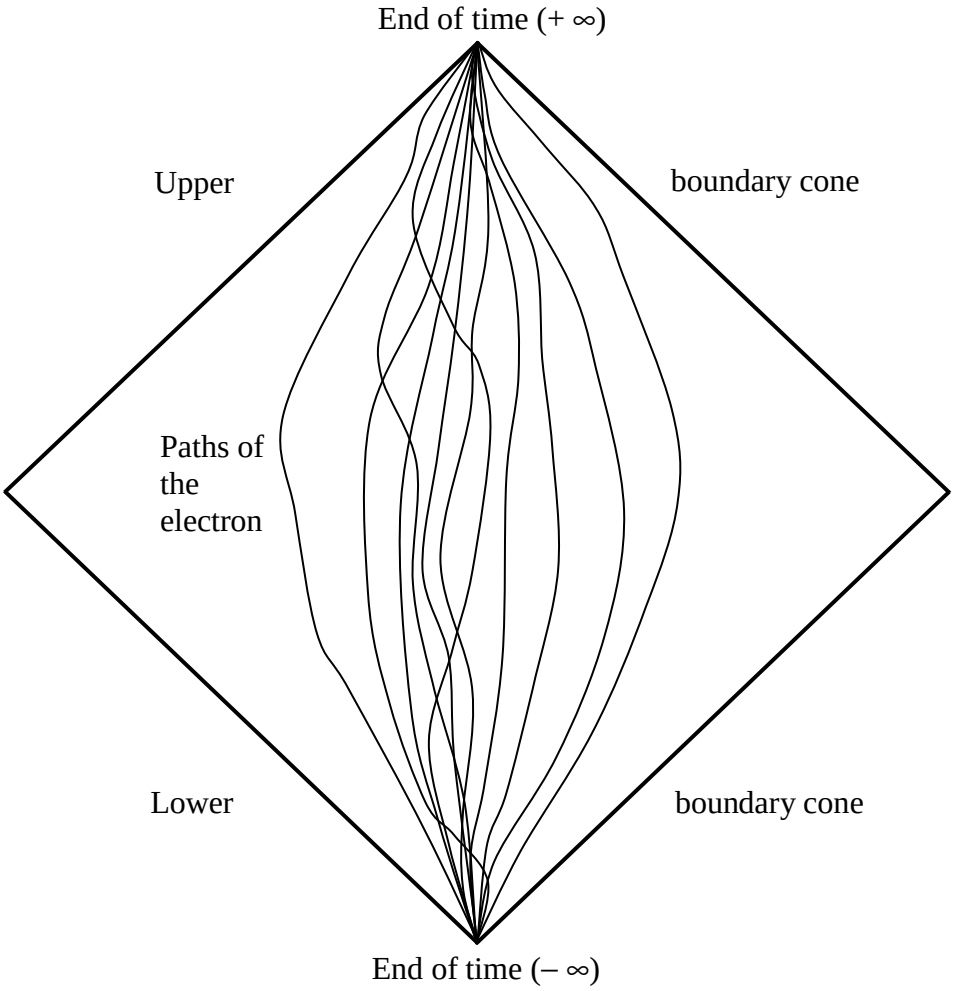
on the one hand, and the interior of the cone:

$$\tau = \tau_2 - \sqrt{x^2 + y^2 + z^2}\,, \tag{6.10}$$

on the other. In this, the axis-cross is laid in such a way that the coordinates of the two limit points will be (0, 0, 0, τ_1) and (0, 0, 0, τ_2), resp. These two cones (cf., Figure 6.1, which is shown spatially) include a well-defined space whose limits cannot be reached or exceed by any sort of means and over which, the four-dimensional integration is taken in HAMILTON's principle, since only the physically-possible space can come under consideration in it. However, nothing stops one from pushing the limit points out to infinity, i.e., exhibiting the universe with no boundary in a temporal, as well as spatial, context. The two vertices of the cones must them be regarded as limits that electron paths come infinitely close to in time, but without reaching the one in the past or the one in the future. The bounding conical surfaces will then naturally go out to infinity, as

([1]) That would not correspond to a sphere of infinitely-large radius, since the fourth variable is not time τ, but $i\tau$.

well. In any case, a direction will be distinguished from the others as the "axis of the universe," in some sense, namely, the line that connects the two limiting conical vertices. It then seems as though the principle of relativity would be introduced *a priori* and then cancelled *a posteriori*. However, it is in the nature of the variational principle that the position of the limits is irrelevant to the behavior of the function. (For example, a geodetic line between two well-defined points will also remain geodetic relative to any other two points along it.)



