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On the derivation of Gauss's principle of least constraint from Lagrange's equations of the second kind

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It is known that the principle of least constraint that **Gauss** presented (1829) can be derived from Lagrange's equations of the first kind for rectangular coordinates in a simple way, in which the expression for the constraint Z – i.e., the function to be minimized – was first formulated by **Scheffler** (1858):

$$Z = \sum_{\nu=1}^{n} \frac{1}{m_\nu} [(m_\nu \ddot{x}_\nu - X_\nu)^2 + (m_\nu \ddot{y}_\nu - Y_\nu)^2 + (m_\nu \ddot{z}_\nu - Z_\nu)^2] .$$

Now, **Lipschitz** (1877) tried to introduce general coordinates into the condition equations that the variable must fulfill identically in place of the rectangular ones that are coupled to each other by the condition equations, and **Wassmuth** (1895) succeeded in performing the transformation of constraint into general coordinates in a very simple way, at least under the assumption that the conditions do not include time explicitly ([1]). In addition, **Radakovich** and other have addressed that problem in detail.

That raises the question of whether it might be possible to derive **Gauss**'s principle for mutually-independent generalized coordinates, as well, and in particular, the general expression for the constraint Z, from the most general Lagrange equations of the *second* kind *directly* in a way that is similar to what one does with rectangular coordinates.

Now, such a direct derivation shall be attempted in what follows in which in total four different cases shall initially be brought under consideration, since the form of the condition equations that exist between the rectangular coordinates, as well as the transformation equations by which the generalized coordinates p_1 , p_2 , …, p_s will be introduced in place of the rectangular ones, can be 1. *holonomic* or 2. *non-holonomic*, and the transformation equations can themselves once more include time t or not in both cases, and thus be *rheonomic* or *scleronomic*, resp.

([1]) **Waßmuth**, "Über die Transformation des Zwanges," These Sitzungsberichte, CIV, Part II.

Meanwhile, it will be shown that the non-holonomity of the generalized coordinates generally exerts no essential influence upon method of derivation, such that actually only *two* main cases will come under consideration in the present article, namely, according to whether the generalized coordinates are scleronomic or rheonomic.

The simpler, and at the same, more common of those two cases shall be attacked first.

A. – Scleronomic generalized coordinates.

At first, the transformation equations shall not include time t explicitly, so they will either read like ([1]):

$$x_\nu = f_\nu (p_1, p_2, \ldots, p_s) \qquad \text{for } \nu = 1, 2, \ldots, 3n,$$

in case they are holonomic, or:

$$dx_\nu = \sum_{h=1}^{s} \pi_h^\nu \, dp_h \qquad \text{for } \nu = 1, 2, \ldots, 3n,$$

in case they are non-holonomic, in which the quantities π_h^ν are any functions of the p_h that do not, however, include time explicitly. The s general or generalized coordinates p_1, p_2, …, p_s that are introduced by these $3n$ equations shall fulfill the τ condition equations that exist between the rectangular coordinates:

$$\varphi_1 (x_1, x_2, \ldots, x_{3n}) = 0, \quad \varphi_2 (x_1, x_2, \ldots, x_{3n}) = 0, \quad \ldots, \quad \varphi_\tau (x_1, x_2, \ldots, x_{3n}) = 0$$

identically, but are completely independent of each other, which is naturally possible only when their number s is equal to the number of degrees of freedom in the system $3n - \tau$. The Lagrange equations of the second kind will then read:

$$Q_h = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} - P_h = 0 \qquad \text{for} \qquad h = 1, 2, \ldots, s, \tag{I}$$

for holonomic coordinates, and for non-holonomic coordinates, they read:

$$Q_h = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} - P_h + \sum_{\nu=1}^{3n} m_\nu \, \dot{x}_\nu \left(\zeta_h^\nu + \sum_{k=1}^{s} \zeta_{hk}^\nu\right) = 0, \qquad \text{for } h = 1, 2, \ldots, s, \tag{II}$$

as **Boltzmann** first found ([2]), in which L is the *vis viva*, and P_h means a generalized force component.

