"Sullo spostamento dell'equilibrio," Atti Ist. Ven. (7) **71** (1911-12), 241-249; *Opere mathematiche*, v. III, art. XVII, pp. 253-260.

On the displacement of equilibrium

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Lord RAYLEIGH ([1]) contributed some general results on the equilibrium of material systems in the proximity of a state of minimum internal energy [whose importance is also revealed in thermodynamical applications ([2])]. Those results were deduced from the principle of virtual work with some algebraic transformations that are quite obvious under the hypothesis that the analytical expressions for the internal energy of the system, its constraints, and the field of external forces were as simple as possible.

In general, expressions of that type are valid approximately, but not exactly, within a sufficiently-small neighborhood of the state of minimum energy (with an approximation that improves as the neighborhood gets smaller). Having said that, one can reasonably expect that RAYLEIGH's qualitative conclusions will continue to be valid, with no further assumptions, in a convenient neighborhood by reason of continuity. In reality, things do not happen in that way, and it is easy to persuade oneself of that by some elementary examples (cf., no. **5** of the present note). However, it is quite simple to exhaust the question: It is enough to invest it with analytical generality from the outset and then specify the importance of the continuity considerations.

One sees that the influence of the terms that are not considered in the typical expression is even zero as far as the internal energy is concerned. It is less important in practice when compared to that of the external field. However, it takes on an essential character when there are constraints.

I. – Review of the calculation of the behavior of a function Ω in the proximity of an effective minimum.

Let $x_1, x_2, \ldots, x_n$ be independent variables, and we adopt the usual hyperspatial language and say "point" to mean a set of values that are attributed to the variables and "origin" to mean the point $x_i = 0$ ($i = 1, 2, \ldots, n$), etc.

Let $\Omega(x_1, x_2, \ldots, x_n)$ be a function that is finite and continuous, along with its derivatives of the first two orders, at least in a neighborhood of the origin, which is meant to limit our considerations.

([1]) "General theorems relating to equilibrium and initial and steady motion," Phil. Mag. **49** (1875), 218-224; or *Scientific Papers*, vol. I, Cambridge, University Press, 1899, pp. 232-237.

([2]) Cf., e.g., P. DUHEM, *Traité d'énergétique*, t. I, Paris, Gauthier-Villars, 1911, chap. XI.

Suppose that Ω assumes an effective minimum at the origin and that the existence of a minimum is recognized from the second differential, or that the quadratic form A_0 whose coefficients are the second derivatives of Ω at the minimum point $x_i = 0$, is positive-definite.

Let A represent the analogous form in which the coefficients refer to another generic point x_i and recall that the entity that was defined is assured for a form of a qualitative character by certain inequalities between the coefficients. If the inequalities are verified for a certain determination of them then that will continue to be true when the coefficients are made to vary sufficiently little.

That implies that the quadric A will continue to be defined in a convenient neighborhood of the origin.

2. – Consideration of another function *U* with a slowly-varying gradient.

Let $U(x_1, x_2, \ldots, x_n)$ denote an arbitrary function of the x that is subject to the single condition that the absolute values of its second derivatives are *sufficiently small* in a neighborhood of the origin. Define the quadric B whose coefficients are those second derivatives. The form $A - B$, like A, will also be positive-definite in a non-zero neighborhood of the origin, which shall be called I. It is intended that the condition that is imposed upon the second derivatives of U that their absolute values should be sufficiently small should be specified in the sense that they do not exceed certain limits (which depend upon Ω).

It is enough to consider the fact that once Ω and U have been fixed, the neighborhood I will certainly exist, under appropriate restrictions.

In order to justify the title of the present section, let us add that supposing that the second derivatives of U are small is equivalent to supposing that the first derivatives are slowly-varying, since they collectively define the gradient of the function.

3. – Static problem and its classical solution based upon the principle of virtual work.

Interpret Ω as the internal energy of a material system S whose state is defined by the variables $x_1, x_2, \ldots, x_n$. The n-tuple $x_i = 0$ then serves to individuate the state of minimum internal energy or the *natural state* C_0.

Suppose that such a system S is found to be subject to holonomic constraints of the type:

$$f_k(x_1, x_2, \ldots, x_n) = 0 \qquad (k = 1, 2, \ldots, l;\ l < n), \tag{1}$$

in which the f are independent functions that are finite and continuous, along with the first and second derivatives, in the neighborhood I.

Now let S be subject to a conservative external force with a potential U, so $\partial U / \partial x_i$ represents the component of the external force along the general coordinate x_i. Considering that external influence will give meaning to the hypothesis of the preceding section (concerning the second derivative of U), as well as exhibit the mechanical significance of the following: The force field in which the system S is supposed to be immersed is *almost uniform* (i.e., the components of $\partial U / \partial x_i$ are slowly varying, at least in the proximity of the natural state C_0 .

