Annali di Mathematica, (1959), 1-95.

Electromagnetic and Gravitational Waves In General Relativity

Memoir of ANDRE LICHNEROWICZ (Paris)

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To Antonio Signorini for his 70th birthday.

Abstract. – *The theory of gravitational radiation based on the analogy that exists between the behavior of the curvature tensor and that of the electromagnetic field in general relativity. Contributions to field quantization.*

INTRODUCTION

One of the most important questions of the relativistic theory of gravitation concerns the definition and properties of gravitational waves and radiation. In latter years this problem has been the object of numerous interesting works.

In the framework of special relativity a satisfactory classical theory of electromagnetic waves and radiation has been elaborated, and it seems to me to be a good method to develop in the framework of general relativity in a manner that is easily adapted to the gravitational case. In this case, it appears that it is the curvature tensor that plays the essential role from either a mathematical or physical viewpoint, and this is found to be plainly in accord with the viewpoint of PIRANI [1, 2]. For a metric that satisfies EINSTEIN'S equations $R_{\alpha\beta} = 0$ the curvature tensor satisfies two groups of relations that formally bear a striking resemblance to the vacuum MAXWELL equations, namely:

$$S\nabla_\gamma R_{\alpha\beta,\lambda\mu} = 0, \qquad \nabla_\alpha R^\alpha_{\beta,\lambda\mu} = 0,$$

where ∇_α is the covariant derivative operator and S indicates a summation over all cyclic permutations of the three indices α, β, γ. These are relations that play a fundamental role in the gravitational part of this work.

Chapter I is dedicated to the theory of electromagnetic waves and radiation in general relativity. After reviewing the classical results the notion of a singular 2-form is defined and applied to the definition of the notion of pure electromagnetic radiation as in the study of the discontinuities of the derivative of the electromagnetic field tensor. In this case, one sees the points of a wave front of a conservative 4th order tensor appear in a natural manner.

In chapter II the discontinuities of the curvature tensor are studied. This study leads us to distinguish tensors that correspond to what we may call a "singular double 2-form" amongst the tensors $H_{\alpha\beta,\lambda\mu}$ that admit the symmetry type of the curvature tensor. Three corresponding conditions imply a remarkable form for the contracted tensor $H_{\alpha\beta}$. An analysis of the differential relations ([1]) that the discontinuities of the curvature tensor are subject to in the case $B_{\alpha\beta} = 0$ lead to conservation identities for a 4th order tensor. This is again the case when there are simultaneously discontinuities in the curvature tensor and the derived tensor of an electromagnetic field that satisfies the vacuum MAXWELL equations.

In chapter III, I give a definition of the notions of total radiation and pure gravitational radiation, notions that correspond to a remarkable particular case of the PIRANI-PETROV classification. The field of isotropic vectors that comes into play admits null geodesics of null length for trajectories. Thanks to some work of BONDI, perfecting an example of ROSEN, one may construct effective examples of such radiation. In the case where $R_{\alpha\beta} = 0$ (or, more generally, $\lambda g_{\alpha\beta}$), I have studied the properties of a 4th order tensor that was introduced by L. BEL [2] in accord with my viewpoint. These properties are formally very similar to those of the MAXWELL tensor of an electromagnetic field that satisfies the vacuum MAXWELL equations, and it seems that this tensor appeals to new and important researches.

Chapter IV is dedicated to the behavior of the relative acceleration of two close particles relative to a wave or radiation according to a viewpoint developed by SYNGE and by PIRANI [2]. The study of the case of charged particles in the presence of an electromagnetic field points to a difference in the behavior of a gravitational wave and an electromagnetic one.

In chapter V, I have adopted the five-dimensional framework in order to translate electromagnetic radiation and gravitational radiation into the same formalism, which leads to the "truncation" of the curvature tensor of the five-dimensional manifold.

In chapter VI, I finally exploit the aforementioned analogies between gravitation and electromagnetism in order to develop the process of quantization of the electromagnetic field in special relativity and the gravitational field in the linear approximation in a parallel fashion. In the first case, the quantization is performed directly on the electromagnetic field tensor and leads naturally to the classical representation of the photon. In the second case, it is performed on the curvature tensor and leads, as I will show, to an equivalent representation of the graviton. Meanwhile, it seems that this quantization process is more satisfactory, both mathematically and physically, to the one put into play by the classical theory of the graviton. The equations written at the beginning of this introduction appear precisely as the fundamental field equations; the condition $R_{\alpha\beta} = 0$ is then presented as a simple initial condition.

[1] Relations that were given by TRAUTMANN and the author independently.

I. – Electromagnetic waves and radiation in general relativity.

1. Generalities. In any relativistic theory of the gravitational field the primitive element is defined by a four-dimensional "space-time" manifold V_4 endowed with a differentiable structure concerning which it is essential to be precise: for reasons entirely related to the covariance of the formalism and which appear in the analysis of the gravitational field (see sec. 17-18) we are led to suppose that in the intersection of the domains of two admissible local coordinate systems the local coordinates of a point in one of the systems are four times differentiable functions – with non-null Jacobian – of the coordinates of this point in the other system, where the first and second derivatives are continuous and the third derivative is only piecewise continuous. We interpret this by saying that the manifold V_4 is of class (C^2, piecewise C^4). Unless stated to the contrary, V_4 is supposed to be *orientable.*

A riemannian metric ds^2 of hyperbolic normal type, with one positive square and 3 negative ones, is defined on V_4. The local expression of this metric in an admissible coordinate system is:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta, \text{ any Greek index} = 0, 1, 2, 3).$$

The metric tensor *gab* – or gravitation tensor – is supposed to be exactly (C^2, piecewise C^4), which is strictly compatible with the differentiable structure imposed on V_4.

The equation $ds^2 = 0$ defines a real cone C_x – called the *elementary* cone at x – at each point x of V_4. For a direction, its interior and its exterior define the time orientation and the space orientation, respectively. A tangent vector to V_4 is called normal if the modulus of its square is equal to 1. An *orthonormal frame* at the point x of V_4 is an ordered set of 4 vectors $(\vec{e}_\alpha)$, at x such that

$$\vec{e}_\alpha \cdot \vec{e}_\beta = 0 \quad \text{for } \alpha \neq b, \qquad \vec{e}_0^{\,2} = 1, \qquad \vec{e}_u^{\,2} = -1, \qquad (u = 1, 2, 3).$$

$\vec{e}_0$ defines a *time direction,* and the perpendicular 3-plane defined by the $\vec{e}_u$ is called the *space* associated with this time direction. We recall that a 2-plan or a 3-plane is called oriented in space if all of its directions are oriented in space; in the contrary case, it is oriented in time.

If a neighborhood U of V_4 is endowed with an orthonormal frame, the ds^2 may be locally written

$$ds^2 = (\theta^0)^2 - \sum_u (\theta^u)^2,$$

where the θ^α are linearly independent local Pfaff forms.