([1]) Following **Boltzmann**, all rectangular coordinates for the system will be denoted with the same symbol x, and likewise, for the sake of simplicity in notation, each mass of each point will be expressed by three symbols; e.g., $m_1 = m_2 = m_3$.

([2]) **Boltzmann**, *Prinzipe der Mechanik*, Part II, pp. 109. See also pp. 8 of this article.

One can now derive Gauss's principle and the general expression for the constraint directly from these generalized Lagrange equation in the present case as follows:

It follows from the equations:

$$Q_1 = 0, \qquad Q_2 = 0, \ldots, \qquad Q_s = 0 \tag{1}$$

that:

$$Q_1\,\delta\ddot{p}_1 + Q_2\,\delta\ddot{p}_2 + Q_3\,\delta\ddot{p}_3 + \cdots + Q_s\,\delta\ddot{p}_s = 0. \tag{2}$$

Furthermore, differentiating the transformation equations will give either:

$$\dot{x}_\nu = \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h}\dot{p}_h \qquad \text{or} \qquad \dot{x}_\nu = \sum_{h=1}^{s} \pi_h^\nu\,\dot{p}_h\,,$$

such that the *vis viva* of the point system will take the form of a quadratic form in the quantities $\dot{p}_h$:

$$L = \sum_{\nu=1}^{3n} \frac{m_\nu}{2}\dot{x}_\nu^2 = \tfrac{1}{2}\sum_{h,k=1}^{s} a_{hk}\,\dot{p}_h\,\dot{p}_k\,. \tag{3}$$

The coefficients $a_{hk} = a_{kh}$ in that form are composed of the quantities $\partial x_\nu / \partial p_k$ (π_h^ν, respectively), so they are functions of the p_h (which do not include time explicitly), and it can be shown that the determinant of the quadratic form L is non-zero ([1]):

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{vmatrix} \neq 0$$

for all values of time t.

One then multiplies equation (2) by $2D$ and gets:

$$\sum_{h=1}^{s} 2Q_h \cdot D \cdot \delta\ddot{p}_h = 0. \tag{4}$$

Now, as is known:

$$D = a_{1h}\,A_{1h} + a_{2h}\,A_{2h} + \ldots + a_{sh}\,A_{sh}\,, \qquad \text{for } h = 1, 2, 3, \ldots, s, \tag{5}$$

in which the A_{rh} shall be the adjoints of D. Equation (4) can then also be written in form:

$$\sum_{h=1}^{s} 2Q_h\,[a_{1h}\,A_{1h} + a_{2h}\,A_{2h} + \ldots + a_{sh}\,A_{sh}]\,\delta\ddot{p}_h = 0. \tag{6}$$

(1) **Boltzmann**, *Prinzipe der Mechanik*, Part II, pp. 35.

Now, there is another theorem from the theory of determinants that when r is different from h :

$$a_{1h}\,A_{1h} + a_{2h}\,A_{2h} + \ldots + a_{sh}\,A_{sh} = 0. \tag{7}$$

If one then adds those terms (which vanish identically) to the left-hand sides of equation (4) or (6) then one will also have:

$$\begin{aligned}\sum_{h=1}^{s} 2Q_h\, \{&[a_{11}\,A_{1h} + a_{21}\,A_{2h} + \ldots + a_{s1}\,A_{sh}]\,\delta\ddot{p}_1 \\ &+ [a_{12}\,A_{1h} + a_{22}\,A_{2h} + \ldots + a_{s2}\,A_{sh}]\,\delta\ddot{p}_2 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ [a_{1s}\,A_{1h} + a_{2s}\,A_{2h} + \ldots + a_{ss}\,A_{sh}]\;\delta\ddot{p}_s\,\} = 0.\end{aligned} \tag{8}$$

