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On transformations of the dynamical equations

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INTRODUCTION

Along with the classical problem of the transformability of two quadratic differential equations, in recent years, **Appell** (*) posed the more-general analogous problem of the transformability of two systems of dynamical equations in the same number of variables.

Afterwards, various authors have carried out research into that subject, notably, **Painlevé** and **R. Liouville**, who deserve the credit for having discovered from interesting general properties. In order to give a brief account of the subject, it would be worthwhile to next specify the question and indicate the definitive character of how it can be formulated.

To begin with, here is what **Appell** said about it:

If one is given two material systems S and S_1 with constraints that are independent of time and have the same degrees of freedom, and one is given x_i and y_i ($i = 1, 2, \ldots, n$) as the Lagrangian coordinates that fix the positions in the two systems, while X_i, Y_i are the forces that act at those coordinates, then one will have:

$$T = \tfrac{1}{2}\sum_{r,s=1}^{n} a_{rs}\, x'_r\, x'_s\,, \qquad T_1 = \tfrac{1}{2}\sum_{r,s=1}^{n} b_{rs}\, \bar{y}'_r\, \bar{y}'_s\,,$$

$$\frac{d}{dt}\frac{\partial T}{\partial x'_i} - \frac{\partial T}{\partial x_i} = X_i\,, \tag{A}$$

$$(i = 1, 2, \ldots, n)$$

$$\frac{d}{dt_1}\frac{\partial T_1}{\partial \bar{y}'_i} - \frac{\partial T_1}{\partial y_i} = Y_i\,, \tag{B}$$

as the *vis vivas* and equations of motion of the two systems. As usual, we have denoted the derivative of x_i with respect to t by x'_i, and in order to avoid any ambiguity, the derivative of y_i with respect to t_1 by $\bar{y}'_i$.

(*) "Sur des transformations de mouvements," Crelle's Journal **109** (1892). In that article, one will find many references to works that relate to that argument or some of its special cases.

We ask under what conditions it will exist and how one can define a transformation of the type:

$$\left.\begin{aligned} & y_i = \varphi_i(x_1, x_1, \ldots, x_1) \\ & dt_1 = \frac{dt}{\mu(x_1, x_1, \ldots, x_1)} \qquad (i = 1, 2, \ldots, n), \end{aligned}\right\} \tag{C}$$

that makes the system of equations (B) coincide with the system (A) when we suppose that:

1. We are given the forces X_i, as well as Y_i .
2. We are given only the forces X_i, and we can assign the Y_i as we please.

As **Appell** has observed, the second part of the question is always soluble when the forces are regarded as also depending upon velocity. That is, one can (and in an infinitude of ways) suitably choose the Y_i and make any motion of the system S that is due to forces that depend upon the coordinates and velocity correspond to an analogous motion of the system S_1, which differential equations (B) are reducible to (A) by means of a transformation of type (C).

The problem, when restricted to that viewpoint, was reprised by **Painlevé**, who presented it (*), but with a felicitous modification: He required only that the trajectories of the system (B) could reproduce those of (A), or in other words, that the $n - 1$ integrals of (B) that are independent of t_1 would be made to coincide with the $n - 1$ integrals of (A) that are independent of t by means of a transformation of the type:

$$y_i = \varphi_i\,(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n). \tag{D}$$

Moreover, he showed how one could split the study of that problem into two parts, only the first of which is essential, since one can reduce the second one to the first by a transformation of the quadratic differential forms. In order to establish that fact, imagine that one has replaced the y_i in (B) with their values $\varphi_i\,(x_1, x_2, \ldots, x_n)$ and has set:

$$T_1 = \sum_{r,s=1}^{n} b_{rs}\,\bar{y}'_r\,\bar{y}'_s = \sum_{r,s=1}^{n} \alpha_{rs}\,\bar{x}'_r\,\bar{x}'_s\,,$$

$$\Xi_i = \sum_{k=1}^{n} Y_k \frac{\partial y_k}{\partial x_i}\,.$$

The equations:

$$\frac{d}{dt_1}\frac{\partial T_1}{\partial \bar{x}'_i} - \frac{\partial T_1}{\partial x_i} = \Xi_i \qquad (i = 1, 2, \ldots, n), \tag{A$_1$}$$

which are, as is known, the transforms of (B), must admit *the same trajectories* as (A). They show, in the first place, from any pair of systems (A), (B) whose trajectories can be made to coincide by