Naturally, V_4 may be the MINKOWSKI spacetime of special relativity. An orthonormal frame is then an ordinary Galilean frame (with $c = 1$). In the Riemannian case, the metric defines the structure of a MINKOWSKI space on any tangent vector space at any point x of V_4. The physical interpretation of a tensor defined at the point x of

V_4 is immediately deduced from the consideration of that tangent vector space: when referred to an orthonormal frame, this space may be identified with the spacetime of special relativity as referred to a Galilean frame, which furnishes the desired physical interpretation directly in terms of the time and space associated to that frame.

2. **The electromagnetic field in the absence of induction.**

In the absence of any induction phenomena, the electromagnetic field is represented by an anti-symmetric tensor field $F_{\alpha\beta}$ of class (C^0, piecewise C^2) on a domain in V_4. One may associate the 2-form:

$$F = \tfrac{1}{2} F_{\alpha\beta} \theta^{\alpha} \wedge \theta^{\beta}, \tag{2-1}$$

with this tensor. If $\eta_{\alpha\beta\gamma\delta}$ is the volume element tensor of the riemannian manifold V_4 then one deduces the *"adjoint"* anti-symmetric tensor $(*F)_{\alpha\beta}$, which is defined by:

$$(*F)_{\alpha\beta} = \tfrac{1}{2} \eta_{\alpha\beta\gamma\delta} F^{\gamma\delta}, \tag{2-2}$$

from F. We denote the associated 2-form by $*F$, which we call the adjoint of F. Note that since the discriminant of ds^2 is negative $**F = - F$.

With respect to an orthonormal frame $(\vec{e}_\alpha)$ at x, the physical interpretation of F and $*F$ is furnished by the following rule: if $\vec{u} = \vec{e}_0$ the electric field vector and the magnetic field relative to time and space defined by the frame are the space vectors (i.e, the vectors orthogonal to $\vec{u}$) determined by

$$\vec{E} : E^{\beta} = F^{\alpha\beta} u_{\alpha} \qquad \vec{H} : H^{\beta} = -(*F)^{\alpha\beta} u_{\alpha}. \tag{2-3}$$

If (X, Y, Z) and (L, M, N) are the components of $\vec{E}$ and $\vec{H}$, respectively, with respect to the frame $(\vec{e}_\alpha)$, one has the table:

$$\begin{cases} X = F^{01} = F_{10} = (*F)_{23} = (*F)^{23} \\ Y = F^{02} = F_{20} = (*F)_{31} = (*F)^{31} \\ Z = F^{03} = F_{30} = (*F)_{12} = (*F)^{12} \end{cases} \qquad \begin{cases} L = (*F)^{10} = (*F)_{01} = F_{23} = F^{23} \\ M = (*F)^{20} = (*F)_{02} = F_{31} = F^{31}. \\ N = (*F)^{30} = (*F)_{03} = F_{12} = F^{12} \end{cases} \tag{2-4}$$

Note that since $(e_\alpha) g_{00} = 1$, $g_{uu} = -1$, in this frame this shows how the lowering of a space index u is carried out with a change of sign, whereas it is not the same for the time index, 0. One may attach two interesting scalars to the electromagnetic field: the scalar product of the form F with itself or with the form $*F$, namely:

$$\Psi = (F, F) = \tfrac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \qquad \Phi = (*F, *F) = \tfrac{1}{2} F_{\alpha\beta} (*F)^{\alpha\beta}. \tag{2-5}$$

$\Phi = 0$ expresses the idea that the form F, and, as a result, the form $*F$, are of rank less than 4. In an orthonormal frame, the scalars Φ and Ψ have the expressions:

(2-6) $$\Psi = L^2 + M^2 + N^2 - X^2 - Y^2 - Z^2 \qquad \Phi = 2(LX + MY + NZ),$$

namely, when the squares and scalar products are evaluated with the help of the positive definite space metric:

$$\Psi = \vec{H}^2 - \vec{E}^2 \qquad \Phi = 2\vec{E}\cdot\vec{H},$$

and the right-hand sides are spacetime invariants.

3. **Maxwell tensor and Poynting vector.**

The study of special relativity has led us to introduce the MAXWELL tensor defined by:

(3-1) $$\tau_{\alpha\beta} = \tfrac{1}{4} g_{\alpha\beta} F_{\lambda\mu} F^{\lambda\mu} - F_{\alpha\rho} F^{\rho}_{\beta} \qquad (\tau^{\alpha}_{\alpha} = 0)$$

to be the energy-momentum tensor of the electromagnetic field. When one refers the electromagnetic field to an orthonormal frame $(\vec{e}_\alpha)$ one sees that this MAXWELL tensor is constructed with the space tensor (τ_{uv}) of electromagnetic tensions, the spatial POYNTING vector with components:

(3-2) $$P_0 = 0 \qquad P_u = \tau_{0u} ,$$

and the electromagnetic field energy:

$$\tau_{00} = \tfrac{1}{2}(E^2 + H^2).$$

If n^u is a normal space vector, the electromagnetic energy flux that traverses a 2-surface element orthogonal to n^u in space is proportional to $P_u n^u$. In order for this flux to be zero for any surface element it is necessary and sufficient that the POYNTING vector that corresponds to $\vec{e}_0$ be null. However, if $\vec{u} = \vec{e}_0$, the corresponding POYNTING vector may be written:

$$P_\alpha = (g^{\gamma}_{\alpha} - u_\alpha u^\lambda)\tau_{\beta\gamma} u^\beta .$$

In order for P_α to be null, it is necessary and sufficient that:

$$\tau_{\alpha\beta} u^\beta = (\tau_{\beta\gamma} u^\beta u^\gamma)\, u_\alpha ,$$

i.e., that $\vec{u}$ be a *proper vector of the Maxwell tensor with respect to the metric tensor.* One is thus led to study the proper vectors of $\tau_{\alpha\beta}$ with respect to $g_{\alpha\beta}$.

4. **The electromagnetic field equations.**

a) The electromagnetic field satisfies MAXWELL'S equations, where the first group expresses that F is locally derivable from a vector potential and the second relates F to the field sources, i.e., to a electric current vector. These equations may be written:

(4-1) $$E^{\alpha} \equiv \tfrac{1}{2}\eta^{\beta\gamma\delta\alpha}\nabla_{\beta}F_{\gamma\delta} = 0 \qquad D^{\alpha} \equiv \nabla_{\beta}F^{\beta\alpha} = J^{a}$$

where ∇ is the covariant derivative operator for the riemannian connection and J^{α} is the electric current vector. In these equations, it results from a classical calculation that:

(4-2) $$\nabla_{\alpha}\tau^{\alpha}_{\beta} = F_{\rho\beta}J^{\rho}.$$

Here I place myself in the *purely electromagnetic* case: in the domain of V_4 envisioned the electric current vector is null and the electromagnetic field contributes only to the energy-momentum. If $S_{\alpha\beta} = R_{\alpha\beta} - (1/2)\, g_{\alpha\beta}R$ is the EINSTEIN tensor of the metric, then the gravitational and electromagnetic fields are related by the EINSTEIN equations:

(4-3) $$S_{\alpha\beta} = \chi\, \tau_{\alpha\beta}.$$

From the vanishing of the current, it results from (4-2) that:

(4-4) $$\nabla_{\alpha}\tau^{\alpha}_{\beta} = 0.$$

This also results from the EINSTEIN equations (4-3) since the tensor $S_{\alpha\beta}$ satisfies conservation identities. (4-3) is therefore compatibility with the vanishing of the current.