For every well-defined h, from (7), all terms that appear in equation (6), except for one, will vanish then. If one adds all of the terms that appear in a column in equation (8) then it will follow that:

$$\begin{aligned}\sum_{h=1}^{s} 2Q_h\, \{&A_{1h}\,(a_{11}\;\delta\ddot{p}_1 + a_{12}\;\delta\ddot{p}_2 + \ldots + a_{1s}\;\delta\ddot{p}_2\,) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ A_{sh}\,(a_{s1}\;\delta\ddot{p}_1 + a_{s2}\;\delta\ddot{p}_2 + \ldots + a_{ss}\;\delta\ddot{p}_2\,)\,\} = 0.\end{aligned} \tag{9}$$

Now equation (3) implies that:

$$\frac{\partial L}{\partial \dot{p}_h} = \sum_{k=1}^{s} a_{hk}\;\dot{p}_k\;, \tag{10}$$

so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) = \sum_{k=1}^{s} a_{hk}\;\ddot{p}_k\;,$$

and therefore $Q_h = a_{h1}\;\ddot{p}_1 + a_{h2}\;\ddot{p}_2 + \cdots + a_{hs}\;\ddot{p}_s\; + \ldots = 0$.

One will then have:

$$\frac{\partial Q_h}{\partial \ddot{p}_1} = a_{h1}\;, \qquad \frac{\partial Q_h}{\partial \ddot{p}_2} = a_{h2}\;, \ldots, \;\frac{\partial Q_h}{\partial \ddot{p}_s} = a_{hs}\;. \tag{11}$$

For that reason, one can also write equation (9) as follows:

$$\sum_{h=1}^{s} 2Q_h \left\{ A_{1h} \left(\frac{\partial Q_1}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_1}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_1}{\partial \ddot{p}_s} \delta \ddot{p}_s \right) \right.$$

$$+ \ A_{2h} \left(\frac{\partial Q_2}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_2}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_2}{\partial \ddot{p}_s} \delta \ddot{p}_s \right) \tag{12}$$

$$+ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left. + \ A_{sh} \left(\frac{\partial Q_s}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_s}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_s}{\partial \ddot{p}_s} \delta \ddot{p}_s \right) \right\} = 0,$$

or in summation form:

$$\sum_{h=1}^{s} 2Q_h \sum_{r=1}^{s} A_{rh} \left(\frac{\partial Q_r}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s} \delta \ddot{p}_s \right) = 0. \tag{12′}$$

If one now expands the first sum over h then that will yield the equation:

$$+ 2Q_1 \sum_{r=1}^{s} A_{r1} \left(\frac{\partial Q_r}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s} \delta \ddot{p}_s \right)$$

$$+ \ 2Q_2 \sum_{r=1}^{s} A_{r2} \left(\frac{\partial Q_r}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s} \delta \ddot{p}_s \right) \tag{13}$$

$$+ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ 2Q_s \sum_{r=1}^{s} A_{rs} \left(\frac{\partial Q_r}{\partial \ddot{p}_1} \delta \ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2} \delta \ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s} \delta \ddot{p}_s \right) = 0.$$

Now one can once more sum over all terms in the same column of this equation, which are all multiplied by the same factor $\frac{\partial Q_r}{\partial \ddot{p}_h} \delta \ddot{p}_h$. Equation (13) will then go to:

$$\sum_{r=1}^{s} \left\{ (2A_{r1}\, Q_1 + 2A_{r2}\, Q_2 + \cdots + 2A_{rs}\, Q_s) \frac{\partial Q_r}{\partial \ddot{p}_1} \delta \ddot{p}_1 \right.$$

$$+ (2A_{r1}\, Q_1 + 2A_{r2}\, Q_2 + \cdots + 2A_{rs}\, Q_s) \frac{\partial Q_r}{\partial \ddot{p}_2} \delta \ddot{p}_2 \tag{14}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ (2A_{r1}\,Q_1 + 2A_{r2}\,Q_2 + \cdots + 2A_{rs}\,Q_s)\frac{\partial Q_r}{\partial \ddot{p}_s}\delta\ddot{p}_s \Big\} = 0.$$

