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FOLIATIONS, ENERGIES, and LIQUID CRYSTALS.

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Riemannian geometry is an important tool in the theory of foliations. If we are given a foliation $\mathfrak{F}$ of a manifold V then we may:

1) Assume that the manifold V is endowed with a metric g and study the Riemannian properties of the leaves of the foliations. This approach, which is borrowed from Plante, Hector, etc., gives results that are all the more interesting since they are independent of the metric, g, if the manifold is compact.

2) Construct functions on a manifold V that is already given a metric g, which are defined with the aid of the Riemannian geometry of the leaves of $\mathfrak{F}$, or the field of planes that are orthogonal to $\mathfrak{F}$. This method generally gives results only when V has constant curvature. Reinhart and Wood have metrically interpreted the Godbillon-Vey invariant (cf. [Re-Wo]) in this way. Rogers in 1912, and then Asimov, Brito, Langevin and Rosenberg, Langevin and Leavitt, have already abundantly exploited this approach.

3) Look for foliations – if they exist – that minimize a curvature integral that is calculated with the aid of the geometry of the leaf within the homotopy class of that plane field or its conjugate.

4) On the contrary, if we are given the manifold V and the foliation $\mathfrak{F}$ then we can investigate whether there exists a metric on M that gives the leaves of $\mathfrak{F}$ some particular metric properties. Gluck, Rummler, Sullivan, Haefliger have given conditions for there to exist a metric on M that makes the leaves minimal. Carriére and Ghys have classified the foliations of codimension one such that there exists a metric that makes the leaves totally geodesic.

Here, I will be concerned with viewpoints 2) and 3). It will be interesting to see whether viewpoints 3) and 4) are related.

I – Symmetric functions of curvature.

Let $\mathcal{P}$ be a field of oriented hyperplanes on a Riemannian manifold M, and let X be the privileged unitary orthogonal vector field that is defined by the orientation of $\mathcal{P}$. We remark that the integral curves of X define a foliation of dimension one on M.

R.P. Rogers [Ro] has studied the matrix of the map DX when M is Euclidian space $\mathbb{R}^3$. We write this matrix at the point, $x\in \mathbb{R}^3$, by using an orthonormal frame that is adapted to the situation, i.e., one such that:

$$e_1 = X(x); \qquad (e_2, e_3) \text{ is an orthonormal basis for } \mathcal{P}(x).$$

One then has:

$$D(x) = \begin{pmatrix} 0 & 0 & 0 \\ k & \dfrac{\partial X_2}{\partial x_2} & \dfrac{\partial X_2}{\partial x_3} \\ 0 & \dfrac{\partial X_3}{\partial x_2} & \dfrac{\partial X_3}{\partial x_3} \end{pmatrix},$$

in which k is the curvature of the integral curve of X that passes through x.

When the field $\mathcal{P}$ is integrable, the block:

$$\mathrm{II}_x = \begin{pmatrix} \dfrac{\partial X_2}{\partial x_2} & \dfrac{\partial X_2}{\partial x_3} \\ \dfrac{\partial X_3}{\partial x_2} & \dfrac{\partial X_3}{\partial x_3} \end{pmatrix}(x),$$

is nothing but the matrix at x of the second fundamental form of the leaf that passes through x of the foliation defined by $\mathcal{P}$.

In this case, and only in this case, is the matrix II_x symmetric.

We remark that we may express k, trace II_x, and $\left|\dfrac{\partial X_2}{\partial x_3} - \dfrac{\partial X_3}{\partial x_2}\right|$, which measures the asymmetry of II_x, "the old way," i.e., in terms of the number div(X), and the vector rot(X).

