Union Française

COMITÉ POUR L'EXPANSION DU LIVRE SCIENTIFIQUE

Secretary General: PAUL GAULTIER, of the Institute

CHARACTERISTICS

OF

DIFFERENTIAL SYSTEMS

AND

WAVE PROPAGATION

BY

TULLIO LEVI-CIVITA

PROFESSOR AT THE UNIVERSITY OF ROME, CORRESPONDENT OF THE INSTITUTE

Lectures edited by G. LAMPARIELLO

French translation from the Italian
by M. BRELOT, DOCTOR OF SCIENCES

English translation
by D H. Delphenich

PARIS
LIBRARIE FÉLIX ALCAN
108, BOULEVARD SAINT-GERMAIN, VI^{e}

—

1932

FOREWORD

TO THE FRENCH TRANSLATION

Now that I am presenting this French translation of my recent book *Caratteristiche e propagazione ondosa* (Bologna, 1931) to the public, I would like to fulfill a very pleasant obligation in vigorously thanking the "Comité pour l'expansion du livre scientifique," and especially its illustrious president Émile Picard, who graciously took the initiative, as well as the eminent director of *la Revue Bleue* and the *Revue Scientifique*, Paul Gaultier, Member of the Institute, who could not have been more friendly nor more obliging in his functions as Secrétaire du Comité. I would also like to thank Marcel Brelot, who accomplished his task as translator with competence and enthusiasm. One must thank him for the presentation, as well as some additions and judicious modifications that made many delicate details much clearer and more precise. Without wishing to enumerate all of them, let me confine myself to pointing out the summary of the interesting notes of Lampariello on elastic waves (cited in the Preface to the Italian edition) that Brelot inserted into the text as a supplementary paragraph (§ **9**).

I would also enjoy this opportunity to emphasize, in a general manner, the elementary character of the mathematical viewpoint of the contents of this little volume. At no point does it deal with difficult questions of existence or the construction of new algorithms, but solely with the consequences that follow easily (by an argument that is entirely analytical) from the notion of characteristic manifold, which permits one to recognize whether this or that type of discontinuity wave is possible, and when that is the case, it provides one with laws of propagation in a simple and elegant form.

Rome, 1 April 1932

TULLIO LEVI-CIVITA

PREFACE

The board of directors of the mathematical seminar at the University of Rome (presided over by Professor ENRIQUES) organized two cycles of conferences for the school year 1930-31 on the theory of characteristics. The first of them, which was entrusted to me, had the goal of briefly reviewing the genesis of that theory in relation to the general existence theorems and pointing out some applications, which are truly grandiose in their simplicity, that began with HUGONIOT and include applications to the propagation of discontinuity waves to acoustic, elastic, optical, electromagnetic, and many other kinds of waves.

The second cycle, which was originally entrusted to VOLTERRA and was developed in his place by ELENA FREDA, was dedicated to the methods of integration by the use of characteristics. They brought to light the extremely substantial contribution of VOLTERRA and the formulas that solved some celebrated problems that he knew how to infer.

The present volume reproduces my lectures, which were carefully transcribed by GIOVANNI LAMPARIELLO.

Having recalled the existence theorems, one then introduces the general notion of characteristics according to the well-known ideas of HADAMARD. There is nothing essentially new in them. Nonetheless, I think that I have made the development simpler and more symmetric, and a result, I have succeeded in endowing the formation of the partial differential equations that define characteristic equations, such as obtaining and discussing the compatibility conditions, with greater algorithmic elegance, which also translates into a certain simplification in the presentation.

That is confirmed in the particular applications to hydrodynamics, electromagnetism, and more especially to the propagation of sound and light that will be studied here ([1]). On the contrary, the other classical meanings to the notion of wave, which are still often considered in mechanics and physics, are hardly mentioned as preliminaries.

Naturally, a study, as summary as it might be, of characteristic manifolds will imply a study of the corresponding bicharacteristic lines. That is why I was led to recall (before passing on to the applications), in general and in a fashion that is more directly associated with canonical systems, CAUCHY's method for the integration of a first-order partial differential equation with an arbitrary number of variables.

Returning to the applications, I would like to point out the general observations in the last paragraph in regard to the characteristics and bicharacteristics that relate to a given differential system (S). Some definitive physical examples will be used there to illustrate and underscore how, in the case where the system (S) permits one to make an adequate analytical representation of an arbitrary physical phenomenon, one can associate the phenomenon itself with a *wave-like aspect* upon crossing the *characteristic manifold* of the system (S) and a *corpuscular aspect* upon traversing the *bicharacteristic lines*. One

---

([1]) LAMPARIELLO adopted the same viewpoint in the study of elastic waves, and that led to several notes to the Rendiconti della R. Accademia dei Lincei (which are collected in § **9** of the French translation).

The case of EINSTEIN's gravitational equations (which present some features that are a bit more complicated) was the first one that I took under consideration in order to apply HADAMARD's theory. Cf., "Caratteristiche e bicaratteristiche delle equazioni gravitazionali di Einstein," Rend. Acc. Lincei (6) **11** (1931), 3-11, 113-121.

will then have a comprehensive mathematical model that is perfectly satisfying in its agnosticism for the duality between waves and corpuscles that inspired the brilliant intuitions of DE BROGLIE, while he himself, along with others, have sought in vain to find a more concrete representation that is truly in accord with the observed facts.

For more precise information on the contents of this book, one can consult the Table of Contents.

Finally, I would also like to express my gratitude to LAMPARIELLO, who has amicably performed the cumbersome task of editing the manuscript and has assisted me in revising the proofs, and to the firm of ZANICHELLI, who undertook and completed this publication with laudable alacrity.

Rome, 20 July 1931

TULLIO LEVI-CIVITA

TABLE OF CONTENTS

___

§ 1. – Review of the existence theorem for the integrals of a system of partial differential equations.

1. Normal systems. – A system of m partial differential equations in m unknown functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ of $n + 1$ independent variables $x_0, x_1, \ldots, x_n$ has the type:

$$E_\mu = 0 \qquad (\mu = 1, 2, \ldots, m), \tag{1}$$

in which E_μ is a function of the x, the φ, and the partial derivatives of the φ with respect to x.

Such a system is called *normal* relative to the variable x_0 if one can put it into the form:

$$\frac{\partial^{r_\nu}\varphi_\nu}{\partial x_0^{r_\nu}} = \Phi_\nu\,(x \mid \varphi \mid \psi \mid \chi) \qquad (\nu = 1, 2, \ldots, m), \tag{1$'$}$$

in which the ψ on the right-hand side are the partial derivatives of each φ_ν with respect to only x_0 that are of order less than r_ν, and the χ are the other partial derivatives of the φ with respect to all of the x, except for φ_ν, which has a global order that is equal to at most r_ν and a partial order in x_0 that is less than r_ν.

Observe that if the system (1′) is normal relative to the variable x_0 then it cannot be normal relative to another variable.

2. Qualitative hypotheses. – The functions Φ_ν are supposed to be analytical and holomorphic in a neighborhood of a system of values for the arguments (viz., the initial values). Under those conditions, one has a fundamental theorem for the existence of unknown functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ that is due to CAUCHY and was made more precise by SOPHIE KOWALEVSKY.

3. Existence theorem for ordinary differential systems. – Before stating the CAUCHY-KOWALEVSKY theorem, and with the goal of understanding its content better, it is convenient to recall the existence theorem for integrals of a system of ordinary differential equations.

If one supposes that the unknown functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ depend upon only the one variable x_0 , which we shall now denote by t, then the differential system (1′) can be written:

$$\frac{d^{r_\nu}\varphi_\nu}{dt^{r_\nu}} = \Phi_\nu\,(t \mid \varphi \mid \psi) \qquad (\nu = 1, 2, \ldots, m). \tag{2}$$

As one knows, the differential system (2) can be put into the form of a system of first-order differential equations, or as one says, normal form (in the strict sense).

Indeed, it suffices to take the derivatives with respect to t up to order $r_\nu - 1$ inclusive to be auxiliary unknowns, along with the φ_ν . Upon setting:

$$\frac{d\varphi_\nu}{dt} = \varphi'_\nu, \qquad \frac{d\varphi'_\nu}{dt} = \varphi''_\nu, \qquad \ldots, \qquad \frac{d\varphi_\nu^{(r_\nu-2)}}{dt} = \varphi_\nu^{(r_\nu-1)},$$

equations (2) can be written:

$$\frac{d\varphi_\nu^{(r_\nu-1)}}{dt} = \Phi_\nu(t \mid \varphi \mid \psi) \qquad (\nu = 1, 2, \ldots, m),$$

and if one lets y_0 denote the general element of the table:

$$\begin{array}{cccc} \varphi_1 & \varphi_2 & \ldots, & \varphi_m \\ \varphi'_1 & \varphi'_2 & \ldots & \varphi'_m \\ \vdots & \vdots & \vdots & \\ \varphi_1^{(r_1-1)} & \varphi_2^{(r_1-1)} & \ldots & \varphi_m^{(r_1-1)} \end{array}$$

then the system (2) will take on the schematic form:

$$(2') \qquad \frac{dy_\rho}{dt} = Y_\rho(t \mid \psi) \qquad (r = 1, 2, \ldots, r\,;\; r = r_1 + r_2 + \ldots + r_m).$$

Under the hypothesis that the y_ρ are analytic and holomorphic in a neighborhood of $t = t_0$, $y_\rho = b_\rho$, there will exist a unique system of analytic functions y_ρ of the variable t that are holomorphic in a neighborhood of $t = t_0$ and take on the values b_ρ for $t = t_0$.

According to CAUCHY, the proof of that celebrated theorem is accomplished by the method of majorants.

First observe that the differential equations permit one to calculate the derivatives of all order for each unknown function y_ρ at the point $t = t_0$ by successive differentiations, and as a result, to write the Taylor development for each y_ρ that relates to that point.

In that development, the term that is independent of t is b_ρ , and the coefficients $\frac{1}{n!}\left(\frac{d^n y_\rho}{dt^n}\right)_{t=t_0}$ $(n = 1, 2, \ldots)$ of the various powers of $t - t_0$ will generally depend upon the b and t_0 .

The essential point of the proof, which was assumed without justification before CAUCHY, consists of showing that those series converge in a suitable neighborhood of $t = t_0$. Upon choosing certain majorizing functions of the y_ρ, the differential system that corresponds to (2′), which can then be integrated by elementary means, will define functions that are analytic and holomorphic in a neighborhood of $t = t_0$ and whose Taylor developments are majorizing for those of the y_ρ .

CAUCHY's theorem for differential systems (2′) is also valid when the right-hand sides of equations (2′) and the initial values b_ρ depend upon a certain (finite) number of parameters that we can denote by $x_1, x_2, \ldots, x_n$ and which vary in the domain where the y_ρ are holomorphic.

One can then state the following theorem, while tacitly assuming in an essential way that everything must behave regularly in a neighborhood of the values considered:

Theorem. – *If one is given a differential system:*

$$\frac{d\varphi_\nu^{(r_\nu-1)}}{dt} = \Phi_\nu(t \mid x \mid \varphi \mid \psi) \qquad (\nu = 1, 2, \ldots, m) \tag{3}$$

then if one chooses the value of each φ_ν *for* $t = t_0$ *arbitrarily, along with its successive derivatives up to order* $r_\nu - 1$ *inclusive as functions of the parameters* $x_1, x_2, \ldots, x_n$ *then there will exist a unique system of functions* φ *that are analytic in* t *and the parameters that satisfy the equations* (3) *and reduce to the chosen functions for* $t = t_0$.

4. – That theorem extends to normal systems (1) of partial differential equations. The novel feature that it presents is that the right-hand sides of equations (3) also include derivatives of the unknown functions with respect to the parameters in such a way that one will also have differential equations that are no longer ordinary, but partial. For reasons of symmetry, we recall the notation x_0 in place of t.

The theorem that was stated in no. **2** asserts that if one is given the values of the φ and ψ (as holomorphic functions of the $x_1, \ldots, x_n$ in a certain domain C) that relate to a value a_0 of x_0 arbitrarily then the functions φ that are holomorphic in the $x_0, x_1, \ldots, x_n$ will be determined (viz., they will exist uniquely) in a neighborhood of $x_0 = a_0$ and in the domain C of the other arguments.

The CAUCHY problem consists precisely in determining the φ that satisfy the normal system (1′) and the preceding initial conditions, which are, we repeat, the values of the unknown functions and their partial derivatives with respect to x_0 of order less than the maximum r_ν for each φ_ν .

That determination – i.e., that of the coefficients in the Taylor developments in a neighborhood of a system of initial values for the x – is obtained by starting from the initial values and successively differentiating equations (1′). Now, the same calculation will apply to the case in which the χ contain derivatives of the φ_ν , always of partial order in x_0 that is less than r_ν , but of total order that is greater than r_ν , except that one can then effectively arrive at the possibility that the developments that one finds will not converge; i.e., one will not have a holomorphic solution.

We shall call a system (1′) *quasi-normal* (relative to x_0) when the ψ are once more partial derivatives of the φ_ν with respect to x_0 and of order less than r_ν , but the χ are the other derivatives with respect to x of arbitrary total order, but of partial order in x_0 less than r_ν for φ_ν .

For the same initial givens, one will not have a multiplicity of holomorphic solutions for a quasi-normal system, but one will not necessarily have that they existence, either.

In what follows, we shall speak of only normal systems. Meanwhile, since the notion of a discontinuity wave that we shall study is more especially linked with the property of *uniqueness* in the CAUCHY problem, as we shall see, it is interesting to point out that some entirely similar considerations can be developed for the analogous questions in which systems that are only quasi-normal are involved essentially.

5. Geometric statement of Cauchy's theorem and its generalization. – Let S be the space of variables $x_0, x_1, \ldots, x_n$. To fix ideas, we suppose that it is endowed with a Euclidian metric upon interpreting the x as Cartesian coordinates. Consider the hyperplane $x_0 = a_0$, which we denote by ϖ.

The existence theorem asserts that one can determine the values of the functions φ in a neighborhood of the hyperplane ϖ (which is called the *support*), when one is given the (initial) values of the φ and the ψ at any point of ϖ arbitrarily.

It is clear that the χ result from the givens on ϖ and the differential equations.

That theorem can be easily generalized by substituting a hypersurface σ in S for the hyperplane ϖ. The generalization can be realized by a simple change of variables, moreover.

Indeed, let:

$$z(x_0, x_1, \ldots, x_n) = x_0 \qquad (x_0 \text{ constant}),$$

for example, be the equation for σ. It will then suffice to replace the x with $(n + 1)$ independent combinations of those x – namely, $z, z_1, z_2, \ldots, z_n$ – one of which (say, z) is rightfully the left-hand side of the equation for σ.

Naturally, in order for the determination of the unknown functions φ to be possible, at least in a neighborhood of σ, it will be necessary that the normal differential system (1′) must become normal relative to z under the change of variables. That is what we shall now address.

6. Change of variables. – Thus, imagine a change of variables $\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ z & z_1 & \cdots & z_n \end{pmatrix}$ under which ϖ will transform into a hypersurface σ.

The normal or quasi-normal differential system (1′) transforms into a system of unknown functions φ of the variables $z, z_1, z_2, \ldots, z_n$. However, one cannot assert its normal or quasi-normal character *a priori*. We then limit ourselves to those particular normal systems for which there is a maximum total order of derivation that is the same for all of the functions.

If we denote that maximum order by s then the differential system, which is assumed to be normal with respect to x_0, can be written more simply:

$$\frac{\partial^s \varphi_\nu}{\partial x_0^s} = \Phi_\nu (x \mid \varphi \mid \chi) \qquad (\nu = 1, 2, \ldots, m). \tag{4}$$

In the right-hand side of this, it is unnecessary to make any distinction between the derivatives ψ of the φ with respect to only x_0 and the derivatives χ of φ with respect to the x_0, x_1, …, x_n, as one does in (1′). If one performs a transformation

$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ z & z_1 & \dots & z_n \end{pmatrix}$ on the variables x then the system (4) will transform into a system of the same maximum order s with respect to z. We shall soon see that it is precisely a normal system, at least, as long as a certain determinant does not vanish.

Upon temporarily assuming that one finds oneself in the case in which that is not the case, one will not have the multiplicity of the functions φ in a neighborhood of the hypersurface σ (which is called the *support*) to begin with when one is given the values of the unknown functions on σ arbitrarily, along with their partial derivatives *with respect to the* x of order less than the maximum s. Without developing the transformation of the CAUCHY problem for σ and the variables x, which would be useless here, we nonetheless once more point out that if one can solve the equation for s for x_0 then its existence and uniqueness relative to σ can be stated with only the derivatives in x_0 as in the case of the hyperplane $x_0 = a_0$.

Finally, we observe that one can always get back to the case in which the derivatives of maximum order s (at most of order s – 1 with respect to x_0) occur linearly in the right-hand side of equations (4).

Indeed, if that were not true then it would suffice to differentiate the two sides of each of equations (4) with respect to x_0. If $\bar{\chi}$ is a general partial derivative of order s then one will have:

$$\frac{\partial^{s+1}\varphi_\nu}{\partial x_0^{s+1}} = \sum \frac{\partial \Phi_\nu}{\partial \bar{\chi}} \frac{\partial \bar{\chi}}{\partial x_0} + \dots$$

The $\partial\Phi / \partial\bar{\chi}$ and the terms that were neglected do not contain partial derivatives of order higher than s; the $\partial\bar{\chi} / \partial x_0$ have order s + 1 and enter linearly.

§ 2. – Characteristic manifolds.

1. In what follows, we shall generally consider only differential systems of the preceding type for which the maximum order of differentiation is $s = 1$ or $s = 2$.

Such a system can be put into the explicit forms:

$$(1) \qquad E_\mu \equiv \sum_{\nu=1}^{m} \sum_{i=0}^{n} E_{\mu\nu}^{i} \frac{\partial \varphi_\nu}{\partial x_i} + \Phi_\mu (x \mid \varphi) = 0 \qquad (\mu = 1, 2, \ldots, m)$$

or

$$(2) \qquad E_\mu \equiv \sum_{\nu=1}^{m} \sum_{i,j=0}^{n} E_{\mu\nu}^{ik} \frac{\partial^2 \varphi_\nu}{\partial x_i \, \partial x_j} + \Phi_\mu (x \mid \varphi) = 0 \qquad (\mu = 1, 2, \ldots, m),$$

respectively.

The $E_{\mu\nu}^{i}$ and Φ_μ in (1) depend upon the x and the φ, while the $E_{\mu\nu}^{ik}$ and Φ_μ in (2) depend upon the x and the φ, along with the first-order partial derivatives of the φ with respect to the x.

We suppose (as one can do with no loss of generality) that:

$$E_{\mu\nu}^{ik} = E_{\mu\nu}^{ki} \qquad (i, k = 0, 1, \ldots, n\,;\ \mu, \nu = 1, 2, \ldots, m).$$

In the particular case of just one unknown function φ, equations (2) will reduce to just one:

$$(3) \qquad E \equiv \sum_{i,k=0}^{n} E^{ik} \frac{\partial^2 \varphi}{\partial x_i \, \partial x_j} + \Phi (x \mid \varphi \mid \chi) = 0,$$

in which the χ denote the first partial derivatives of φ with respect to $x_0, x_1, \ldots, x_n$.

A remarkable equation of type (3) is ([1]):

$$(4) \qquad \Box\varphi = \frac{1}{V^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta_2\, \varphi = 0,$$

in which V is a constant, and:

$$\Delta_2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}.$$

The operator:

$$\Box = \frac{1}{V^2} \frac{\partial^2}{\partial t^2} - \Delta_2$$

is called the *d'Alembertian* or *Lorentzian.*

([1]) We have put the symbol t in place of x_0 ; we shall sometimes do that in what follows without making note of that fact.

Equation (4) occurs in many equations of mathematical physics, and it is called the *canonical equation of small motions* or D'ALEMBERT's equation; we shall develop its genesis a bit later.

2. Conditions for the systems (1) and (2) to be normal. – The equations that constitute the systems (1) and (2) are not solved for the partial derivatives of first or second order, resp., relative to the variable x_0 .

We propose to determine the conditions for such a solution to be possible, which will be conditions under which those systems will be normal with respect to x_0 .

First consider the system (1).