If one now divides equation (14) by D and then sets:

$$\frac{1}{D}(2A_{r1}\,Q_1 + 2A_{r2}\,Q_2 + \cdots + 2A_{rs}\,Q_s) = \frac{\partial Z}{\partial Q_r} \quad \text{for } r = 1, 2, 3, \ldots, s \tag{15}$$

then Z will likewise be defined to be a quadratic function of the quantities Q_1, Q_2, …, Q_s by those s equations that will take the form:

$$Z = \frac{1}{D}\sum_{\mu,\nu=1}^{s} A_{\mu,\nu}\,Q_\mu\,Q_\nu + \varphi(p_1,\ldots,p_s,\dot{p}_1,\ldots,\dot{p}_s) \tag{16}$$

after one integrates, as one easily convinces oneself by differentiating the latter equation. The function φ enters into it because only the quantities $\ddot{p}_h$ were regarded as variable in equations (11).

Equation (14) will now go to:

$$\sum_{r=1}^{s}\frac{\partial Z}{\partial Q_r}\left(\frac{\partial Q_r}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) = 0. \tag{17}$$

However, the last expression is nothing but the first variation of Z when only the quantities $\ddot{p}_h$ are varied (so in the Gaussian sense), and everything else is considered to be constant. Therefore, from equation (17):

$$\delta Z = 0. \tag{18}$$

The second variation is obviously positive, such that the last equation says that when one varies the motion in question in the Gaussian sense, the constraint:

$$Z = \frac{1}{D}\sum_{\mu,\nu=1}^{s} A_{\mu,\nu}\,Q_\mu\,Q_\nu + \varphi(p_1,\ldots,p_s,\dot{p}_1,\ldots,\dot{p}_s)$$

must be a *minimum* for the actual motion, which results from just the general Lagrangian equations (1).

It is probably immediately obvious that this derivation will be valid for holonomic, as well as non-holonomic, *scleronomic* coordinates, when one now understands the Q_h to mean the left-hand sides of the Lagrangian equations in the form (I) in the former case, but in the form (II) in the latter case, because the argument will remain the same in either case.

B. – Rheonomic, generalized coordinates.

The second, and significantly more difficult, of the two main cases that were cited above shall now be brought under consideration, namely, the case in which *time also enters explicitly* into the transformation equations. Those transformation equations will then read simply:

$$x_\nu = f_\nu (t, p_1, p_2, \ldots, p_s) \qquad \text{for } \nu = 1, 2, 3, \ldots, 3n$$

when they are holonomic, or:

$$dx_\nu = \vartheta_\nu \, dt + \sum_{h=1}^{s} \pi_h^\nu \, dp_h \qquad \text{for } \nu = 1, 2, 3, \ldots, 3n$$

when they are non-holonomic ([1]), but in which the functions ϑ_ν and π_h^ν also include time explicitly. The rheonomic generalized coordinates p_1 , p_2 , …, p_s shall once more fulfill the conditions on the system:

$$\varphi_1 (x_1 , x_2 , \ldots, x_{3n}) = 0, \qquad \varphi_2 (x_1 , x_2 , \ldots, x_{3n}) = 0, \quad \ldots, \qquad \varphi_\tau (x_1 , x_2 , \ldots, x_{3n}) = 0$$

identically, and just as before, they must be *completely-independent* of each other.

The Lagrangian equations of the second kind do not change under that assumption, so they will also appear in the form (I) and (II) here according to whether the coordinates p_h are holonomic or non-holonomic, because a change in those equations would occur only when certain relations exist between the coordinates p_h themselves.

However, in order to be able to go deeper into that second case at all, from the mathematical standpoint, it will be necessary to make some assumption about the nature of the functions that appear here, and for that reason, one will perhaps assume that all of the functions that enter are *single-valued*, *analytical* functions, which is an assumption that can probably be regarded as too broad, rather than too narrow, for the physical problem, in general.

