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Integral invariance relations and their applications to dynamics

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Introduction.

Since their introduction by Henri Poincaré ([1]), the role that integral invariants play in the theory of differential systems and Pfaff systems has not ceased to expand. One knows about all of that theory due to the work of Élie Cartan ([2]).

It seems to me that along with "invariant forms" for a differential system and "invariant equations," there is some interest in considering exterior differential forms whose associated system can be verified by taking into account the proposed differential system in their own right. Here, one will find the elements of a theory of integral invariance relations to which one will be led and some applications to the dynamics of non-conservative systems. That theory is closely related to a special case of my theory of generalized variational spaces ([3]). However, I shall try to present it in an autonomous manner here.

Some the most elementary results of that work have appeared in a note to the *Comptes rendus de l'Académie des Sciences* ([4]). In what follows, $d\,\Omega$ will denote the exterior differential of the form Ω.

I. – INTEGRAL INVARIANCE RELATIONS.

1. Definition of integral invariance relations. – Suppose that we are given a system of first-order differential equations in m variables:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_m}{X_m}, \tag{1.1}$$

in which the functions X_i ($x_1, x_2, \ldots, x_m$) are not simultaneously zero and have class $C^{(2)}$ in a certain region R of the representative m-dimensional space.

([1]) H. POINCARÉ, [**5**], pp. 4.
([2]) Cf., in particular, E. CARTAN [**1**].
([3]) Cf., LICHNEROWICZ [**4**].
([4]) LICHNEROWICZ [**3**].

Let us consider an arbitrary p-dimensional domain ($p < m$) $D_0^{(p)}$ of the region R and suppose that we displace the various points of $D_0^{(p)}$ along the integral curves of the system (1.1) arbitrarily in such a way that we obtain another p-dimensional domain $D_1^{(p)}$. We let $\mathcal{T}_0^1$ denote the (p + 1)-dimensional domain that is generated by the arcs of the integral curves that are limited by the points of $D_0^{(p)}$ and $D_1^{(p)}$. Naturally, we suppose that the various points of $\mathcal{T}_0^1$ are interior to the region R.

Finally, let Ω denote an exterior differential form of degree (p + 1) and class $C^{(2)}$ that belongs to R .

Definition. – We say that Ω generates an *absolute integral invariance relation* for the system (1.1) when for any domain $D_0^{(p)}$ and any domain $\mathcal{T}_0^1$ that one deduces from it, one has:

(1.2) $$\int_{\mathcal{T}_0^1} \Omega = 0 .$$

Ω generates a *relative integral invariance relation* if one has the relation (1.2) when the domain $D_0^{(p)}$ is restricted to being closed.

We shall ultimately establish that, contrary to what happens in the case of integral invariants, that distinction between absolute and relative relations, which seems natural *a priori*, is, in fact, superfluous.

2. Absolute integral invariance relation. – We propose to look for the condition under which a form Ω can generate an absolute integral invariance relation for the system (1.1). Let ($y_1, y_2, \ldots, y_{m-1}$) be a system of ($m$ – 1) independent first integrals of (1.1). The differential system (1.1) is locally equivalent to the differential system:

(2.1) $$dy_1 = dy_2 = \ldots = dy_{m-1} = 0.$$

Suppose that X_m (for example) is non-zero in the region considered and adopt y_1, y_2, …, y_{m-1} and x_m as new variables. When expressed in terms of these new variables, the form Ω will include two types of terms: Ones that contain the differential dx_m and ones from which that differential is absent. Hence, the form Ω can be written:

$$\Omega = \Omega_1 + \Omega_2 ,$$

in which Ω_1 involves only the differentials of the first integrals as differentials:

(2.2) $$\Omega_1 = \sum A_{i_1 i_2 \cdots i_{p+1}} dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_{p+1}} ,$$

and Ω_2 has the form:

$$\Omega_2 = \sum B_{i_1 i_2 \cdots i_{p+1}} dx_m \wedge dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_p} = dx_m \wedge \Pi \,, \tag{2.3}$$

in which Π denotes an exterior differential form of order p.