Since only the first partial derivatives with respect to x_0 are important, we write:

$$\sum_{\nu=1}^{m} E_{\mu\nu}^{0} \frac{\partial \varphi_\nu}{\partial x_0} + \ldots = 0 \qquad (\mu = 1, 2, \ldots, m).$$

That system is soluble for the $\partial\varphi / \partial x_0$ if the determinant of the $E_{\mu\nu}^{0}$ is non-zero:

$$\Omega = \left\| E_{\mu\nu}^{0} \right\| \neq 0 \qquad (\mu, \nu = 1, 2, \ldots, m), \tag{5}$$

and one will observe that this determinant contains the independent variables x_0 , x_1, …, x_n and (generally) the unknown functions φ_1 , φ_2 , …, φ_n , as well.

We now pass on to the system (2).

Equations (2) are written:

$$\sum_{\nu=1}^{m} E_{\mu\nu}^{00} \frac{\partial^2 \varphi_\nu}{\partial x_0^2} + \ldots = 0 \qquad (\mu = 1, 2, \ldots, m).$$

and can be solved for the $\partial^2\varphi / \partial x_0^2$ if the determinant of the $E_{\mu\nu}^{00}$ is non-zero:

$$\Omega = \left\| E_{\mu\nu}^{00} \right\| \neq 0 \qquad (\mu, \nu = 1, 2, \ldots, m), \tag{6}$$

In order for (3) to be normal, it is necessary and sufficient that one must have:

$$E^{00} \neq 0. \tag{6′}$$

The determinant $\left\| E_{\mu\nu}^{00} \right\|$ contains the x and, in general, the φ and the first derivatives of the φ with respect to the x.

If the conditions that were found previously are satisfied then one can apply CAUCHY's theorem to given a (supporting) hyperplane $x_0 = a_0$ and the unique determination of the functions φ_ν (or the single function φ, in particular) in a neighborhood of the hyperplane will result.

We shall now seek the conditions under which the normal character will be preserved under a change of variables $\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ z & z_1 & \dots & z_n \end{pmatrix}$ that transforms the hyperplane $x_0 = a_0$ into a hypersurface σ in the space S whose equation is:

$$z\,(x_0\,,\,x_1\,,\,\dots,\,x_n) = z_0\,,$$

and starting from which, it should be possible to determine the functions φ (at least in a certain neighborhood).

3. Conditions for having a normal character relative to the argument *z*. – Set:

$$p_i = \frac{\partial z}{\partial x_i} \qquad (i = 0, 1, \dots, n).$$

One has:

$$\frac{\partial \varphi_\nu}{\partial x_i} = \frac{\partial \varphi_\nu}{\partial z}\,p_i + \sum_{j=1}^{n} \frac{\partial \varphi_\nu}{\partial z_j}\frac{\partial z_j}{\partial x_i} \qquad (\nu = 1, 2, \dots, m),$$

which we abbreviate in the form:

$$\frac{\partial \varphi_\nu}{\partial x_i} = \frac{\partial \varphi_\nu}{\partial z}\,p_i + \dots \qquad (\nu = 1, 2, \dots, m), \tag{7}$$

upon exhibiting only the derivative with respect to z.

Upon substituting that into equations (1), they will become:

$$\sum_{\nu=1}^{m} \frac{\partial \varphi_\nu}{\partial z} \sum_{i=0}^{n} E_{\mu\nu}^{i}\,p_i \; + \dots = 0 \qquad (\mu = 1, 2, \dots, m).$$

Upon setting:

$$\omega_{\mu\nu} = \sum_{i=0}^{n} E_{\mu\nu}^{i}\,p_i\,, \tag{8}$$

the condition for the transformed system to be normal will be written:

$$\Omega = |\,\omega_{\mu\nu}\,\| \neq 0 \qquad (\mu,\, \nu = 1, 2, \dots, m). \tag{9}$$

As far as the system (2) is concerned, one will have:

$$\frac{\partial^2 \varphi_\nu}{\partial x_i\,\partial x_k} = \frac{\partial^2 \varphi_\nu}{\partial z^2}\,p_i\,p_k + \dots,$$

in an analogous fashion, and equations (2) will transform into:

$$\sum_{\nu=1}^{m}\frac{\partial^2\varphi_\nu}{\partial z^2}\sum_{i,k=0}^{n}E_{\mu\nu}^{ik}\,p_i\,p_k+\ldots=0 \qquad (\mu,\ \nu=1,2,\ldots,m)$$

Upon setting:

$$\omega_{\mu\nu}=\sum_{i,k=0}^{n}E_{\mu\nu}^{ik}\,p_i\,p_k\,, \tag{10}$$

the condition for the system to be normal will be expressed by:

$$\Omega = |\,\omega_{\mu\nu}\,| \neq 0 \qquad (\mu,\ \nu=1,2,\ldots,m). \tag{11}$$

In the determinant (9), the $\omega_{\mu\nu}$ are linear forms in $p_0, p_1, \ldots, p_n$, and as a result, Ω will be a form of degree m in its arguments. In the determinant (11), the $\omega_{\mu\nu}$ are quadratic forms in the p in such a way that Ω will be a form of degree $2m$ in its arguments $p_0, p_1, \ldots, p_n$.

In the case of a single equation (3), the determinant will reduce to the single element:

$$\Omega=\sum_{i,k=0}^{n}E^{ik}\,p_i\,p_k\,.$$

We conclude: Any function $z\,(x_0, x_1, \ldots, x_n)$ for which Ω is not identically zero corresponds to a family of hypersurfaces $z = z_0$ such that if one starts from any of them then the CAUCHY problem will admit a unique solution and, in particular, there will not be a multiplicity of holomorphic integral functions φ for the given values of φ in the hypersurface, as well as the first derivatives in the second case (2). Moreover, that will be true by virtue of the fact that the transformed system is normal with respect to z.

4. – When the function $z\,(x_0, x_1, \ldots, x_n)$ satisfies the equation:

$$\Omega = 0, \tag{12}$$

one can no longer apply CAUCHY's theorem upon starting from the supporting hypersurfaces $z = z_0$ for any z_0. One then says that those hypersurfaces are *characteristic manifolds.*

Equations (12) encompass the manifolds for which the unknown functions (if they exist) are determined in a unique manner when one is given their values on the manifold, along with the values of their partial derivatives with respect to x_0 whose order is less than the maximum. It even permits one to specify them completely in certain cases that we shall examine.

In the case of equation (3), the characteristic manifolds are the ones that satisfy the equation:

$$\sum_{i,k=0}^{n} E^{ik} p_i p_k = 0.$$

If one supposes that the coefficients are real E^{ik} then they can be real or imaginary. They will necessarily be imaginary when the quadratic form on the left-hand side is well-defined; otherwise, they will be real if the initial givens are real.

In particular, consider the characteristic manifolds of the canonical equation of small motions. They are the integrals of the partial differential equation:

$$\Omega = \frac{1}{V^2} p_0^2 - \sum_{i=1}^{3} p_i^2 = 0,$$

whose left-hand side is an indefinite quadratic form.

5. Partial differential equations for the characteristic manifold in a particular case. – The determination of the characteristic manifolds is identified with the problem of integrating the first-order partial differential equation $\Omega = 0$, in which the unknown function is z.

That problem will present some special difficulties when the coefficients E_μ^i or $E_{\mu\nu}^{ik}$ in the determinant Ω also depend upon the unknown functions φ of the differential system considered.

The question will simplify when one can narrow down the search for z to the integration of the given normal system. That situation presents itself when the equations of the system are linear in the derivatives of maximum order, because the E will then depend upon only the x.

In that case, Ω will contain only the x and the p, and the equation will have the type:

$$\Omega\,(x \mid p) = 0,$$

in which $p_i = \partial z / \partial x_i$ ($i = 0, 1, \ldots, n$). We stress the fact that the function z does not enter explicitly; we shall return to that equation much later.

6. Mechanical genesis of the canonical equation of small motions. – The fundamental equation of pure hydrodynamics in the case of an irrotational motion of a (perfect) fluid under the action of conservative forces is written ([1]):

$$\frac{\partial \varphi}{\partial t} + \tfrac{1}{2} v^2 - (U - V) = c, \tag{13}$$

([1]) Cf., T. LEVI-CIVITA and U. AMALDI, *Compendio Meccanica razionale*, Part 2a, Chap. XII, Zanichelli, Bologna, 1928 or P. APPELL, *Traité de mécanique rationelle*, t. III, Chap. XXIV, no. 733, Gauthier-Villars, Paris, 1921.

upon denoting the (vectorial) velocity of a particle by $\mathbf{v} = \text{grad}\ \varphi$, as usual, the time by t, the Cartesian coordinates by x_1, x_2, x_3, the velocity potential by $\varphi(t \mid x_1, x_2, x_3)$, and the force function per unit mass by U. The right-hand side is constant in the x_1, x_2, x_3. Finally:

$$P = \int \frac{dp}{\rho},$$

in which p and ρ are the pressure and density, resp., at the same arbitrary point of the fluid, and by hypothesis they satisfy a relation that is called characteristic for the fluid (or the supplementary equation or the equation of state).

In order to determine the motion of the fluid, one must consider not only equation (13) and the characteristic equation, but also the continuity equation:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0,$$

which translates analytically (according to the EULERian viewpoint) into the conservation of mass during the motion. In that equation, the term $d\rho / dt$ denotes the substantial derivative (i.e., the one that follows the particle) of the density with respect to time.

In regard to that, recall that in the study of the motion of a continuous system, one will be led to consider the manner by which some scalar or vectorial quantities depend upon either the position of the point in the domain where the particle exists (the EULERIAN viewpoint) or that of the moving particle M of the system (the LAGRANGIAN viewpoint) at each instant. If q is such a quantity then its local derivative will be defined to be the derivative of q with respect to t by considering P to be fixed; one denotes it by $\partial q / \partial t$.

On the contrary, one defines the substantial derivative of q like the derivative of q with respect to t by considering the same particle M that one follows.

In the first case, one envisions the local variation of q with time. In the second case, one envisions the fashion by which q varies when it is referred to the same particle.

One sees immediately that the two derivatives are linked by the relation:

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \sum_{i=1}^{3} \frac{\partial q}{\partial x_i} u_i ,$$

in which u_i are the components of $\mathbf{v}$ along the x_i-axis.

Having said that, consider, more especially, the case of a perfect gas in the adiabatic regime. Each particle of the gas (in which, the temperature can vary) will then be isolated from any exchange of heat with the neighboring particles, and as one knows from thermodynamics, one will have the relation:

$$p = c_1\, \rho^{\gamma}$$

between p and ρ , in which c_1 depends exclusively upon the initial state of the particle considered (it will reduce to a constant if the temperature and the density are initially uniform), and in which γ is the ratio of the two specific heats at constant pressure and constant volume (γ= 1.41 approximately for air and the most common gases).

The system of equations that serves to determine the motion is then:

$$(14)\qquad \begin{cases} \dfrac{\partial\varphi}{\partial t}+\frac{1}{2}v^2-(U-P)=c, \\ \dfrac{d\rho}{dt}+\rho\operatorname{div}\mathbf{v}=0, \\ P=\displaystyle\int\frac{dp}{\rho}, \quad p=c_1\rho^\gamma, \end{cases}$$

in which the unknown functions are φ, ρ, p.

Now suppose that the gas is removed from any action of forces, and that p and φ differ little from their values under normal conditions; in particular:

$$\rho = \rho_0\,(1 + \sigma),$$

in which σ is a pure number (i.e., a dimensionless quantity) that one considers to be a first-order infinitesimal. Since $\sigma = (\rho - \rho_0) / \rho$, one quite naturally calls it the *concentration* of the gaseous particle.

In addition, we suppose that the differences between substantial derivatives and the local derivatives (with respect to t) of the functions φ and ρ are negligible at any point of the gaseous mass.

It will then follow, in particular, that one can neglect the term $\frac{1}{2}v^2$ in the first equation in the system (14). Indeed, since $\mathbf{v}$ is the gradient of φ:

$$v^2 = \sum_{i=1}^{3}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 .$$

Now:

$$\frac{d\varphi}{dt}-\frac{\partial\varphi}{\partial t} = \sum_{i=1}^{3}\frac{\partial\varphi}{\partial x_i}u_i ,$$

so

$$\frac{d\varphi}{dt}-\frac{\partial\varphi}{\partial t} = v^2$$

will be negligible in comparison to $\partial\varphi/\partial t$.

One will then have:

$$\operatorname{div}\mathbf{v} = \operatorname{div}\operatorname{grad}\varphi = \Delta_2\,\varphi,$$

and as a result, by virtue of the hypotheses that were made, the differential system will take the form:

$$\text{(14}'\text{)}\qquad \begin{cases} \dfrac{\partial\varphi}{\partial t} + P = c, \\ \dfrac{d\rho}{dt} + \Delta_2\varphi = 0, \\ P = \displaystyle\int \dfrac{dp}{\rho} \quad (p = c_1\rho^\gamma). \end{cases}$$

Now:

$$dp = c_1\,\gamma\rho^{\gamma-1}\,d\rho, \qquad \frac{dp}{\rho} = c_1\,\gamma\rho^{\gamma-2}\,d\rho,$$

$$P = c_1\,\frac{\gamma}{\gamma-1}\rho^{\gamma-1} + \text{const.} = \frac{\gamma}{\gamma-1}\frac{p}{\rho} + \text{const.}$$

On the other hand, if one neglects the terms in *s* order higher than 1 then one will deduce from $\rho = \rho_0\,(1 + \sigma)$ that:

$$\begin{aligned} \frac{p}{\rho} &= c_1\,\rho^{\gamma-1} = c_1\,\rho_0^{\gamma-1}(1+\sigma)^{\gamma-1} = c_1\,\rho_0^{\gamma-1}[1+(\gamma-1)\sigma] \\ &= \frac{p_0}{\rho_0}[1 + (\gamma - 1)\,\sigma]. \end{aligned}$$

One will then find that:

$$P = V^2\,\sigma + k,$$

in which *k* is an irrelevant constant, and:

$$V^2 = \gamma\,\frac{p_0}{\rho_0}.$$

With the same approximation, one will find that:

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} = \frac{\partial\log\rho}{\partial t} = \frac{\partial\log(1+\sigma)}{\partial t} = \frac{\partial\sigma}{\partial t}.$$

Observe once more that φ is defined only up to an additive constant with respect to *x*. One can then replace φ with $\varphi + \varphi_0\,(t)$ in equations (14′), in which $\varphi_0\,(t)$ is an arbitrary function of only *t*. $\Delta_2\varphi$ will not change then, while the left-hand side of the first equation will be augmented by $\dfrac{\partial\varphi_0}{\partial t} = \dfrac{d\varphi_0}{dt}$.

In particular, if one chooses φ_0 in such a way that:

$$\frac{d\varphi_0}{dt} = c - k$$

then the system (14′) will reduce to the final form:

$$\text{(15)} \qquad \begin{cases} \dfrac{\partial \varphi}{\partial t} + V^2 \sigma = 0, \\ \dfrac{\partial \sigma}{\partial t} + \Delta_2 \varphi = 0. \end{cases}$$

Upon eliminating σ, one will find that:

$$\frac{1}{V^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta_2\,\varphi = 0,$$

which is the canonical equation for small motions; V^2 has the *constant* $\gamma p_0 / \mu_0$ in it.

---

§ 3. – The canonical equation of small motions. Notion of wave. Velocities of displacement and propagation of a wave surface or discontinuity.

1. Acoustic interpretation. – The equation that was previously established:

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta_2\,\varphi = 0 \tag{1}$$

is applicable to sound vibrations in air or any other gaseous mass, in particular, because one can neglect all dissipative action, to a first approximation that is already quite good, so one can suppose that the motion is irrotational and that there is no exchange of heat between particles (viz., the adiabatic regime).

Suppose that the velocity potential φ relates to sound vibrations in air. $\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}$ then represent the components of the velocity of the air molecule that is at the point (x_1, x_2, x_3) at the instant t.

Furthermore, suppose, more precisely, that a certain layer of air that is found between two surfaces:

$$z\,(t \mid x) = c_1\,, \qquad z\,(t \mid x) = c_2 \tag{2}$$

is in vibration at the arbitrary instant t.

It is at rest [which corresponds to the zero solution $\varphi^* = 0$ for (1)] outside the layer. The phenomenon is characterized by a non-zero solution $\varphi\,(t \mid x)$ inside of it.

2. – We shall now leave aside the acoustic interpretation of the solutions to equation (1) and suppose that $\varphi\,(t \mid x)$ and $\varphi^*\,(t \mid x)$ are solutions of (1) inside and outside the layer that is determined by the surfaces (2), resp. The phenomenon that is represented by equation (1) is characterized by two distinct functions depending upon whether one is located inside or outside the layer. The derivatives of φ of various order are generally subject to sharp variations across the surfaces (2), and that is why they are called *discontinuity* surfaces.

Now, it can happen that such a surface varies with time. One will then say that the discontinuity *propagates*, and it will take on the name of a *wave*, more specifically.

Therefore, if one interprets equation (1) as being capable of characterizing the propagation of a wave then the discontinuity surfaces (or, as we also say, the *wave surfaces*) bound a layer that displaces and possibly deforms with time.

If one assumes that no molecular interpenetrations or cavitations are produced during the motion then the normal components of the velocity of a particle cannot be subject to any discontinuity upon crossing a wave surface. We shall also exclude the phenomenon of molecule sliding across such a surface, which would imply tangential discontinuities for the velocities.

We remark here that from the postulate of the forces (in particular, the pressures) upon which the mechanics of continuous media is based, under normal conditions, the pressure cannot be subject to any sharp jump, even if the regime of the motion varies sharply.

Observe further that the density ρ is coupled with the pressure by the characteristic equation (which is the same on both sides of the discontinuity surface).

The continuity in ρ will then result from that of p. On the other hand, from the first equation (15) of the preceding paragraph, the derivatives of φ and φ^* with respect to t represent the density up to a constant factor. They must therefore be exempt from any discontinuity upon crossing the wave surface.

The preceding considerations lead us to conclude that in order for equation (1) to define the propagation of a wave, one must assume that the two solutions φ and φ^*, which are assumed to exist and characterize the phenomena inside and outside the layer, agree; i.e., that their first derivatives in space and time must be equal to each other on the wave surfaces that bound the layer at each instant.

On the contrary, the second derivatives *are* subject to sharp variations. We shall address them later when we extend the present considerations to an arbitrary normal system of partial differential equations. We can also see then how the wave surfaces are characterized from the analytical viewpoint.

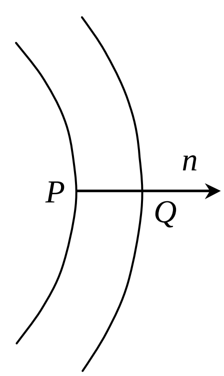

Figure 1

3. Velocities of displacement and propagation. – Consider a wave surface that bounds a layer that is the site of a perturbation at the instant t, and let n be the normal at an arbitrary point P that is oriented outward (Fig. 1).

The surface displaces, and at the instant $t + dt$, it cuts that normal n at a point Q.

Let dn be the algebraic measure of the segment PQ, which is regarded as positive outward.

The ratio $a = dn / dt$ is called the *displacement velocity* of the wave surface at the point P at the instant considered. In ordinary situations, $a > 0$ at all points of one of the surfaces that bound the layer, and $a < 0$ at all points of the other one. The surfaces are then called the *leading* and *trailing* wave fronts.

Much later, we shall give explicit expressions for a by utilizing the equation of the wave surface σ_t .

The difference $c = a - d\varphi / dn$ between the displacement velocity and the normal component to σ_t of the velocity of the fluid particle that if found at P at the instant considered is called the *(normal) velocity of propagation* of σ_t at the point P.

From the principle of relative motion, that difference obviously measures the velocity with which the surface displaces, not with respect to the fixed axes, but with respect to the medium.

If it is at rest outside the layer then one will have $\varphi^* = 0$, and since, as one sees, φ and φ^* must agree on σ, one will have $d\varphi / dn = 0$, so $c = a$.

In that case, the velocity of propagation will be identical with that of displacement.

4. – Now consider the hypersurface σ: $z\,(t \mid x)$ = const. in space-time that corresponds to the wave surface σ_t in the space of only the x. It is essential to remark that σ is a characteristic manifold relative to equation (1); i.e., that z is an integral of the equation:

$$\frac{1}{V^2}\,p_0^2 - \sum_{i=1}^{3} p_i^2 = 0. \tag{3}$$

Indeed, assume that one can argue as if the functions φ in question were holomorphic on σ; if σ is not a characteristic then there will be a contradiction between the uniqueness property in CAUCHY's theorem and the existence of two solutions φ that take the same values on σ, as well as their first-order partial derivatives, but present discontinuities in the higher-order derivatives on σ.