Now, as differentiating the transformation equations will show, in the present case, one will have either:

$$\dot{x}_\nu = \frac{\partial x_\nu}{\partial t} + \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h} \dot{p}_h \tag{1}$$

or

$$\dot{x}_\nu = \vartheta_\nu + \sum_{h=1}^{s} \pi_h^\nu \, \dot{p}_h \, ,$$

so the *vis viva* of the point-system L will no longer take the form of a quadratic function of the quantities $\dot{p}_h$ now, but of a function of degree two in those quantities of the type:

([1]) Cf., **Boltzmann**, *Prinzipe der Mechanik*, Part II.

$$L = \sum_{\nu=1}^{3n} \frac{m_\nu}{2} \dot{x}_\nu^2 = \tfrac{1}{2} \sum_{h,k=1}^{s} a_{hk}\, \dot{p}_h\, \dot{p}_k + \sum_{h,k=1}^{s} b_h\, \dot{p}_h + c, \tag{2}$$

in which the coefficients a_{hk} , b_h , and c will themselves include time and the coordinates p_h in any way that depends upon the form of the functions f_ν (ϑ_ν and π_h^ν , resp.).

If one now attempts to actually combine the expressions Q_h here again then one will get from equation (2) that:

a) $$\frac{\partial L}{\partial \dot{p}_h} = \sum_{k=1}^{s} a_{hk}\, \dot{p}_k + b_h ,$$

b) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) = \sum_{k=1}^{s}\left[\frac{\partial a_{hk}}{\partial t}\, \dot{p}_k + \sum_{l=1}^{s} \frac{\partial a_{hk}}{\partial p_l}\, \dot{p}_k\, \dot{p}_l + a_{hk}\, \ddot{p}_k\right] + \frac{\partial b_h}{\partial t} + \sum_{l=1}^{s} \frac{\partial b_h}{\partial p_l}\, \dot{p}_l , \tag{3}$$

c) $$\frac{\partial L}{\partial p_h} = \tfrac{1}{2} \sum_{\rho,k=1}^{s} \frac{\partial a_{\rho k}}{\partial p_h}\, \dot{p}_k\, \dot{p}_\rho + \sum_{\rho=1}^{s} \frac{\partial b_\rho}{\partial p_h}\, \dot{p}_\rho + \frac{\partial c}{\partial p_h} .$$

Moreover, it is known that:

d) $$P_h = \sum_{\nu=1}^{3n} X_\nu \frac{\partial x_\nu}{\partial p_h} ,$$

or for non-holonomic coordinates:

$$P_h = \sum_{\nu=1}^{3n} X_\nu\, \pi_h^\nu ,$$

and finally, from **Boltzmann** ([1]):

e) $$\zeta_h^\nu + \sum_{k=1}^{s} \zeta_{hk}^\nu = \frac{\partial \dot{x}_\nu}{\partial p_h} - \frac{d}{dt}\left(\frac{\partial x_\nu}{\partial p_h}\right).$$

If one then forms the expression Q_h from these relations 3*b*, *c*, *d*, and possibly *e*, according to the forms (I) or (II) that were presented to begin with, then one will see immediately that for the rheonomic coordinates, as well as for the scleronomic ones, one will have in full generality:

$$\frac{\partial Q_h}{\partial \ddot{p}_r} = a_{hr} , \qquad \text{for} \quad \begin{cases} h = 1, 2, \ldots, s, \\ r = 1, 2, \ldots, s. \end{cases} \tag{4}$$

For that reason, one should also consider the determinant D of the quadratic parts in the expression for L [equation (2)] here; i.e., the determinant of degree s :

([1]) **Boltzmann**, *Prinzipe der Mechanik*, Part II, pp. 106.