One has:

$$\begin{aligned} \mathrm{Div}(X) &= \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3}\right) \\ &= \mathrm{trace}\ \mathrm{II}_x \end{aligned}$$

$$\mathrm{rot}(X) = \begin{pmatrix} \dfrac{\partial X_1}{\partial x_3} - \dfrac{\partial X_3}{\partial x_2} \\ \dfrac{\partial X_3}{\partial x_1} - \dfrac{\partial X_1}{\partial x_3} \\ \dfrac{\partial X_1}{\partial x_2} - \dfrac{\partial X_2}{\partial x_1} \end{pmatrix},$$

from which $|k| = |X \wedge \mathrm{rot}\, X|$ and $\left|\dfrac{\partial X_2}{\partial x_3} - \dfrac{\partial X_3}{\partial x_2}\right| = |X \cdot \mathrm{rot}\, X|$ (the non-integrability term); we shall encounter these three terms later on.

More generally, when $\mathcal{P}$ is a transversally oriented hyperplane field on the Riemannian manifold M, we define the second fundamental form of $\mathcal{P}$ with the aid of the unitary vector N that is normal to $\mathcal{P}$.

$$\mathrm{II}_x(X, Y) = \langle \nabla_X N, Y \rangle(x) \qquad (X \text{ and } Y \text{ are vectors of } \mathcal{P}).$$

Remark 1: Upon calculating the matrix of the map, $W: X \to \nabla_X N$, in a “Frenet frame,” $e_1 = N$, $e_2 = \dfrac{\nabla_N N}{\| \nabla_N N \|}$, e_3 , …, $e_{n+1} \in \mathcal{P}$, one obtains, moreover, information about the orbits of the field N, because:

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ k & & & \\ 0 & & II_x & \\ 0 & & & \end{pmatrix} \qquad \text{where } k = \| \nabla_N N \|$$

(if $\nabla_N N = 0$ for some convenient frame e_2 , …, e_{n+1} of $\mathcal{P}$).

Remark 2: Like Rogers, B.L. Reinhart [Re] was interested in the symmetry of the form, II_x; of course, the symmetry of II_x at every point is equivalent to the integrability of the field $\mathcal{P}$.

Remark 3: [Re] The plane field $\mathcal{P}$ will be called *totally geodesic* if every geodesic that is tangent to a point x of $\mathcal{P}$ is an integral curve of $\mathcal{P}$. Reinhart proved that $\mathcal{P}$ is totally geodesic if and only if the symmetry of the second fundamental form of $\mathcal{P}$ is identically null.

Remark 4: Upon following, step-by-step, the proof that was given by Do Carmo of the fact that a developable surface admits generatrices at its non-flat points, one may prove the following proposition:

Proposition ([1]): Let $\mathcal{P}$ be a hyperplane field of $\mathbb{R}^{n+1}$ whose Gaussian curvature $K(x) = \det \mathrm{II}_x(x)$ is identically null. Let x be a point such that trace $\mathrm{II}_x \neq 0$. There exists a line D_x that passes through x such that the field $\mathcal{P}$ is constant along D_x in a neighborhood of x.

We define functions that are symmetric in the curvature $\sigma_i^+(x)$ of $\mathcal{P}$ by:

$$\mathrm{Det}(\mathrm{Id} + t\mathrm{II}_x) = \sum_{i=0}^{n} \binom{n}{i} \sigma_i^+(x) t^i .$$

Theorem: [BLR] When the manifold M^n is compact and has constant curvature C the integrals $\int_{M^n} \sigma_i^+(x)$ do not depend on the hyperplane field $\mathcal{P}$; more precisely:

$$\int_{M^n} \sigma_i^+(x) = \begin{cases} C^{i/2} \binom{n/2}{i/2} vol\, M^n & if\ i\ is\, even \\ 0 & if\ i\ is\, odd \end{cases} .$$

(We remark that if n is even and M^n admits an oriented hyperplane field then one has $X(M^n) = 0$, and therefore $C = 0$.)

This result may be partially extended to singular foliations when the codimension of the singular set is sufficiently small (A singular foliation is a field $\mathcal{P}$ defined on $(M^n - \Sigma)$, where Σ is a reasonable subset of M^n, for example, a stratified set whose strata of maximal dimension are of dimension at least $(n - 2)$).