In order to define the domain of integration $\mathcal{T}_0^1$ that is associated with Ω, we introduce an auxiliary variable v. To that effect, we take an arbitrary non-zero function ρ of the variables $(x_1, x_2, \ldots, x_m)$ and complete the system (1.1) by writing:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_m}{X_m} = \rho\, dv \,. \tag{2.4}$$

Upon integrating the differential system (2.4), we will obtain a parametric representation of the integral curves of (1.1) in terms of the parameter v.

Having said that, define the domain $D_0^{(p)}$ by a parametric representation of the variables x_i as functions of the p parameters $(\alpha_1, \alpha_2, \ldots, \alpha_p)$. On the manifold $\mathcal{T}$ that is generated by the integral curves that issue from the various points of $D_0^{(p)}$, the variables x_i will be expressed by relations of the form:

$$x_i = f_i\,(v_1, \alpha_1, \alpha_2, \ldots, \alpha_p) \,.$$

We can choose that domain to be the portion of $\mathcal{T}$ that is bounded by two domains v = const.; we thus restrict v to vary within an interval (v_0, v_1).

It is clear that one has:

$$\int_{\mathcal{T}_0^1} \Omega_1 = 0. \tag{2.5}$$

Indeed, upon passing to the parametric representation in terms of the variables $(v_1, \alpha_1, \alpha_2, \ldots, \alpha_p)$, each term:

$$dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_{p+1}}$$

of Ω_1 will correspond to the determinant whose first row is:

$$\frac{dy_{i_1}}{dv} = \frac{dy_{i_2}}{dv} = \ldots = \frac{dy_{i_{p+1}}}{dv} \,,$$

which will be composed of only zeroes, from (2.1).

If Ω generates an absolute integral invariance relation then one must have:

$$\int_{\mathcal{T}_0^1} \Omega_2 = \int_{\mathcal{T}_0^1} dx_m \wedge \Pi = 0.$$

Suppose that v_1 varies while v_0 remains fixed and consider the integral function of v_1 :

$$I(v_1) = \int_{\mathcal{T}_0^1} \Omega_2 = \int_{v_0}^{v_1} dv \int_{D^{(p)}} \rho X_n \Pi ,$$

in which the domain $D^{(p)}$ corresponds to the value v of the auxiliary variable. When v tends to v_0, the function I will obviously admit the integral:

$$\left(\frac{dI}{dv}\right)_{v=v_0} = \int_{D^{(p)}} \rho X_n \Pi \tag{2.6}$$

for its derivative at $v = v_0$. When that derivative is zero, it will result that for any domain $D_0^{(p)}$, one must have:

$$\int_{D_0^{(p)}} \rho X_n \Pi = 0.$$

One deduces from the continuity conditions that:

$$\rho X_n \Pi = 0,$$

and as a result:

$$\Pi = 0.$$

Hence, in order to Ω to generate an absolute integral invariance relation, it is necessary and sufficient that it can be put into the form (2.2), in which the $A_{i_1, i_2, \cdots i_{p+1}}$ are functions of the m variables (x_i) . *The form* Ω *only includes the differentials of the first integrals as differentials in the sum.*

Theorem:

If the forms Ω *and* $d\Omega$ *generate integral invariance relations of the differential system* (1.1) *then the form* Ω *will define an absolute integral invariant of* (1.1), *and conversely.*

Indeed, it is the union of the associated systems for the forms Ω and $d\Omega$ that constitutes the characteristic system of the form Ω. As a result, it will be an invariant for the differential system (1.1).

Geometrically, suppose that Ω has degree p and apply Stokes's formula to the domain V_0^1 that is bounded by $D_0^{(p)}$, $D_1^{(p)}$, and $\mathcal{T}_0^1$. That domain is generated by arcs of the integral curves that issue from the points of $D_0^{(p)}$. If Ω satisfies the stated hypotheses then one can deduce that:

$$\int_{D_1^{(p)}} \Omega - \int_{D_0^{(p)}} \Omega = \int_{\mathcal{T}_0^{(p)}} \Omega + \int_{V_0^1} d\,\Omega = 0.$$

5. Uniqueness theorems for differential systems that admit certain integral invariance relations. – It is often convenient to seek to characterize a differential system of the type (1.1) by the fact that it admits a certain integral invariance relation. To that effect, we shall make use of the following two theorems:

Theorem:

There exists one and only one differential system in $(2n + 1)$ *variables that admits a quadratic integral invariance relation of maximum rank.*

Indeed, let Ω be a quadratic exterior form in $(2n + 1)$ variables. Since the rank r of that form is even, it will be a maximum for $r = 2n$. In that case, Ω can be put into the canonical form:

$$\Omega = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 + \ldots + \omega_{2n-1} \wedge \omega_{2n} ,$$

in which the Pfaff forms ω_i are linearly independent. The associated system for Ω then consists of the Pfaff system:

$$\omega_1 = \omega_2 = \ldots = \omega_{2n} = 0,$$

which is equivalent to one and only one differential system of the form (1.1).