In order to characterize the equilibrium of the system under the indicated conditions, it is naturally sufficient to recall the principle of virtual work, according to which the first differential of $\Omega - U$:

$$\sum_{i=1}^{n} \frac{\partial(\Omega - U)}{\partial x_i} \delta x_i$$

must be annulled for any displacement δx_i that is compatible with the constraints.

If one introduces the LAGRANGE multipliers then one will have the n equations:

$$\text{(2)} \qquad \frac{\partial(\Omega - U)}{\partial x_i} + \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n) ,$$

from which one will get the values of x_i [upon combining the system with (1) and eliminating the auxiliary functions]. That will define the equilibrium configuration C, provided that the solution of (1), (2) is actually possible.

Suppose that this has been done, while the configuration C once more falls in the neighborhood I.

It should be noted that in the absence of constraints and an external field, C will be identified with the state C_0 of minimum energy and that, on the other hand, one can (and in general, one must, in expectation of the restrictions that will be introduced) limit oneself to a convenient neighborhood that circumscribes that state, so it will appear perfectly natural to represent the equilibrium at C as a phenomenon of *displacement of equilibrium* (from C_0 to C), where that displacement is due to the two facts that one imposes (holonomic) constraints and makes a conservative force act: Of course, one should not exclude the particular cases in which one of the two perturbing influences intervene.

4. – Stability of displaced equilibrium. Case of linear constraints.

In order for the equilibrium at C [which was defined by (1), (2)] to also be stable, it is enough to know that the function $\Omega - T$ has an effective minimum at C with respect to the other configurations that are compatible with the constraints. Moreover, that is ensured whenever the second derivative of $\Omega - U$, as calculated with regard to (1), proves to be essentially positive.

Let us do that explicitly by differentiating:

$$\sum_{i=1}^{n} \frac{\partial(\Omega - U)}{\partial x_i} \delta x_i$$

a second time and taking into account the fact that one cannot set $\delta^2 x_i = 0$ with no further conditions, because the x_i must be treated as variables that are (not completely independent, but) coupled by (1).

It obviously results that:

$$\delta^2(\Omega - U) = A - B + \sum_{i=1}^{n} \frac{\partial(\Omega - U)}{\partial x_i} \delta^2 x_i ,$$

where the arguments of the quadratic form $A - B$ [nos. **1** and **2**] are the increments δx_i and its coefficients are the second derivatives of $\Omega - U$ that relate to the configuration C.

Let α denote the additional term:

$$\sum_{i=1}^{n} \frac{\partial(\Omega - U)}{\partial x_i} \delta^2 x_i ,$$

which is provided by the constraints, and write:

$$\delta^2(\Omega - U) = A - B + \alpha ,$$

accordingly.

Imagine that just as many x are eliminated by means of the l equations of constraint as there are equations (or more generally, that all of the x are expressed by means of $n - l = \nu$ Lagrangian parameters $q_1, q_2, \ldots, q_n$), so A, B, and α will become quadratic forms in $n - l$ arguments (the δx remain independent, while the increments $\delta q_1, \delta q_2, \ldots, \delta q_n$ will be arbitrary).

Since $A - B$ is essentially positive when it is considered to be an n-ary form of the δx, that will still remain true when one reduces the degrees of freedom. However, one can say nothing *a priori* about the additional term α. Nonetheless, an obvious observation presents itself, namely, that α is annulled where one has $\delta^2 x_i = 0$ ($i = 1, 2, \ldots, n$). In particular, that will happen when the constraints (1) are represented by linear (and also inhomogeneous) equations in the x.

Indeed, when the $f_k = 0$ ($k = 1, 2, \ldots, l$) are linear, one will also get expressions for l of the x, as solved in terms of the remaining $n - l$ of them.

Due to the linearity of those expressions with respect to the variables, which are all independent, their second differentials will be identically zero. Therefore, the second differentials of the l variables x, which are considered to be functions of the other ones, will also be annulled. Q.E.D.

That will imply the stability of displaced equilibrium in any quasi-uniform force field and in the face of linear constraints.

5. – Example.

Let the system S consist of a single material point P in a plane x, y that is endowed with energy (e.g., elastic) that returns it to the origin O of the coordinates and is expressed by:

$$\Omega = \frac{\mu}{2}(x^2 + y^2) ,$$

in which μ denotes a positive constant.

Things clearly happen as if *P* were attracted to *O* by a force that is proportional to the distance $\overline{OP}$.

In the absence of constraints and other forces, *O* is a stable equilibrium position.

Consider a displacement of the equilibrium that is limited, for the sake of simplicity, to the case in which one introduces a constraint without make an external force act (*U* = const.).