The propagation of waves is possible then only as long as the wave surfaces σ_t correspond to characteristic manifolds σ.

Moreover, a particular case of equation (1) shows that in order for one to be able to once more solve the CAUCHY problem upon starting with a characteristic manifold, certain conditions must be satisfied; there will not be a single solution then, but an infinitude of them.

In order to explain that, suppose that φ depends upon t and just x_1, which we now write as x. Equation (1) will become:

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0 \tag{1′}$$

Recall how one integrates that celebrated equation. One remarks that it can be written:

$$\left(\frac{1}{V}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{1}{V}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\varphi = 0,$$

in which the left-hand side naturally amounts to the product of operators that is applied to φ.

Introduce the variables z, z_1, which are linked with the old ones t, x by the relations:

$$z = x - V\,t, \qquad z_1 = x + V\,t,$$

so

$$x = \tfrac{1}{2}(z + z_1), \quad t = \frac{1}{2V}(z_1 - z).$$

From the theorem on the derivation of composed functions, one has:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{V}\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{V}\frac{\partial}{\partial t}\right),$$

and equation (1′) transforms into:

$$\frac{\partial^2 \varphi}{\partial z\, \partial z_1} = 0,$$

which is integrated by inspection.

The general integral is:

(4) $$\varphi = \alpha(z) + \beta(z_1),$$

in which α and β are two arbitrary differentiable functions of z and z_1, respectively. One will see forthwith (and as one might expect, moreover) that one cannot generally solve the CAUCHY problem for a supporting line $z = c$, but it is necessary that the givens must satisfy a compatibility condition. When it is verified, there will be an infinitude of solutions.

Indeed, it follows from (4) that:

$$\begin{cases} \varphi(c, z_1) = \alpha(c) + \beta(z_1), \\ \left(\dfrac{\partial \varphi}{\partial z}\right)_{z=c} = \alpha'(c). \end{cases}$$

One cannot give the functions φ_0 and φ_1 of the variable z_1 arbitrarily then, which must reduce to φ and $\partial\varphi / \partial z$ for $z = c$. The function φ_1 (z_1) must be a constant, and in that case, there will be an infinitude of forms for the solution φ to the problem.

Those remarks show how essential the consideration of characteristic manifolds is.

Up to now, we have only addressed the negative aspects of such things, but it is appropriate to point out that their importance is also very great from the constructive point of view. Indeed, they serve to solve the CAUCHY problem precisely for supporting manifolds that are not characteristic.

That idea is due to B. RIEMANN, who successfully treated the problem of integrating the second-order linear equation of hyperbolic type in two independent variables:

$$\frac{\partial^2 z}{\partial x\, \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + c\, Z = 0$$

in a celebrated presentation to the Göttingen Academy of Science (1860).

RIEMANN's method was reprised by DARBOUX ([1]) and others. Some important research on partial differential equations of hyperbolic type in three or more variables, as well as on the mathematical expression for HUYGHENS's principle ([2]), which was formulated for the first time by KIRCHHOFF for the canonical equation of small motions, was done by VOLTERRA ([3]) and HADAMARD ([4]) since 1892.

([1]) Cf., G. DARBOUX, *Leçons sur la théorie des surfaces*, v. II, Gauthier-Villars, Paris, 1889.

([2]) One will find some bibliographic references, and especially for the Italian contributions, in the *Lezioni di meccanica razionale* by LEVI-CIVITA and AMALDI, vol. II, Part Two, pp. 468. Zanichelli, Bologna, 1927.

([3]) Cf., V. VOLTERRA, *Leçons sur l'intégration des équations différentielles aux dérivées partielles*, taught in Stockholm, Paris, Hermann, 1912. *Lectures delivered at the celebration of the twentieth anniversary of the foundation of Clark University*, second lecture, 1912.

([4]) Cf., HADAMARD, *Leçons sur la propagation des ondes*, Hermann, Paris, 1903. *Lectures on Cauchy's Problem in linear partial differential equations*, New Haven, 1921. A French edition is currently in press at Hermann.

For the bibliography of the subject, the reader can consult the interesting pamphlet by R. D'ADHEMAR, *Les des équations aux dérivées partielles à caractéristiques réelles*, Coll. Scientia, Gauthier-Villars, Paris, 1907.

§ 4. – Extension of the concept of wave propagation to an arbitrary normal system.

1. – The considerations that were originally developed for equation (1) of the preceding paragraph can be easily extended to the systems of equations that were considered in no. **1** of § **2**.

Once more, introduce the variables t, x_1, x_2, …, x_n in the space S, and suppose that inside and outside the layer that is bounded by two *hypersurfaces* (which we shall even call simply *surfaces* when it will create no ambiguity):

$$z = c_1 , \qquad z = c_2 , \tag{1}$$

one of the two systems:

$$E_\mu \equiv \sum_{\nu=1}^{m} \sum_{i=0}^{n} E_{\mu\nu}^{i} \frac{\partial \varphi_\nu}{\partial x_i} + \Phi_\mu (x \mid \varphi) = 0 \qquad (\mu = 1, 2, \dots, m) \tag{2}$$

or

$$E_\mu \equiv \sum_{\nu=1}^{m} \sum_{i=0}^{n} E_{\mu\nu}^{ik} \frac{\partial \varphi_\nu}{\partial x_i \, \partial x_k} + \Phi_\mu (x \mid \varphi \mid \chi) = 0 \qquad (\mu = 1, 2, \dots, m), \tag{3}$$

is satisfied by two groups of functions $\varphi_1 , \varphi_2 , \dots, \varphi_m$ and $\varphi_1^* , \varphi_2^* , \dots, \varphi_m^*$, respectively.

Upon using the considerations that were introduced in the context of the canonical equation for small motions as our basis, we shall suppose that upon crossing some hypersurfaces (1) that bound a layer that displaces and even deforms in the course of time in the space S' of only (x_1, x_2, x_3), certain first-order partial derivatives [in the case of s = 1 – i.e., equation (2)] or second-order ones [in the case of s = 2 – i.e., equation (3)] will be subject to sharp variations (i.e., jumps).

We also suppose that the functions φ and φ^* are continuous upon traversing the hypersurfaces (1) and that in the case of s = 2, the same thing is also true for the first derivatives.

Those hypotheses correspond to a type of wave phenomenon for which the wave surfaces are the ones that bound the layer.

In the case of a general system of maximum order s, the functions φ and φ^* must agree on the wave surfaces, along with their derivatives of order less than s.

On the contrary, there *will* be discontinuities for the derivatives of order s.

Upon assuming, as above, that one can argue as if the φ and φ^* were holomorphic in ($x \mid t$) on the surfaces (1), they must characteristic manifolds, due to the uniqueness property of CAUCHY's theorem.

We shall address the problem of determining the two groups of unknown functions φ and φ^* here. Such a study would oblige us to discuss the CAUCHY problem that relates to the characteristic manifolds.

That study was carried out, at least in certain special cases, by HADAMARD and advanced by RIQUIER ([1]) and DELASSUS. Following CARTAN, it will bring us back to the PFAFF equation ([2]).

On the contrary, we assume the existence of functions φ and φ^* at the same time as the existence of a propagation of waves and propose to illuminate some properties.

2. – If $z = c$ is a characteristic surface σ then the function z must satisfy the equation:

$$\Omega\,(x \mid p) = 0,$$

in which:

$$p_i = \frac{\partial z}{\partial x_i} \qquad (i = 0, 1, \dots, n).$$

In reality, that was established only on σ; i.e., for z equal to a particular value c. However, since the argument z does not enter into Ω explicitly, the restriction to $z = c$ is not essential; i.e., Ω must be zero as long as one takes the p_i to be equal to the derivatives of that function z. One will then be dealing with a true (first-order) partial differential equation for z.

When the functions E that figure in system (2) or (3) depend upon only the x, that will characterize one and only one z. However, before advancing the study of that case, we shall consider the wave surface σ_t in the Euclidian space S' with Cartesian coordinates $(x_1, x_2, \dots, x_n)$ that corresponds to σ and extend the notion of the velocity of displacement.

The surface σ_t of the equation $z\,(t \mid x) = c$ divides the neighboring space into two regions I and II, which generalize the interior and exterior of the layer to the case where σ_t is the wave surface of a vibrating wave that moves in a medium that is at rest. Orient the normal in the direction that points from the region I to the region II. Since one can always replace z with $-z$, one can suppose that the direction for the normal that is > 0 is that of increasing z.

Now consider two wave surfaces at the instants t and $t + dt$:

$$(4) \qquad z\,(t \mid x) = c, \qquad\qquad z\,(t + dt \mid x) = c.$$

The normal n to σ_t at P meets the second surface σ_{t+dt} at a point Q. If dn is the algebraic measure of the segment PQ on the oriented normal then the ratio $a = dn / dt$ is called the displacement velocity of the wave surface at the point P at the instant considered.

3. Calculating the displacement velocity. – We seek an expression for a that involves the elements of the surface σ.

([1]) Cf., Ch. RIQUIER, *Les systèmes d'équations aux dérivées partielles*, Gauthier-Villars, Paris, 1910. Furthermore, M. JANET, *Leçons sur les systèmes d'équations aux dérivées partielles*, Gauthier-Villars, Paris, 1929.

([2]) Cf., E. GOURSAT, *Leçons sur le problème de Pfaff*, Hermann, Paris, 1922.

One knows that the quantities:

(5) $$\alpha_i = \frac{p_i}{g} \qquad (i = 1, 2, \ldots, n),$$

in which g is the positive determination of the square root of:

(5′) $$g^2 = \sum_{i=1}^{n} p_i^2 ,$$

constitute a system of direction cosines for the normal n to σ_t at P; that is the one that corresponds to the normal that is oriented in the sense of increasing z.

If x_i and $x_i + dx_i$ are the coordinates of the points P, Q, resp., then from equations (4), one must have:

$$z\,(t \mid x) = c, \qquad z\,(t + dt \mid x + dx) = c,$$

so when one takes the difference:

(6) $$dz = p_0\, dt + \sum_{i=1}^{n} p_i\, dx_i = 0.$$

Since the dx_i are the components of the vector PQ and the normal is oriented in the sense of increasing z:

(7) $$dx_i = \alpha_i\, dn \qquad (i = 1, 2, \ldots, n).$$

Upon substituting those expressions in (6) and then taking (5) and (5′) into account, one will get:

$$p_0\, dt + dn \sum_{i=1}^{n} \alpha_i\, p_i \equiv p_0\, dt + g\, dn = 0,$$

so

$$a \equiv \frac{dn}{dt} = -\frac{p_0}{g}$$

and

(8) $$|\, a \,| = \left| \frac{dn}{dt} \right| = \frac{|\, p_0 \,|}{g}.$$

That is the formula that we have in mind. It exhibits the manner by which the displacement velocity varies on each surface with P and time.

4. An application of formula (8). – Let us apply the formula that was just found to equation (1) of § **3**, which corresponds, as we said, to the phenomenon of the propagation of sound.

From equation (5′) of the preceding no., equation (3) of § **3** is written:

$$\frac{1}{V^2}\,p_0^2 = g^2,$$

so

$$V = \frac{|\,p_0\,|}{g}.$$

One then finds that the constant V is nothing but the propagation velocity of a wave surface that bounds a layer that is the site of sound vibrations at the instant t.

For perfect gases in the adiabatic regime, we have seen that:

$$V^2 = \gamma\frac{p_0}{\rho_0},$$

in which γ, p_0, ρ_0 have the significance that given in (§ **2**, no. **6**), and in particular, p_0 is the rest pressure.

Upon considering the case in which there is equilibrium outside of a certain vibrating layer, one can conclude that the formula:

$$V = \sqrt{\gamma\frac{p_0}{\rho_0}}$$

must give the propagation velocity for sound.

Let us adopt the practical system of units (meters, seconds, kg-weight): 1 m^3 of air weighs 1.29 kg. p_0 , viz., atmospheric pressure, is around 1 kg per cm^2, so it amounts to 10^4 units in the system. The acceleration of gravity is 9.8, and:

$$V^2 = 1.41\,\frac{10^4 \times 9.8}{1.29}.$$

One will then find that the velocity V is around 331 m /s, which is in good agreement with experiments.

The calculation of the propagation velocity of sound (when we imagine the simplest case of plane waves) was done for the first time by NEWTON, who found that:

$$V = \sqrt{\frac{p_0}{\rho_0}}$$

when one assumes that the phenomenon is isothermal.

In the case of air, that expression will give V = 280 meters per second at 0°. On the contrary, experiments yield the value of 333 m / s at 0°.

LAPLACE gave the reason for the disagreement between theory and experiment by noting that the variations of pressure that are to the propagation of waves produce variations in temperature that imply warming in the compressed layers and cooling in the dilated ones.

By taking that into account, he then showed that in order to obtain the true propagation velocity theoretically that agrees with experiment, it would suffice to regard the phenomenon considered to be adiabatic, which would lead one to multiply the ratio p_0 / ρ_0 by γ.

§ 5. – Digression on the general conception of wave motion ([1]).

1. What is a wave motion? – One can perhaps restrict the motion of a fluid to one for which the displacements of its particles imply an even more marked motion for some particular elements that are present, such as a free surface or a separation surface.

However, that would not be a property that clearly discriminates, as one can show in a classical example.

Consider a rectangular channel with a horizontal base and vertical walls, and take the case in which the motion of the gravitating liquid that is contained in the channel (say, water, to be precise) always takes place parallel to the ends and in an identical fashion in all of the longitudinal sections of the channel; i.e., in the various vertical planes that are parallel to the ends. The study of the phenomenon will then come down to the two-dimensional case in an arbitrary longitudinal section.

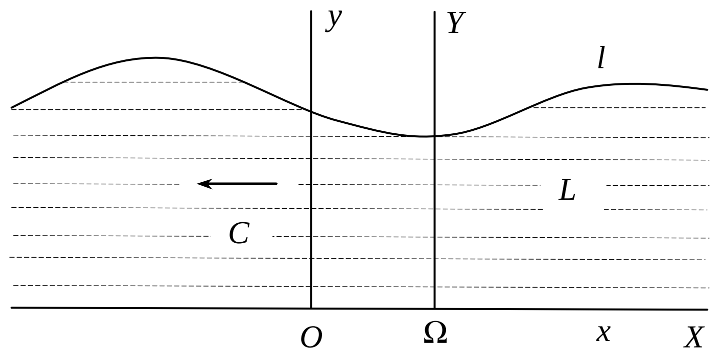

Figure 2.

The base (Fig. 2) will be represented by a horizontal line ΩX, and the free surface by a line l, which generally varies with time, but in such a way that it is only slightly different from a horizontal line $y = h$ (at least, under ordinary conditions); that will be the level line under static conditions, when h is the (mean) depth of the channel.

Let L denote the domain of the motion – i.e., the indefinite band (which generally also varies in time) that is found between the base and the line l.

Having said that, the general problem of hydrodynamics for those moving planes can obviously be formulated in the following way: At the instant $t = 0$, one is given a perturbation; i.e., the configuration of l and the distribution of the velocities in L. Characterize the appearance of the later motion, and in particular, the law of variation of l.

The question thus posed (all details aside) belongs to the general problem of waves in channels if one says "wave propagation" to mean, more precisely, the evolution of motion according to a certain law when one starts with a given perturbation. From such a viewpoint, one can focus on the general integral directly, and it is only then that it can present a wave-like aspect in the ordinary sense of the word, and almost by accident in certain applications. That is what one sees in the early research by LAGRANGE, who

---

([1]) Cf., T. LEVI-CIVITA, "Questioni di meccanica classica e relativista," *II Conferenze. Le onde dei liquidi. Propagazione nei canali.* Zanichelli, Bologna, 1924.

reduced problem of the equation of the vibrating string by neglecting the vertical acceleration of the motion of each particle in comparison to g (the acceleration of gravity). The most important application that he made was concerned with the tides.

POISSON and CAUCHY proceeded in an analogous fashion while abandoning the too-restrictive hypothesis on the acceleration and treating small motions in deep channels in general. The notion of wave appeared by itself in a manner that was at least very expressive in regard to question whose physical nature imposed such a notion, if not quite clear.

That is what will happen, for example, in the case of what one calls emersion waves, which are produced when a solid, such as a floating body, is raised briefly and removed from contact with the fluid mass, which then tends to recover its equilibrium.

The proper motion (at least, theoretically – i.e., the ideal case of absolute incompressibility) of the liquid will begin immediately in the entire mass of water and change in the height of the free surface will displace along the channel with an acceleration that is reasonable constant (if one is indeed dealing with acceleration and not velocity). There is something that propagates, but although that constitutes a highlight, it does not seem to be a law that can clearly characterize the motion as a wave motion. Things are entirely different for the propagation of discontinuities that we are addressing systematically here.

2. – It is important to emphasize that although the case in which discontinuities are involved is indisputably the most striking one, the quantitative study of wave phenomena in fluids and elastic media was not originally posed in that form, but was developed without relinquishing the principle of continuity.

In reality, one takes the simplest cases as models, in which one can limit oneself to the consideration of just one dimension, which is what happens for vibrating strings.

Let s denote the position parameter of the vibrating particle in the one-dimensional region in question (e.g., the initial rectilinear configuration of the vibrating string), and let t denote time, while φ denotes the displacement.

First of all, one considers the solutions $\varphi(s, t)$ to the differential equation that models the phenomenon that depend upon a unique argument $s_1 = s - Vt$, in which V is a constant. The binomial $s_1 = s - Vt$ is called the *phase* of the corresponding phenomenon.

When the phase is constant – i.e., when one imagines a relation of the type:

$$s_1 = s - Vt = \text{const.}$$

between the arguments s and t, which are independent *a priori*, the characteristic $\varphi(s_1)$ of the vibratory phenomenon will remain constant. In other words, for an observer that displaces along the string (or more generally, along the support of the argument s) with the *constant velocity V*, the phenomenon will appear to be stationary. That is why the constant V can be interpreted as a *propagation velocity* for the vibratory state of the string relative to the solutions of the particular type $\varphi(s_1)$. That is precisely the sense in which one refers to *waves that propagate with velocity V.*

More generally, even in the case of three-dimensional sound, elastic, or electromagnetic phenomena, the study of waves is developed by the search for particular classes of solutions (of the systems of partial differential equations that correspond to the phenomena) that depend upon a single argument that is a linear function of the three spatial coordinates x_1, x_2, x_3, and time $t \equiv x_0$; i.e., just one argument of the type:

$$\xi = \sum_{i=0}^{3} c_i\, x_i\ ,$$

in which the c_i are constants that are arbitrary *a priori*.

We suppose that we are dealing with solutions that depend effectively upon the point (x_1, x_2, x_3) (i.e., which are not just functions of time). It will then be necessary that one of the three coefficients c_1, c_2, c_3 must be non-zero, or rather, that the vector **c** whose components along the coordinate axes are c_1, c_2, c_3 must be non-zero. One can then regard $s_1 = \xi / c$ (c is the length of the vector **c**) as the unique spatial element upon which the solution in question depend, instead of ξ. We remark that the $c_i / c = \alpha_i$ ($i = 1, 2, 3$) are the direction cosines of the vector **c** and set $- c_0 / c = V$. The unique argument upon which the determining parameters of the phenomenon are supposed to depend is then once more present in the form $s_1 = s - Vt$, in which:

$$s = \sum_{i=1}^{3} \alpha_i\, x_i$$

is virtually a spatial coordinate along the direction $(\alpha_1, \alpha_2, \alpha_3)$ and can then be denoted more simply by x_0 with no essential restriction (and by taking the x_3-axis for its direction, for example).

One will then be dealing with plane waves, in the sense that the vibratory state depends upon only s for any value of t, and as a result, it will be identical to the same plane s_1 = const. at all points.

It will follow further that the phenomenon will be stationary for an observer with respect to which s_1 = const.; i.e., for which s displaces with velocity V, etc.

One poses a more general problem by taking s to be an arbitrary function (and not necessarily a linear one) of x_1, x_2, x_3 and supposing that the determining parameters of the phenomenon are functions of not only $s_1 = s - Vt$, but also of another purely-spatial argument.

The latter type includes the waves that one calls spherical waves. Some types of waves that are even more general, but conceived in an analogous fashion, have been studied from various viewpoints by BATEMAN and MAGGI ([1]).