Theorem: Let $\mathcal{P}$ be a hyperplane field on a compact manifold of constant curvature, M^n. If the codimension of the singular set is greater than or equal to i then one has:

$$\int_{M^n} \sigma_i^+(x) = \begin{cases} C^{i/2} \binom{n/2}{i/2} vol\, M^n & if\ i\ is\, even \\ 0 & if\ i\ is\, odd \end{cases}$$

(C is the sectional curvature of M) such that the integral:

$$\int_{M^n} | \sigma_i^+(x) | ,$$

is convergent.

Some complementary partial results that imply the singular set are given in [La_1] and [LS].

[1] This proposition was undoubtedly known to the ancients. I would like to find a reference, preferably one dating back more than fifty years.

These first results are somewhat deceiving since they do not depend on the geometry of the field $\mathcal{P}$ or its integrability.

Nevertheless, we retain the result that foliations of codimension one of manifolds of constant curvature verify a family of theorems that are analogous to the Gauss-Bonnet theorem that is verified by hypersurfaces of even dimension in Euclidian space. Similarly, since the integral:

$$\int_V K(x)dx = \text{const. } K(V)$$

of the Gaussian curvature of V does not depend on the embedding $V^n \subset \mathbb{R}^{n+1}$, but only on the topology of V, the associated curvature of the foliation $\mathfrak{F}$ ($\mathfrak{F}$ and M are oriented if the symmetric curvature function considered is σ_i) do not depend on $\mathfrak{F}$ when the curvature of M is constant.

The integral $\int_V |K(x)|\,dx$ of the modulus of the Gaussian curvature of a surface in $\mathbb{R}^3$ contains considerable information about the geometry of the embedding $V^2 \subset \mathbb{R}^3$; in particular, the fact that this integral attains its minimum value:

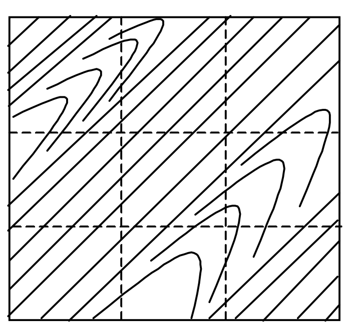
$$m(V) = \min \int_V |K(x)| \qquad (V \text{ has a given topology}).$$

The same type of result is verified by foliations of surfaces.

Theorem: Let $\mathbb{F}$ be the homotopy class of a direction field on the torus $T^2 = \mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z}$ (endowed with the flat metric derived from the Euclidian metric on $\mathbb{R}^2$). The *taut* leaves, i.e., the ones that minimize the integral $\int_{T^2} |\sigma_1(x)|dx$ in the homotopy class $\mathbb{F}$ are:

1) the linear foliations;

2) the ones that are obtained from a linear foliation whose leaves have rational slope by adding a finite number of Reeb components, in whose interior the sign of the geodesic curvature is constant:



Remark: All of the leaves of the foliation $p^{-1}(\mathfrak{F})$ of the universal covering $p: \mathbb{R}^2 \to T^2$ are convex (i.e., the boundary of a convex subset of $\mathbb{R}^2$).

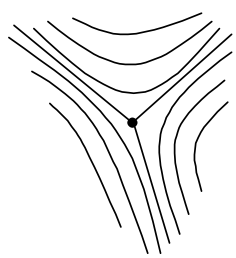
Proof: cf. [La_1] or [La_2].

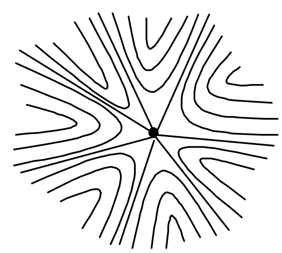
Theorem: Let $\mathfrak{F}$ be a foliation of a closed surface of constant curvature (–1) that has only isolated points for singularities. Then:

$$\int_S |k(x)| \leq (6 \log 2 - 3 \log 3)\, |\chi(S)|,$$

in which k is the curvature of the leaves.

Remark: The singularities of isolated type are:





… …

Proof: cf. [La-Le].