Relativistic boundary of the universe in the sense of Hamilton's principle

Figure 6.1 Relativistic boundary of the universe.

The dimension of HAMILTON's integral is a so-called *quantity of action* in regard to its dimension. We can then refer to the integrand – viz., the square of the field intensity – as the quantity of action that is contained in a unit volume, and thus rephrase HAMILTON's principle as: *The total quantity of action that is contained in the universe must be an extreme value amongst all possible values.* It should be remarked here that auxiliary conditions in the form of prescriptions for the boundary cones can be added to that demand. For example, one such condition would be that the field strengths should be

zero throughout the upper cone. The consequence of that requirement is that the electrons can radiate only into the future, but not into the past (so no contracting spherical waves would be possible).

We would now like to turn to the explicit calculation of HAMILTON's integral. It shows that the integration over a regular space can be replaced by a surface integral. It will then follow from the two equations:

$$\frac{\partial \bar{\mathcal{F}}}{\partial x}1_x + \frac{\partial \bar{\mathcal{F}}}{\partial y}1_y + \frac{\partial \bar{\mathcal{F}}}{\partial z}1_z + \frac{\partial \bar{\mathcal{F}}}{\partial t} = 0, \tag{6.11}$$

$$\mathcal{F} = 1_x \frac{\partial \bar{\Phi}}{\partial x} + 1_y \frac{\partial \bar{\Phi}}{\partial y} + 1_z \frac{\partial \bar{\Phi}}{\partial z} + \frac{\partial \bar{\Phi}}{\partial t} \tag{6.12}$$

that:

$$\frac{\partial}{\partial x}(\bar{\mathcal{F}}1_x\bar{\Phi}) + \frac{\partial}{\partial y}(\bar{\mathcal{F}}1_y\bar{\Phi}) + \frac{\partial}{\partial z}(\bar{\mathcal{F}}1_z\bar{\Phi}) + \frac{\partial}{\partial t}(\bar{\mathcal{F}}\bar{\Phi}) = \bar{\mathcal{F}}\mathcal{F}. \tag{6.13}$$

We can apply GAUSS's theorem to the left-hand side immediately and if we denote the four-dimensional volume element by *dv* and the surface element on the boundary surface by *df* then we will get the relation:

$$\int \bar{\mathcal{F}}\mathcal{F}\,dv = \int \bar{\mathcal{F}}\,n\bar{\Phi}\,df\,. \tag{6.14}$$

That is also identical to:

$$\int \Phi\,\bar{n}\,\mathcal{F}\,df\,, \tag{6.15}$$

in which *n* again means the surface normal of unit length that points outward.

The singularities require a special treatment in the integration over all space. One must exclude them from the integration domain that is enveloped by closed surfaces that are closely nestled and will unite with the singularities in the limit. The spatial domain of integration will then be bounded, on the one hand, by these surfaces that surround singularities, and on the other, by the boundary cones, so one must also take the surface integral over them in place of the original spatial integral. We assume that the outer boundary is at infinity, and we would like to temporarily address the totality of the singularities, and thus the paths of the electrons.

We can generally say nothing specific about its special construction and structure, since we still do not know whether those paths should be regarded as simple lines (i.e., point-like electrons) or more complicated structures. Fortunately, part of the integral can be calculated (and indeed, as it seems, in many cases, it will be the definitive one) without having to go into any deeper details. We can divide the field strengths into two parts in the vicinity of a singularity: An external one $\mathcal{F}_e$, which is due to the external singularities (i.e., the remaining electrons), and an internal one $\mathcal{F}_i$, which represents the "eigen-field" of the electron in question; the same thing will be true of the potential. We then have to compute the following expression:

$$\int (\Phi_e + \Phi_i)\,\bar{n}\,(\mathcal{F}_e + \mathcal{F}_i)\,df\ . \tag{6.16}$$

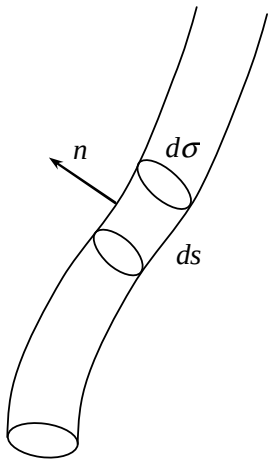


Figure 6.2. Tubular integration surface.