____

([1]) H. BATEMAN, *Electrical and optical wave motion*, Cambridge University Press, 1915.
G. A. MAGGI, "Sulla propagazione delle onde di forma qualsivoglia nei messi isotropi," Rend. Acc. Lincei (5) **29** (2nd sem, 1920), pp. 371-378.

§ 6. – The Cauchy method for integrating a first-order partial differential equation.

1. – As we saw in § **2**, (no. **4**), for the two systems that were considered there (no. **1**), the characteristic manifolds:

$$z\ (x_0, x_1, \ldots, x_n) = \text{const.}$$

annul a certain determinant Ω that generally contains the unknown functions φ, in addition to the x and $p = \partial z / \partial x$. However, as was pointed out (§ **2**, no. **5**), there is an important class of normal systems for which Ω contains only the x and the p. It is comprised of the systems of order $s = 1$ and $s = 2$ whose coefficients, $E_{\mu\nu}^{i}$ or $E_{\mu\nu}^{ik}$, respectively, are functions of only the x.

Similarly, for the normal systems of maximum order (which is the same for all variables) $s > 1$, one will have an equation of the same type for the determination of the characteristic manifolds provided that the coefficients of the derivatives of maximum order depend upon the x exclusively.

Since we propose to study the equation:

$$\Omega\ (x \mid p) = 0, \tag{1}$$

in which:

$$p_i = \frac{\partial z}{\partial x_i} \qquad (i = 0, 1, 2, \ldots, n), \tag{2}$$

we shall present CAUCHY's method for the integration of a first-order partial differential equation, and in particular, equation (1), in which the unknown z does not enter explicitly. However, we are sure that at least one of the p figures in Ω – for example, p_0. Upon solving (1) for p_0, we can write:

$$p_0 + H\ (t, x_1, \ldots, x_n \mid p_1, p_2, \ldots, p_n) = 0, \tag{3}$$

in which:

$$p_i = \frac{\partial z}{\partial x_i} \qquad (i = 0, 1, 2, \ldots, n).$$

It is convenient to first treat the linear case.

2. Case of the linear equation. – It is well-known that if H is a linear function of the p then the problem of the integration of (3) will amount to the integration of an ordinary differential system.

It is nevertheless good to recall that result, which likewise applies to the general case.

Equation (3) will then have the type:

$$p_0 + A_0 + \sum_{i=1}^{n} A_i\, p_i = 0, \tag{4}$$

in which the A are functions of only the variables $t, x_1, \ldots, x_n$. Consider the space S_{n+2} of $n + 2$ variables $t, x_1, \ldots, x_n, z$, and a hypersurface $z = \varphi(t \mid x)$, namely, σ, that is an integral of equation (4).

Draw reference axes in the space S_{n+2} (which we assume to be Euclidian, for the sake of convenience), while exhibiting just one variable x for more clarity.

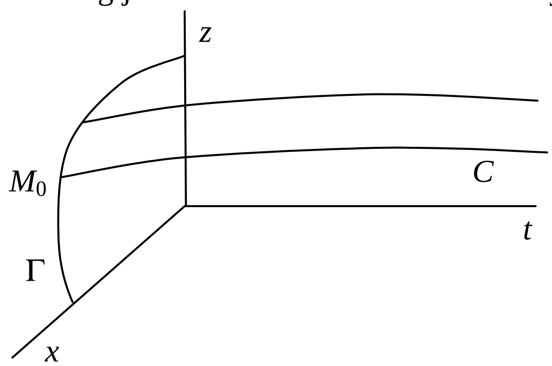

Figure 3.

Let Γ be the section of the hypersurface s by the hyperplane $t = 0$; i.e., the locus of points of $t = 0$ that are defined by the equation:

$$z = \varphi(0 \mid x) \qquad \text{or, more briefly} \qquad z = \varphi_0(x).$$

The fundamental idea that will guide us in what follows consists of regarding σ as the locus of ∞^n curves that are obtained by integrating a convenient ordinary system of the type:

$$\frac{dx_i}{dt} = X_i(t \mid x) \qquad (i = 1, 2, \ldots, n), \tag{5}$$

$$\frac{dz}{dt} = Z(t \mid x) \tag{6}$$

of rank $(n + 1)$, whose unknown functions of t are $x_1, \ldots, x_n$, and z. The system (5), (6) introduces $n + 1$ arbitrary constants, but it will diminish their number by 1 if one wishes that the system should be compatible with the equation $z = \varphi(t \mid x)$ for σ.

The essential hypothesis that justifies the consideration of that system is that it must be independent of the previous integration of equation (6).

Upon regarding z as a function of t and x, one will deduce from (6) and (5) that:

$$\frac{dz}{dt} = Z = p_0 + \sum_{i=1}^{n} p_i \frac{dx_i}{dt} = p_0 + \sum_{i=1}^{n} p_i X_i,$$

so upon taking (4) into account:

$$Z = -A_0 + \sum_{i=1}^{n} p_i\,(X_i - A_i).$$

Since one desires that the differential system (5), (6) should be independent of the integration of (4) – i.e., valid for any integral hypersurface – the coefficients of p_i must be zero, so:

$$X_i = A_i\,,$$

and therefore, it will follow that:

$$Z = -A_0\,.$$

The desired differential system is then:

(5′) $$\frac{dx_i}{dt} = A_i \qquad (i = 1, 2, \ldots, n),$$

(6′) $$\frac{dz}{dt} = -A_0\,,$$

or, if one prefers the classical form:

$$\frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \ldots = \frac{dx_n}{A_n} = -\frac{dz}{A_0} = dt,$$

which permits one to determine the integral hypersurfaces of (4).

Indeed, in order to solve the CAUCHY problem that relates to a given curve Γ in the hyperplane $t = 0$, it will suffice to first consider the totality of the (∞^n) integral curves of the system (5′), in which z does not occur.

The integration of the remaining differential equation (6′), which amounts to a simple quadrature when one has already integrated the system (5′), then completes the determination of the curves in the space S_{n+2} [of $(t \mid x \mid z)$]. If one wishes that among those curves there are ones that are supported by Γ then one must write out that z takes the value $\varphi_0(x)$ for $t = 0$, in which the x correspond to the same value $t = 0$ and can then be identified with the n arbitrary constants that are introduced by the integration of the system (5′). Hence, there will be n arbitrary constants, and each integral hypersurface σ of (4) will appear to be the locus of (∞^n) integral curves of (5′), (6′) that issue from the points of Γ.

3. General case. – The process that consists of converting the integration of a linear first-order partial differential equation into that of an ordinary differential system, which is due to LAGRANGE, was generalized to nonlinear equations by LAGRANGE himself, and then by CHARPIT, CAUCHY, and JACOBI. Here, we shall give CAUCHY's method in a form that will best show the principle (somewhat better than what appears in the usual presentations).

We recall the general equation:

$$p_0 + H(t, x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0 \tag{3}$$

and investigate whether it is possible to determine the general integral hypersurface (viz., the one that is provided by CAUCHY's theorem with arbitrary initial data) as the locus of integral curves of a suitable differential system.

One easily recognizes that, in general, one can no longer associate (3) with a congruence of curves in the space S_{n+2} that agrees with any integral hypersurface. However, one must pass to an auxiliary space with a larger number of dimensions. It will be precisely useful to consider the arguments to be the $p_0, p_1, \ldots, p_n$ that define the tangent element at P geometrically, in addition to the coordinates x of the running point on the integral hypersurface σ. If one wishes that the p should have a concrete metric significance then it will suffice (as one has done already for ease of description, moreover) to attribute a Euclidian metric on the space S_{n+2} and regard the t, x, and z as Cartesian coordinates. Hence $p_0, p_1, \ldots, p_n$, and -1 are proportional to the direction cosines of the normal to σ with respect to the axes of the $t, x_1, x_2, \ldots, x_n, z$, respectively.

Having said that, we seek to associate (3) with a differential system of the type:

$$\left\{\begin{aligned} \frac{dx_i}{dt} &= X_i(t, x \mid p), \\ \frac{dp_i}{dt} &= P_i(t, x \mid p), \end{aligned}\right. \qquad (i = 1, 2, \ldots, n), \tag{7}$$

$$\frac{dz}{dt} = Z(t, x \mid p). \tag{8}$$

If one knows the X_i as functions of t, x, p then one can easily define the expression for Z. Indeed, since z is a function of t by the intermediary of $x_0 \equiv t$ and the other x, one will have:

$$\frac{dz}{dt} = p_0 + \sum_{i=1}^{n} p_i \frac{dx_i}{dt},$$

so, thanks to the first of equation (7):

$$\frac{dz}{dt} = Z(t, x \mid p) = p_0 + \sum_{i=1}^{n} p_i X_i. \tag{9}$$

Observe that equation (8), in which Z is given by (9), must be considered only after integrating the system (7) because z will then be given as a function of t by a simple quadrature.

Once more, consider the space S_{n+2} and a hypersurface Γ of the hyperplane $t = 0$. Let M_0 and ω_0 be a point of Γ and the hypersurface that is tangent at M_0 to the hypersurface s, which is the integral of (3) that passes through Γ.

We shall express the idea that the integral curve C_0 of the system (7), (8) that issues from M_0 and is tangent to ω_0 belongs to the integral hypersurface σ while respecting the equations:

$$p_i = \frac{\partial z}{\partial x_i} \qquad (i = 0, 1, \ldots, n;\ x_0 \equiv t),$$

and that this is true for any Γ that passes through M_0 .

Upon passing from t to $t + dt$, p_i will be increased by dp_i , in such a way that:

$$dp_i = P_i\, dt. \tag{10}$$

On the other hand, in order for the relations:

$$p_i = \frac{\partial z}{\partial x_i}$$

to persist, it is necessary that one must have:

$$dp_i = \sum_{j=0}^{n} p_{ij}\, dx_j \qquad (i = 0, 1, \ldots, n), \tag{11}$$

in which:

$$p_{ij} = p_{ji} = \frac{\partial^2 z}{\partial x_i\, \partial x_j} \qquad (i, j = 0, 1, \ldots, n).$$

One must realize the equality of the expressions for the dp_i that are provided by (10) and (11). Observe that the quantities p_{ij} with non-zero indices i, j depend upon the choice of Γ (which is arbitrary, by hypothesis), while the p_{ij} that have at least one zero index satisfy some relations that are deduced from (3) by differentiation, namely, the $(n + 1)$ relations:

$$p_{0i} + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} p_{ji} + \frac{\partial H}{\partial x_i} = 0 \qquad (i = 0, 1, \ldots, n). \tag{12}$$

Since there are $\frac{1}{2}(n + 1)(n + 2)$ quantities p_{ij} , in total:

$$\tfrac{1}{2}(n + 1)(n + 2) - (n + 1) = \tfrac{1}{2} n\,(n + 1)$$

of them will remain arbitrary, while the quantities that are available are:

$$X_1 , X_2 , \ldots, X_n , \qquad P_1 , P_2 , \ldots, P_n ,$$

which are $2n$ in number, which will be less than $\frac{1}{2} n\,(n + 1)$ when $n > 3$.

The preceding conditions will then lead one to think that it would be impossible to determine the P_i in such a way:

$$P_i\, dt = \sum_{j=0}^{n} p_{ij}\, dx_j$$

are independent of the p_{ij} .

Nevertheless, the following developments will assure the success of CAUCHY's idea:

Upon differentiating with respect to t, the $p_i = \partial z / \partial x_i$ will become:

$$\frac{dp_i}{dt} = P_i = p_{i\,0} + \sum_{j=0}^{n} p_{ij} \frac{dx_j}{dt} = p_{i\,0} + \sum_{j=0}^{n} p_{ij}\, X_j\ .$$

Upon eliminating the $p_{i\,0} = p_{0i}$ by means of the relations (12) and taking into account the symmetry of the p_{ij} in their indices, the preceding relations will become:

$$P_i = -\frac{\partial H}{\partial x_i} + \sum_{j=1}^{n} \left(X_i - \frac{\partial H}{\partial p_j} \right) p_{ij} \quad (i = 1, 2, \ldots, n).$$

These will also be satisfied independently of the p_{ij} if:

$$X_i = \frac{\partial H}{\partial p_j},$$

$$P_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2, \ldots, n).$$

It will then seem that if one starts from the arbitrary point M_0 of the integral hypersurface σ and attributes increments to the t, x, p, z that satisfy the differential system (7), (8), which will henceforth be characterized in the form:

$$(13) \qquad \begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_j}, \\[2mm] \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x_i}, \end{cases} \qquad (i = 1, 2, \ldots, n),$$

$$(14) \qquad \frac{dz}{dt} = \sum_{j=1}^{n} p_n \frac{\partial H}{\partial p_i} - H,$$

then one will pass to an infinitely-close point M_1 that again belongs to σ and for which the $p_i + dp_i$ will determine the direction of the normal to σ at that point.

The same considerations can be repeated immediately when one starts from M_1, and that will exhibit the essential fact that the system (13), (14) was formed in such a fashion that is will be valid for all integral hypersurfaces σ that pass through M_0 with a given orientation to the normal – i.e., with given p.

We then find (for the integral hypersurface σ) the same conditions at M_1 that we found at M_0 .

One then deduces that the entire curve C that is defined unambiguously by (13), (14) under the condition that the x, p, z take values for $t = 0$ that correspond to M_0 belongs to the integral hypersurface in question, which one should note well is any of the integral hypersurfaces that pass through M_0 and admit ϖ_0 as a tangent hyperplane there.

One will then obtain the important geometric corollary that:

If two integral manifolds touch at a point then they will touch all along a curve C that passes through that point.

CAUCHY called the curves C "characteristics." Following HADAMARD, we shall call them *bicharacteristics*, while reserving the word "characteristics" for the hypersurfaces (in the space S of t, x) that behave in an exceptional way in regard to the CAUCHY problem.

4. Solving the Cauchy problem. – The method that was just presented permit one to solve the CAUCHY problem; i.e., to determine the integral hypersurface s in the space S_{n+2} that passes through a given hypersurface Γ in the plane $t = 0$.

Indeed, it will suffice to consider the integral curves of the system (13), (14) that issue from the points of Γ. They will constitute an integral hypersurface σ of equation (3).

5. The Hamiltonian system that is associated with the equation $\Omega = 0$. – The system (13) has the Hamiltonian form. The characteristic function H depends upon t, x, p, in general.

Now, one sees that Ω is a form of degree m or $2m$ with respect to the p according to whether $\Omega = 0$ is the equations of the characteristic manifold in the case $s = 1$ or $s = 2$, respectively.

From that homogeneity, if the p verify the equation $\Omega = 0$ then the same thing will be true for the $\lambda\, p_i$, in which λ is arbitrary. Upon solving for p_0, one will then see that if one multiplies $p_1, p_2, \ldots, p_n$, and also p_0 by λ then the same thing will be true for H. In other words, H is a homogeneous function of degree one with respect to the p.

The Hamiltonian system for which the function H is homogeneous of degree one with respect to the p enters into some questions of geometrical optics ([1]).

It is important to observe that in this case, from EULER's theorem on homogeneous functions, the right-hand side of (14) will be identically zero. Hence:

$$\frac{dz}{dt} = 0, \qquad \text{so} \qquad z = \text{const.}$$

([1]) Cf., T. LEVI-CIVITA and U. AMALDI, *loc. cit.* (see above, pp. 19), pp. 456-469.

That must say that, in this case, the integral curves of the system (13), (14) belong to the hypersurfaces z = const. In particular, they are effectively plane curves if $n = 1$ (i.e., if there is only one variable x besides t).

6. Applications. – Suppose that $E^i_{\mu\nu}$ or $E^{ik}_{\mu\nu}$ are constants, which will physically correspond to the case of a *homogeneous medium* in the case of $n = 3$.

Ω will then depend upon only the p, and as a result, the function H will depend upon only the $p_1, p_2, \ldots, p_n$.

The Hamiltonian system becomes:

$$\begin{cases} \dfrac{dp_i}{dt} = 0, \\ \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i} \end{cases} \qquad (i = 1, 2, \ldots, n).$$

The first of them gives the n integrals:

$$p_i = p_i^0 \qquad (i = 1, 2, \ldots, n),$$

which, when substituted in the $\partial H / \partial p_i$, will render them constant in such a way that if one lets x_i^0 denote the initial values of the x_i then the second equation will give:

$$x_i = t\frac{\partial H}{\partial p_i^0} + x_i^0 \qquad (i = 1, 2, \ldots, n) \tag{13}$$

upon integration.

Hence, it will emerge that the bicharacteristics are lines in either the space S of the variables $t \equiv x_0 , x_1 , \ldots, x_n$ or, upon eliminating t, in the geometrical space S' of only the $x_1 , \ldots, x_n$.

As far as the determination of the wave surfaces are concerned (always under the hypothesis of a homogeneous medium), we fix our attention upon their configuration, instant-by-instant, in the geometric space S' (of only x).

In fact, we are dealing with a particular case of the geometric solution of the CAUCHY problem, which was pointed out already in no. **4**, upon taking into account the two facts that t no longer has the significance of a geometric coordinate, but that of time, and that the bicharacteristics are lines.

Let us see what a wave surface σ_0 that was given arbitrarily at the instant $t = 0$ will become at the instant t.

Draw the line through each point x_i^0 of σ_0 (which corresponds to the instant $t = 0$) that is defined parametrically by equations (15). One sees that its direction will depend upon the manner by which H is a function of the p.

The point M_0 with coordinates x_i^0 at the instant $t = 0$ will go to the points M whose coordinates are (15) at the instant t.

The locus of points M is the wave hypersurface σ_t at the instant t.

7. Plane waves. – Formulas (15) highlight the fact that if the wave surface is planar at the instant $t = 0$ then that will continue to be true in the course of time.

It will then follow that plane waves are always possible in an arbitrary homogeneous medium and for a phenomenon of an arbitrary nature.

8. Epicentral waves. – In particular, suppose that σ_0 is infinitely small around a point O (which we take to be the origin, so it will follow that $x_i^0 = 0$) at the instant $t = 0$.

That is the case of a perturbation that is initially limited to a very small neighborhood of the point O. If one extends a term that is used in seismology then the point O will be called the *epicenter* and the waves that emanate from it will be called *epicentral*.

Now take σ_0 to be infinitely small and $x_i^0 = 0$. From (15), one will see that the wave surfaces will be enlarged homothetically around O in the course of time.

Furthermore, recall that H is homogeneous and of degree one with respect to the p, so dH / dp_i will be homogeneous of degree zero. One will then see that the dH / dp_i depend upon only the direction cosines:

$$\alpha_i = \frac{p_i}{g} \qquad (i = 1, 2, \ldots, n).$$

Equations (15) will then give:

$$x_i = t\ \psi_i\,(\alpha_1, \alpha_2, \ldots, \alpha_n) \qquad (i = 1, 2, \ldots, n),$$

and for each value of t, that will constitute a parametric representation of the wave surface as a function of the n variables α_i, when they are coupled by the relation $\sum_{i=1}^{n} \alpha_i^2 = 1$, and will provide some other ones with $n - 1$ independent parameters.

---

§ 7. – Geometrico-kinematical and dynamical compatibility conditions.

1. Geometrico-kinematical compatibility conditions. – Suppose that $z(t \mid x) =$ const. is a wave surface in the space of x at the instant t, and consider the corresponding surface σ in the space-time S of (t, x). Let φ, φ^* be two groups of functions that satisfy the normal system of partial differential equations.

By analogy with the linking conditions, which are formulated in conformity with the mechanical problem, in the case of the canonical equation of small motions, we suppose that the φ and φ^* take the same values on σ, as well as their partial derivatives up to order $s - 1$, but that some of the partial derivatives of order σ present discontinuities upon traversing σ. The φ and φ^* will then define a wave phenomenon on one side of σ and the other.

We shall determine the compatibility relations that those jumps must verify upon crossing the surface.

Case of $s = 1$. Suppose that f is a continuous and differentiable function of the variables $t \equiv x_0, x_1, \ldots, x_n$, and set:

$$f_i = \frac{\partial f}{\partial x_i} \qquad (i = 0, 1, \ldots, n).$$

In the case of $s = 1$, the first derivatives of f – i.e., the f_i – will generally be subject to jumps.

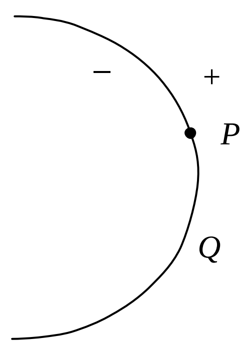

Figure 4.