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{vmatrix} .$$

However, one can no longer assert now, as perhaps one would in the first case, that this determinant D must be non-zero for all values of time t, because the proof that was given before was essentially based upon the assumption that L was a homogeneous form, which is indeed no longer the case here. In order to examine the determinant D more closely, it is therefore necessary to actual construct it, and indeed initially under the assumption that the transformation equation are *holonomic* in form. One will then have:

$$\dot{x}_\nu = \frac{\partial x_\nu}{\partial t} + \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h}\,\dot{p}_h ,$$

and the *vis viva* will be:

$$L = \sum_{\nu=1}^{3n} \frac{m_\nu}{2}\,\dot{x}_\nu^2 = \sum_{\nu=1}^{3n} \frac{m_\nu}{2}\left[\left(\frac{\partial x_\nu}{\partial t}\right)^2 + 2\frac{\partial x_\nu}{\partial t}\sum_{h=1}^{s}\frac{\partial x_\nu}{\partial p_h}\,\dot{p}_h + \left(\sum_{h=1}^{s}\frac{\partial x_\nu}{\partial p_h}\,\dot{p}_h\right)^2\right]$$

or also:

$$L = \tfrac{1}{2}\sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_1}\,\dot{p}_1 + \cdots + \frac{\partial x_\nu}{\partial p_s}\,\dot{p}_s\right)^2 + \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial t}\sum_{h=1}^{s}\frac{\partial x_\nu}{\partial p_h}\,\dot{p}_h + \tfrac{1}{2}\sum_{h=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial t}\right)^2 .$$

If one then compares that to the general form [equation (2)]:

$$L = \tfrac{1}{2}a_{11}\,\dot{p}_1^2 + \tfrac{1}{2}a_{22}\,\dot{p}_2^2 + \cdots + \tfrac{1}{2}a_{ss}\,\dot{p}_s^2 + a_{12}\,\dot{p}_1\,\dot{p}_2 + \cdots + a_{s-1,s}\,\dot{p}_{s-1}\,\dot{p}_s + b_1\,\dot{p}_1 + \ldots$$

then one will see immediately that the determinant of the coefficients a_{hk} will take the following form in the case of rheonomic, holonomic coordinates:

$$D = \begin{vmatrix} \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_1}\right)^2 & \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_2} & \cdots & \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_s} \\ \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_2} & \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_2}\right)^2 & \cdots & \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_2}\frac{\partial x_\nu}{\partial p_s} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_s} & \cdots & \cdots & \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_s}\right)^2 \end{vmatrix} .$$

However, there is nothing more that one can say about the vanishing of that determinant from its form. It therefore important to point out that this symmetric determinant D can also be regarded as the product of two rectangular matrices ([†]):

$$\begin{bmatrix} m_1 \dfrac{\partial x_1}{\partial p_1} & m_2 \dfrac{\partial x_2}{\partial p_1} & \cdots & m_{3n} \dfrac{\partial x_{3n}}{\partial p_1} \\ m_1 \dfrac{\partial x_1}{\partial p_2} & m_2 \dfrac{\partial x_2}{\partial p_2} & \cdots & m_{3n} \dfrac{\partial x_{3n}}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ m_1 \dfrac{\partial x_1}{\partial p_s} & m_2 \dfrac{\partial x_2}{\partial p_1} & \cdots & m_{3n} \dfrac{\partial x_{3n}}{\partial p_s} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1}{\partial p_1} & \dfrac{\partial x_1}{\partial p_2} & \cdots & \dfrac{\partial x_1}{\partial p_s} \\ \dfrac{\partial x_2}{\partial p_1} & \dfrac{\partial x_2}{\partial p_2} & \cdots & \dfrac{\partial x_2}{\partial p_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_{3n}}{\partial p_1} & \dfrac{\partial x_{3n}}{\partial p_2} & \cdots & \dfrac{\partial x_{3n}}{\partial p_s} \end{bmatrix}$$