Let d be the operator of exterior differentiation on forms ($d^2 = 0$), and let δ be the operator of codifferentiation on a form of degree p defined by $\delta = (-1)^{p-1} * d *$ ($\delta^2 = 0$). The MAXWELL equations may be written:

(4-6) $$dF = 0 \qquad \text{or} \qquad \delta(*F) = 0,$$

and in the purely electromagnetic case:

(4-6) $$d(*F) = 0 \qquad \text{or} \qquad \delta F = 0.$$

b) In this section, we do not avail ourselves of the EINSTEIN equations. We remark only that if one substitutes a material fluid without pressure for the pure electromagnetic field, then one must substitute the energy-momentum tensor:

(4-7) $$T_{\alpha\beta} = \rho u_{\alpha}u_{\beta} = (\sqrt{\rho}u_{\alpha})(\sqrt{\rho}u_{\beta}),$$

for the tensor $\tau_{\alpha\beta}$, where ρ is the proper matter density and u_α is the unitary velocity vector $(\vec{u}^2 = 1)$ of the fluid. From the conservation conditions that are satisfied by the energy momentum tensor $\nabla_\alpha T^\alpha_\beta = 0$, and the unitary character of $\vec{u}$ one deduces:

(4-8) $$\nabla_\alpha(\rho u^\alpha) = 0,$$
and
(4-9) $$u^\alpha \nabla_\alpha u^\beta = 0.$$

(4-8) is the continuity equation; from (4-9), the current lines of the fluid, which are trajectories of the vector field $\vec{u}$ are time-oriented geodesics in the manifold V_4.

5. Proper vectors of the Maxwell tensor ([2]). The study of the vanishing of the POYNTING vector leads to the study of the proper values and proper vectors of $\tau_{\alpha\beta}$ with respect to $g_{\alpha\beta}$. Here, it comes down to a purely algebraic study conducted at a given point of V_4 and which leads to reduced expressions for the pair of forms (F, $*F$) in an orthonormal frame $(\vec{e}_\alpha)$. In this section, F may be an arbitrary 2-form that is interpreted in terms of the electromagnetic field.

We start with an arbitrary time-oriented vector for the $\vec{e}_0$ of $(\vec{e}_\alpha)$. There are electric and magnetic field vectors $\vec{E}$ and $\vec{H}$ that correspond to it. It is possible to choose the 2-plane $(\vec{e}_2, \vec{e}_3)$ to be parallel to both $\vec{E}$ and $\vec{H}$. The vector $\vec{e}_1$ is then fixed up to sign, and one has $X = L = 0$. An orthonormal frame that satisfies this condition will be called an *adapted frame* for the form F. In the fixed 2-plane $(\vec{e}_2, \vec{e}_3)$, one may then choose the vectors $\vec{e}_2$ and $\vec{e}_2$ in such a manner that they are proper vectors of the matrix (τ_{AB}) (A, B = 2, 3) with respect to the identity. One then has $YZ + MN = 0$. An adapted frame that satisfies this condition will be called *simple* for F. If one introduces two numbers ξ, η that satisfy

(5-1) $$Y^2 + M^2 = \xi^2, \qquad Z^2 + N^2 = \eta^2, \qquad ZM - YN = \xi\eta$$

then one can show that the matrix $(\tau_{\alpha\beta})$ takes the form:

(5-2) $$(\tau_{\alpha\beta}) = \begin{pmatrix} \dfrac{\xi^2+\eta^2}{2} & 0 & 0 & 0 \\ & \dfrac{\xi^2+\eta^2}{2} & 0 & 0 \\ & & \dfrac{\eta^2-\xi^2}{2} & 0 \\ & & & \dfrac{\eta^2-\xi^2}{2} \end{pmatrix}$$

[2] Here, we summarize the results discussed by LICHNEROWICZ in *Théorie relativistes de la gravitation et de l'electromagnétisme*. Chap. 1.

for a simple frame. Conversely, if the matrix ($\tau_{\alpha\beta}$) takes the form:

$$\begin{pmatrix} \tau_{00} & \tau_{01} & 0 & 0 \\ & \tau_{11} & 0 & 0 \\ & & \tau_{22} & 0 \\ & & & \tau_{33} = -\tau_{22} \end{pmatrix},$$

relative to a frame, then one easily establishes that the frame is simple.

This said, it is easy to obtain the proper values of $\tau_{\alpha\beta}$ with respect to $g_{\alpha\beta}$ by means of (5-2). The equation of the proper values is then written:

$$\left[s^2 - \left(\frac{\xi^2 - \eta^2}{2} \right)^2 \right]^2 = 0.$$

It then results that:

THEOREM. – *The Maxwell tensor of an electromagnetic field admits 4 real proper values that are pairwise equal and opposite k, k, –k, –k.*

It is easy to relate k to the invariants Ψ and Φ. One has:

$$k^2 = [(H^2 - Z^2) - (N^2 - Y^2)]^2 = [M^2 - Z^2 + N^2 - Y^2]^2 - 4(M^2 - Z^2)(N^2 - Y^2).$$

Now, from (2-6) and the simplicity conditions, it results that:

$$[M^2 - Z^2 + N^2 - Y^2]^2 = \Psi^2, \qquad - 4(M^2 - Z^2)(N^2 - Y^2) = 4(MY + NZ)^2 = \Phi^2.$$

Therefore:

(5-4) $$4k^2 = \Psi^2 + \Phi^2.$$

6. **The regular case.** It is now convenient to distinguish the $k \neq 0$ case from the $k = 0$ case. In the first case, we say that the form F is *regular;* in the second case, it is *singular.* Let us examine the regular case; there then exist two distinct proper values k and $-k$, and, as a result, two 2-planes of orthogonal proper vectors, where one is necessarily oriented in time and the other in space. If $\vec{e}_0$ is a time-oriented normal vector of the first 2-plane, $\vec{e}_3$ is the normal vector that is orthogonal to this plane, and $\vec{e}_1$ and $\vec{e}_2$ are two orthogonal normal vectors of the space-oriented 2-plane, then the frame $(\vec{e}_\alpha)$ is an orthonormal frame composed of proper vectors. One has:

(6-1) $$g_{\alpha\beta} = e_{(0)\alpha}e_{(0)\beta} - e_{(1)\alpha}e_{(1)\beta} - e_{(2)\alpha}e_{(2)\beta} - e_{(3)\alpha}e_{(3)\beta},$$

and

(6-2) $$\tau_{\alpha\beta} = k[e_{(0)\alpha}e_{(0)\beta} + e_{(1)\alpha}e_{(1)\beta} + e_{(2)\alpha}e_{(2)\beta} - e_{(3)\alpha}e_{(3)\beta}].$$

When referred to the $(\vec{e}_\alpha)$ frame, the MAXWELL tensor gives a null POYNTING vector, and is necessarily represented by a matrix of the form (5-2), with $\xi\eta = 0$, since, from a preceding remark, $(\vec{e}_\alpha)$ is a simple frame. Since $\vec{e}_0$ and $\vec{e}_3$ have the same proper value, one has $\xi = 0$, hence $Y = M = 0$.