Theorem:

There exists one and only one differential system in m variables that admits an integral invariance relation of degree $(m - 1)$.

Indeed, let Ω be an exterior differential form of degree $(m - 1)$ in m variables. The rank of that form is equal to $(m - 1)$, and Ω can be put into the canonical form:

$$\Omega = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{m-1} ,$$

in which the Pfaff forms ω_i are linearly independent. The associated system to Ω is then written:

$$\omega_1 = \omega_2 = \ldots = \omega_{m-1} = 0,$$

which then proves the theorem.

II. – APPLICATIONS.

6. The dynamical equations for non-conservative holonomic systems. – Consider a not-necessarily-conservative material system (M) that has perfect holonomic constraints and admits n degrees of freedom (q_i). The forces to which the system is subject *cannot be derived from a potential and can even depend upon the velocities of the material elements of* (M).

In their Lagrangian form, the equations of motion of the material system (M) can be written:

$$\frac{dq_i}{dt} = q_i', \qquad \frac{d}{dt}\frac{\partial L}{\partial q_i'} - \frac{\partial L}{\partial q_i} = Q_i \qquad (i = 1, 2, \ldots, n), \tag{6.1}$$

in which L denotes a Lagrangian function can possibly amount to one-half the *vis viva* T of the system, and the Q_i are functions of q_i, q_i', and time t. We let (M) denote a material system that is mechanically identical to (M), but is subject to forces that make the Q_i zero.

Let $\mathcal{E}$ be the $(2n + 1)$-dimensional state space of the system (M). A trajectory of (M) is a curve of $\mathcal{E}$ that is a solution of the differential system (6.1). Consider the action integral:

$$S = \int_{t_0} L\,dt,$$

which is evaluated along an arc of a trajectory in (M). Upon looking for the variation of S for an arbitrary variation of that arc, including the extremities, one will get by a classical calculation ([1]):

$$\delta S = [\omega(\delta)]_1 - [\omega(\delta)]_0 - \int_{t_0}^{t_1} \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial q_i'} - \frac{\partial L}{\partial q_i}\right)\delta q_i\,dt,$$

into which one has introduced the elements that relate to (M'):

$$\omega(\delta) = \sum_i p_i\,\delta q_i - H\,dt, \qquad p_i = \frac{\partial L}{\partial q_i'}, \qquad H = \sum_i p_i\,q_i' - L.$$

One deduces from the differential system (6.1) that:

$$\delta S = [\omega(\delta)]_1 - [\omega(\delta)]_0 - \int_{t_0}^{t_1} \sum_i Q_i\,\delta q_i\,dt. \tag{6.2}$$

Let $\mathcal{T}$ then be a tube that is generated by a closed continuous sequence of trajectories of (M). On each trajectory that is limited by an arc $\overset{\frown}{P_0 P_1}$ whose extremities generate two closed curves C_0 and C_1, and let $\mathcal{T}_0^1$ denote the portion of $\mathcal{T}$ that is bounded by C_0 and C_1. Upon integrating the two sides of (6.2) over the closed continuous sequence of trajectories, one will get the relation ([2]):

([1]) Cf., E. CARTAN [**1**], pp. 10.

([2]) That relation is equivalent to a relation that was given by D. C. Lewis [**2**] [pp. 280, formula (10)] in a considerably more complicated, purely analytical form that does not lend itself well to applications.