However, since the number of rows s is smaller than the number of rows $3n$ in any event, from a theorem in the theory of determinants ([1]), this symbolic product can also be represented as a sum of $\binom{3n}{s}$ squares, each of which has the general form:

$$m_{r_1} \cdot m_{r_2} \cdots m_{r_s} \cdot \begin{vmatrix} \dfrac{\partial x_{r_1}}{\partial p_1} & \dfrac{\partial x_{r_1}}{\partial p_2} & \cdots & \dfrac{\partial x_{r_1}}{\partial p_s} \\ \dfrac{\partial x_{r_1}}{\partial p_1} & \dfrac{\partial x_{r_2}}{\partial p_2} & \cdots & \dfrac{\partial x_{r_2}}{\partial p_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_{r_1}}{\partial p_1} & \dfrac{\partial x_{r_2}}{\partial p_2} & \cdots & \dfrac{\partial x_{r_s}}{\partial p_s} \end{vmatrix}^2 ,$$

in which $r_1, r_2, r_3, \ldots, r_s$ mean any combination of class s of the elements 1, 2, 3, …, $3n$ without repetition. However, the coordinates $x_1, x_2, \ldots, x_{3n}$ are not independent of each other, but are coupled by the τ condition equations:

$$\varphi_1 (x_1, x_2, \ldots, x_{3n}) = 0, \qquad \varphi_2 (x_1, x_2, \ldots, x_{3n}) = 0, \qquad \ldots, \qquad \varphi_\tau (x_1, x_2, \ldots, x_{3n}) = 0,$$

which is why one can think of representing τ of the quantities x in these equations as functions of the remaining $3n - \tau = s$ quantities x, which will then be independent of each other – say, in the form:

$$x_{s+1} = \psi_1 (x_1, x_2, \ldots, x_s), \qquad x_{s+2} = \psi_1 (x_1, x_2, \ldots, x_s), \quad \ldots, \quad x_{3n} = \psi_1 (x_1, x_2, \ldots, x_s) .$$

([†]) [Translator: I have taken the liberty of modernizing the matrix notation in this product, so not all of the discussion that follows in regard to the theory of determinants is not applicable to it. I have included the remarks only the for the sake of completeness.]

([1]) **Balzer**, *Determinanten*, § **6**, pp. 48, *et seq.* – **E. Pascal**, *Determinanten*, I, § **7**.

However, one will then have:

$$\frac{\partial x_{s+\lambda}}{\partial p_h} = \frac{\partial \psi_\lambda}{\partial x_1}\frac{\partial x_1}{\partial p_h} + \frac{\partial \psi_\lambda}{\partial x_2}\frac{\partial x_2}{\partial p_h} + \cdots + \frac{\partial \psi_\lambda}{\partial x_s}\frac{\partial x_s}{\partial p_h} \tag{5}$$

for all values of $\lambda = 1, 2, \ldots, \tau$, and $h = 1, 2, \ldots, s$.

If one now introduces this representation (5) into the individual determinant-squares in the development above then, as is clear immediately, a factor that is combined with $\left(\frac{\partial \psi_\lambda}{\partial x_h}\right)^2$ will emerge from each of them, with exception of the first one, and the remaining determinant-squares will then coincide with the first one. For that reason, one can also represent the entire original determinant D as the square of those individual first sub-determinants in the form:

$$D = \left[m_1\, m_2 \cdots m_s + \Phi\left(m_1, \cdots, m_s, \left(\frac{\partial \psi_1}{\partial x_1}\right)^2, \ldots, \left(\frac{\partial \psi_\tau}{\partial x_s}\right)^2 \right)\right] \begin{vmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_s}{\partial p_1} \\ \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_s}{\partial p_2} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial p_s} & \cdots & \frac{\partial x_s}{\partial p_s} \end{vmatrix}^2 .$$

As would emerge immediately from its form, the expression in square brackets cannot vanish, and can only be positive, because the m_ν are positive quantities, since they are material masses, and the function Φ can also be only positive or zero, since it is a sum of quadratic terms. Therefore, the vanishing of the determinant D is possible only when the determinant of degree s :

$$\Delta \equiv \begin{vmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_2}{\partial p_1} & \cdots & \frac{\partial x_s}{\partial p_1} \\ \frac{\partial x_1}{\partial p_2} & \frac{\partial x_2}{\partial p_2} & \cdots & \frac{\partial x_s}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial p_s} & \frac{\partial x_2}{\partial p_s} & \cdots & \frac{\partial x_s}{\partial p_s} \end{vmatrix}$$

vanishes.