First, let us treat a linear constraint. Namely, let *P* be constrained to remain along a straight line. The foot *M* of the perpendicular that is based at *O* along the line obviously constitutes the new equilibrium position. Since *M* is closest to *O* of all points on the line, Ω will assume the minimum value with respect to all positions that are consistent with the constraints here, and the equilibrium will again be stable.

Let us move on to the general case in which *P* is constrained to remain along a line *L* that is not straight. The equilibrium can very well become unstable.

In order to explain that intuitively, we begin by fixing a point *M* (which is distinct from *O*) at will, and suppose that γ is a circle with center *O* that passes through *M* and *L* is a curve that is tangent to γ at *M*. Suppose (with the sole purpose of abbreviating the discussion) that the radius of curvature of *L* at *M* is different from $\overline{OM}$. That will permit us to assert that either *L* will be completely external to γ or completely internal to it in the immediate proximity of *M*.

In the first case, the constrained equilibrium will obviously prove to be stable (as in the case of the straight line). However, it is unstable in the second case, and that will also make the displacement infinitely small, i.e., one must choose *M* to be as close as one wishes to *O*.

Analogous conclusions can follow quite simply from the analytical approach that is connected with the general criterion that was recalled in the preceding section.

Imagine that the equation of constraint expresses *y* as a function of *x* (that is regular in a neighborhood of the origin) in the form:

(3) $$y = a + b\,x + c\,x^2 + \ldots,$$

in which *a*, *b*, *c*, … likewise denote constants.

One infers from this that:

$$\delta y = \{b + 2cx + \ldots\}\ \delta x ,$$
$$\delta^2 y = \{2c + \ldots\}\ \delta x^2 ,$$

in which the omitted terms contain at least x^2 as a factor in the expression for δy, and therefore at least *x* in the expression for $\delta^2 y$.

The quadric that must be considered is then:

$$\delta^2\Omega = \mu\{\delta x^2 + \delta y^2\} + \mu\, y\,\delta^2 y = \{1 + b^2 + 2ac + \cdots\}\,\delta x^2 ,$$

and the terms that were not specified contain x as a factor.

Therefore, if the displacement from equilibrium is sufficiently small (in particular, it might oscillate about the abscissa x of the new equilibrium position) then the discriminant of stability will be the sign of the trinomial $1 + b^2 + 2ac$. c is zero for a linear constraint, and one then has stability for any value of a (but not that of b). However, in general, not matter how small a is, i.e., no matter how closely the curve (3) passes near the origin, one can always attribute a value to c such that the aforementioned trinomial proves to be negative, etc.

6. – Rayleigh's second theorem.

Turning to a generic system S, suppose that the constraints that are imposed consist of assigning values to some of the x, e.g., to $x_1, x_2, \ldots, x_l$. That can be expressed by saying that l of the displacements that make the natural configuration pass to the one that will be the new equilibrium position are assigned. One obviously has a special case of a linear constraint (which is inhomogeneous, in general).

Furthermore, suppose that no external forces are acting.

One can then assert (no. **4**) that $\Omega - U$, and therefore (when U is constant) *the internal energy* Ω, *has a minimum for the configuration C in which the effect of the constraints is to establish equilibrium when compared to all other configurations that are consistent with those constraints, i.e., in the present case, the l prescribed displacements.*

7. – The first theorem.

Now consider a system whose internal energy Ω has the typical expression of a quadratic form in the x (with constant coefficients) and whose constraints are linear and *homogeneous*. In addition, suppose that the external force is constant (i.e., a rigorously uniform field), which also allows one to suppose that U is a linear and *homogeneous* function of the x.

In that case, the qualitative limitations (that were referred to in nos. **1** and **2**) are certainly verified for arbitrary values of the x, so the neighborhood I of the origin is comprised of all of space. One then has from (2), after multiplying by x_i in the ordinary way (the x_i means the equilibrium configuration C), summing, and recalling EULER's theorem on homogeneous functions:

$$2\Omega - U + \sum_{k=1}^{l} \lambda_k \, f_k = 0 .$$

By virtue of (1), what remains is:

$$2\Omega - U = 0 ,$$

or, if one prefers:

$$\Omega - U = - \Omega . \tag{4}$$

This last relation, which is verified for C, would be just as valid for any other configuration $\bar{C}$ in which equilibrium is established, constraints are added, and the external influences are left unaltered.

On the other hand (no. **4**), $\Omega - U$ has a minimum for C compared to all other configurations that are consistent with the ordinary constraints (1), and therefore, *a fortiori*, compared to all $\bar{C}$. However, (4) persist in an arbitrary $\bar{C}$. The minimum of $\Omega - U$ (with respect all varied $\bar{C}$) therefore translates into a maximum of Ω, which implies: Under the supposed conditions, *the displaced equilibrium configuration C will make the internal energy Ω a maximum with respect to the values that its energy assumes in any new equilibrium state $\bar{C}$ that is a consequence of the introduction of the final constraints* (which are linear and homogeneous).

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