II – Energies.

a) *Foliations of Surfaces.* Now let S be a closed surface of constant curvature C. It is natural to study the integral $\int_S (k_1^2(x) + k_2^2(x))dx$, in which k_1 and k_2 are the geodesic curvatures of $\mathfrak{F}$ and $\mathfrak{F}^\perp$, which is more similar to an energy than the integral $\int_S |k(x)|$.

Being given a transversally orientable foliation $\mathfrak{F}$ on the flat torus is equivalent to being given a function $\theta: T^2 \to S^1$. The function $\theta(x)$ is the angle that the leaf that passes through x makes with the horizontal. The norm $k_1^2(x) + k_2^2(x)$ is the energy $e(\theta)(x)$, of the function θ that was defined by Eells and Samson [Ee-Sa]. This implies that the maps θ that minimize the integral:

$$E(\theta) = \int_{T^2} [k_1^2(x) + k_2^2(x)]dx,$$

in their homotopy class are harmonic, i.e., linear, in the present case. (This would make them the quotient of functions $q: \mathbb{R}^2 \to \mathbb{R}$ of the form:

$$\tilde{\theta}(x, y) = ax + by, \qquad a, b \in \mathbb{Z}).$$

We have proved the:

Proposition: The foliations of the flat torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ that minimize the integral $E(\mathfrak{F}, \mathfrak{F}^{\perp}) = \int_{T^2} k_1^2 + k_2^2$ in their homotopy class are described by an angle $\theta(x)$ = (angle F, horizontal) that is a linear function on T^2.

E. Ghys has remarked that the same question may be posed in the interior of a conjugacy class of foliations. He has also conjectured that there will be no representative that minimizes $E(\mathfrak{F}, \mathfrak{F}^{\perp})$ for a foliation of T^2 that is the suspension of a diffeomorphism of S^1 that is different from the identity and has fixed points.

It will be interesting to define the energy of a foliation of a Riemannian manifold more generally.

Questions: If $\mathfrak{F}$ is transversally oriented and of codimension one, can one, with the aid of an atlas adapted to the foliation, define a notion of energy that coincides with the energy of the section of the (unitary?) tangent bundle to M that is defined by the vector normal to $\mathfrak{F}$?

The energy that was defined by Eells and Samson already answers the question if $\mathfrak{F}$ is given by the fibers of a Riemannian submersion.

A first step towards generalizing this construction is something pointed out by Kamber and Tondeur [Ka-To]$_1$:

Recall: A ("bundle-like") Riemannian foliation is a foliation of a Riemannian manifold M such that one may give any submanifold that is transverse to the foliation a Riemannian metric in such a way that:

1) The holonomy diffeomorphisms are isometries.
2) The submersions of the charts $U_i \to T_i$ are Riemannian submersions.

(U_i is a distinguished open subset and T_i is transversal to U_i; it is given the invariant transverse metric.)

Just as one may define a harmonic Riemannian submersion, a Riemannian foliation will be called *harmonic* if all of the submersions of the charts U_i onto the transversals T_i, which are given a metric that satisfies 1) and 2), are harmonic maps.

Let Q be the normal bundle to the foliation. The orthogonal projection $T_xM \to Q_x$ (Q_x is the normal space at x to the leaf of $\mathfrak{F}$ that passes through x), may be seen as a form on M with values in Q. A scalar product may be defined on this space of forms. Cf. [Ka-To]$_1$, [Ka-To]$_2$.

One sets:

$$E(\mathfrak{F}) = \tfrac{1}{2}\,\|\pi\|^2.$$

Proposition: [Ka-To]$_1$, [Ka-To]$_2$. For a Riemannian foliation $\mathfrak{F}$ of a compact manifold M the following three conditions are equivalent:

1) $\mathfrak{F}$ is a critical point of $E(\mathfrak{F})$,
2) $\mathfrak{F}$ is harmonic,
3) The leaves of $\mathfrak{F}$ are minimal submanifolds of M.

b) *Curvature integrals and sections.* Define the intersection of a vector field on $\mathbb{R}^3$ with an affine plane H to be the following field on H:

$$X|_H(x) = p_H X(x),$$

in which p_H is the orthogonal projection onto H and x is a point of H.