Only Z and N are non-null in general. As a result, F and $*F$ admit the expressions:

$$\text{(6-3)} \qquad \begin{cases} F = -Z\theta^0 \wedge \theta^2 + N\theta^1 \wedge \theta^2 \\ *F = N\theta^0 \wedge \theta^3 + Z\theta^1 \wedge \theta^2 \end{cases}.$$

F and $*F$ are linearly independent, and it is clear that there does not exist any non-null linear form such that the exterior products of this form with F and $*F$ are both null.

We observe that $\tau_{\alpha\beta}$ admits *two isotropic proper vectors*, which one may define by:

$$\text{(6-4)} \qquad \vec{l} = \vec{e}_0 + \vec{e}_3, \qquad \vec{l}^{\,*} = \vec{e}_0 - \vec{e}_3.$$

These vectors are also proper vectors of F and $*F$. For example, from (6-3), one deduces:

$$l^\alpha F_{\alpha\beta} = -Z\, l_\beta, \qquad l^\alpha (*F)_{\alpha\beta} = N\, l_\beta.$$

Conversely, any proper vector of F is obviously a proper vector of $\tau_{\alpha\beta}$; as a result, if it is isotropic, then it is collinear to $\vec{l}$ or $\vec{l}^{\,*}$.

7. The singular case. For $k = 0$, one has $\Psi = \Phi = 0$. *The electric field and the magnetic field are orthogonal and have the same length relative to any orthonormal frame.*

a) In this case, $\eta^2 = \xi^2$, and, as a result, $\eta = \pm\xi$. In a simple frame, $\tau_{01} = \pm\xi^2$. Upon changing $\vec{e}_1$ into $-\vec{e}_1$, and, simultaneously, $\vec{e}_2$ into $-\vec{e}_2$, in order to not change the orientation of the frame, one may do this in such a way that $\eta = -\xi$. For such a frame, the MAXWELL tensor admits the components:

$$\text{(7-1)} \qquad (\tau_{\alpha\beta}) = \begin{pmatrix} \xi^2 & -\xi^2 & 0 & 0 \\ & \xi^2 & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}.$$

Introduce the null-length vector:

$$\vec{l} = \vec{e}_0 + \vec{e}_1,$$

which admits the covariant components:

$$l_0 = 1, \qquad l_1 = -1, \qquad l_2 = l_3 = 0,$$

(7-1) translates into the relation:

(7-2) $$\tau_{\alpha\beta} = \xi^2\, l_\alpha l_\beta .$$

$\tau_{\alpha\beta}$ admits proper vectors that are vectors in the 3-plane tangent to the elementary cone $ds^2 = 0$ along $\vec{l}$. Except for the isotropic direction defined by $\vec{l}$, they define space-oriented directions. It is therefore impossible here to find a time-oriented proper vector, and, as a result, to *annihilate the Poynting vector*. The form (7-2) of the energy-momentum tensor for the electromagnetic field, where $\vec{l}$ is an isotropic vector, is in agreement with the form (4-7) of the energy-momentum tensor for a material fluid without pressure.

b) Let $(\vec{e}_\alpha)$ be an *adapted* frame. If $\vec{E}$ has components (Y, Z) in the 2-plane $(\vec{e}_2, \vec{e}_3)$ then the vector $\bar{H}$ that is orthogonal to $\vec{E}$ and has the same length will have the components $M = -\varepsilon Z$ and $N = \varepsilon Y$ (where $\varepsilon = \pm 1$). Since $YZ + MN = 0$, the frame is simple. Therefore in the singular case *any adapted frame is simple.* From (5-1), one has

$$ZN - YN = -\,\xi^2$$

in a simple frame. It follows that:

$$-\,\varepsilon(Y^2 + Z^2) = -\,\xi^2,$$

and, as a result, $\varepsilon = 1$. Therefore, with our sign conventions, one has:

$$M = -\,Z, \qquad N = Y$$

in an adapted frame. Upon using these expressions for the components of F and $*F$, it follows that:

$$F = Y\,\theta^2 \wedge (\theta^0 - \theta^1) + Z\,\theta^3 \wedge (\theta^0 - \theta^1)$$
$$*F = Z\,\theta^2 \wedge (\theta^0 - \theta^1) - Y\,\theta^3 \wedge (\theta^0 - \theta^1).$$

Upon introducing the linear form λ that is defined by l_α:

$$\lambda = l_\alpha \theta^\alpha = \theta^0 - \theta^1,$$

one obtains:

(7-3) $$\begin{cases} F = (Y\theta^2 + Z\theta^3) \wedge \lambda \\ *F = (Z\theta^2 - Y\theta^3) \wedge \lambda. \end{cases}$$

In particular, one may choose $\vec{e}_2$ parallel to the electric field and $\vec{e}_3$ parallel to the magnetic field in the $(\vec{e}_2, \vec{e}_3)$ 2-plane; one will then have $Z = 0$.

It is clear that in order for a form to annihilate both F and $*F$ it is necessary and sufficient that they be proportional to λ. The existence of such forms that simultaneously annihilate F and $*F$ *characterizes the singular case.* Note that $\lambda \wedge (*F) = 0$ is equivalent to $l^\alpha F_{\alpha\beta} = 0$. Therefore:

(7-4) $l^\alpha F_{\alpha\beta} = 0 \qquad l^\alpha (*F)_{\alpha\beta} = 0.$

c) Let V_{m+1} be an $(m+1)$-dimensional manifold endowed with a Riemannian metric of hyperbolic normal type. If $F \neq 0$ is a 2-form on this manifold we say that *F is a singular 2-form if there exists a vector* $\vec{l}$ *such that*

(7-5) $l_\alpha F_{\beta\gamma} + l_\beta F_{\gamma\alpha} + l_\gamma F_{\alpha\beta} = 0 \qquad (\alpha, \beta, \gamma = 0, 1, \ldots, m)$

and

(7-6) $l^\alpha F_{\alpha\beta} = 0.$

The vector $\vec{l}$ which is defined up to a scalar factor, will be called the *fundamental vector* of *F*. In the case of general relativity ($m = 3$), (7-5) and (7-6) completely characterize the 2-forms that correspond to the singular case since these relations express that there exists a linear form that simultaneously annihilates *F* and $*F$ (which is a 2-form here).

In the general case, *the vector* $\vec{l}$ *is necessarily isotropic*. Indeed, if this is not true then one may choose an orthonormal frame $(\vec{e}_\alpha)$ such that one of the vectors, namely $\vec{e}_0$, is collinear to $\vec{l}$. Let us temporarily designate the indices that take the values 1, …, *m* by *u*, *v*. If we set $\alpha = 0$, $\beta = u$, $\gamma = v$ in (7-5) then we get $F_{uv} = 0$. On the other hand, from (7-6), $F_{\alpha\beta} = 0$. Therefore if $\vec{l}$ is not isotropic then we necessarily have $F = 0$.

It is easy to deduce expressions for *F* from (7-5) and (7-6) that will be useful in what follows.

