"Sur les ondes et radiations gravitationelles," C. R. Acad. Sci. Paris **246** (1958), 893-896.

On gravitational waves and radiation

Note by **ANDRÉ LICHNEROWICZ**

Presented by Georges Darmois

Translated by D. H. Delphenich

Relations verified by the discontinuity tensor of the curvature tensor. General geometric study of those relations. Definition of a state of total and purely gravitational radiation. Deviation of the trajectories of charged particles.

1. In the case of a penta-dimensional Riemannian manifold V_5 that is endowed with a one-parameter group of isometries that is interpreted in terms of gravitational and electromagnetic fields, the study of discontinuities of the first derivatives of the electromagnetic field is equivalent to that of the discontinuities in one part of the curvature tensor of V_5 .

Therefore, let V_{m+1} be a differentiable manifold of class (C^2, piecewise C^4) that is endowed with a Riemannian metric:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = 0, 1, \ldots, m)$$

that has normal hyperbolic type and class (C^2, piecewise C^4). If a neighborhood U in V_{m+1} is referred to local coordinates (x^α) then let $f(x^\alpha) = 0$ be the local equation of a hypersurface S that will produce discontinuities in the curvature tensor upon crossing it (viz., a *wave front).* If $l_\alpha = \partial_\alpha f$ ([1]) then one will have:

$$(1) \qquad l_\alpha\, [R_{\beta\gamma,\, \lambda\mu}] + l_\beta\, [R_{\gamma\alpha,\, \lambda\mu}] + l_\gamma\, [R_{\alpha\beta,\, \lambda\mu}] = 0,$$

in which the symbol [...] denotes the discontinuity upon crossing the wave front.

Suppose that the metric on V_{m+1} satisfies the "generalized" Einstein equations with a continuous right-hand side. One will then have $[R_{\alpha\beta}] = 0$. If:

$$[\partial_{\lambda\mu}\, g_{\alpha\beta}] = a_{\alpha\beta}\, l_\lambda\, l_\mu$$

([1]) $\partial_\alpha = \partial_\alpha / \partial x^\alpha$.

then in order for one to have $[R_{\alpha\beta}] = 0$, from a result of Bel, it will be necessary and sufficient that l_α should be *isotropic* and satisfy $a_{\alpha\beta}\, l^\beta = (a / 2)\, l_\alpha$ (with $a = g^{\alpha\beta} a_{\alpha\beta}$). One will immediately infer from the preceding relations that on a wave front:

(2) $$l^\alpha [R_{\alpha\beta,\lambda\mu}] = 0.$$

2. Consider a tensor $H_{\alpha\beta,\lambda\mu}$ at the point x of the manifold V_{m+1} that presents the same type of symmetry as the curvature tensor:

$$H_{\alpha\beta,\lambda\mu} = - H_{\beta\alpha,\lambda\mu} = - H_{\alpha\beta,\mu\lambda}\,, \qquad H_{\alpha\beta,\lambda\mu} = H_{\lambda\mu,\alpha\beta}\,,$$

and suppose that there exists a vector l_α such that:

(3) $$l_\alpha H_{\beta\gamma,\lambda\mu} + l_\beta H_{\gamma\alpha,\lambda\mu} + l_\gamma H_{\alpha\beta,\lambda\mu} = 0$$
and
(4) $$l^\alpha H_{\beta\gamma,\lambda\mu} = 0.$$

If is non-zero then l_α will be *necessarily isotropic.* An algebraic study of the contracted tensor:

$$H_{\alpha\beta} = g^{\rho\sigma} H_{\alpha\rho,\beta\sigma}$$

will show that there will then exist a scalar τ such that:

(5) $$H_{\alpha\beta} = \tau\, l_\alpha\, l_\beta\,.$$

Conversely, (3) and (5) imply (4). Finally, *for* $m = 3$ (viz., the case of general relativity), one can establish that (4) and (5) imply (3). In that case, there exists an orthonormal frame $(\mathbf{e}_\alpha)$ whose vector $\mathbf{e}_0$, which is time-oriented, can be chosen arbitrarily, such that:

$$H_{\alpha\beta,\lambda\mu} = a\, m_{\alpha\beta}\, m_{\lambda\mu} + b\, n_{\alpha\beta}\, n_{\lambda\mu}\,,$$

in which m and n are bivectors that are defined by $\boldsymbol{l}$ and $\mathbf{e}_2$, $\boldsymbol{l}$ and $\mathbf{e}_3$, respectively. In order to have $H_{\alpha\beta} = 0$, it will be necessary and sufficient that $a + b = 0$. One then recovers, in particular, the reduced form for the matrix of discontinuities in the curvature tensor that Pirani pointed out ([2]).