Let us label the two parts of space that are separated by the surface s by + and – and let f^+ and f^- denote the limiting values of a function f whose point-argument tends to a point of the surface from each side of it. In general, set:

$$\Delta f = f^+ \text{ and } f^-.$$

In particular, if one is dealing with a function f that is continuous upon crossing σ then:

$$f_P^+ = f_P^-,$$

in which P is a point on the surface.

If Q is another point of the surface then one will also have:

$$f_Q^+ = f_Q^- ,$$

so

$$f_Q^+ - f_P^+ = f_Q^- - f_P^- .$$

Upon taking Q infinitely close to P, one will get:

$$df_P^+ = df_P^- ,$$

or since the derivatives have limits and if we denote the coordinate differentials by dx_i then we will have:

$$\sum_{i=0}^{n} f_i^+ \, dx_i = \sum_{i=0}^{n} f_i^- \, dx_i$$

upon passing from P to Q, so:

$$\sum_{i=0}^{n} (f_i^+ - f_i^-) \, dx_i = \sum_{i=0}^{n} \Delta f_i \, dx_i = 0$$

for all dx_i that correspond to infinitely-small displacements that are tangent to the surface; i.e., ones for which:

$$dz = \sum_{i=0}^{n} p_i \, dx_i = 0.$$

Upon applying LAGRANGE's classical procedure (viz., the method of undetermined multipliers), the condition will become:

$$\sum_{i=0}^{n} (\Delta f_i - \lambda \, p_i) \, dx_i = 0.$$

Upon supposing that $p_0 \neq 0$, one can choose λ in such a fashion that:

$$\Delta f_0 - \lambda \, p_0 = 0,$$

so

$$\lambda = \frac{\Delta f_0}{p_0},$$

and as a result, since the $dx_1, dx_2, \ldots, dx_n$ are arbitrary, one will have:

$$\Delta f_i = \lambda \, p_i \qquad (i = 1, 2, \ldots, n).$$

If the f are assumed to be continuous then one will then remember that the $n + 1$ jumps in the first derivatives of f upon crossing the surface are coupled to the p by the relations:

(1) $$\Delta f_i = \lambda\, p_i \qquad (i = 1, 2, \ldots, n),$$

in which λ is undetermined *a priori*.

Case of $s = 2$. The function f and its first derivatives must be continuous upon crossing the surface, in such a way that the preceding formulas will apply to the second derivatives.

Since each f_i is continuous, one will then have:

$$\Delta f_{ij} = \lambda_i\, p_j = \lambda_j\, p_i\,,$$

if λ_i denotes the multiplier (which is characteristic of the discontinuity in the derivatives) that corresponds to f_i, and:

$$f_{ij} = \frac{\partial^2 f}{\partial x_i\, \partial x_j} = \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}\,.$$

The coefficients λ generally vary when one passes from one first derivative to the other. One infers, moreover, that:

$$\frac{\lambda_i}{p_i} = \frac{\lambda_j}{p_j} = \rho,$$

so

(2) $$\Delta f_{ij} = \rho\, p_i\, p_j \qquad (i, j = 0, 1, \ldots, n).$$

We shall give the name of *geometrico-kinematical compatibility conditions* to the conditions (1) or (2) (which are independent of the fact that we are dealing with solutions to a given normal system, so in the physical interpretation, it will be independent of the special mechanism of the phenomenon that is governed by that system).

2. Dynamical compatibility conditions. – On the contrary, the dynamical compatibility conditions are deduced from the partial differential equations directly, and their name comes from the fact that one considers the differential system to be one that defines a certain physical phenomenon (in particular, a dynamical one).

For $s = 1$, the equations are written:

$$E_\mu \equiv \sum_{\nu=1}^{m} \sum_{i=0}^{n} E_{\mu\nu}^{i} \frac{\partial \varphi_\nu}{\partial x_i} + \Phi_\mu = 0 \qquad (\mu = 1, 2, \ldots, m).$$

Since the $E_{\mu\nu}^{i}$ and Φ_μ are continuous, the jumps in the partial derivatives $\partial\varphi_\nu / \partial x_i$ upon crossing the surface σ must satisfy the relations:

$$\sum_{\nu=1}^{m}\sum_{i=0}^{n} E_{\mu\nu}^{i} \Delta \frac{\partial \varphi_\nu}{\partial x_i} = 0 \qquad (\mu = 1, 2, \ldots, m).$$

However (no. **1**), if λ_ν is the multiplier that corresponds to φ_ν then:

$$\Delta \frac{\partial \varphi_\nu}{\partial x_i} = \lambda_\nu p_\nu \qquad (\nu = 1, 2, \ldots, m).$$

It will then result that:

$$\sum_{\nu=1}^{m}\sum_{i=0}^{n} E_{\mu\nu}^{i} \lambda_\nu p_i = 0,$$

or, from the notation (8) of § **2** (cf., pp. 8):

$$\sum_{\nu=1}^{m} \omega_{\mu\nu} \lambda_\nu = 0 \qquad (\mu = 1, 2, \ldots, m). \tag{3}$$

Those relations constitute a system of m homogeneous linear equations in m parameters:

$$\lambda_1, \lambda_2, \ldots, \lambda_n,$$

which characterize the discontinuities in the first derivatives upon crossing the surface σ.

Such a system admits non-zero solutions because the determinant of the $\omega_{\mu\nu}$ is zero for a characteristic manifold (viz., the surface σ).

In the concrete applications, one must often specify not only the nature of the wave surfaces, but also the dynamical compatibility conditions. One will then form the linear equations (3), and when their determinant is equal to zero, that will permit one to determine the wave surfaces. One will then deduce the following rule:

Practical rule: The partial differential equation of the characteristic manifolds is obtained by annulling the determinant of the system of dynamical compatibility conditions.

____

§ 8. – Applications to the equations of hydrodynamics.

1. – The fundamental equations of hydrodynamics are:

$$(1)\qquad \begin{cases} \dfrac{d\mathbf{v}}{dt} = \mathbf{a} = \mathbf{F} - \dfrac{1}{\rho}\,\text{grad}\,p, \\[2mm] \dfrac{d\rho}{dt} + \rho\sum_{i=1}^{3}\dfrac{\partial u_i}{\partial x_i} = 0. \end{cases}$$

In this system, the unknown functions are the u_i (i = 1, 2, 3), which are the components of the velocity **v** of the fluid particle and the density ρ. The independent variables are t, x_1, x_2, x_3, while p denotes the pressure, and **F** is the force per unit mass.

As one knows, the substantial derivative d / dt is expressed by:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_j u_j \frac{\partial}{\partial x_j}.$$

Upon excluding the case of homogeneous liquids for which ρ is a constant, one can regard p as a function of ρ.

The first of equations (1) is equivalent to three scalar equations:

$$\frac{du_i}{dt} + \frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial\rho}{\partial x_i} = X_i \qquad (i = 1, 2, 3),$$

in which the X_i are the components of the force **F** along the axes.

The system (1) can then be written:

$$(2)\qquad \begin{cases} \dfrac{du_i}{dt} + \dfrac{1}{\rho}\dfrac{dp}{d\rho}\dfrac{\partial\rho}{\partial x_i} = X_i \quad (i = 1, 2, 3) \\[2mm] \dfrac{d\rho}{dt} + \rho\sum_{i=1}^{3}\dfrac{\partial u_i}{\partial x_i} = 0. \end{cases}$$

Form the corresponding equation $\Omega = 0$. Under the change of variables $\begin{pmatrix} t \equiv x_0, & x_1, & x_2, & x_3 \\ z, & z_1, & z_2, & z_3 \end{pmatrix}$, the system (2) will become normal only if $\Omega \neq 0$.

Since:

$$\frac{\partial\varphi}{\partial x_j} = \frac{\partial\varphi}{\partial z}p_j + \ldots \qquad (j = 0, 1, 2, 3)$$

for an arbitrary function $\varphi(t, x)$, the transformed equations of (2) will be written:

$$\frac{\partial u_i}{\partial z}(p_0 + \sum_{j=1}^{3} u_j p_j) + \frac{1}{\rho}\frac{dp}{d\rho}\, p_i \frac{\partial \rho}{\partial z} + \ldots = 0 \qquad (i = 1, 2, 3),$$

$$\frac{\partial \rho}{\partial z}(p_0 + \sum_{j=1}^{3} u_j p_j) + \rho \sum_{i=1}^{3} p_i \frac{\partial u_i}{\partial z} + \ldots = 0,$$

or rather, since $dz / dt = p_0 + \sum_j u_j p_j$:

$$(2') \qquad \begin{cases} \dfrac{\partial u_i}{\partial z}\dfrac{dz}{dt} + \dfrac{1}{\rho}\dfrac{dp}{d\rho}\, p_i \dfrac{\partial \rho}{\partial z} + \cdots = 0 \quad (i = 1, 2, 3), \\ \dfrac{\partial \rho}{\partial z}\dfrac{dz}{dt} + \rho \displaystyle\sum_{i=1}^{3} p_i \dfrac{\partial u_i}{\partial z} + \cdots = 0. \end{cases}$$

The determinant Ω is that of the coefficients of the three $\partial u_i / \partial z$ and $\partial \rho / \partial z$.

Hence, the equation that must be satisfied by any wave surface is:

$$(3) \qquad \begin{vmatrix} \dfrac{dz}{dt} & 0 & 0 & \dfrac{1}{\rho}\dfrac{dp}{d\rho}\, p_1 \\ 0 & \dfrac{dz}{dt} & 0 & \dfrac{1}{\rho}\dfrac{dp}{d\rho}\, p_2 \\ 0 & 0 & \dfrac{dz}{dt} & \dfrac{1}{\rho}\dfrac{dp}{d\rho}\, p_3 \\ \rho p_1 & \rho p_2 & \rho p_3 & \dfrac{dz}{dt} \end{vmatrix} = 0,$$

or, after developing:

$$\left(\frac{dz}{dt}\right)^2 \left[\left(\frac{dz}{dt}\right)^2 - g^2 \frac{dp}{d\rho}\right] = 0,$$

upon once more setting:

$$g^2 = \sum_i p_i^2 \,.$$

The equation the splits into:

$$(\mathrm{I}) \qquad \frac{dz}{dt} = 0,$$

$$(\mathrm{II}) \qquad \left(\frac{dz}{dt}\right)^2 - g^2 \frac{dp}{d\rho} = 0.$$

Equation (I) expresses the idea that one is dealing with a discontinuity surface that is fixed *with respect to the medium*; i.e., it always involves the same fluid particles.

As for equation (II), if we suppose (as is always the case for real fluids) that the pressure increases with the density and set:

$$\frac{dp}{d\rho} = V^2, \qquad V \text{ real} > 0$$

then we will get:

$$\frac{dz}{dt} = \pm\, g\, V.$$

Now:

$$\begin{aligned}\frac{dz}{dt} &= p_0 + \sum_i u_i\, p_i \\ &= g\left(\frac{p_0}{g} + \sum_j u_j \frac{p_j}{g}\right),\end{aligned}$$

so if we preserve our conventions (§ **4**, no. **3**) and let a and v_n denote the displacement velocity and the normal component of **v** then:

$$\frac{dz}{dt} = g\,(= a + v_n) = -\, g\,(a - v_n),$$

in which $a - v_n$ is the propagation velocity.

The two possible signs of $dz / dt = \mp\, g\, V$ then corresponds to the two cases in which the velocities of propagation and displacement do or do not have the same sign, resp.; i.e., in which those two velocity vectors have the same or opposite sense, resp. Moreover, one will have:

$$V = |\, a - v_n\,|,$$

which shows that V is the absolute value of the propagation velocity.

Hence, that propagation velocity V will have the formula:

$$V = \sqrt{\frac{dp}{d\rho}}\,, \tag{4}$$

and in the adiabatic case (§ **1**, no. **6**), one will have:

$$p = c_1\, \rho^\gamma, \qquad \frac{dp}{d\rho} = \gamma\frac{p}{\rho}, \quad \text{which will give:} \qquad V = \sqrt{\gamma\frac{p}{\rho}}\,.$$

These results were stated for the first time by HUGONIOT and presented systematically by HADAMARD in his *Leçons sur la propagation des ondes*, which we cited above on pp. 19.

Here, one can get a new simplification, thanks to the representation in the space S of t, x, which will permit one to treat the four independent variables on the same basis.

2. The dynamical compatibility conditions. Discontinuity parameters. – If we let h_1, h_2, h_3, k denote the parameters that characterize the discontinuity in the first partial derivatives of z as functions of u_1, u_2, u_3, μ then the first equations in (2) will give:

$$h_i \frac{dz}{dt} + \frac{1}{\rho}\frac{dp}{d\rho} p_i k = 0 \qquad (i = 1, 2, 3), \tag{5}$$

from the preceding §, no. **2**.

The condition that one deduces from the fourth equation (2′) is a consequence of the preceding ones, from (II).

Since $dz / dt \neq 0$, one will infer from (5):

$$h_i = -\frac{k}{\rho} p_i \frac{dp / d\rho}{dz / dt}.$$

If one takes into account that $V^2 = dp / d\rho$ and $dz / dt = \pm g\, V$ then one will get:

$$h_i = \mp \frac{k}{\rho} V \frac{p_i}{g} = \mp \frac{kV}{\rho} \alpha_i \qquad (i = 1, 2, 3),$$

in which:

$$\alpha_i = \frac{p_i}{g},$$

as always.

As a result, if **n** denotes the unit vector along the oriented normal then one can condense the preceding formulas into the single vectorial relation:

$$\mathbf{h} = \mp \frac{kV}{\rho} \mathbf{n} = -\frac{k}{\rho\, g} \cdot \frac{dz}{dt} \mathbf{n}, \tag{6}$$

in which **h** denotes the vector whose components are h_1, h_2, h_3 .

Let us also calculate the discontinuity in the vector **a** that represents acceleration. Its components a_i are given by:

$$a_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \sum_j \frac{\partial u_i}{\partial x_j} u_j,$$

and since:

$$\Delta \frac{\partial u_i}{\partial t} = h_i\, p_0 \qquad (i = 1, 2, 3),$$

$$\Delta \frac{\partial u_i}{\partial x_j} = h_i\, p_j \qquad (i, j = 1, 2, 3),$$

$$\Delta a_i = h_i \Big(p_0 + \sum_j u_j p_j\Big) = h_i \frac{dz}{dt},$$

so

$$\Delta \mathbf{a} = \pm\, gV\, \mathbf{h} = -\frac{k}{\rho}\, gV^2\, \mathbf{n}. \tag{7}$$

This shows that the discontinuity in the acceleration vector is parallel to **h**, so from (6), it will be normal to the wave surface; i.e., it will be *longitudinal.*

3. – We have excluded the case of liquids from this discussion. The study of that case can be deduced from general considerations by passing to the limit when $d\rho / dp$ tends to zero; i.e., when $dp / d\rho$, which is the square of the speed of propagation, goes to infinity. If we then recall equation (1) then we will see that only the two extreme cases are possible in liquids: Fixed discontinuity or instantaneous propagation. In reality, even in the case of liquids, there is a finite propagation speed, since they are also compressible.

4. Viscous fluids. Impossibility of wave propagation. – In order to show that impossibility, which goes back to P. DUHEM, we shall follow LAMPARIELLO ([1]) in our application in the application of the preceding general principles.

We shall show that the viscosity is incompatible with the presence of discontinuities that vary in time.

The differential equations of slow motion in viscous fluids are ([2]):

$$\begin{cases} \dfrac{du_i}{dt} = X_i - \dfrac{1}{\rho}\dfrac{\partial p}{\partial \rho}\dfrac{\partial \rho}{\partial x_i} + \dfrac{1}{3}\nu\dfrac{\partial}{\partial x_i}\sum_k \dfrac{\partial u_k}{\partial x_k} + \nu\,\Delta_2 u_i \qquad (i = 1, 2, 3), \\ \dfrac{d\rho}{dt} + \rho\sum_k \dfrac{\partial u_k}{\partial x_k} = 0, \end{cases} \tag{8}$$

in which u_1, u_2, u_3, ρ are the components of the velocity and density of the fluid particle, p is the mean pressure, ν is the coefficient of kinematic viscosity, and X_i are the components of the force per unit mass along the x_i axes, resp. The system (8) in the unknown functions u_1, u_2, u_3, ρ of the four variables $t \equiv x_0$, x_1, x_2, x_3 is quasi-normal with respect to t. One then performs an arbitrary *real* transformation on the independent variables and examines whether the transformed system is quasi-normal with respect to the new variable z.

Let z = const. be a wave surface and further set:

$$p_i = \frac{\partial z}{\partial x_i} \qquad (i = 0, 1, 2, 3).$$

([1]) Cf., G. LAMPARIELLO, "Sull' impossibilità di propagazioni ondose nei fluidi viscosi," Rend. della R. Accad. dei Lincei (6), vol. XIII, 1st sem. (1931), 688-691.

([2]) Cf., e.g., H. LAMB, *Hydrodynamics*, 5th ed., Cambridge University Press, 1924, pp. 546. – M. BRILLOUIN, *Leçons sur la viscosité, etc.*, Part I, chap. II, Gauthier-Villars, Paris, 1907.

From the known relations:

$$\frac{\partial}{\partial x_i} = p_i \frac{\partial}{\partial z} + \dots, \qquad\qquad (i, k = 0, 1, 2, 3),$$

$$\frac{\partial^2}{\partial x_i \, \partial x_k} = p_i p_k \frac{\partial^2}{\partial z^2} + \dots,$$

we will find that:

$$\frac{\partial}{\partial x_i} \sum_k \frac{\partial u_k}{\partial x_k} = \sum_k \frac{\partial^2 u_k}{\partial x_i \, \partial x_k} = \sum_k p_i p_k \frac{\partial^2 u_k}{\partial z^2} + \dots = p_i \sum_k p_k \frac{\partial^2 u_k}{\partial z^2} + \dots,$$

$$\Delta_2 \, u_i = \sum_k \frac{\partial^2 u_k}{\partial x_k^2} = \sum_k \frac{\partial^2 u_i}{\partial z^2} p_k^2 + \dots = \frac{\partial^2 u_i}{\partial z^2} \sum_k p_k^2 \; + \dots,$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_k u_k \frac{\partial}{\partial x_k} = (p_0 + \sum_k p_k u_k) \frac{\partial}{\partial z} + \dots$$
$$= \frac{dz}{dt} \cdot \frac{\partial}{\partial z} + \dots$$

Hence, upon neglecting to write the terms that do not contain one of the derivatives $\frac{\partial^2 u_k}{\partial z^2} \cdot \frac{\partial \rho}{\partial z}$, so they do not influence the quasi-normal character, the transformed system of (8) will take the form:

$$\begin{cases} \frac{1}{3} \nu \, p_i \sum_k p_k \frac{\partial^2 u_k}{\partial z^2} + \nu \frac{\partial^2 u_i}{\partial z^2} \sum_k p_k^2 - \frac{1}{\rho} \frac{dp}{d\rho} \frac{\partial \rho}{\partial z} p_i + \dots = 0 \quad (i = 1, 2, 3), \\ \frac{dz}{dt} \frac{\partial \rho}{\partial z} + \dots = 0. \end{cases}$$

Upon discarding the case of $dz / dt = 0$, in which the wave surface is fixed in the medium, the fourth equation of the system can be solved for $\partial\rho / \partial z$, in such a way that it would suffice to consider the third-order determinant that is formed from the coefficients of the three derivatives $\frac{\partial^2 u_k}{\partial z^2}$. It will be written:

$$\begin{vmatrix} p_1^2 + 3\sum_k p_k^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & p_2^2 + 3\sum_k p_k^2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & p_3^2 + 3\sum_k p_k^2 \end{vmatrix} = 36 \left(\sum_k p_k^2 \right)^2,$$

up to a trivial factor.

If z = const. represents a wave surface that propagates then one must have:

$$\sum_k p_k^2 = 0,$$

which will imply that:

$$p_1 = p_2 = p_3 = 0,$$

which will yield the impossibility of wave propagation.

Meanwhile, one should not believe that it is only the viscosity that is at fault. One should consider the following example:

The vibration of a string in a medium that exerts viscous resistance (air, for example) obeys a second-partial differential equation of the type:

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + \lambda\frac{\partial\varphi}{\partial t} = 0. \tag{9}$$

The unknown function φ of the variables t, x, y denotes the displacement of the particle (at the instant t) of the vibrating string. The term $-\lambda\dfrac{\partial\varphi}{\partial t}$ ($\lambda > 0$), which has the dimensions of the inverse of a length, translates analytically into the resistance of the medium.