The entire integral can thus be decomposed into four sums, which we would like to consider individually. The integration extends over a tubular surface that surrounds the entire singularity. We decompose them into nothing but elementary cylinders of the type in figure 6.2 whose height shall be denoted by *ds*. We now perform the integration in such a way that we first integrate around the base line of the cylinder – i.e., the cross-section of the tubular surface – and then along the length. In the first integration, the external field strengths – and likewise, the external potential – can be regarded as homogeneous when one neglects the possible dimensions of the electrons in comparison to their mutual distances, and that is ordinarily the case. We can then remove $\mathcal{F}_e$ and Φ_e from the integration sign and then obtain the two expressions:

$$\int \Phi_e \left(\int \bar{n}\,\mathcal{F}_i\,d\sigma \right) ds \tag{6.17}$$

and

$$\int \bar{\mathcal{F}}_e \left(\int n\bar{\Phi}_i\,d\sigma \right) ds\,. \tag{6.18}$$

We set:

$$L = \lim \int \bar{\mathcal{F}}_i\,n\,d\sigma\,, \tag{6.19}$$

$$\Lambda = \lim \int n\bar{\Phi}_i\,d\sigma\,, \tag{6.20}$$

and ultimately:

$$H = \lim \int \bar{\Phi}_i\,\bar{n}\,\mathcal{F}_i\,d\sigma\,. \tag{6.21}$$

$d\sigma$ is the element of a surface in three-dimensional space that surrounds the electron. Certain values of L, Λ, and H belong to each point of the path-line. The electron under scrutiny will then contribute three integrals to the action integral:

$$\int \Phi_e \bar{L}\, ds + \int \bar{\mathcal{F}}_e \Lambda\, ds + \int H\, ds\,. \tag{6.22}$$

The fourth part:

$$\lim \int \Phi_e \bar{n}\, \mathcal{F}_e\, ds \tag{6.23}$$

is, in fact, zero.

We then get the quantity of action over all space from three sums of the form:

$$\sum \int \Phi_e \bar{L}\, ds + \sum \int \bar{\mathcal{F}}_e \Lambda\, ds + \sum \int H\, ds \tag{6.24}$$

that are distributed over the sources of the field strengths, and one must add the integral over the external boundary surface to that:

$$\int \Phi \bar{n}\, \mathcal{F}\, df\,, \tag{6.25}$$

in addition.

One can show that the two limits L and Λ actually exist in general, and indeed for L (we would like to call it the *allocation* [Ger. *Belegung*]), only the coefficient of $1 / r$ comes under consideration in the infinite series development of the potential, while for Λ, only the coefficient of $1 / r^2$ gets considered. If we truncate the series at the first terms, for example, then:

$$\Phi = \frac{\varphi(\tau')\sqrt{1-v^2}}{r(1+\dot{r})}, \tag{6.26}$$

so the allocation will be:

$$L = 4\pi\, \varphi(\tau), \tag{6.27}$$

while $\Lambda = 0$. τ should be employed as the time coordinate of the point of the world-line being considered.

Things are different for the limiting value H, which we would like to call the *HAMILTONian function of the electron.* Generally, that limiting value does not have to exist, at least not as a finite quantity. The value of H will already become infinitely large in the example above of the potential Φ. In the following chapter, I would like discuss an electron model for which the HAMILTONian function possesses zero as a limiting value without examining the general conditions for staying finite.

CHAPTER 7

The circular electron.

The potential:

$$\Phi = \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} \tag{7.1}$$

generates a temporally-stationary field with the point (ξ, η, ζ) as a singularity. Since we have introduced potentials and field strengths as complex quantities at the fundamental level, it would make intuitive sense to regard the constants (ξ, η, ζ) as complex numbers. We arrange the coordinate displacement and rotation in such a way that two of the constants – e.g., ξ and η – are equal to zero, and the third one can be set to something purely imaginary:

$$\zeta = -i\rho, \tag{7.2}$$

such that we will have:

$$\Phi = \frac{1}{\sqrt{(x^2+y^2+(z+i\rho)^2}}. \tag{7.3}$$

Here, the denominator will not just be zero at an isolated point, but along an entire circle; namely, for:

$$\left.\begin{array}{rc} & z=0 \\ \text{and} & \\ & x^2+y^2=\rho^2. \end{array}\right\} \tag{7.4}$$

However, not only is this circle singular for the function, but also the entire circular surface that it surrounds. Namely, the square roots will change their signs when one crosses through that surface.