(6.3) $$\int_{C_1} \omega(\delta) - \int_{C_0} \omega(\delta) = \iint_{\mathcal{T}_0^1} \sum Q_i \, dq_i \wedge dt .$$

By applying Stokes's formula to the left-hand side of (6.3) and the domain $\mathcal{T}_0^1$, it will become:

(6.4) $$\iint_{\mathcal{T}_0^1} \sum dp_i \wedge dq_i - \left(dH - \sum Q_i \, dq_i \right) \wedge dt = 0.$$

Hence, the system of dynamical equations (9.1) admits the integral invariance relation that is generated by the quadratic form:

(6.5) $$\Omega = \sum dp_i \wedge dq_i - \left(dH - \sum Q_i \, dq_i \right) \wedge dt .$$

That obviously generalizes Cartan's integral invariant for the dynamics of conservative systems.

If one takes C_0 and C_1 in the relation (6.3) to be closed sequences of simultaneous states that relate to the instants t_0, t_1 then one will get:

(6.6) $$\int_{C_1} \sum_i p_i \, dq_i - \int_{C_0} \sum_i p_i \, dq_i = \int_{t_0}^{t_1} dt \int_C \sum Q_i \, dq_i ,$$

in which the closed curve C corresponds to the instant t. If the Q_i are derived from a potential $U(q_i, t)$ then one will immediately get Poincaré's relative integral invariant.

If the equations of motion (6.1) admit infinitesimal transformations then one can deduce some new integral invariance relations from the integral invariance relation that is generated by (6.5) . In particular, suppose that the constraints and forces that are given for the system (M) are independent of time. The equations of motion admit the infinitesimal transformation:

$$X f = \frac{\partial f}{\partial t} .$$

It will then result that the Pfaff form:

$$\Omega_X = dH - \int_C \sum Q_i \, dq_i$$

generates a new integral invariance relation. In other words, for any instants t_0, t_1, the integral:

$$\int_{t_0}^{t_1} \left(dH - \sum_i Q_i \, dq_i \right)$$

will be zero when it is evaluated along an arc of a trajectory. We thus recover a classical result of Painlevé.

7. Characterization of the dynamical equations. – By its very structure, the quadratic form (6.5) in $(2n + 1)$ variables admits a maximum rank of $2n$. By virtue of the theorem in paragraph **5**, it will then result that there exists one and only one differential system in $(2n + 1)$ variables that admits the corresponding integral invariance relation.

Explicitly construct the associated system to the form Ω. Upon successively equating the coefficients of the differentials dq_i , dp_i , dt in Ω, which have previously been moved to the first position, to zero, one will get the Pfaff system:

$$dp_i + \left(\frac{\partial H}{\partial q_i} - Q_i\right) dt = 0, \tag{7.1}$$

$$dq_i - \frac{\partial H}{\partial p_i} dt = 0, \tag{7.2}$$

$$\sum \frac{\partial H}{\partial p_i} dp_i + \sum_i \left(\frac{\partial H}{\partial q_i} - Q_i\right) dq_i = 0. \tag{7.3}$$

That system reduces to the differential system:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + Q_i\,, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \tag{7.4}$$

since equation (7.3) is a consequence of equations (7.1) and (7.2). Hence, the only differential system of type (1.1) that admits the integral invariance relation that is generated by (6.5) is the pseudo-canonical differential system (7.4) that is equivalent to the Lagrangian system (6.1). From now on, the Q_i will be considered to be functions of p_i , q_i , t.

We are thus led to state the following theorem:

Theorem:

The motions of a material system with perfect holonomic constraints are governed by first-order differential equations that involve the parameters of position, velocity, and time. Those equations are characterized by the property that they admit the integral invariance relation that is generated by the form:

$$\Omega = \sum_i dp_i \wedge dq_i - \left(dH - \sum_i Q_i\, dq_i\right) \wedge dt\,.$$

In my paper [**4**], one will find a study of the extended cases in which that statement can be put into a form that is independent of the framing that is adopted for the configuration space-time of the system.

8. An extension of Liouville's integral invariant. – If the exterior differential form of even degree Ω generates an integral invariance relation for the differential system (1.1) then it will be clear, by virtue of the theorems in paragraphs **1** and **2**, that the same thing will be true for the forms Ω^p (p = 1, 2, …) .