At the point *x* of V_{m+1}, we denote an isotropic direction defined by a vector $\vec{l}$, and denote a system of $(m-1)$ orthogonal normal vectors that are tangent to the elementary cone along the generatrix defined by $\vec{l}$ by $(\vec{n}^{(i)})$ ($I = 1, \ldots, m-1$). There is a time-oriented 2-plane that contains $\vec{l}$ and is orthogonal to the $(m-1)$-plane determined by the $\vec{n}^{(i)}$. We choose an arbitrary unitary vector $\vec{e}_0$ $(\vec{e}_0^{\,2} = 1)$ in this 2-plane and let $\vec{e}_m$ be the normal vector orthogonal to $\vec{l}_0$ so that we may take $\vec{e}_0 + \vec{e}_m = \vec{l}$. Consider the orthonormal frame $(\vec{e}_\lambda)$ defined by $\vec{e}_0, \vec{e}_i = \vec{n}^{(i)}, \vec{e}_m$; if *F* is a singular 2-form with the fundamental isotropic vector $\vec{l}$ then (7-5) translates into:

(7-7) $F_{ij} = 0 \qquad F_{0i} + F_{mi} = 0,$

and (7-6) into:

(7-7) $F_{0\alpha} + F_{m\beta} = 0,$

and the form *F* is determined by the knowledge of $(m-1)$ numbers $a_i = F_{0i}$.

Introduce the $(m-1)$ singular 2-forms $\varphi^{(i)}$ that are defined by:

$$\varphi^{(i)}_{\alpha\beta} = l_\alpha n^{(i)}_\beta - l_\beta n^{(i)}_\alpha .$$

The squares and scalar products of these forms are obviously null. In the frame in question $(\vec{e}_\lambda)$, we have:

$$\varphi_{j0}^{(i)} = -n_j^{(i)} = \delta_j^i .$$

It results from this that if F is a singular 2-form with fundamental vector $\vec{l}$ then:

(7-9) $$F_{\alpha\beta} = \sum_i a_i \varphi_{\alpha\beta}^{(i)} = \sum_i a_i (l_\alpha n_\beta^{(i)} - l_\beta n_\alpha^{(i)}).$$

If F and $\vec{l}$ are given, then the a_i may depend only on the system of $\vec{n}^{(i)}$. This system may be subjected to the transformation $\vec{n}^{(i)} \to \vec{n}^{(i)} + \vec{k}^{(i)}\vec{l}$ or to a rotation in the (m–1)-plane that it determines. In the first case, one may leave the $\vec{e}_0, \vec{e}_m$ fixed and the F_{i0} determine a vector in the plane envisioned; the a_i are thus the components of a vector in the (m–1)-plane $\vec{n}^{(i)}$.

If one sets $b_\alpha = \sum a_i n_\alpha^{(i)}$ one sees that there exists a vector b_α orthogonal to l_α, such that:

(7-10) $$F_{\alpha\beta} = l_\alpha b_\beta - l_\beta b_\alpha ,$$

Such a vector is defined up to the transformation $b_\alpha \to b_\alpha + k l_\alpha$. One notes that the positive scalar:

$$| b^2 | = -b^\alpha b_\alpha = \sum (a_i)^2$$

depends only on the form F, and on the choice of vector $\vec{l}$.

d) Finally, we establish the following lemma for an arbitrary 2-form F on V_4:

LEMMA. – *In order for a proper vector* $\vec{l}$ *of the form* $F \neq 0$ *to also be a proper vector of the form* $*F$, *it is necessary and sufficient that it be isotropic.*

Indeed, if l is the common proper vector of F and $*F$ then:

$$l^\alpha F_{\alpha\beta} = a l_\beta \qquad l^\alpha (*F)_{\alpha\beta} = b\, l_\beta .$$

From this, one deduces:

$$a\, l_\beta l^\beta = 0 \qquad b\, l_\beta l^\beta = 0.$$

If $\vec{l}$ is not isotropic one will have $a = b = 0$, and if relations (7-5) and (7-6) are satisfied then one can deduce that $F = 0$. Conversely, if l^α is an isotropic proper vector of F then it is an isotropic proper vector of $\tau_{\alpha\beta}$ – hence of ($*F$) – whether F is regular or singular.

8. **Discontinuities of the derivatives of the electromagnetic field.**

a) We consider an electromagnetic field in the absence of any induction phenomena that satisfies MAXWELL'S equations *with a continuous electric current vector* in the domain in question. Since the field $F_{\alpha\beta}$ if of class (C^0, piecewise C^2) by hypothesis we are led to study the crossing of hypersurfaces S, where the first derivatives of that field present discontinuities, as well as the structure of those discontinuities themselves.

We designate the local equation of a hypersurface S that produces discontinuities when traversed by $f(x^\alpha) = 0$. Since the metric tensor if a class (C^0, piecewise C^2) the associated riemannian connection is continuous and, if we notate the discontinuity of a quantity as its traverses S by the sign, [], then one has:

$$[\nabla_\gamma F_{\alpha\beta}] = [\partial_\gamma F_{\alpha\beta}] \qquad (\partial_\gamma \text{ is the Pfaffian derivative}).$$

One immediately deduces from the HADAMARD conditions on wave propagation that there exists an anti-symmetric tensor $\varphi_{\alpha\beta}$ on the points of S such that:

(8-1) $$[\nabla_\gamma F_{\alpha\beta}] = \varphi_{\alpha\beta}\, l_\gamma,$$

where $l_\gamma = \partial_\gamma f$. We let φ denote the 2-form defined on the points of S by the tensor $\varphi_{\alpha\beta}$. Upon symmetrizing (8-1), it results from $[dF] = 0$ that:

(8-2) $$l_\alpha \varphi_{\beta\gamma} + l_\beta \varphi_{\gamma\alpha} + l_\gamma \varphi_{\alpha\beta} = 0.$$

Upon contracting (8-1), it results from $[\delta F] = 0$ that:

(8-3) $$l^\gamma \varphi_{\alpha\beta} = 0.$$

Therefore *the form φ is singular at any point of S is necessarily isotropic.* The hypersurface S satisfies the first order partial differential equation:

(8-4) $$\Delta_1 f = g^{\alpha\beta}\partial_\alpha \partial_\beta = 0.$$

The *electromagnetic wave fronts,* or characteristic hypersurfacess of the Maxwell equations, are the hypersurfaces that are tangent at each of their points to the elementary cone at that point. On the other hand, since $l_\alpha = \partial_\alpha f$ is a gradient:

$$\nabla_\beta l_\alpha - \nabla_\alpha l_\beta = 0.$$

As a result:

$$l^\beta(\nabla_\beta l_\alpha - \nabla_\alpha l_\beta) = 0,$$

and, since $\vec{l}$ is of null length, one has:

(8-5) $$l^\beta \nabla_\beta l_\alpha = 0,$$

which expresses that the trajectories of the vector field $\vec{l}$ on S are geodesics of null length. Therefore the *electromagnetic rays*, or bicharacteristics, which are characteristics of (8-4), are *null-length geodesics* of the metric.