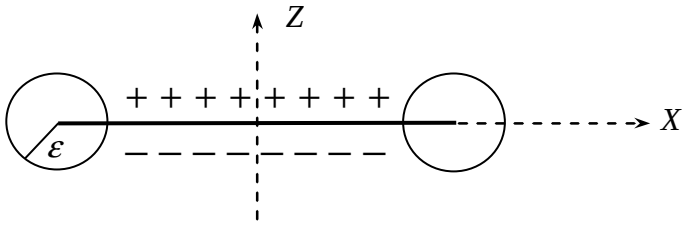


Figure 7.1. Toral surface that wraps around a circular electron.

The calculation of the HAMILTONian function can be accomplished in two steps. We integrate along the circular disc above and below it up to a radius of $\rho - \varepsilon$, and then along a toral surface of radius ε that surrounds the circle, and then pass to the limit $\varepsilon = 0$. However, since Φ, as well as $\mathcal{F}$, change signs on the circular disc, their product will remain regular, and the integral over the circular disc will, in turn, be zero. Due to its

omni-directional symmetry around the *Z*-axis, it will suffice to perform the calculation in the projection onto $y = 0$ for the circular disc in order to then multiply by $2\pi\rho$. We set:

$$\left.\begin{aligned} x &= \rho + \varepsilon\cos\varphi, \\ z &= \varepsilon\sin\varphi, \end{aligned}\right\} \tag{7.5}$$

and obtain:

$$H = \frac{2\pi\rho}{\varepsilon}\int_0^{2\pi} \frac{\varepsilon + \rho(\cos\varphi + i\sin\varphi)}{[\varepsilon + 2\rho(\cos\varphi + i\sin\varphi)]^2}\, d\varphi, \tag{7.6}$$

as a result. One easily convinces oneself that this integral has the value zero (and is indeed independent of ε).

The circular electron gives us a simple example in which the HAMILTONian function of the electron does not need to be infinite. In a stationary field, the HAMILTONian function means the difference between the electric and magnetic energy. The field energy will be infinitely large for point-like electrons, which generate a purely-electric field. The electric energy will also be infinite for the circular electron, but a magnetic field will also appear whose energy (which will be infinite in any event) will compensate for the electric energy precisely, in such a way that the difference of the two will tend to the limiting value of zero. The magnetic forces will be noticeable only in the vicinity of the electron. At a larger distance, where the radius ρ of the electron can be neglected, the field of the circular electron will differ from the point electron only infinitely little.

CHAPTER 8

Dynamics of the electron in a gravitational field and in a electromagnetic field.

The outer surface integral into which the action integral over space will be converted is to be spread over the external bounding surface of space, in addition to the singularities. The contribution of that surface can by no means be simply set to zero, even when the boundary lies at infinity. Rather, it is likely that the boundary surface will take on a characteristic role. We have probably justified the boundary cones of the possible space relativistically, but when the standpoint of the theory of functions is valid, those limits must also have a meaning in the theory of functions. That is the case when all points of the boundary surface are singular, so the function cannot be continued over the domain that it encloses. "Such" functions with natural limits do not belong to the exceptions here, but rather they define the natural transition from real space (with the variable t) to the imaginary one (with the variable $i\tau$), to some extent. A point-like singularity in real four-dimensional space whose basic type is defined by the function:

$$\Phi = \frac{1}{r^2 + t^2}, \tag{8.1}$$

in fact, corresponds to a cone as a singular surface in imaginary space, as the function:

$$\Phi = \frac{1}{r^2 - \tau^2} \tag{8.2}$$

shows, and indeed to the two cones $r = t$ or $r = -t$. The same thing will be true for the infinite series that arises by differentiating that function with respect to the individual variables and summing. If we place such a series at both limit points of the universe and add them to the series that arise from the internal singularities then we will already have a function before us whose natural limits will actually come from the two boundary cones. However, those developments already go too far beyond the existing context of physical speculation, so their closer discussion here would have to take place at an embryonic stage of the theory. I shall, in turn, pass over that question and leave the boundary surface unconsidered, for the time being.