In particular, consider the quadratic form Ω that is defined by (6.5) and whose associated system is given by (7.4). As a result, that form can be written:

$$\Omega = \sum_i \left[dp_i + \left(\frac{\partial H}{\partial q_i} - Q_i \right) dt \right] \wedge \left[dq_i - \frac{\partial H}{\partial p_i} dt \right]. \tag{8.1}$$

The differential system of dynamical equations that admit the integral invariance relation that is generated by Ω will likewise admit integral invariance relations that are generated by Ω^p (p = 1, 2, …) . In particular, for $p = n$, one will obtain the form of degree $2n$:

$$\Omega^n = \prod_i \left[dp_i + \left(\frac{\partial H}{\partial q_i} - Q_i \right) dt \right] \wedge \left[dq_i - \frac{\partial H}{\partial p_i} dt \right]. \tag{8.2}$$

We then arrive at the following theorem:

Theorem:

The dynamical equations for systems with perfect holonomic constraints admit the integral invariance relation of degree $2n$ *that is generated by the form* (8.2).

Under what condition will the form (8.2) define an integral invariant for the dynamical equations? In order for that to be true, it is necessary and sufficient that the associated system of the form $d\,\Omega^n$ should be verified when one takes the system (7.4) into account. Now, that will be possible only when:

$$d\,\Omega^n = 0.$$

Evaluate the exterior differential:

$$d\,\Omega^n = n\,\Omega^{n-1} \wedge d\,\Omega .$$

One will first have:

$$d\,\Omega = \sum_i dQ_i \wedge dq_i \wedge dt .$$

One will then deduce that:

$$d\,\Omega^n = n \left(\sum_i \frac{\partial Q_i}{\partial p_i} \right) dp_1 \wedge dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge dq_1 \wedge \ldots \wedge dq_n \wedge dt .$$

One is then led to the condition:

$$\sum_i \frac{\partial Q_i}{\partial p_i} = 0.$$

We state:

Theorem:

In order for the form (8.2) *of degree* 2*n to define an absolute integral invariant for the dynamical equations, it is necessary and sufficient that one should have:*

(8.3) $$\sum_i \frac{\partial Q_i}{\partial p_i} = 0.$$

It is equivalent to say that under that hypothesis, the dynamical equations admit the multiplier 1, which one can verify directly.

If we confine ourselves to the consideration of simultaneous states then the form Ω^n will reduce to the form in 2n variables:

$$dp_1 \wedge dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge dq_2 \wedge \ldots \wedge dq_n .$$

Hence, the material systems that satisfy the condition (8.3) will admit the Liouville integral invariant:

$$\int dp_1 \wedge dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge dq_2 \wedge \ldots \wedge dq_n ,$$

whose importance is well-known in its applications to statistical mechanics and ergodic theory.

9. Case of non-holonomic systems. – The case in which the material system considered is non-holonomic will immediately reduce to the case of holonomic systems. Suppose that the system (M) is subject to the non-holonomic constraints:

$$\sum_i a_{ki}\, \delta q_i + b_k\, \delta t = 0 \qquad (k = 1, 2, \ldots, q < n),$$

in which the a_{ki} and the b_k denote functions of q_i and time. The Lagrangian equations of motion can be put into the form:

(9.1) $$\frac{dq_i}{dt} = q_i' , \qquad \frac{d}{dt}\frac{\partial L}{\partial q_i'} - \frac{\partial L}{\partial q_i} = Q_i + Q_i^* ,$$

with:

(9.2) $$Q_i^* = \sum_k a_{ki}\, \lambda_i .$$

The Lagrange multipliers λ_k can be considered to be functions of the parameters of position, velocity, and time that are chosen in such a fashion that equations (9.1) will admit the first integrals ([1]):

$$\sum_i a_{ki}\, q_i' + b_k \,.$$

The results that are established for holonomic systems will persist if one replaces the Q_i with the quantities $Q_i + Q_i^*$. Hence, the equations of motion will admit the integral invariance relation that is generated by the form:

$$\Omega = \sum_i dp_i \wedge dq_i - \left[dH - \sum \left(Q_i + \sum_k a_{kl}\, \lambda_k \right) dq_i \right] \wedge dt \,, \tag{9.3}$$

which coincides (up to notation) with the result of A. E. Taylor [**6**]. In conclusion, I will point out that the various results that are established for the equations of dynamics easily extend to the equations of the hydrodynamics of perfect fluids with non-conservative forces, as well as to viscous fluids.

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([1]) Cf., for example, D. C. LEWIS [**2**], pp. 281.