However, although we are dealing with a dissipative system here, there is still a possibility of wave propagation, since the characteristics of equation (9) coincide with those of the equation:

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0.$$

____

§ 9. – Application to elastic media.

1. – In a more general way than the one that was followed by BELTRAMI and others, we shall adopt the same guiding principle and follow LAMPARIELLO ([1]) in our study of the propagation of waves in an elastic medium for infinitely-small deformations.

2. Wave propagation in an isotropic medium (homogeneous or not). – We shall see that one can have two types of waves – viz., longitudinal and transverse – that displace with velocities $\sqrt{\dfrac{\lambda+2\mu}{\rho}}$ and $\sqrt{\dfrac{\mu}{\rho}}$, resp., in which ρ denotes the density, and λ, μ are the Lamé parameters (which are possibly constant in a homogeneous medium), which satisfy the conditions $\mu > 0$, $3\lambda + 2\mu > 0$.

3. – If u, v, w denotes the displacement of the point (x, y, z) at the instant t then the deformation will be characterized by the parameters:

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \qquad \varepsilon_2 = \frac{\partial v}{\partial y}, \qquad \varepsilon_3 = \frac{\partial w}{\partial z},$$

$$\gamma_1 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad \gamma_2 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_3 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

and the elastic energy, within the limits of validity for Hooke's law, will be expressed by the positive-definite quadratic form:

$$W = \tfrac{1}{2}[\lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + \mu(2\varepsilon_1^2 + 2\varepsilon_2^2 + 2\varepsilon_3^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2)]. \tag{1}$$

The differential equations of elastic motion are then written:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial \varepsilon_1}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial W}{\partial \gamma_3}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\partial W}{\partial \gamma_2}\right) + \rho\left(X - \dfrac{\partial^2 u}{\partial t^2}\right) = 0, \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial \gamma_3}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial W}{\partial \varepsilon_2}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\partial W}{\partial \gamma_1}\right) + \rho\left(Y - \dfrac{\partial^2 v}{\partial t^2}\right) = 0, \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial W}{\partial \gamma_2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial W}{\partial \gamma_1}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\partial W}{\partial \varepsilon_3}\right) + \rho\left(X - \dfrac{\partial^2 w}{\partial t^2}\right) = 0. \end{cases} \tag{2}$$

([1]) Cf., G. LAMPARIELLO, Rend. della R. Acc. dei Lincei, vol. XIII, fasc. 11 (June 1931); vol. XIV, fasc. 7-8 (October 1931); vol. XIV, fasc. 9 (November 1931).

4. – Let x_i, u_i, X_i (i = 1, 2, 3) denote the quantities (x, y, z), (u, v, w), (X, Y, Z), to abbreviate. Upon writing out only the terms in the second derivatives, equations (2) will take the form:

$$(2') \qquad (\lambda + \mu)\,\frac{\partial}{\partial x_i}\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k} + \mu\,\Delta_2\,u_i - \rho\frac{\partial^2 u_i}{\partial t^2} + \ldots = 0 \qquad (i = 1, 2, 3).$$

The wave surfaces are given by the characteristics of this system in the unknown functions u_i of the variables $x_0 \equiv t$, x_1, x_2, x_3.

If one sets $p_j = \dfrac{\partial z}{\partial x_j}$ (j = 0, 1, 2, 3) then the change of variables $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ z & z_1 & z_2 & z_3 \end{pmatrix}$ will give:

$$(\lambda + \mu)\,p_i\sum_{k=1}^{3}p_k\,\frac{\partial^2 u_k}{\partial z^2} + \left(\mu\sum_{k=1}^{3}p_k^2 - \rho p_0^2\right)\frac{\partial^2 u_i}{\partial z^2} + \ldots = 0 \qquad (i = 1, 2, 3),$$

by a calculation that is entirely analogous to the one in the preceding paragraph no. **4**, so the differential equation of the characteristics will be:

$$\Omega \equiv \| (\lambda + \mu)\,p_i\,p_k + \varepsilon_{ik}\left(\mu\sum_{k=1}^{3}p_k^2 - \rho p_0^2\right) \| = 0,$$

upon setting $\varepsilon_{ik} = 0$ if $i \neq k$ and $\varepsilon_{ik} = 1$ if $i = k$, and can then be put into the form ([1]):

$$\Omega \equiv \left[(\lambda + 2\mu)\sum_{k=1}^{3}p_k^2 - \rho p_0^2\right]\left(\mu\sum_{k=1}^{3}p_k^2 - \rho p_0^2\right)^2 = 0,$$

The characteristics will then be given by one or the other of the two equations:

$$(3) \qquad \begin{cases} (\lambda + 2\mu)\displaystyle\sum_{k=1}^{3}p_k^2 - \rho p_0^2 = 0, \\ \mu\displaystyle\sum_{k=1}^{3}p_k^2 - \rho p_0^2 = 0, \end{cases}$$

which have the type:

$$\frac{1}{V^2}\,p_0^2 - \sum_{k=1}^{3}p_k^2 = 0,$$

which is nothing but the characteristic equation for the canonical equation of small motions:

([1]) One appeals to the following property: The determinant of order n : $a = \| a_{ik} + \varepsilon_{ik}\,x \|$, in which $\varepsilon_{ik} = 0$ if $i \neq k$ and $\varepsilon_{ik} = 1$ if $i = k$, is developed into:

$$a = x^n + \mu_1\,x^{n-1} + \mu_2\,x^{n-2} + \ldots + \mu_{n-1}\,x + \mu_n,$$

in which μ_s is the sum of the principal minors of order s in the determinant $\| a_{ik} \|$.

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta_2\,\varphi = 0,$$

in which V is a function of position (which is constant in a homogeneous medium, moreover) that is always the displacement velocity of the wave.

One then concludes the possibility of waves displacing in an isotropic elastic medium with the velocities $\sqrt{\dfrac{\lambda+2\mu}{\rho}}$, $\sqrt{\dfrac{\mu}{\rho}}$ from that.

5. – It remains for us to see the longitudinal character of the former kind of wave and the transverse character of the latter. One will succeed in that by looking for the dynamical compatibility conditions for the two motions that agree along a wave surface σ_t but have second-order discontinuities.

The u_i and their first derivatives are continuous. Introduce the second-order discontinuity parameters h_1, h_2, h_3, which correspond to u_1, u_2, u_3. From formula (2) in § **7**, no. **1**:

$$\Delta\frac{\partial^2 u_\nu}{\partial x_i\,\partial x_k} = h_\nu\, p_i\, p_k \qquad (\nu = 1, 2, 3).$$

Now, one infers from (2′) that:

$$(\lambda+\mu)\sum_{k=1}^{3}\Delta\frac{\partial^2 u_k}{\partial x_i\,\partial x_k} + \mu\sum_{k=1}^{3}\Delta\frac{\partial^2 u_i}{\partial x_k^2} - \rho\,\Delta\frac{\partial^2 u_i}{\partial t^2} = 0,$$

so:

$$(\lambda+\mu)\sum_{k=1}^{3} h_k\, p_i\, p_k + h_i\Big(\mu\sum_{k=1}^{3} p_k^2 - \rho\, p_0^2\Big) = 0 \qquad (i = 1, 2, 3),$$

which is a system of linear equations in h_i whose determinant is rightfully $\Omega = 0$.

Set $g^2 = \sum_{k=1}^{3} p_k^2$, let **h** and **n** be the vectors whose components are h_k and $\alpha_k = p_k / g$. h_n will then be the normal component to **h**:

$$\sum_{k=1}^{3} h_k\, p_k = g\sum_{k=1}^{3} h_k\alpha_k = g\, h_n\,.$$

The compatibility conditions condense into the single vectorial relation:

$$(\mu\, g^2 - \rho\, p_0^2)\,\mathbf{h} + (\lambda+\mu)\, g^2\, h_n\,\mathbf{n} = 0.$$

For the first type of wave [first equation (3): velocity $\sqrt{\mu/\rho}$], the compatibility condition reduces to $h_n = 0$. It expresses the idea that the discontinuity vector **h** is transverse.

6. Case of an anisotropic medium with three rectangular symmetry planes. – With the notations of no. **3**, the equations of motion will once more be the equations (2), with the condition that one must take the expression:

$$W = \tfrac{1}{2}[A\varepsilon_1^2 + B\varepsilon_2^2 + C\varepsilon_3^2 + 2A'\varepsilon_2\varepsilon_3 + 2B'\varepsilon_3\varepsilon_1 + C'\varepsilon_1\varepsilon_2 + A''\gamma_1^2 + B''\gamma_2^2 + C''\gamma_3^2] \tag{4}$$

for the elastic energy, in which the nine coefficients A, B, …, C'' are functions of (x, y, z, t) that reduce to constants when the medium is homogeneous.

If we keep only the second derivatives then the equations of motion will be written:

$$(5)\quad\left\{\begin{aligned}
&A\frac{\partial^2 u}{\partial x^2} + C''\frac{\partial^2 u}{\partial y^2} + B''\frac{\partial^2 u}{\partial z^2} + (C'+C'')\frac{\partial^2 v}{\partial x\,\partial y} + (B'+B'')\frac{\partial^2 w}{\partial x\,\partial z} - \rho\frac{\partial^2 u}{\partial t^2} + \cdots = 0,\\
&C''\frac{\partial^2 v}{\partial x^2} + B\frac{\partial^2 v}{\partial y^2} + A''\frac{\partial^2 v}{\partial z^2} + (C'+C'')\frac{\partial^2 u}{\partial x\,\partial y} + (A'+A'')\frac{\partial^2 w}{\partial y\,\partial z} - \rho\frac{\partial^2 v}{\partial t^2} + \cdots = 0,\\
&B''\frac{\partial^2 w}{\partial x^2} + A''\frac{\partial^2 w}{\partial y^2} + C\frac{\partial^2 w}{\partial z^2} + (B'+B'')\frac{\partial^2 w}{\partial x\,\partial z} + (A'+A'')\frac{\partial^2 v}{\partial y\,\partial z} - \rho\frac{\partial^2 w}{\partial t^2} + \cdots = 0,
\end{aligned}\right.$$

and when we set:

$$p_0 = \frac{\partial\zeta}{\partial t}, \qquad p_1 = \frac{\partial\zeta}{\partial x}, \qquad p_2 = \frac{\partial\zeta}{\partial y}, \qquad p_3 = \frac{\partial\zeta}{\partial z},$$

the change of variables $\begin{pmatrix} x & y & z & t \\ \zeta & \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix}$ will lead to the transformed system:

$$(5')\quad\left\{\begin{aligned}
&(A\,p_1^2 + C''p_2^2 + B''p_3^2 - \rho p_0^2)\frac{\partial^2 u}{\partial\zeta^2} + (C'+C'')p_1p_2\frac{\partial^2 v}{\partial\zeta^2} + (B'+B'')p_1p_3\frac{\partial^2 w}{\partial\zeta^2} + \cdots = 0,\\
&(C'+C'')\frac{\partial^2 u}{\partial\zeta^2} + (C''\,p_1^2 + B\,p_2^2 + A''p_3^2 - \rho p_0^2)\frac{\partial^2 v}{\partial\zeta^2} + (A'+A'')p_2p_3\frac{\partial^2 w}{\partial\zeta^2} + \cdots = 0,\\
&(B'+B'')\frac{\partial^2 u}{\partial\zeta^2} + (A'+A'')p_2p_3\frac{\partial^2 v}{\partial y^2} + (B''\,p_1^2 + A''\,p_2^2 + C\,p_3^2 - \rho p_0^2)\frac{\partial^2 w}{\partial z^2} + \cdots = 0.
\end{aligned}\right.$$

The characteristic equations and wave surfaces $\zeta(x, y, z, t) = \zeta_0$ are obtained by annulling the determinant of the coefficients of $\frac{\partial^2 u}{\partial\zeta^2}$, $\frac{\partial^2 v}{\partial\zeta^2}$, $\frac{\partial^2 w}{\partial\zeta^2}$, namely:

$$\Omega\ (p_0, p_1, p_2, p_3) = 0.$$

It is interesting to remark that *this equation is, up to a change of symbols, the equation for S* ($S = \rho\, p_0^2$) that corresponds to the search for the axes of the *ellipsoid* (of propagation).

$$\begin{aligned} E\ (x, y, z) - 1 \equiv\ & (A\, p_1^2 + C'' p_2^2 + B'' p_3^2)\ x^2 \\ & + (C' p_1^2 + B\, p_2^2 + A'' p_3^2)\ y^2, \\ & + (B'\, p_1^2 + A''\, p_2^2 + C\, p_3^2)\ z^2, \\ & \quad + 2\ (A' + A'')\ p_2\, p_3\, y\, z \\ & \quad + 2\ (B' + B'')\ p_3\, p_1\, z\, x \\ & \quad + 2\ (C' + C'')\ p_1\, p_2\, x\, y - 1 = 0. \end{aligned}$$

BELTRAMI started out by considering that ellipsoid in his remarkable paper on the theory of waves ([1]). In it, he supposed that the waves were planar and that p_1, p_2, p_3 denoted the direction cosines of the normal to the planes of those waves.

The geometric interpretation of $\Omega = 0$ shows that the equation in p_0^2 has degree three and its three roots are positive. Solving it for p_0 will then lead us to conclude that there is a triple infinitude of possible wave surfaces, with two directions of propagation.

Moreover, the displacement velocities of the discontinuity waves will be identical for a homogeneous medium (and in particular, an isotropic one), as well as the displacement velocities of the plane waves of a vibratory character that BELTRAMI studied. The same thing is not true in the most general case of elastic media, for which BELTRAMI showed the impossibility of such vibratory plane waves. However, as we shall see, the preceding results for (second order) discontinuity waves are once more true; in particular, one can have plane waves in a homogeneous medium.

7. The case of the most general elastic media. – The differential equations of motion are once more given by equations (3) when one takes the following expression for the elastic energy:

$$W = \tfrac{1}{2}\Big(\sum_{r,s} A_{rs}\varepsilon_r\varepsilon_s + \sum_{r,s} B_{rs}\gamma_r\gamma_s + 2\sum_{r,s} C_{rs}\varepsilon_r\gamma_s\Big)\,.$$

Upon arguing in exactly the same way as in the preceding, one will obtain a characteristic equation:

$$\Omega\ (p_0, p_1, p_2, p_3) = 0,$$

which will once more have degree three in p_0^2, and its three roots will be real and positive. Each of them will correspond to an infinitude of wave surfaces with two possible directions of displacement.

___

([1]) Cf., E. BELTRAMI, *Opera*, t. IV, pp. 224-235.

§ 10. – Application to Maxwell's equations for electromagnetic phenomena.

1. – The functions φ of the system of partial differential equations are six in number in this case, namely, the three components of the electrical force **E** and the three components of the magnetic force **H**.

Further introduce the electric displacement **D** and the magnetic induction **B**. One has **E** = **D**, **H** = **B**, *in vacuo*. In a homogeneous and isotropic dielectric **D** = ε **E**, **B** = μ **H**, in which ε and μ are two positive constants. (ε is the dielectric constant, and μ is the magnetic permeability.)

In general, in an arbitrary medium (at rest), the components of **D** and **B** are linear forms of the components of **E** and **H**, respectively. We then write:

$$\mathbf{D} = \varepsilon\,\mathbf{E}, \qquad \mathbf{B} = \mu\,\mathbf{H},$$

while agreeing this time that the symbols ε and μ represent two vectorial homographies.

Be that as it may, the differential equations of the electromagnetic field are written:

$$\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \text{rot}\,\mathbf{H} + \ldots, \tag{1}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\,\text{rot}\,\mathbf{E} + \ldots, \tag{2}$$

$$\text{div}\,\mathbf{D} = \ldots, \tag{3}$$

$$\text{div}\,\mathbf{B} = 0, \tag{4}$$

in which c is the speed of light, and the omitted terms can depend upon charges, currents, electromotive forces, etc. In summary, they are quantities that are either completely independent of the field (i.e., the vectors **E** and **H**) or at the very least (if they do depend upon them essentially), they are independent of the derivatives of those vectors ([1]).

Since one supposes that the medium (viz., the ether) is at rest, one can use the terms "displacement velocity" and "propagation velocity" interchangeably.

We shall first address only the first two equations, which constitute a normal system of order $\varepsilon = 1$. One will recognize that the results to which we will arrive are compatible with the last two equations of the differential system.

We must consider the components E_i, H_i of the electrical and magnetic forces, as well as the homographies ε and μ, and as a result, the polarization vectors **B** = **D**, to be continuous upon crossing the surface σ in space-time that corresponds to a possible wave surface σ_t.

As far as the homographies are concerned, we also assume that their coefficients and all of the first derivatives will remain continuous upon crossing σ.

([1]) Cf., H. HERTZ, *Gesammelte Werke*, Bd. II, pp. 220.

On the contrary, one will have to presume that there are discontinuities in the first derivatives of **E**, **H** (hence, in **D**, **B**, as well).

Let e_i , h_i (i = 1, 2, 3) be the six discontinuity parameters upon crossing σ_t , which correspond to the components E_i, H_i, which are parameters that characterize the discontinuities in the derivatives of those functions, from § **7**, no. **1**. It will be useful to consider them to be the components of two vectors **e**, **h** (relative to an arbitrary point of the discontinuity surface σ_t).

Having said that, we seek the dynamical compatibility conditions that the vectors **e**, **h** must satisfy.

Since $\mathbf{D} = \varepsilon\,\mathbf{E}$, one will get:

$$\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\varepsilon\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\frac{\partial \varepsilon}{\partial t}\mathbf{E} \tag{5}$$

by derivation, so if one lets ε^{-1} denote the inverse homography to ε (which will reduce to arithmetic inverse of the constant ε in the isotropic case) and refers to equation (1) then one will have:

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\left(\varepsilon^{-1}\frac{\partial \varepsilon}{\partial t}\right)\mathbf{E} = \varepsilon^{-1}\,\mathrm{rot}\,\mathbf{H} + \ldots \tag{5′}$$

Similarly, it results from $\mathbf{B} = \mu\,\mathbf{H}$ that:

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{c}\mu\frac{\partial \mathbf{H}}{\partial t} + \frac{1}{c}\frac{\partial \mu}{\partial t}\mathbf{H}, \tag{6}$$

so, thanks to (2) and with the obvious notation μ^{-1}:

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \frac{1}{c}\left(\mu^{-1}\frac{\partial \mu}{\partial t}\right)\mathbf{H} = -\frac{1}{c}\mu^{-1}\,\mathrm{rot}\,\mathbf{E} + \ldots \tag{6′}$$

Now introduce the limiting values of **E**, **H** on σ_t into equations (1), (2), namely, $\mathbf{E}^+$, $\mathbf{H}^+$, relative to one side and $\mathbf{E}^-$, $\mathbf{H}^-$ relative to the other side. Upon subtracting the corresponding sides of the equations, and taking into account (5), (6) and the fact that the continuous terms (in particular, the unwritten ones) disappear in the subtraction, we will get:

$$\frac{1}{c}\varepsilon\left(\Delta\frac{\partial \mathbf{E}}{\partial t}\right) - \Delta\,\mathrm{rot}\,\mathbf{H} = 0, \tag{7}$$

$$\frac{1}{c}\mu\left(\Delta\frac{\partial \mathbf{H}}{\partial t}\right) + \Delta\,\mathrm{rot}\,\mathbf{E} = 0. \tag{8}$$

Having said that, apply the formulas (1) of § **7** to the various functions E_i , H_i , while replacing the factor λ in those formulas with e_i , h_i , respectively. One will then obtain the scalar relations:

$$\text{(9)} \qquad \begin{aligned} \Delta\frac{\partial E_i}{\partial t} &= e_i\,p_0\,, & \Delta\frac{\partial E_i}{\partial x_j} &= e_i\,p_j\,, \\ \Delta\frac{\partial H_i}{\partial t} &= h_i\,p_0\,, & \Delta\frac{\partial H_i}{\partial x_j} &= h_i\,p_j\,. \end{aligned} \qquad (i, j = 1, 2, 3),$$

The two groups on the left can each be condensed into a single vectorial equation:

$$\text{(10)} \qquad \Delta\frac{\partial \mathbf{E}}{\partial t} = p_0\,\mathbf{e}\,,$$

$$\text{(11)} \qquad \Delta\frac{\partial \mathbf{H}}{\partial t} = p_0\,\mathbf{h}\,.$$

Thanks to (10), if one agrees to regard indices that differ by three as identical then (7) will yield the equivalent scalar equations:

$$\text{(12)} \qquad \frac{p_0}{c}(\varepsilon e)_i + \Delta\frac{\partial H_{i+1}}{\partial x_{i+2}} - \Delta\frac{\partial H_{i+2}}{\partial x_{i+1}} = 0 \qquad (i = 1, 2, 3).$$

Similarly, (8) will give the three equations:

$$\text{(13)} \qquad \frac{p_0}{c}(\mu\,h)_i + \Delta\frac{\partial E_{i+2}}{\partial x_{i+1}} - \Delta\frac{\partial E_{i+1}}{\partial x_{i+2}} = 0 \qquad (i = 1, 2, 3).$$

Upon taking the relations (9) and replacing the p_i ($i = 1, 2, 3$) by the products $\alpha_i\,g$, in which:

$$g = \left|\sqrt{\sum_{i=1}^{3} p_0^2}\right|,$$

and in which the α_i are the direction cosines of the vector **n** that is normal to σ_t , equations (12), (13) will be written:

$$\frac{p_0}{c}\,\varepsilon\,e_i - g\,(h_{i+2}\,\alpha_{i+1} - h_{i+1}\,\alpha_{i+2}) = 0,$$

$$\frac{p_0}{c}\,\eta\,h_i + g\,(e_{i+2}\,\alpha_{i+1} - e_{i+1}\,\alpha_{i+2}) = 0,$$

or, in vectorial form:

$$(14) \qquad \begin{cases} \dfrac{p_0}{g}\varepsilon\,\mathbf{e} - g\,\mathbf{n}\wedge\mathbf{h} = 0, \\ \dfrac{p_0}{g}\mu\,\mathbf{H} + g\,\mathbf{n}\wedge\mathbf{e} = 0. \end{cases}$$

The equations that must be satisfied by the characteristic vectors **e**, **h** of the discontinuities in the derivatives of the electric and magnetic force exhibit the fact that, contrary to what happens in hydrodynamics, the vectors ε **e**, μ **h** are normal to **n**; i.e., they are *tangent* to the discontinuity surfaces. One will then be dealing with *transverse discontinuities*, as one is accustomed to say. However, more precisely, it is not the vectors **e** and **h** (which characterize the discontinuities in the derivatives of the electric and magnetic forces) that are transverse, but the vectors ε **e** and μ **h**, which relate to the derivatives of the electric polarization and magnetic induction.