By contrast, I would like to show the applicability of the intuitions that were developed in an entirely special example, namely, by deriving the equation of motion for a free electron that is found in a gravitational field or an electromagnetic field, in addition. For the physical application of the variational principle, one can employ the usual fortunate artifice of replacing a hidden mechanics with empirically-established condition equations – i.e., replacing the unknown system with a postulated reduction of the degrees of freedom, in some sense. We will shrink the general problem very considerably in a similar way. First, we would like consider only the first term in the infinite series development, second, the allocation shall be postulated empirically such

that the only degree of freedom that remains is the course of the path-line of the electron, which will be considered to be point-like. The electron will then be a system with three degrees of freedom here, and its spatial coordinates shall be determined as functions of time.

One must set $\Lambda = 0$ in the three partial integrals that make up the entire action integral with the assumed restriction; that part will then drop out. From now on, the HAMILTONian functions of the electrons shall be equal to zero, or at least vanishingly small in comparison to the quantities that come under considerations. Only the single sum:

$$\sum \int \Phi_e \bar{L}\, ds \tag{8.3}$$

then remains, in which:

$$\Phi = \sum \frac{L\sqrt{1-v^2}}{4\pi r\left(1+\dfrac{\partial r}{\partial \tau'}\right)} \qquad (t = t - r)\,. \tag{8.4}$$

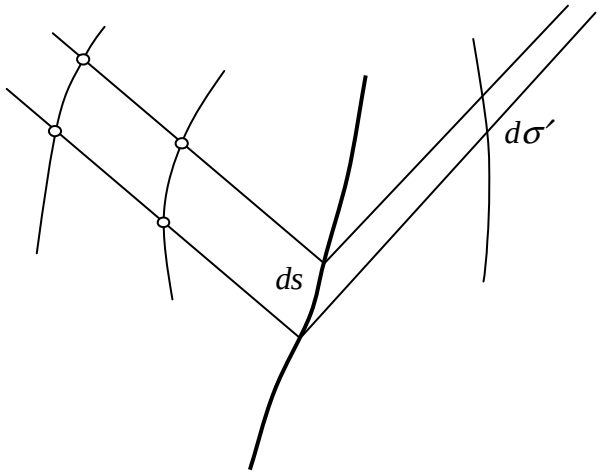


Figure 8.1. Electron path-lines.

Before everything else, one asks how, in general, the integration is to be performed in order to be able to obtain the present EULER equations as the solution. We choose a definite electron. In order to perform the variation, it will suffice to know only that part of the action integral into which the data of that electron enter. Above all, one must integrate over its proper path then. In addition, however, it is also influenced by all of the remaining electrons in the external potential Φ_e, such that all other paths must be considered at the same time. The points of the path lines of all remaining electrons shall now be associated with the points of the proper path as follows: Each point of the remaining electrons path shall always belong to that point of the path line that the wave that hastens away from the reference point considered with unit speed will arrive at precisely. The time point of the coordinate point will then be connected with the time t of the reference point by the equation:

$$\tau' = \tau + r, \tag{8.5}$$

in which r is taken to be the distance from the reference point at time τ to the coordinate point at time τ'. r then depends upon τ', as well as τ, so one will have:

$$r^2 = [\xi'(\tau') - \xi(\tau)]^2 + [\eta'(\tau') - \eta(\tau)]^2 + [\zeta'(\tau') - \zeta(\tau)]^2, \tag{8.6}$$

and furthermore:

$$d\tau' = d\tau + \frac{\partial r}{\partial \tau'} d\tau' + \frac{\partial r}{\partial \tau} d\tau, \tag{8.7}$$

so:

$$d\tau' = \frac{1 + \dfrac{\partial r}{\partial \tau}}{1 - \dfrac{\partial r}{\partial \tau'}}\, d\tau. \tag{8.8}$$

We now combine all terms that are multiplied by the differential of the path length ds and depend upon the position of the reference point (ξ, η, ζ). It can be written in the form:

$$(\Phi_e + \Phi'_e)\, \bar{L}, \tag{8.9}$$

in which:

$$\Phi'_e = \sum \frac{1}{4\pi} \frac{L'\sqrt{1-v^2}}{r\left(1 - \dfrac{\partial r}{\partial \tau'}\right)} \qquad (\tau' = \tau + \dot{r}). \tag{8.10}$$

This potential replaces the retarded potential Φ_e with one that is equivalent to it, but radiates backward from the future, and which is, however, reducible to the retarded potential.