Observation: The trace of $\mathcal{P} = X^\perp$ on the plane H is a line field that is orthogonal to the intersection of X with H.

Proof: It suffices to apply the theorem of three perpendiculars.

The trace of $\mathcal{P}$ on H is transversally oriented by the projection of the vector X. We let k_g notate the geodesic curvature of the leaves of the foliations that is defined by $\mathcal{P}$.

Theorem: [La]2: If W is an open subset of $\mathbb{R}^3$ that is given a plane field $\mathcal{P}$ then one has:

$$\int_{A_{3,2}} \int_{H\cap W} k_g = \int_W \sigma_1 ,$$

in which $A_{3,2}$ is the space of affine planes in $\mathbb{R}^3$.

Moreover, the integral $\int_W \sigma_1 = \int_W div\, X$ depends only on the boundary conditions. In particular, if $\mathcal{P}$ is periodic and W is a fundamental domain then one has $\int_W \sigma_1 = 0$, since $\mathcal{P}$ is the lift of a plane field on the torus T^3 in this case.

Physical observations of a liquid crystal that is described by the field X do not give us the sign of k_g. Although one may also ([La]$_1$) interpret the integral, $\int_{A_{3,2}} \int_{H\cap W} | k_g |$, we would rather try to interpret the integral:

$$\int_{A_{3,2}} \int_{H\cap W} k_g^2 ,$$

which seems more like an energy integral.

Unfortunately, for a general plane field $\mathcal{P}$ there exists a subset of the affine plane H of non-zero measure such that the trace of $\mathcal{P}$ on H admits singularities. This implies that the integral:

$$\mathrm{I} = \int_{A_{3,2}} \int_{H\cap W} k_g^2 ,$$

diverges, since the foliation traced by $\mathcal{P}$ admits a quadratic singularity for the integral, $\int_{H\cap W} k_g^2$.

It will be interesting to find a relation between the integrals:

$$I_H = \int_W \sigma_1^2 , \qquad I_P = \int_W | T \cdot rot\, T |^2 ,$$

and a convergent integral that is obtained by starting with I.

We may hope for such a relation to exist if the field $\mathcal{P}$ has null Gaussian curvature since the set of planes of $\mathcal{P}$ has measure zero in $A_{3,2}$ in that case, which implies that almost any H intersects $\mathcal{P}$ in a foliation without singular points.

An example of such a field is the following one:

$$X = \begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix},$$

for which:

$$DX = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix},$$

in which $|\,\alpha\,| = \dfrac{\partial X}{\partial z}$, with z being the rotational axis of the structure.

If X_α is such that $\alpha(z)$ = const. = α then the plane field $\mathcal{P}$ has a trace on H that consists of the foliation $\mathfrak{F}_P$ whose leaves are convex. (This foliation is the lift of a taut foliation. Cf. [La]$_1$).

The free energy of a cholesteric liquid crystal has the form:

$$A|\dim X|^2 + B|\,X \cdot \mathrm{rot}\, X - \alpha\,|^2 + C|\,X_1 \cdot \mathrm{rot}\, X\,|^2.$$

The field X_α that was constructed above corresponds to a configuration with zero free energy; it essentially represents the equilibrium state. α is then the vertical period of the field X_α. Cf., [C.P.K], [De Ge].

The observations made by Cladis, Kleman, and Pieranski, [C-K-P], probably exhibit the intersection of the field X_1, which, as we have remarked, is orthogonal to the trace of $\mathcal{P}$, and is therefore the lift of a taut foliation of T^2.

BIBLIOGRAPHY

[A] D. ASIMOV, "Average Gaussian Curvature of Leaves of Foliations," Bull. of the Amer. Math. Soc., Vol. 48, no. 1, January 1978, pp. 131-133.

[B.L.R.] F. BRITO, R. LANGEVIN, H. ROSENBERG, "Intégrales de courbure sur une variété feuilletée," Note C.R. Acad. Sc., series A, October 1971, pp. 533-536.