b) Let $(\vec{e}_\alpha)$ be a frame at $x \in S$ that is adapted to the form φ. If:

$$\lambda = \theta^0 - \theta^1,$$

then there exists a scalar a such that $df = a\lambda$. From (7-3), one may find two numbers Y and Z such that φ is expressed by:

$$\varphi = \frac{1}{a}(Y\theta^2 + Z\theta^3) \wedge \lambda.$$

From (8-1), it results that the only non-null components of the tensor $[\nabla_\gamma F_{\alpha\beta}]$ relative to the frame in question, are given by:

(8-6) $$[\nabla_0 F_{20}] = -[\nabla_0 F_{21}] = -[\nabla_1 F_{20}] = [\nabla_1 F_{21}] = Y,$$

and

(8-7) $$[\nabla_0 F_{30}] = -[\nabla_0 F_{31}] = -[\nabla_1 F_{30}] = [\nabla_1 F_{31}] = Z.$$

One may then choose the frame $(\vec{e}_\alpha)$ in such a fashion that $Z = 0$, so that the only non-null discontinuities are then given by (8-6). We thus come down to a canonical form for the components of the tensor $[\nabla_\gamma F_{\alpha\beta}]$.

c) One may observe that it is possible to choose a system of local coordinates in a neighborhood of a point x_0 of S, with respect to which, S admits a simple equation, and which has the property that its natural frame at x_0 coincides with a given adapted frame $(\vec{e}_\alpha)$. Indeed, let θ_0^α be the linear forms at x_0 that have the property:

$$\theta_0^\alpha(\vec{e}_\beta) = \delta_\beta^\alpha,$$

and let u be a variable such that the local equation of S is $u = 0$; from the previous study, it results that there exists a number a such that:

$$(du)_0 = a(\theta^0 - \theta^1)_0.$$

Upon modifying a by a constant factor, one may make $a = 1$. This said, if we are given a linear form x_0 there always exists a local function of class (C^2, piecewise C^4) in a neighborhood of x_0 such that its differential coincides with the linear form at x_0. We may thus find functions v, x^2, x^3 such that at x_0:

$$(dv)_0 = (\theta^0 + \theta^1)_0 \qquad (dx^2)_0 = \theta_0^2 \qquad (dx^3)_0 = \theta_0^3.$$

If we set:

$$2x^0 = v + u \qquad 2x^1 = v - u$$

then we see that:

(8-8) $$(dx^\alpha)_0 = \theta_0^\alpha .$$

From the independence of the forms θ_0^α, the 4 functions x^α have a non-null Jacobian at x_0. We have thus defined local coordinates (x^α) such that S admits the equation $x_0 - x_1 = 0$, and for which the natural frame at x_0 is the given adapted frame. With these coordinates:

$$[\nabla_0 F_{\alpha\beta}] = - [\nabla_2 F_{\alpha\beta}] = \varphi_{\alpha\beta} \quad [\nabla_2 F_{\alpha\beta}] = [\nabla_3 F_{\alpha\beta}] = 0.$$

9. Differential relation obeyed by $[\nabla_\alpha F_{\beta\gamma}]$. Suppose that when we cross S *the curvature tensor of* V_4 *remains continuous.* The tensor $[\nabla_\gamma F_{\alpha\beta}]$ then satisfies a remarkable differential relation on S that we shall derive *on the assumption that* J^α *is null.*

Let $f(x^\alpha) = 0$ be the local equation of S, where f is supposed to be of class C^2; the vector $l_\alpha = \partial_\alpha f$ is, as a result, of class C^1, and $\nabla_\beta l_\alpha$ is continuous when we cross S. Moreover, since l_α is a gradient:

(9-1) $$\nabla_\alpha l_\beta - \nabla_\beta l_\alpha = 0.$$

a) We adopt local coordinates on a neighborhood U such that S admits the equation $x^0 = 0$. One then has $g^{00} = 0$ and the vector l_α, which is the gradient of x^0, admits the covariant components:

$$l_0 = 1 \qquad l_u = 0 \qquad (u = 1, 2, 3).$$

Its contravariant components are therefore:

$$l^0 = 0 \qquad l^u.$$

Since l_α has null length:

$$l_\alpha \nabla_\beta l^\alpha = l_0 \nabla_\beta l^0 + l_u \nabla_\beta l^u = \nabla_\beta l^0 = 0.$$

In particular:

(9-2) $$\nabla_0 l^0 = 0.$$

On S, the tensor $[\nabla_\gamma F_{\alpha\beta}]$ satisfies the relations:

(9-3) $$l_\alpha [\nabla_\sigma F_{\alpha\beta}] + l_\beta [\nabla_\sigma F_{\gamma\alpha}] + l_\gamma [\nabla_\sigma F_{\alpha\beta}] = 0$$

and

(9-4) $$l_\alpha [\nabla_\sigma F_\beta^\alpha] = 0.$$

In the adopted local coordinates, (9-3) may be written:

(9-5) $$[\nabla_\gamma F_{uv}] = 0,$$

and

(9-6) $$[\nabla_\sigma F_\beta^0] = 0.$$

Finally, we note that $\nabla_u[] = [\nabla_u]$, and that if F and the curvature tensor are continuous when we cross S then it results from the RICCI identity that:

(9-7) $$[\nabla_\rho \nabla_\sigma F_{\beta\gamma}] = [\nabla_\sigma \nabla_\rho F_{\beta\gamma}].$$

b) Since the relation (9-3) is satisfied on S, one may differentiate it on that hypersurface and we obtain:

$$\nabla_u (l^\alpha [\nabla_\sigma F_{\beta\gamma}]) + \nabla_u (l_\beta [\nabla_\sigma F_\gamma^\alpha]) + \nabla_u (l_\gamma [\nabla_\sigma F_\beta^\alpha]) = 0,$$

namely, upon giving the value u to α and summing:

$$l^u \nabla_u[\nabla_\sigma F_{\alpha\beta}] + (\nabla_u l^u)[\nabla_\sigma F_{\beta\gamma}] + Q_{\sigma,\beta\gamma} = 0,$$

where one has set:

$$Q_{\sigma,\beta\gamma} = \nabla_u l_\beta [\nabla_\sigma F_\gamma^u] + \nabla_u l_\gamma [\nabla_\sigma F_\beta^u] + l_\beta [\nabla_u \nabla_\sigma F_\gamma^u] + l_\gamma [\nabla_u \nabla_\sigma F_\beta^u].$$

Upon taking (9-2) and (9-6) into account, one has:

(9-8) $$l^\rho \nabla_\rho [\nabla_\sigma F_{\beta\gamma}] + (\nabla_\rho l^\rho)[\nabla_\sigma F_{\beta\gamma}] + Q_{\sigma,\beta\gamma} = 0,$$

with

$$Q_{\sigma,\beta\gamma} = [\nabla_\beta l_\rho \nabla_\sigma F_\gamma^\gamma] + \nabla_\gamma l_\rho \nabla_\sigma F_\beta^\rho] + l_\beta [\nabla_\sigma \nabla_u F_\gamma^u] + l_\gamma [\nabla_\sigma \nabla_u F_\beta^u].$$