To that end, we add another integral to the variational integral that has the value zero in its totality. It is the integral that is taken over the cone:

$$\tau = \tau_0 - r, \tag{8.11}$$

namely:

$$2 \int \left[\Xi \left(\frac{\partial \Xi}{\partial x} a + \frac{\partial \Xi}{\partial y} b + \frac{\partial \Xi}{\partial y} c - \frac{\partial \Xi}{\partial t} d \right) + \cdots \right] df, \tag{8.12}$$

in which τ_0 shall tend to the limit $+\infty$, Ξ, H, Z, Θ shall mean the components of the total potential, and a, b, c, d shall mean the components of the surface normal. If we select the part that depends upon the X-component of the allocation of the electron under consideration then we will find the magnitude:

$$\int \frac{L_x}{4\pi R} \int \left[\left(\frac{\partial \Xi_e}{\partial x} a + \frac{\partial \Xi_e}{\partial y} b + \frac{\partial \Xi_e}{\partial y} c - \frac{\partial \Xi_e}{\partial t} d \right) d\sigma \right] ds, \tag{8.13}$$

such that one must add:

$$A_x = \frac{1}{4\pi R}\int\left(\frac{\partial \Xi_e}{\partial x}a + \frac{\partial \Xi_e}{\partial y}b + \frac{\partial \Xi_e}{\partial y}c + \frac{\partial \Xi_e}{\partial t}d\right)d\sigma \tag{8.14}$$

to the *X*-component of the potential Φ'_e when (*a*, *b*, *c*) is the normal to the infinitely-distant cone with radius *R*, $d\sigma$ is the surface element, and the values of the differential quotients are taken at time $\tau + R$. However, the sum $\Phi'_e + A_x$ is nothing but the *X*-component of the external potential at the point (ξ, η, ζ). We can then introduce Φ_e into the formula in place of Φ'_e and obtain:

$$2\int \bar{\Phi}_e L\, ds \tag{8.15}$$

as the single quantity the comes under consideration in the variation.

We have already mentioned that certain requirements on the boundary surface must be added as auxiliary conditions. We now introduce such an auxiliary condition when we demand that the total allocation of the boundary surface should have a prescribed value; i.e., the surface integral:

$$\int \bar{\mathcal{F}}\, n\, df \tag{8.16}$$

that is taken over it. By applying the fundamental equations in the conversion by GAUSS's theorem, it will follow that this limiting allocation will be equal to the sum of all allocations of the electrons, namely:

$$\sum\int L\, dt\,. \tag{8.17}$$

From the method of LAGRANGE multipliers, we must add the auxiliary condition, multiplied by a constant, to the variation integral. Let that constant be -2Γ in our case. It has the effect that the external potential will be reduced by the value Γ, such that ultimately for a particular electron the quantities to be varied will be (up to a constant factor and introducing time as the independent variable in place of time):

$$\int_{\tau_1}^{\tau_2}(\Phi_e - \Gamma)\bar{L}\sqrt{1-v^2}\, d\tau\,. \tag{8.18}$$

Now, the EULER equations can be applied immediately when only the allocation *L* is known.

For the gravitational field, we now assume that all allocations point in a well-defined direction that does not change in time, either (it is plausible that the "world axis" – i.e., the line that links the two limit points – has that universal direction). We lay the time axis of the coordinate system in that direction. The allocation and the potential have only temporal components, so they are regarded as scalars. We also assume that the magnitude of the allocation – i.e., the mechanical "mass" – is unchanging in time; let its notation be μ. We will then have:

$$\int_{\tau_1}^{\tau_2} (\Theta_e^g - \Gamma)\mu\sqrt{1-\dot{\xi}^2-\dot{\eta}^2-\dot{\zeta}^2}\,d\tau\,. \tag{8.19}$$

The constant Γ is, in an event, a scalar quantity. It is quite large in comparison to the gravitational potentials that we know of; namely, when it is expressed in the C. G. S. system in terms of the speed of light c and ordinary gravitational constant γ, it will be:

$$\Gamma = \frac{c^2}{4\pi\gamma} = 10 \times 10^{27}\ \frac{\mathrm{g}}{\mathrm{cm}}, \tag{8.20}$$

while the potential on the solar surface is merely 3×10^{22}.