[C.K.P] P. CLADIS, M. KLEMAN, P. PIERANSKI, Note C.R. Acad. Sc., series B, 1971, no. 273, pp. 275.

[Ee-Sa]$_1$ EELLS and SAMSON, "Harmonic Mappings of Riemannian Manifolds," Amer. Journal of Math. (86) 1964.

[Ee-Sa]$_2$ EELLS and SAMSON, "Variational Theory in Fiber Bundles," Proc. US-Japan Seminar on Differential Geometry, Kyoto (1965), pp. 61-69.

[De Ge] P.G. DE GENNES, *Liquid Crystals*, Clarendon Press, Oxford University Press, 1974.

[Ka-To]$_1$ F.W. KAMBER and P. TONDEUR, "Feuilletages harmoniques," Note C.R. Acad. Sc. Paris, t. 291 (13 October 1980), series A, pp. 409-411.

[Ka-To]$_2$ F.W. KAMBER and P. TONDEUR, "Infinitesimal Automorphisms and Second Variation of the Energy for Harmonic Foliations," preprint.

[La]$_1$ R. LANGEVIN, Thesis, Orsay Mathematical Publication no. 431.

[La]$_2$ R. LANGEVIN, "Feuilletages tendus," Bulletin de la société mathématiques de France, no. 107 (1979), pp. 271-281.

[La-Le] R. LANGEVIN and G. LEVITT, "Courbure totale des feuilltages des surfaces," Commentarii Mathematici Helvetici, 57 (1982), pp. 175-195.

[L-S] R. LANGEVIN and T. SHIFRIN, "Polar Varietes," American Journal of Mathematics, Vol. 104, no. 3, pp. 553-605.

[Re] B.L. REINHART, "The Second Fundamental Form of a Plane Field," Journal of Differential Geometry, Vol. 12 (1977), pp. 619-627.

[Re-Wo] B.L. REINHART and J.W. WOOD, "A Metric Formula for the Godbillon-Vey Invariant for Foliations," Proc. of the Amer. Math. Soc., Vol 38 (1973), pp. 427-430.

[Ro] R.P. ROGERS, "Some Differential Properties of the Orthogonal Trajectories of a Convergence of Curves, with an Application to Curl and Divergence of Vectors," Proceedings of the Royal Irish Academy, Vol. 29, sect. A, no. 6 (1932).

COMPLEMENTARY BIBLIOGRAPHY (Texts cited in the Introduction)

[Ca-Gh] Y. CARRIERE and E. GHYS, "Feuilletages totalement géodésique," to appear in Anais Brasileiras de Ciencias.

[Gh] E. GHYS, "Classification des feuilletages totalement géodésiques de codimension 1," Publ. IRMA, Lille, vol. 4, fasc. 1, no. 9 (1982).

[Gl] H. GLUCK, "Can Space be Filled by Geodesics, and, If So, How?" letter, to appear (?).

[He] G. HECTOR, "Croissance des feuilletages presque sans holonomie," Springer Lecture Notes, no. 652, School of Topology, P.U.C., Rio de Janeiro (1976).

[Ha] A. HAEFLIGER, "Some Remarks on Foliations With Minimal Leaves," Journal of Differential Geometry, 15 (1980), pp. 269-284.

[Mo-Pe] R. MOUSSU and F. PELLETIER, "Sur le théorème de Poincaré-Bendixon," Annales de l'Institut Fourier, t. 24, fasc. 1 (1974), pp. 131-148.

[Pl] J.F. PLANTE, "Foliations with Measure-preserving Holonomy," Annals of Math. 102 (1975), pp. 327-361.

[Ru] H. RUMMLER, "Quelques notions simples en géométrie riemannienne et leurs applications aux feuilletages compactes," Commentarii Mathematici Helvetici, 54 (1979), pp. 224-239.

[Su] D. SULLIVAN, "A Homological Characterization of Foliations Consisting of Minimal Surfaces," Commentarii Mathematici Helvetici 54 (1979), pp. 218-223.

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