2. – Now let **d** = μ **h**, **h** = ε **e**, denote the characteristic vectors of the derivatives of **D** and **B**, and apply then to the (conservation) equations (3), (4), so the preceding process will yield the dynamical compatibility condition upon starting from (1) and (2); one will then get:

$$\mathbf{d}\times\mathbf{n} = 0,$$
$$\mathbf{h}\times\mathbf{n} = 0.$$

Now, those relations can also be deduced from (14) upon scalar-multiplying them by **n**. They show that the transversal character of the vectors **d** and **b** that was recalled and underlined above contains the compatibility conditions that are derived from equations (3) and (4), which we have left aside, but which must be associated with the normal system (1), (2) in order to produce the complete representation of electromagnetic phenomena according to the MAXWELL-HERTZ theory.

3. Forming the equation $\Omega = 0$ in a magnetically-isotropic medium. Application to the electromagnetic theory of light. – In general, the preceding considerations will be valid even when the homographies ε and μ depend upon the electromagnetic field – i.e., upon the electric and magnetic forces. Meanwhile, in view of the ultimate developments, we shall suppose from now on that those homographies are constants and even that the magnetic homography reduces to an ordinary multiplication.

We also suppose that the homography ε is a dilatation that reduces to its canonical form by a convenient choice of reference axes. Recall that the dilatation ε is associated with a quadric that is called the *indicatrix* and that one calls the planes of the indicatrix quadric the *principal planes of the dilatation* ([1]). We shall take them to be *reference planes in what follows.*

The coefficients of the homography will then reduce to three: ε_1, ε_2, ε_3 .

([1]) Cf., R. MARCOLONGO, *Meccanica razionale*, vol. I, 3rd ed., Hoepli, Milan, 1922, pp. 24-25.

The equations of the electromagnetic theory of light in crystalline media are included within this schema, in particular. We shall limit ourselves to considering the case of media that are called *biaxial*, in which the constants ε are distinct, and we can then suppose that:

$$\varepsilon_3 > \varepsilon_2 > \varepsilon_1 > 0.$$

One gets from the second equation (14) that:

$$\mathbf{h} = -\frac{g}{\mu p_0 / c}(\mathbf{n} \wedge \mathbf{e}),$$

so when one substitutes this in the first one, one will get:

$$\frac{p_0^2}{c^2}\mu\,\varepsilon\,\mathbf{e} + g^2\,\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{e}) = 0. \tag{13}$$

Decompose the vector **e** into two vectors: **e′**, which is normal **n**, and $\mathbf{e}'' = (\mathbf{e} \times \mathbf{n})\,\mathbf{n}$, which is parallel to **n**, in such a way that $\mathbf{e} = \mathbf{e}' + \mathbf{e}''$. Hence:

$$\mathbf{n} \wedge \mathbf{e} = \mathbf{n} \wedge (\mathbf{e}' + \mathbf{e}'') = \mathbf{n} \wedge \mathbf{e}'.$$

The vector $\mathbf{n} \wedge \mathbf{e}'$ is nothing but the vector that is obtained when one starts from **e′** and rotates it 90° around **n**. As a result:

$$\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{e}') = -\mathbf{e}' = -(\mathbf{e} - \mathbf{e}') = -[\mathbf{e} - (\mathbf{e} \times \mathbf{n})\,\mathbf{n}],$$

and equation (15) will become:

$$\frac{p_0^2}{c^2}\mu\,\varepsilon\,\mathbf{e} - g^2\,\mathbf{e} + (\mathbf{e} \wedge g\,\mathbf{n})\,g\,\mathbf{n} = 0. \tag{16}$$

Now set:

$$\frac{\mu\,\varepsilon_i}{c^2}p_0^2 - g^2 = \rho_i \qquad (i = 1, 2, 3) \tag{17}$$

and

$$V_i^2 = \frac{c^2}{\mu\,\varepsilon_i}, \tag{18}$$

in such a way that:

$$\rho_i = \frac{p_0^2}{V_i^2} - g^2. \tag{19}$$

Since the components of $g\,\mathbf{n}$ are nothing but p_1, p_2, p_3, equation (16) is equivalent to three scalar equations:

$$\begin{cases} (\rho_1 + p_1^2)\, e_1 + p_1 p_2 e_2 + p_1 p_3 e_3 = 0, \\ p_2 p_1\, e_1 + (\rho_2 + p_2^2)\, e_2 + p_2 p_3 e_3 = 0, \\ p_3 p_1\, e_1 + p_3 p_2 e_2 + (\rho_3 + p_3^2)\, e_3 = 0. \end{cases}$$

From the practical rule of § **7**, no. **2**, the differential equation of the wave surfaces is obtained by equating the determinant of the coefficients of e_1, e_2, e_3 to zero.

One will then find the equation:

$$\Omega\,(p) = \begin{vmatrix} \rho_1 + p_1^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & \rho_2 + p_2^2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & \rho_3 + p_3^2 \end{vmatrix} = 0,$$

and after developing this:

$$\text{(20)} \qquad \Omega\,(p) = \rho_2\,\rho_3\,p_1^2 + \rho_3\,\rho_1\,p_2^2 + \rho_1\,\rho_2\,p_3^2 + \rho_1\,\rho_2\,\rho_3 = 0.$$

Upon replacing the ρ_i with their values (19), that equation will be the desired partial differential equation for the unknown function z that defines the wave surfaces.

If the α_i are direction cosines of the normal, as always, then upon dividing both sides of (20) by g^2, one will get:

$$\text{(20}'\text{)} \qquad \frac{1}{g^2}\,\Omega\,(p) = \rho_2\,\rho_3\,\alpha_1^2 + \rho_3\,\rho_1\,\alpha_2^2 + \rho_1\,\rho_2\,\alpha_3^2 + \frac{1}{g^2}\,\rho_1\,\rho_2\,\rho_3 = 0.$$

That equation will shed light upon the important characteristics of the phenomenon, and independently of any integration of (20), moreover, insofar as it will permit us to show, as we shall see, how the speed of propagation (in the normal sense) of an arbitrary element of the wave surface will vary with the orientation of that element.

4. Law of variation for the speed of propagation. – In order to make the speed of propagation V appear, it will suffice to replace p_0 with the product $\pm\, V\, g$ in the expressions ρ that are defined by (17).

However, it will first be useful to examine the case in which equation (20) is found to be verified, due to the fact that certain ρ are annulled.

One will remark immediately that since the ε are distinct, by hypothesis, it will not be possible for two of the ρ to be zero simultaneously, from their expressions in (17).

It will then suffice to examine only the case in which ρ_1 and ρ_2 are annulled. Equation (20) then implies that $\alpha_1 = 0$, i.e., that the normal to the wave surface must be parallel to the plane (x_2, x_3). On the other hand, $\rho_1 = 0$ implies that:

$$\frac{p_0^2}{V_1^2} - g^2 = 0,$$

so $V_1 = \left|\frac{p_0}{g}\right|$ (in which p_0 is a function of position), which permits one interpret the constant V_1 as a possible velocity of propagation for light in any direction that is normal to the x_1-axis.

One can make some analogous considerations for the cases $\rho_2 = 0$ or $\rho_3 = 0$. It is then established in that way that V_1, V_2, V_3 are the propagation velocities in the directions parallel to the coordinates place, respectively, which are the principal planes of the electric homography (viz., dilatation), since that is how one chose them.

Having treated the case in which **V** is annulled at the same time as one of the ρ, we shall now move on to the general case in which $\Omega = 0$, while all of the ρ are non-zero.

The left-hand side of (20) can then be written:

$$\Omega = \rho_1\,\rho_2\,\rho_3\left(1+\sum_{i=1}^{3}\frac{p_0^2}{\rho_i}\right).$$

Now:

$$\frac{p_0^2}{\rho_i} = \frac{g^2\alpha_i^2}{\rho_i} = \frac{\alpha_i^2}{\dfrac{p_0^2}{g^2}\dfrac{1}{V_i^2}-1} \qquad (i = 1, 2, 3),$$

in which p_0^2 / g^2 represents the square of the propagation speed of the wave surface in question. Hence:

$$\Omega = \rho_1\,\rho_2\,\rho_3\left(1+\sum_{i=1}^{3}\frac{\alpha_i^2}{\dfrac{V^2}{V_i^2}-1}\right).$$

Upon taking the identity $\sum_i \alpha_i^2 = 1$ into account, one can finally write our equation in the form:

$$\Omega\,(p) \equiv V^2\rho_1\,\rho_2\,\rho_3\sum_i\frac{\alpha_i^2}{V^2-V_i^2} = 0. \tag{21}$$

In the first place, this is satisfied for $V = 0$ – i.e., since $p_0 = V\,g$, if:

$$p_0 = \frac{\partial z}{\partial t} = 0.$$

That is the case of a fixed discontinuity surface.

However, from now on, we shall consider the equation that we obtain upon annulling another factor:

(22) $$\sum_i \frac{\alpha_i^2}{V^2 - V_i^2} = 0.$$

Set:

(23) $$f(V^2) = \tfrac{1}{2}\sum_i \frac{\alpha_i^2}{V^2 - V_i^2},$$

and examine the issues for equation (22), which can also be written:

(22′) $$f(V^2) = 0.$$

When put into entire form, that equation will have degree two in V^2, and will then admit two roots that will both be real and positive, as we shall see.

The quantities V_i will satisfy the inequalities:

$$V_1 > V_2 > V_3,$$

from their expressions (18) and the order of magnitude of the ε ($\varepsilon_1 > \varepsilon_2 > \varepsilon_3$).

First consider the general case of a direction α_i that is not parallel to any of the principal (coordinate) planes.

The function $f(V^2)$ will then be everywhere regular, except for the values of V^2 that are equal to one of the V_i^2, and the ones for which it is infinite.

If we give V^2 values from the interval (V_3^2, V_2^2) and close to V_3^2 then the term $\frac{\alpha_3^2}{V^2 - V_3^2}$ will have a positive sign and dominate the other ones, so $f(V^2)$ will take on positive values. On the contrary, if we give values to V^2 from the same interval, but closer to V_2^2 then the term $\frac{\alpha_2^2}{V^2 - V_2^2}$ will have a negative sign and dominate the others, so $f(V^2)$ will take on a negative sign. Equation (22′) will then admit a root in the interval (V_3^2, V_2^2); one proves that it will admit a root in the interval (V_2^2, V_1^2) in the same way.

One will then see that there are two possible propagation speeds (in absolute value), and they will be found between V_1 and V_2 and V_2 and V_3, respectively.

Now, if one of the direction cosines α_i is zero – for example, α_1 – then equation (20′) will be satisfied for $\rho_1 = 0$, and one of the possible velocities of propagation will be V_1. The same equation (20′), when stripped of the factor ρ_1, will then show that the equation that defines the possible velocities, other than V_1, V_2, V_3, will reduce to:

$$\frac{\alpha_2^2}{V^2 - V_2^2} + \frac{\alpha_3^2}{V^2 - V_3^2} = 0,$$

which will have one root V that is found between V_2 and V_3.

5. Geometric construction of the roots of the equation $f(V^2) = 0$. – Consider the ellipsoid:

(24) $$\rho \equiv \sum_i V_i^2 x_i^2 = 1,$$

and let:

$$\psi \equiv \sum_i \alpha_i \, x_i = 0$$

be the equation of an arbitrary plane that passes through the origin, in which the coefficients α_i denote the direction cosines of the normal to the plane, when oriented arbitrarily.

In order to find the lengths of the semi-axes of the ellipse that is the intersection of the ellipsoid $\varphi = 1$ and the plane $\psi = 0$, it will obviously suffice to look for the maximum and minimum of the distance $\rho = \sqrt{\sum_i x_i^2}$ (or, what amounts to the same thing, the square of the distance $\rho^2 = \sum_i x_i^2$) when the point x_i varies in the ellipse – i.e., when the variables x_i are linked by the two relations:

$$\varphi = 1, \qquad \psi = 0.$$

Upon applying the classical method of LAGRANGE multipliers, we will be led to write:

(25) $$\delta[\rho^2 + \lambda(\rho - 1) + \lambda_1 \psi] = 0,$$

in which λ and λ_1 are undetermined *a priori*, and the variation must be zero for any choice of the δx_i.

Upon dividing by 2, in addition, it will result that:

(26) $$x_i (1 + \lambda V_i^2) + \tfrac{1}{2}\lambda_1 \alpha_i = 0 \qquad (i = 1, 2, 3).$$

Upon multiplying this by α_i and summing, from (24) and the fact that $\psi \equiv \sum_i \alpha_i \, x_i = 0$, one will get:

$$\rho^2 + \lambda = 0.$$

We note that ρ (viz., a semi-axis of an effective ellipse) is essentially supposed to be greater than zero.

On the other hand, upon first taking the general case in which $1 / \rho^2$ is different from each of the V_i^2, equations (26) can be solved for the x_i, and when one replaces λ with the value $-\rho^2$ that is found for it, that will give:

(27) $$x_i = -\frac{1}{2}\frac{\lambda_1 \alpha_i}{1 - \rho^2 V_i^2} \qquad (i = 1, 2, 3)$$

Hence, upon substituting this in $\psi \equiv \sum_i \alpha_i x_i = 0$ (and exhibiting the factor $1 / \rho^2$):

$$\frac{1}{2}\frac{\lambda_1}{\rho^2}\sum_i \frac{\alpha_i}{1/\rho^2 - V_i^2} = 0.$$

Observe that λ_1 cannot be zero, because from (27), the same thing would be true for all of the x_i, and thus, the ρ, which is not true. We can then neglect the factor $-\lambda_1 / \rho^2$, and what will remain is:

$$\frac{1}{2}\sum_i \frac{\alpha_i}{1/\rho^2 - V_i^2} = 0,$$

which will be identified with the equation $f(V^2) = 0$ that defines the possible propagation velocities, on the condition that one must set:

$$V^2 = \frac{1}{\rho^2}.$$

Thus, one gets the geometric construction of the propagation velocities that relate to an arbitrary direction α_i, which will remain valid even in the previously-excluded case in which $1 / \rho^2$ takes one of the values V_i^2.

One draws the plane that is normal to the direction α_i through the center of the ellipsoid $\varphi = 1$. The inverses of the semi-axes of the ellipse that is its section will give the absolute values of the two possible propagation velocities.

6. Fresnel wave surfaces. – We refer to the general considerations that were developed in nos. **6-8** of § **6** on the subject of integrating the equation $p_0 + H = 0$ by means of bicharacteristics. Suppose that $n = 3$, to begin with. The parametric equations of the configuration that is taken at the instant t by the wave surface that reduces to an epicenter O, which is chosen to be the coordinate origin, at the instant $t = 0$, will then be:

$$x_i = t\left(\frac{\partial H}{\partial p_i}\right)_0 \qquad (i = 1, 2, 3), \tag{28}$$

in which one can take the ratios of the p to one of them to be the parameters. (The left-hand sides depend upon only those ratios because H is homogeneous and of degree one with respect to the p.)

Upon regarding the x_i as functions of t, equations (28) will be those of light rays and will exhibit a rectilinear progression (in a homogeneous medium).

As we have already seen, the wave surfaces (28) at the various instants t are mutually homothetic to each other. It will then suffice to consider any of them. Ordinarily, one chooses the one that corresponds to $t = 1$. One calls it the wave surface, more especially, and its parametric equations will be written:

(29) $$x_i = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2, 3).$$

In the case that we are presently addressing, we will then find the celebrated FRESNEL wave surface (discovered in 1827), whose analytical study led HAMILTON to discover the phenomenon of conical refraction. We shall denote it by F in what follows.

We now propose to determine the Cartesian equation of F, which is obtained theoretically by starting from (29) and eliminating the parameters ([1]).

To that end, it is convenient to first evaluate the distance δ from the origin to the tangent plane at a running point P (x_i) on F.

If α_i is the direction cosine of the normal to F at P then that will give:

$$\delta = \sum_i \alpha_i \, x_i \, ;$$

i.e., upon taking into account the parametric equations (29) and the values $\pm p_i / g$ of the α_i (which correspond to the chosen positive sense along the normal):

$$\delta = \sum_i \alpha_i \frac{\partial H}{\partial p_i} = \pm \frac{1}{g} \sum_i p_i \frac{\partial H}{\partial p_i}$$

(with the usual convention on the sign of the distance δ).

Due to the homogeneity of degree 1 in H and the equation $p_0 + H = 0$, one will have:

$$\delta = \pm \frac{H}{g} = \mp \frac{p_0}{g} = \pm V.$$

In order to simplify the writing, take one of the two signs (the + sign, for example), but observe immediately that one will arrive at the same result by taking the other sign. The equation of the tangent plane will then be written:

(30) $$\sum_i \alpha_i \, x_i - V = 0,$$

in which the x_i represent the running coordinates, this time, while the α_i and V are coupled by the equation:

(22′) $$f(V^2) = \frac{1}{2} \sum_i \frac{\alpha_i^2}{V^2 - V_i^2} = 0.$$

Equation (30) will give:

(31) $$\sum_i x_i \, d\alpha_i - dV = 0$$

([1]) M. BOGGIO gave an ingenious way of obtaining the Cartesian equation by vectorial methods quite recently. See his note "Sulle superficie d'onda di Fresnel," Rend. Acc. Lincei (6) **14** (1932), 551-556.

upon differentiation with respect to the parameters α_i and V.

Set:

(32) $$f_i = \frac{\partial f}{\partial \alpha_i} = \frac{\alpha_i^2}{V^2 - V_i^2} \qquad (i = 1, 2, 3),$$

(33) $$f_0 = \frac{\partial f}{\partial V} = - V \sum_i f_i^2 .$$

Upon differentiating (22′), one will get:

(34) $$\sum_i f_i \, d\alpha_i + f_0 \, dV = 0.$$

The relations (30), (31), (34) permit us to eliminate the parameters α_i , V and thus obtain the Cartesian equation for F.