We now apply the EULER equation to the integrand I for each of the components of the path; e.g., for the X-component:

$$\frac{d}{d\tau}\frac{\partial I}{\partial \dot{\xi}} - \frac{\partial I}{\partial \xi} = 0. \tag{8.21}$$

When we neglect Θ_e^g, along with Γ, we will get:

$$\Gamma\frac{d}{d\tau}\frac{\mu\dot{\xi}}{\sqrt{1-v^2}} = \mu\sqrt{1-v^2}\ \frac{\partial\Theta_e^g}{\partial x}. \tag{8.22}$$

This equation includes NEWTON's law of gravitation, correspondingly extended for η and ζ, in its relativistic formulation – viz., the coupling of the MINKOWSKI force vector with the gradient of the retarded potential – and, at the same time, the equivalence of gravitating and inertial mass.

In an electromagnetic field, the allocation falls along the present direction of the path, its magnitude is proportional to the four-velocity, with an imaginary proportionality factor, and the coefficient of i is called the *charge* of the electron. One will then have:

$$L = ie\left(\frac{\partial\xi}{\partial s}1_x + \frac{\partial\eta}{\partial s}1_y + \frac{\partial\zeta}{\partial s}1_z + \frac{\partial\vartheta}{\partial s}\right), \tag{8.23}$$

or also:

$$L = \frac{e}{\sqrt{1-v^2}}(\dot{\xi}1_x + \dot{\eta}1_y + \dot{\zeta}1_z + i)\,. \tag{8.24}$$

The electromagnetic field is superimposed with the gravitational field; i.e., their allocations must be added. We would like to direct our attention to just the real part of the variational integral (what happens with the imaginary part must remain undecided, for the time being). It is extended to the electromagnetic field with the following quantity:

$$e\left(\Xi_e\,\dot{\xi} + \mathrm{H}_e\dot{\eta} + \mathrm{Z}_e\,\dot{\zeta} + \Theta_e\,i\right). \tag{8.25}$$

Here, Ξ, H, Z, Θ mean the components of the electromagnetic "vector potential." The auxiliary condition drops out, due to the integrability of the allocation. If we apply the EULER equation then we will see that "inertial force" of the gravitational field will remain unchanged, while the following expression in the *X*-component must be added to the moving force:

$$e\left(\frac{\partial \Xi_e}{\partial x}\dot{\xi}+\frac{\partial \mathrm{H}_e}{\partial x}\dot{\eta}+\frac{\partial \mathrm{Z}_e}{\partial x}\dot{\zeta}+\frac{\partial \Theta_e}{\partial x}i\right)-\frac{d}{d\tau}(e\,\Xi_e) \tag{8.26}$$

However, one has:

$$\frac{d}{d\tau}(e\,\Xi_e)=e\left(\frac{\partial \Xi_e}{\partial x}\dot{\xi}+\frac{\partial \mathrm{H}_e}{\partial x}\dot{\eta}+\frac{\partial \mathrm{Z}_e}{\partial x}\dot{\zeta}+\frac{\partial \Theta_e}{\partial x}i\right), \tag{8.27}$$

and the result will then be:

$$e\left(i\frac{\partial \Theta_e}{\partial x}-\frac{\partial \Xi_e}{\partial \tau}\right)+e\dot{\eta}\left(\frac{\partial \mathrm{H}_e}{\partial x}-\frac{\partial \Xi_e}{\partial y}\right)+e\dot{\zeta}\left(\frac{\partial \mathrm{Z}_e}{\partial x}-\frac{\partial \Xi_e}{\partial z}\right). \tag{8.28}$$

However, by splitting the real part from the imaginary one, we will now find the following connection between these quantities and the electric and magnetic field strengths:

$$\left.\begin{aligned} i\frac{\partial \Theta}{\partial x}-\frac{\partial \Xi}{\partial \tau}&=-\mathcal{E}_x,\\ \frac{\partial \mathrm{H}}{\partial x}-\frac{\partial \Xi}{\partial y}&=-\mathcal{H}_x,\\ \frac{\partial \mathrm{Z}}{\partial x}-\frac{\partial \Xi}{\partial z}&=-\mathcal{H}_y. \end{aligned}\right\} \tag{8.29}$$

We then obtain precisely the same expression for the moving force of the electromagnetic field as the one that is posed in the theory of the electron, namely, when written vectorially:

$$-e\,(\mathcal{E}_e+v\,\mathcal{H}_e). \tag{8.30}$$

The remarkable change in sign in COULOMB's law compared to NEWTON's law, by which electricity of the same kind repels, while masses of the same kind attract, arises here by the appearance of the factor i^2. Namely, we have regarded the mechanical mass as the source of the magnetic field strength, and therefore as a real quantity, while electricity represents the source of the electric field strength, and is therefore imaginary.