However, the determinant Δ, whose vanishing then represents a necessary and sufficient condition for the vanishing of the original determinant D, is now nothing but the functional determinant of the single-valued, analytic functions:

$$x_1 = f_1\,(t, p_1, p_2, \ldots, p_s), \quad x_2 = f_2\,(t, p_1, p_2, \ldots, p_s), \quad \ldots, \quad x_s = f_s\,(t, p_1, p_2, \ldots, p_s) .$$

For that reason, Δ itself is a single-valued, analytic function of the quantities p_1 , p_2 , …, p_s , and as such, must vanish identically when it also vanishes identically in any arbitrarily-small time-interval t' to t''. In that way, however, one would obviously define a relation between the quantities p_1 , p_2 , …, p_s , which would contradict the complete independent of the generalized coordinates from each other that was assumed. One will then have the *identical* vanishing of the original determinant D in any time interval that is also not too small is *excluded*, and all that will remain is that the determinant vanishes for only isolated special moments t_0 , t_1 , t_2 , …, while it will otherwise *be non-zero, in general*.

However, as long as that determinant D is non-zero, as a result of the results that were obtained in equation (4), one can, with no further analysis, also repeat precisely the same conclusions for the case of rheonomic coordinates that one reached in the case of scleronomic coordinates, such that one will get the same expression for the constraint Z (which must be a minimum for the actual motion) as in the first case, namely:

$$Z = \frac{1}{D}\sum_{\mu,\nu=1}^{s} A_{\mu\nu}\, Q_\mu\, Q_\nu + \varphi(p_1,\dots,p_s,\dot{p}_1,\dots,\dot{p}_s),$$

for all intervals in which the assumption $D \neq 0$ is fulfilled.

Naturally, time will also appear *explicitly* in that expression for the constraint Z now, just as it does in the quantities $A_{\mu\nu}$, and D.

However, as far as each individual moment is concerned in which the determinant D actually vanishes (and for which the derivation that was given above would no longer be meaningful then), from a purely-mathematical standpoint, one can probably find the modified general expression for the constraint at those moments by a passing to the limit under special assumptions on the nature of the functions that appear. However, such detailed mathematical investigations must be skipped over here, since more detailed specializations of the assumptions are not permissible with no further discussion for the present general physical problem, and since on the other hand the mechanical principles will no longer be actually applicable when one selects individual moments at which the time and coordinates do not vary, but are regarded as fixed. In general, one also cannot decide when that exceptional case can actually occur, or how often, without going into special cases.

Finally, in regard to the *non-holonomity* of the rheonomic coordinates, which has been excluded up to now, it should be pointed out that it will exert no essential influence on the course of the investigation here either, just as in the first case, except that the single-valued analytic functions π_k^ν (which are generally not better known) will enter in place of the differential quotients $\partial x_\nu / \partial p_k$, and the Lagrangian equations will again be employed in the form (II). However, all conclusions will remain correct, with the exception of the decomposition of the determinant D, which can no longer be performed here, in general. Nonetheless, that fact, which will become clear later, has no further influence on the entire argument, because just like the functional determinant Δ, D is obviously also itself a single-valued analytic function of the quantities p_1 , p_2 , …, p_s that cannot vanish identically without the disturbing the assumed mutual-independence of the generalized coordinates. Under the assumption of single-valued, analytic functions, *the* *general expression for the constraint Z is precisely the same for rheonomic coordinates, and indeed for holonomic, as well as non-holonomic, as it is for sceleronomic coordinates* (except for individual moments that might possibly arise when $D = 0$).

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