(*) "Sur la transformation des équations de la dynamique," Journal de Liouville **10** (1894).

means of a transformation (D), one can deduce a pair of systems (A), (A_1) that are also dynamical equations and *have the same trajectories*. On the other hand, if one knows all of the (A_1) that have the same trajectories as any system (A), i.e., all of the *corresponding* systems, to use **Painlevé**'s terminology, then the original question that was posed of deciding whether and how one can define a transformation (D) that serves to take the trajectories of (B) to those of (A) can be regarded as solved. As long as one such transformation exists, then and only then will it be possible to establish an identity of the type:

$$\sum_{r,s=1}^{n} b_{rs}\, dy_r\, dy_s = \sum_{r,s=1}^{n} \alpha_{rs}\, dx_r\, dx_s ,$$

in which the b_{rs} are the coefficients of the *vis viva* of (B) and the α_{rs} are the coefficients of the *vis viva* of a system (A_1) that corresponds to (A).

If one supposes, in addition, that all of the systems (A_1) are known, then nothing will remain but to examine whether the quadratic differential forms $\sum_{r,s=1}^{n} b_{rs}\, dy_r\, dy_s$, $\sum_{r,s=1}^{n} \alpha_{rs}\, dx_r\, dx_s$ are equivalent in the ordinary sense for each of them and to determine the transformation formulas in the affirmative case. In order for that criterion to be indeed applicable obviously demands that the possible expressions for the α_{rs} reduce to a finite number of types, and that is precisely true in our case, as long as the problem of determining all of the correspondent to a given (A) is that of finding a general integral that depends upon only a finite number of arbitrary constants, as we will see, and as **Painlevé** had observed before.

We therefore conclude that, with no loss of generality, it is enough to limit ourselves to the question of *determining all* (A_1) *that correspond to a given system* (A).

The most noteworthy results that have been achieved up to now can be summarized as follows (*):

1. Whenever a system (A) admits a correspondent (A_1) that is *not ordinary* [i.e., not of a certain type that is very easy to define and belongs to any (A) that is fixed arbitrarily], there will exist a quadratic first integral for both of them.

2. If one supposes that forces in the system (A) are zero then the same thing must be true for any of its correspondents (A_1), so if one sets $ds^2 = 2T\, dt^2$, $ds_1^2 = 2T_1\, dt_1^2$, the search for the systems that correspond to (A) will be equivalent in that case to the determination of all manifolds ds_1 that have the same geodetics as ds. (For $n = 2$, the problem was solved by **Dini**, as is known.)

3. **R. Liouville**'s theorem (**): Two systems (A) and (A_1) that are subject to no forces and define the same geodetics will both admit n quadratic integrals that can, however, coincide, and they can also reduce to just the *vis viva* integral.

(*) I have presented these results in a way that seems simplest to me, but their order of presentation does not respect their natural progression, as we will have occasion to confirm later on.

(**) "Sur les équations de la dynamique," Acta Math. **19** (1895). What is noteworthy about it is that some of the results that are indicated here are extended to a more-general class of dynamical equations.

4. The transformation that permits one to take a system (A_1) to its correspondent (A) will be of the type $dt_1 = \dfrac{dt}{\mu(x_1, x_2, \ldots, x_n)}$ when no forces are active, and more generally of the type $dt_1^2 = \dfrac{dt^2}{\mu^2}\{1 - \sum_{r,s=1}^{n} c_{rs} x_r' x_s'\}$ (the μ and c_{rs} are functions of the x) in all other cases. From that, it would seem that, in general, the condition that two systems of equations should admit the same trajectories is somewhat less restrictive than that of their transformability according to **Appell** [see the second of equations (C)]. The two conditions will be equivalent only when the forces are absent.

Despite those noteworthy propositions, the general problem of determining all correspondents to a given (A) has still not been addressed: **Liouville** established the differential equations (although he established the suggested property in a different way from **Painlevé**) without, at the same time, proposing how one could integrate them, which would not, in truth, be easy if one were to address his formulas directly.

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§ 1. Corresponding systems.
Their equivalence up to quadrature.

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§ 2. – Method for passing from one to another of two corresponding systems.

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(**§ 2** duplicated in file)

§ 3. – Characters of the functions *f*.

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§ 4. – Proof that the forces in two corresponding systems must be simultaneously zero.
Form of the relation between t and t_1 when no forces are active.
Equations that couple the α_{rs} to the a_{rs} in that case.

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