3. On a space-time manifold V_4 of general relativity, we will then be led to turn our attention to metrics for which there exists a vector l_α such that the curvature tensor $R_{\alpha\beta,\lambda\mu} \neq 0$ satisfies the relations:
(6) $$l_\alpha R_{\beta\gamma,\lambda\mu} + l_\beta R_{\gamma\alpha,\lambda\mu} + l_\gamma R_{\alpha\beta,\lambda\mu} = 0$$
and

([2]) PIRANI, Phy. Rev. **105** (1957), 1089-1099.

(7) $$l^{\alpha} R_{\alpha\beta,\lambda\mu} = 0,$$

with l_{α} is isotropic and $R_{\alpha\beta} = \tau\, l_{\alpha}\, l_{\beta}$, in which the right-hand side can be identified with the Maxwell tensor of a singular electromagnetic field.

If that were true at a point x of V_4 then we would say that the metric describes a state of *pure total radiation* at that point. It can then be true for all points of a domain of V_4. If $l_{\alpha}\, dx^{\alpha} = 0$ is completely integrable then the radiation envisioned will said to have *integrable type.* The metric:

$$ds^2 = \exp(2\varphi)\,(dt^2 - dx^2) - (\xi^2\, dy^2 + \eta^2\, dz^2),$$

in which φ, $\xi > 0$, $\eta > 0$ are three arbitrary functions of the single variable $u = t - x$, will provide an example of pure total radiation of integrable type. For $\xi = u \exp(-\beta)$, $\eta = u \exp(\beta)$, $2\varphi' = u\, \beta'^2$, one will have the Rosen metric ($R_{\alpha\beta} = 0$). If one is given a function $\beta'(u)$ of class C^1 that is non-zero on only the interval $0 < u_0 < u < u_1$ then one can define ([3]) an everywhere-regular exterior metric on the numerical space R^4 that is non-Euclidian for $u_0 < u < u_1$ (pure gravitational radiation).

Consider a vector field $\boldsymbol{l}$ on V_{m+1} such that (6) and (7) are satisfied. For $\tau \neq 0$, it results from the conservation identities that *the trajectories of* $\boldsymbol{l}$ *are isotropic geodesics.* The same thing will also be true for $\tau = 0$. Indeed, from the Bianchi identities, one can deduce that:

$$P_{\alpha\beta,\lambda\mu} = l^{\rho}\, \nabla_{\rho}\, R_{\alpha\beta,\lambda\mu}$$

satisfies (3) and (4). Upon differentiating (6) and (7), one will see that $m^{\alpha} = l^{\rho}\, \nabla_{\rho}\, l^{\alpha}$ once more satisfies (6) and (7) and is therefore isotropic; since it is orthogonal to l^{α}, it will be collinear with it.

4. The trajectories Γ of charged test particles in V_4 that are subject to a gravitational field and an electromagnetic field $F_{\alpha\beta}$ satisfy the equation of deviation:

$$\frac{\nabla^2 v^{\alpha}}{ds^2} + R^{\alpha}{}_{\rho,\lambda\mu}\, u^{\rho}\, v^{\alpha}\, u^{\mu} = \frac{e}{m}\left(\nabla_{\rho} F^{\alpha}{}_{\beta}\, u^{\beta} v^{\rho} + F^{\alpha}{}_{\beta}\, \frac{\nabla v^{\alpha}}{ds}\right),$$

in which $\mathbf{u}$ is the unit vector that is tangent to Γ, and the corresponding points on two trajectories will be the ones with the same s. $\mathbf{v}$ can be assumed to be orthogonal to $\mathbf{u}$ at a point x that is determined along a trajectory Γ. Suppose that wave front that is both gravitational and electromagnetic passes through x. For a convenient choice of orthonormal frame at x, the relations:

$$\left[\frac{\nabla v^0}{ds^2}\right] = \left[\frac{\nabla^2 v^1}{ds^2}\right] = 0, \qquad \left[\frac{\nabla^2 v^2}{ds^2}\right] = \sigma v^2 + \frac{e}{m}\mu\, v^1, \qquad \left[\frac{\nabla^2 v^3}{ds^2}\right] = -\,\sigma v^3 + \frac{e}{m}\upsilon\, v^1$$

([3]) H. BONDI, Nature **179** (1957), 1072-1073.

will give the components of the discontinuity in the relative acceleration (with respect to *s*) ([2]). We can apply the preceding results to the *penta-dimensional theories.*

———