Upon differentiating the identity:

$$\sum_i \alpha_i^2 = 1,$$

one will deduce that:

(35) $$\sum_i \alpha_i \, d\alpha_i = 0.$$

On the other hand, upon replacing dV by its expression that one infers from (31) in (34), one will get:

$$\sum_i (f_i + f_0 \, \alpha_i) \, d\alpha_i = 0.$$

That must be true for any direction cosines α_i – i.e., for all $d\alpha_i$ that satisfy (35). It will result that:

(36) $$f_i + f_0 \, \alpha_i = k \, \alpha_i ,$$

in which k is a proportionality factor that is undetermined *a priori*. It is easy to calculate, because from the expressions (32) for f_i and due to (22′), one will have:

(37) $$\sum_i f_i \, d\alpha_i = 0.$$

Multiply the two sides of (36) by α_i and sum. That will give:

$$k \sum_i \alpha_i^2 = \sum_i f_i \, d\alpha_i + f_0 \sum_i \alpha_i \, x_i ;$$

i.e., from (30) and (37):

(38) $$k = f_0 \, V.$$

Now observe that:

$$\sum_i f_i^2 V_i^2 = \sum_i f_i^2 (V_i^2 - V^2) + V^2 \sum_i f_i^2 .$$

The first of the terms on the right-hand side is zero, due to equations (32) and (22′), so:

(39) $$\sum_i f_i^2 V_i^2 = V^2 \sum_i f_i^2 .$$

Moreover, one has:

$$\sum_i f_i \alpha_i V_i^2 = \sum_i f_i \alpha_i (V_i^2 - V^2) + V^2 \sum_i f_i \alpha_i ,$$

and from (37) and (32):

(40) $$\sum_i f_i \alpha_i V_i^2 = \sum_i f_i \alpha_i (V_i^2 - V^2) = - \sum_i \alpha_i^2 = -1.$$

Return to equations (26). Multiply the two sides by $f_i V_i^2$ and sum; that will give:

$$\sum_i f_i^2 V_i^2 + f_0 \sum_i x_i f_i V_i^2 = k \sum_i f_i \alpha_i V_i^2 ,$$

so

$$f_0 \sum_i x_i f_i V_i^2 = k \sum_i f_i \alpha_i V_i^2 - \sum_i f_i^2 V_i^2 .$$

The right-hand side is annulled, as one will recognize directly when one takes (39), (33), (40), and the value (38) of k into account; since $f_0 \neq 0$, what will then remain is:

(41) $$\sum_i x_i f_i V_i^2 = 0.$$

Now, one infers from (36) that:

$$f_0 x_i = k \alpha_i - f_i ,$$

so upon squaring both sides of this and summing:

$$f_0^2 \rho^2 = k^2 - 2k \sum_i f_i \alpha_i + \sum_i f_i^2 , \qquad \text{in which} \qquad \rho^2 = \sum_i x_i^2 .$$

By virtue of (37), (33), (38), that relation will become, in turn:

$$f_0^2 \rho^2 = k^2 - \frac{f_0}{V} = -\frac{f_0}{V} + k f_0 V = \frac{f_0}{V}(k V^2 - 1),$$

so

(42) $$1 - k V^2 = - f_0 V \rho^2 = - k \rho^2.$$

From (36) and (32), one will also have:

$$f_0\, x_i = k\, \alpha_i - f_i = k\, f_i\, (V^2 - V_i^2) - f_i\,,$$

so

$$f_0\, x_i = - f_i\, (1 - kV^2 + k\, V_i^2).$$

Upon replacing $1 - k\, V^2$ with its value (42), one will finally get:

$$f_0\, x_i = f_i\, k\, \rho^2 - f_i\; k\, V_i^2 = k\, f_i\, (\rho^2 - V_i^2),$$

so

$$f_i = -\frac{f_0}{k}\frac{x_i}{V_i^2 - \rho^2}\,.$$

Upon substituting this in (41), one will finally have:

$$\sum_i \frac{V_i^2\, x_i^2}{V_i^2 - \rho^2} = 0, \tag{43}$$

which is the point-wise equation for the FRESNEL wave surface.

As one will see immediately upon clearing the denominators, it is a fourth-degree algebraic surface.

7. Tangent planes to the surface *F*. – We saw above that the tangent planes that relate to an arbitrary direction α_i (i.e., the ones that admit the α_i for the direction cosines of their normals) are at (algebraic) distances of $d = \pm V$ from the origin, in which V is one of the propagation speeds.

The geometric construction of V that was pointed out in no. **5** will permit one to determine only the four tangent planes that are perpendicular to an arbitrary direction upon taking those distances to O to be equal to the two propagation velocities ([1]). As one sees, the FRESNEL surface enjoys the special property of having both order four and class four [while an algebraic surface of order n has class n (n – 1), *in general*, and *vice versa* ([2])].

8. Optical axes. – One calls the direction for which the two corresponding propagation velocities are equal (cf., no. 4) the *optical axes.*

([1]) For a geometric study of the FRESNEL surface, the reader can consult G. SALMON, *Traité de géométrie analytique à trois dimensions* (French translation by O. CHEMIN), part three, Gauthier-Villars, Paris, 1892, Chap. XVI, pp. 117-119. G. DARBOUX, *Leçons sur la théorie des surfaces*, t. IV, pp. 466, Gauthier-Villars, Paris, 1986. D'OCAGNE, *Cours de géométrie pure et appliqée de l'École polytechnique*, *ibidem*, 1930. DRUDE, *Précis d'optique*, t. II, Chap. IV., Gauthier-Villars, Paris, 1912. An extensive bibliography can be found in a book by GINO LORIA, *Il passato e il presente delle principali teorie geometriche*, 2nd ed., Cedam, Padua, 1931, pp. 99-102, and also in the Enz. der Math. Wiss., Bd. III, 10b, pp. 1740-1744.

([2]) Cf., e.g., ENRIQUES and CHISINI, *Teoria geometrica delle equazioni e delle funzioni algebriche*, vol. II, Zanichelli, Bologna, 1918, pp. 152.

Before everything else, we shall show that such axes will belong to a principal plane. Indeed, for any optical axis that has a direction α_i that is not parallel to the principal planes, the relation between its direction cosines and the possible propagation speeds will be expressed by (22′):

$$f(V^2) = 0,$$

and for the two roots to coincide, their common value must also satisfy the equation:

$$f_0 = \frac{\partial f}{\partial V} = 0.$$

Now, by virtue of (33):

$$f_0 = -V \sum_i f_i^2 ,$$

it will follow immediately that:

$$f_i = 0 \qquad (i = 1, 2, 3);$$

i.e.:

$$\alpha_i = 0,$$

which is absurd.

Hence, one must seek the optical axes only in the principal planes.

Consider the plane that is perpendicular to the x_i-axis ($\alpha_i = 0$), with the usual convention for the indices $i + 1$, $i + 2$.

The equation:

$$\Omega(p) = \rho_2\,\rho_3\,p_1^2 + \rho_3\,\rho_1\,p_2^2 + \rho_1\,\rho_2\,p_3^2 + \rho_1\,\rho_2\,\rho_3 = 0,$$

when one annuls the α_i (or p_i) and divides by g^2, will give:

$$\rho_1 \left(\rho_{i+2}\,\alpha_{i+1}^2 + \rho_{i+1}\,\alpha_{i+2}^2 + \frac{1}{g^2}\,\rho_{i+1}\,\rho_{i+2} \right) = 0.$$

As we know, one of the two roots V^2 is already V_i^2, which annuls the factor ρ_i, while the second one must annul the other factor, and furthermore, since we are dealing with an optical axis, it must also be equal to V_i^2.

From that, upon considering the expressions (19) that provide the ρ, dividing the left-hand side by $\rho_1\,\rho_2\,\rho_3 / g^2$, and replacing p_0^2 / g^2 with V_i^2, one will get:

$$\frac{\alpha_{i+1}^2}{\dfrac{V_i^2}{V_{i+1}} - 1} + \frac{\alpha_{i+2}^2}{\dfrac{V_i^2}{V_{i+2}} - 1} + 1 = 0;$$

i.e., upon taking the identity $\sum_i \alpha_i^2 = 1$ into account, which will then reduce to:

$$\alpha_{i+1}^2 + \alpha_{i+2}^2 = 1,$$

one will get:

$$V_i^2 \left\{ \frac{\alpha_{i+1}^2}{V_i^2 - V_{i+2}^2} + \frac{\alpha_{i+2}^2}{V_i^2 - V_{i+1}^2} \right\} = 0.$$

That relation will be satisfied by real values of the ratio $\alpha_{i+1} / \alpha_{i+2}$ only if the two denominators have opposite signs. By virtue of the inequalities:

$$V_1^2 > V_2^2 > V_3^2 ,$$

which can be true only for $i = 2$; i.e., for the principal plane that corresponds to the propagation velocity V_2 that is intermediate between the largest and the smallest ones.

There are effectively two directions in the $x_3\, x_1$-plane that correspond to the two values $\pm\sqrt{\dfrac{V_2^2 - V_3^2}{V_1^2 - V_2^2}}$ of the ratio α_3 / α_1 .

9. Case in which the Fresnel surface degenerates. – The case that we have excluded in which two (and only two) of the propagation velocities V_i are equal corresponds to the media that are called *uniaxial*, in which there is only one optical axis. From the algorithmic viewpoint, one must recall the preceding calculations upon taking into account the fact that two of the V_i are equal. However, if one envisions a well-defined result or a geometric relation that is valid when all of the V_i are distinct then one will have the right to pass to the limit that makes two of those V_i coincide.

For example, one can assert that in that case, the auxiliary ellipsoid $\varphi = 1$ (no. **5**) will become one of revolution and will then have an equatorial radius R that is inverse to the common value of the two equal velocities V_i .

An arbitrary semi-diameter will always remain between that equatorial radius and the inverse of the third velocity V_i .

For any section by a diametral plane, one of the semi-axes will necessarily coincide with the equatorial radius R, in such a way that any plane that is at a distance of R from the center will belong to the set of tangent planes to F. In other words, the surface F must contain the sphere of radius R.

Since F has order four, it will then decompose into that sphere and a quadric (viz., an ellipsoid). That is easy to verify by means of the Cartesian equation of F by supposing that two of the V_i are equal and clearing the denominators.

____

§ 11. – The wave-corpuscle duality of modern physics according to de Broglie.

1. – Ever since YOUNG and FRESNEL, all light phenomena that were known for some times seemed to take place in a wave-like schema, first, by means of an elastic representation, and then by means of MAXWELL's electromagnetic equations (viz., the electromagnetic theory of light). However, one could still not succeed in reconciling the wave theory in any simple way with the observed facts that pertained to photoelectric phenomena, which go back to HERTZ.

Here is essentially what one is dealing with: When a beam of light strikes a metallic surface, it will very often liberate electrons. Qualitatively, one models that phenomenon by supposing that part of the incident light energy is utilized to do a certain amount of work l (which depends upon the metal in question) that is necessary to liberate the electron and part of it also communicates kinetic energy.

The intensity of the incident light will be included in the energetic evaluation, but not its frequency. Now, one can experimentally exhibit the fact that below a certain frequency, and for any intensity, of the incident light, the photoelectric effect will not be produced (LENARD), while the maximum velocity that is communicated to the electrons will depend upon its frequency exclusively (MILLIKAN).

That aspect of the phenomenon, which is inexplicable from the standpoint of wave optics, has, on the contrary, found a brilliant quantitative representation with EINSTEIN's quantum, corpuscular hypothesis (1905), according to which, any sheaf of light rays of frequency v must be considered to be composed of a cloud of photons, or light quanta (viz., particles of energy) that each possess an energy E that is proportional to the frequency v, and is precisely:

$$E = hv,$$

in which h is the celebrated PLANCK constant.

To the extent that the photoelectric effect is attributed to the collisions of those photons, it is clear that whereas hv will remain less than the work l that is done by the extraction that we spoke of, it will not produce the emission of any electrons, no matter how large the intensity of the light considered, and the various observed facts agree remarkably with that corpuscular hypothesis.

It likewise served to account for a phenomenon that was discovered by COMPTON in 1923, according to which a beam of X-rays that meets up with material elements will generally be scattered with a reduction in its frequency, while there will once more be an emission of electrons. All of that is explained in the following fashion, as was shown by COMPTON, DEBYE, FERMI, and PERSICO, by associating EINSTEIN's hypothesis with the principle of the conservation of the quantity of motion, in addition to the principle of the conservation of energy.

How the wave theory and the return to the corpuscular hypothesis might be founded in some advanced theory cannot be explained completely at present. It will suffice for us to remark that if modern physics wishes to explain certain optical phenomena then it will need to include both wave concepts and corpuscular ones at the same time. An analogous situation presents itself in the study of electrons, which is based, above all, upon the properties of cathode rays and some celebrated experiments at the end of the Nineteenth

Century that are due, above all, to J. J. THOMSON, KAUFMANN, and H. A. WILSON, who characterized electrons completely as pure electric charges of the same value.

However, that exclusively corpuscular viewpoint will not account for the phenomenon of the diffraction of electrons in crystals that was discovered by DAVISSON and GERMER in 1927, and ultimately confirmed by the experiments of RUPP and G. P. THOMSON.

The inverse of what we saw previously in the interpretation of phenomena will present itself here; i.e., the electronic phenomena that could, up to these latter years, take place in the context of an exclusively-corpuscular theory now seem to demand some complementary developments of wave type as a result of new experimental observations.

That sort of duality for which the most remarkable facts of modern physics demand the simultaneous intervention of waves and corpuscles was recognized and proposed as a general law of nature by the physicist LOUIS DE BROGLIE, and that was even before it was so admirably illustrated by the diffraction of electrons.

He first of all sought to give a more concrete form to its conception by associating any moving corpuscle with a well-defined group or packet of waves. However, he himself recognized the difficulty in such an association ([1]).

Another remarkable idea regarding the correspondence was pointed out by MAGGI ([2]) as an application of HAMILTON's principle of least action. However, no matter how seductive it might be theoretically, it does not seem to permit a quantitative representation of the many observed facts.

We again point out that a very interesting dynamico-optical reconciliation was proposed by PERSICO ([3]) in order to justify the SCHRÖDINGER equation, although he did not provide the true law of correspondence between the well-defined corpuscular and wave aspects of a given phenomenon either.

The considerations that were developed before for normal systems that associate them with, on the one hand, characteristic manifolds (viz., wave surfaces) and on the other hand, characteristic lines (viz., trajectories) offer a very broad paradigm that will reflect both the wave and corpuscular aspects of the same phenomenon as soon as one is in possession of a differential system that is appropriate to it.

That is what appears clearly in the case of the SCHRÖDINGER equation (from the admirable spectroscopic verifications of SCHRÖDINGER himself).

Recall the equation in question, by way of example ([4]):

$$S \equiv \frac{2}{E^2}(U+E)\frac{\partial^2 \varphi}{\partial t^2} - \Delta_2\,\varphi = 0, \tag{1}$$

([1]) Cf., L. DE BROGLIE, *Introduction à l'étude de la mécanique ondulatoire*, Hermann, Paris, 1930 (Preface).

([2]) Cf., G. A. MAGGI, "Sul significato nel passato e nell'avennire delle equazioni dinamiche," Rend. del Sem. mat. e Fis. di Milano **3** (1930), 53-72.

([3]) Cf., E. PERSICO, *Lezioni di Meccanica ondulatoria* (lith.), 2nd ed., Cedam, Padua, 1930, pp. 29-40.

([4]) Cf., E. SCHRÖDINGER, *Abhandlungen zur Wellenmechanik*, Barth, Leipzig, 1927, pp. 38. See also pp. 40 of the lectures of prof. PERSICO that were cited in the preceding footnote.

in which the constant E represents a unitary energy and will take on a quantum form *a posteriori* by means of the eigenvectors (i.e., characteristic vectors) of (1), which are defined by convenient regularity conditions. (U is the unitary electrostatic potential.)

Recall once more that the solutions φ to equation (1) that are utilized in wave mechanics are generally complex and that it is only $|\varphi|^2$, and not φ, that has a direct physical interpretation, moreover, as a quantity that is proportional to a certain local probability (viz., the probability of the presence of the electron in a neighborhood of a given point).

From the mathematical viewpoint, which has been at the basis of the considerations, or better still, the *divinations* that led SCHRÖDINGER to equation (1) for the first time, we shall retain only the fact that a very important set of phenomena, such as the distribution of the BALMER spectroscopic lines and the "fine structure" of the hydrogen atom, are admirably interpreted and condensed by equation (1).

If suffices to denote the coefficients $2(E + U) / E^2$ of $\frac{\partial^2 \varphi}{\partial t^2}$ by $1 / V^2$ in order to convert it into the form of the canonical equation of small motions ([1]) (§ **2**, nos. **1** and **6**), whose characteristic manifolds:

$$z(x_0, x_1, x_2, x_3) = \text{const.}$$

are defined, as we saw (§ **3**, no. **4**) by the homogeneous equation of degree two:

$$\Omega = \frac{1}{V^2} p_0^2 - \sum_{i=1}^{3} p_i^2 = 0.$$

Upon solving this for p_0 in the form:

$$p_0 + H = 0,$$

one will get:

$$H = -V \sqrt{\sum_i p_i^2} ,$$

which constitutes the Hamiltonian function of the bicharacteristics, as we know.

All of that is quite simple. We wished to state it explicitly in order to draw attention to the following general fact, thanks to that characteristic example: If one knows a theoretical representation of a phenomenon as a normal system of partial differential equations in the parameters φ (viz., the SCHRÖDINGER equation $S = 0$, in the present case) then one can immediately deduce the equations that define the characteristics and bicharacteristics from it, i.e., the partial wave and corpuscular aspects that they are linked to. On the contrary, if one knows only one or the other of those aspects in some situation (i.e., Ω or H, analytically) then one cannot get back to the complete law of the phenomenon– in other words, to the normal system that represents it – without knowing more.

([1]) In truth, the coefficient V denotes a constant in this. However, the manner by which one obtains the characteristics would not suffer any modification even if V were an arbitrary function of space and time.

If one consider the SCHRÖDINGER equation, more especially, then one will observe that knowing Ω will not suffice to determine S, since that would result easily from the fact that if one adds a function F to S that depends arbitrarily upon the x, the φ, and the first derivatives of φ, then, from the rule in § **7**, no. **2**, the equation $S + F = 0$ will possess the same characteristics and bicharacteristics.

In certain cases for which one knows one of the two partial aspects of a particular phenomenon from ordinary macroscopic physics – i.e., analytically, the function Ω of the p and the x – it can suffice to replace each p_k with the operator:

$$\frac{h}{2\pi i}\frac{\partial}{\partial x_k} \qquad (k = 0, 1, 2, 3;\ i = \sqrt{-1})$$

for the equation:

$$\Omega\left(x \;\middle|\; \frac{h}{2\pi i}\frac{\partial}{\partial x}\right)\varphi = 0$$

to provide the corresponding partial differential equation of micro-mechanics, but such a rule is not general.

Indeed, it will suffice to think that a term of the type $a\,p_0\,p_1$, in which a is a function of position and time, can just as well give rise to one of the four expressions:

$$a\frac{\partial^2\varphi}{\partial x_0\,\partial x_1}, \qquad \frac{\partial^2(a\,\varphi)}{\partial x_0\,\partial x_1}, \qquad \frac{\partial}{\partial x_0}\left(a\frac{\partial\varphi}{\partial x_1}\right), \qquad \frac{\partial}{\partial x_1}\left(a\frac{\partial\varphi}{\partial x_0}\right),$$

which all have the second-order term $a\dfrac{\partial^2\varphi}{\partial x_0\,\partial x_1}$ in common, but differ by terms that depend upon the x, the φ, and first derivatives, and which have no influence upon Ω, as was remarked above.

The formal rule that was given previously can then have only a heuristic value ([1]) [which is still admirably in the work of SCHRÖDINGER and DIRAC ([2])], but it does not seem possible to infer a systematic method of construction from it that will reflect a true physical reality.

As for the purely mathematical paradigm that provides the theory of characteristics, we shall further point out a remarkable application that M. RACAH ([3]) made to the DIRAC equations, which generalize that of SCHRÖDINGER and which constitute what one can presently consider to be the most complete mathematical synthesis of electromagnetic and optical micro-phenomena. He deduced an instructive justification of HEISENBERG's uncertainty principle as a consequence of the equation $\Omega = 0$ that defines the characteristics in a very expressive special case.

———————

([1]) Especially if the normal system in question must satisfy some special conditions, such as invariance under a group or even conditions that relate to any transformation of the x.

([2]) *The Principle of Quantum Mechanics*, Clarendon Press, Oxford, 1930.

([3]) "Caratteristiche delle equazioni di Dirac e principio di indeterminazione," Rend. Acc. Lincei (6) **13** (1931), 424-427.