CHAPTER 9

Concluding remarks.

It would be a self-evident achievement of the theory if one could also succeed in regarding the allocation as a degree of freedom and arrive at the electrical allocation and mechanical mass by variation, while also finding the mutual connection between the two. For now, the theory that was presented here does not go appreciably beyond the theory of electrons in practice. It remains problematic why the charge of the electron, like its mass, must be universal constants. Similarly, the essential difference between positive and negative electricity, as well as the quantum character of radiation ([1]), remains unexplained for now. Still, one cannot forget that one cannot immediately count on convincing evidence, due to the complexity of the problem, as well as its mathematical treatment. I hope that the theory that was sketched here will make a contribution to the structural design of modern theoretical physics, as was inaugurated by the work of Einstein, especially. For that reason, its merits or demerits will not be assessed in terms of practical positivistic-economic principles – since it is no mere "work hypothesis." Its persuasiveness (if that does not amount to merely my own subjective delusion) lies not in "convincing evidence," but in the consistency and lack of arbitrariness in its construction, by which, it captures the true soul of MAXWELL's equations, and leads to electrons in a natural way when MAXWELL's theory is fused with the theory of relativity. In my opinion, its superiority over the ordinary theory of electrons lies in this systematic simplicity and necessity. I will not go into the working-out of details here, but only address the broad outline. Therefore, it would seem to me to be more indicative of realizing the initiation of a path that might possibly open up new perspectives into the unreachable depths of nature when explored.

It would affect my speculations in a very challenging way, and for just that reason, it would be greatly appreciated if my highly-valued colleagues that might possibly find these thoughts interesting would communicate their kind remarks and criticisms to be by letters – perhaps recommended – and indeed to the address: Kornél Lánczos, Assistant an der technischen Hochschule, Institut für Experimentalphysik, Budapest.

Date of this manuscript: October 1919.

([1]) Here, I would like to remark that the riddle of positive electricity and the riddle of the quantum possibly have a common root. Namely, a connection very likely exists that allows one to the express the universal quantity action h in terms of nothing but electrical quantities. I write down the purely-empirical equation:

$$\frac{2hc}{e^2} = \frac{\mu_+}{\mu_-}, \tag{9.1}$$

in which μ_+ denotes the mass of the positive electron, μ_- denotes that of the negative electron, and e denotes the elementary charge in electrostatic units. If that equation also requires a numerical correction, since it is endowed with an error of several percent in this form, then it would be very unlikely that two dimensionless numbers that are both defined in terms of constants of nature would be almost identical to each other by mere chance.

Addendum.

I must unfortunately make a belated remark that seems to have characteristic importance for the unity of the theory and its entire conceptualization. At the same time, it provides the actual proof of the essential relationship that was suggested in the text between the equations that were referred to as "fundamental equations" – which defined quaternion functions – and the CAUCHY-RIEMANN equations. The latter can be derived from the variational principle by which one looks for a minimum of the integral:

$$\int (X^2 + Y^2)\, dx\, dy \tag{9.2}$$

when a given regular integration domain is present, and the values of:

$$\int \mathcal{F} \nabla \left(\log \frac{1}{R^2} \right) dx\, dy\,, \tag{9.3}$$

with:

$$\mathcal{F} = X + iY, \qquad \nabla = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\,, \qquad R^2 = (x - \xi)^2 + (y - \eta)^2 \tag{9.4}$$

are prescribed for each point (ξ, η) of the boundary. The fundamental equations in four dimensions will then follow from the principle:

$$\delta \int (X^2 + Y^2 + Z^2 + T^2)\, dx\, dy\, dz\, dt = 0 \tag{9.5}$$

in a corresponding way for prescribed values of the quantities:

$$\int \bar{\mathcal{F}} \left(\nabla \frac{1}{R^2} \right) dx\, dy\, dz\, dt\,. \tag{9.6}$$

(∇ and $\mathcal{F}$, as well as R, have their current four-dimensional meanings.) However, we have applied this same variational principle (with complex components for $\mathcal{F}$ and imaginary t) to the world as a whole in the form of "HAMILTON's principle" (without specifying the boundary conditions) in order to arrive at the equations of motion of the electron, when the domain of integration also contained singularities. That principle then defines the *universal basis* for the entire theory, as well as the field equations and dynamics. However, the corresponding boundary conditions for the universe must be found if one is to perform the variation exactly.