From the vanishing of the current vector, it results that:

$$Q_{\sigma,\beta\gamma} = [\nabla_\beta l_\rho \nabla_\sigma F_\gamma^\gamma] + \nabla_\gamma l_\rho \nabla_\sigma F_\beta^\rho] - l_\beta [\nabla_\sigma \nabla_0 F_\gamma^0] - l_\gamma [\nabla_\sigma \nabla_0 F_\beta^0].$$

Now:

$$\nabla_\beta l_\rho \nabla_\sigma F_\gamma^\gamma + \nabla_\gamma l_\rho \nabla_\sigma F_\beta^\rho = \nabla_\beta (l_\rho \nabla_\sigma F_\gamma^\rho) + \nabla_\gamma (l_\rho \nabla_\sigma F_\beta^\rho) - l_\rho (\nabla_\beta \nabla_\sigma F_\gamma^\rho + \nabla_\gamma \nabla_\sigma F_\beta^\rho),$$

where the discontinuity in the last term may be evaluated upon taking into account that $dF = 0$. One therefore has:

$$Q_{\sigma,\beta\gamma} = l^\rho \nabla_\rho [\nabla_\sigma F_{\beta\gamma}] + [\nabla_\beta (l_\rho \nabla_\sigma F_\beta^\rho) + \nabla_\gamma (l_\rho \nabla_\sigma F_\beta^\rho) - l_\beta [\nabla_0 \nabla_\sigma F_\gamma^0] - l_\gamma [\nabla_0 \nabla_\sigma F_\beta^0].$$

Now, for $\beta = u$:

$$[\nabla_u (l_\rho \nabla^\sigma F_\gamma^\rho)] = \nabla_u (l_\rho [\nabla_\sigma F_\gamma^\rho]) = 0.$$

From this, it results that:

(9-10) $$Q_{\sigma,uv} = l^\rho \nabla_\rho [\nabla_\sigma F_\gamma^\rho]) = 0.$$

On the other hand, for $\beta = 0$, $\gamma = u$:

$$[\nabla_0 (l_\rho \nabla^\sigma F_u^{\ \rho})] = [\nabla_0 l^\rho \nabla_\sigma F_{u\rho} + l_\rho \nabla_0 \nabla_\sigma F_u^{\ \rho}] = [\nabla_0 \nabla_\sigma F_u^{\ 0}],$$

since $\nabla_0 l^0 = 0$. One therefore has:

$$Q_{\sigma,0u} = l^\rho \nabla_\rho [\nabla_\sigma F_{0u}].$$

Namely, from (9-10)

$$Q_{\sigma,\beta\gamma} = l^\rho \nabla_\rho [\nabla_\sigma F_{\beta\gamma}].$$

Substituting this into (9-8), one sees that on S the tensor $[\nabla_\gamma F_{\alpha\beta}]$ satisfies the differential relation:

$$2l^\rho \nabla_\rho [\nabla_\gamma F_{\alpha\beta}] + (\nabla^\rho l_\rho)\, [\nabla_\gamma F_{\alpha\beta}] = 0,$$

which clearly show the propagation of discontinuities of $\nabla_\gamma F_{\alpha\beta}$ along the null length geodesics of *S*. In particular, if $[\nabla_\gamma F_{\alpha\beta}] = 0$ at a point *x* of *S*, it is the same all along the null-length geodesic that issues from *x* and is situated on *S*.

Upon setting $[\nabla_\gamma F_{\alpha\beta}] = \varphi_{\alpha\beta} l_\gamma$ in (9-11), and taking into account that $l^\rho \nabla_\rho l_\gamma = 0$, it results that:

$$2l^\rho [\nabla_\rho \varphi_{\alpha\beta}] + (\nabla^\rho l_\rho)\, \varphi_{\alpha\beta} = 0.$$

Let Σ be a *space-oriented* hypersurface and cut *S* ($x^0 = 0$) along a 2-surface *U*. We associate the points of Σ with a 2-form $(F_{\alpha\beta})_\Sigma$ whose first derivatives present discontinuities $[\partial_0 F_{\alpha\beta}]_U = (\varphi_{\alpha\beta})_U$ upon crossing *S*, where $(\varphi_{\alpha\beta})_U$ is a 2-form that is defined on the points of *U*, and is singular with an arbitrarily give fundamental vector $(l_\alpha)_U$. By means of the CAUCHY data $(F_{\alpha\beta})_\Sigma$ on Σ, the vacuum MAXWELL equations determine an electromagnetic field outside of Σ whose first derivatives are discontinuous upon crossing *S*. The corresponding discontinuity tensor $\varphi_{\alpha\beta}$ is necessarily the obviously singular solution of (9-12) that corresponds to the initial data, $(\varphi_{\alpha\beta})_U$ on *U*.

10. A conservation identity. Since $\varphi_{\alpha\beta}$ is a singular 2-form with fundamental vector l_ρ, there exists a vector b_ρ that is orthogonal to l_ρ, and is such that:

$$\varphi_{\alpha\beta} = l_\alpha b_\beta - l_\beta b_\alpha .$$

Upon substituting this expression for $\varphi_{\alpha\beta}$ into (9-12), it becomes, on account of the fact that $l^\rho \nabla_\rho l_\alpha = 0$,

$$l_\alpha \{2l^\rho \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)\, b_\beta\} - l_\beta \{2l^\rho \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)\, b_\alpha\} = 0,$$

namely:

$$2\, l^\rho \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)\, b_\beta = k\, l_\alpha,$$

where *k* is a scalar. Upon multiplying both sides of this relation by b^α, one obtains:

$$2\, l^\rho b^\alpha \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)\, b^\alpha b_\beta = 0,$$

namely,

$$l^\rho \nabla_\rho | b^2 | + (\nabla_\rho l^\rho) | b^2 | = 0.$$

From this, one deduces the identity:

(10-1) $$\nabla_\rho (| b^2 | \, l^\rho) = 0.$$

If F is given then the vector l_ρ is defined up to a constant factor λ along the isotropic geodesic trajectories of l_ρ. If $l^\rho \to \lambda l^\rho$, $\varphi_{\alpha\beta} \to \lambda^{-1} \varphi_{\alpha\beta}$, and, as a result, $b_\alpha \to \lambda^{-2}\, b_\alpha$. Therefore, $| b^2 | \to \lambda^{-4} | b^2 |$. The tensor:

(10-2) $$\tau_{\alpha\beta\gamma\delta} = | b^2 | \, l_\alpha l_\beta l_\gamma l_\delta = -[\nabla_\alpha F_\gamma^{\ \rho}][\nabla_\beta F_{\rho\delta}],$$

depends only on $F_{\alpha\beta}$, and, from (10-1), is conservative:

(10-3) $$\nabla_\alpha \tau^\alpha_{\ \beta\gamma\delta} = 0.$$

11. Pure electromagnetic radiation.

a) The study made in sec. 8 showed us that the presence of discontinuities in the derivatives of the electromagnetic field as you cross a hypersurface *S* defines a singular 2-form on the points of *S*. We are thus led to represent a pure electromagnetic radiation field in vacuo by a field *F*, which is defined by a *singular* form. From (7-2), for such a field there exists an isotropic vector field $\vec{m}$, such that the MAXWELL tensor may be written:

(11-1) $$\tau_{\alpha\beta} = m_\alpha m_\beta .$$

We express the conservation conditions:

(11-2) $$\nabla_\alpha \tau^{\alpha\beta} = 0$$

by means of (11-1). They become:

$$(\nabla_\alpha m^\alpha)\, m^\beta + m^\alpha \nabla_\alpha m^\beta = 0,$$

namely:

(11-2) $$m^\alpha \nabla_\alpha m^\beta = -(\nabla_\alpha m^\alpha)\, m^\beta.$$

Equation (11-3) expresses that the absolute differential of the vector $\vec{m}$ in its proper direction is collinear to $\vec{m}$. From this, it results that the trajectories of the vector field $\vec{m}$ are auto-parallel, i.e., are *null-length geodesics of the metric.*

The study of the geodesics of a linear connection shows that there exists an "affine parameter" σ along each geodesic, which is defined up to a transformation $\sigma \to a\sigma + b$, such that for the corresponding velocity vector $\vec{l} = dx / d\sigma$ one has:

(11-4) $$l^\alpha \nabla_\alpha l^\beta = 0.$$

The vector dx / ds is therefore defined up to a scalar factor, which is constant along each geodesic.

We are thus led to substitute for $\vec{m}$, a vector $\vec{l}$ that is collinear to $\vec{m}$ and satisfies (11-4). Set

$$m_\alpha = \pi\, l_\alpha,$$

where π is a scalar. We then have:

(11-5) $$\tau_{\alpha\beta} = \pi^2\, l_\alpha l_\beta,$$

and the conservation conditions may be written:

$$\nabla_\alpha(\pi^2\, l_\alpha)\, l_\beta + \pi^2\, l^\alpha \nabla_\alpha l_\beta = 0.$$

From this, by the introduction of the vector field l^α in question, it results that the conservation conditions for the MAXWELL tensor translate into (11-4) and the relation:

(11-6) $$\nabla_\alpha(\pi^2\, l_\alpha) = 0.$$

Therefore, a singular electromagnetic field is a field whose MAXWELL tensor may be put into the form (11-5), where $\vec{l}$ is an isotropic vector field such that $i(\vec{l}) \cdot F = i(\vec{l})(*F) = 0,$ (where $i(\vec{l})$ is the operator defined by the "interior product" with $\vec{l}$), and which may be constrained to satisfy (11-4). This field may be associated with a *photon fluid* whose "current lines" – i.e., trajectories of $\vec{l}$ – are null-length geodesic electromagnetic rays that admit an equation of continuity (11-6). Such a field translates into electromagnetic radiation whose propagation is governed by the elementary cone.

b) If $\vec{l}$ defines an isotropic vector field such that (11-5) is satisfied then the LIE derivative $L(\vec{l})F$ of F with respect to the field $\vec{l}$ is given by:

$$L(\vec{l})F = di(\vec{l})F + i(\vec{l})dF.$$

Since $i(\vec{l})F = 0,$ $dF = 0$, we have:

(11-7) $$L(\vec{l})F = 0.$$

Therefore F *is invariant with respect to* $\vec{l}$. If F is null at a point x of V_4 then it is null all along the isotropic geodesic trajectory of $\vec{l}$ that issues from x. The explicit form of (11-7) is:

(11-8) $$l^\rho \nabla_\rho F_{\alpha\beta} - \nabla_\alpha l_\rho F_\beta^\rho - \nabla_\beta l_\rho F_\alpha^\rho = 0.$$

One may combine this relation with a slightly different one that we proceed to derive. By hypothesis:

$$l^\rho F_{\alpha\beta} + l_\alpha F_\beta^\rho + l_\beta F_\alpha^\rho = 0.$$

By derivation, and taking MAXWELL'S vacuum equation into account, one deduces:

(11-9)
$$\nabla_\rho (l^\rho F_{\alpha\beta}) + \nabla_\rho l_\alpha F_\beta^\rho + \nabla_\rho l_\beta F_\alpha^\rho = 0.$$

c) Since F is singular, one knows that there exists a vector b_ρ that is orthogonal to l_ρ, such that:

(11-10)
$$F_{\alpha\beta} = l_\alpha b_\beta - l_\beta b_\alpha .$$

The MAXWELL tensor $\tau_{\alpha\beta}$ therefore admits the following expression:

$$\tau_{\alpha\beta} = -F_\alpha^\rho F_{\beta\rho} = -(l_\alpha b^\rho - l^\rho b_\alpha)(l_\beta b_\rho - l_\rho b_\beta),$$

namely:

$$\tau_{\alpha\beta} = -(b^\rho b_\rho)\, l_\alpha l_\beta = |\, b^2 \,|\, l_\alpha l_\beta .$$

Therefore, for the same choice of vector $\vec{l}$ that satisfies (11-4) $\pi^1 = |\, b^2 \,|$ and (11-6) may be written:

(11-11)
$$\nabla_\rho (|\, b^2 \,|\, l_\rho) = 0.$$

d) The identity (11-11) may be established by a direct method that does not involve the MAXWELL tensor, and which may be extended to the gravitational case. Upon substituting (11-10) into (11-8), we obtain:

$$l_\alpha l^\rho \nabla_\rho b_\beta - l_\beta l^\rho \nabla_\rho b_\alpha - \nabla_\alpha l_\rho l_\beta b^\rho + \nabla_\beta l_\rho l_\alpha b^\rho = 0.$$

Upon multiplying this by b^α, one obtains:

(11-12)
$$l^\rho b^\alpha \nabla_\rho b_\alpha + \nabla_\alpha l_\rho b^\alpha b^\rho = 0.$$

Upon proceeding in a similar manner after staring with (11-9), one first obtains:

$$l_\alpha l^\rho \nabla_\rho b_\beta - l_\beta l^\rho \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)(l_\alpha l_\beta - l_\beta l_\alpha) + \nabla_\rho l_\alpha l_\beta b^\rho - \nabla_\rho l_\beta l_\alpha b^\rho = 0.$$

After multiplying by b_α, it becomes:

(11-13)
$$l^\rho b^\alpha \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)\, b^\alpha b_\alpha - \nabla_\rho l_\alpha b^\alpha b^\rho = 0.$$

By term-wise addition of (11-12) and (11-13), one obtains:

$$2 l^\rho b^\alpha \nabla_\rho b_\alpha + (\nabla_\rho l^\rho)\, b^\alpha b_\alpha = 0;$$

i.e., identity (11-11).

12. **Electromagnetic field of integrable type.** Suppose we have an arbitrary electromagnetic field F on a neighborhood U of V_4. At a point x, where F is regular it admits two distinct isotropic proper directions, and, at a point where it is singular, only one. Consider the 3-plane $\Pi(x)$ that is tangent to the elementary cone C_x at x along one such generatrix of the cone. We therefore have two ways of associating F with a field of 3-planes $\Pi(x)$ that is tangent to the elementary cones. We say that *the electromagnetic field is of integrable type* if the field Π is completely integrable. For this to be the case, it is necessary and sufficient that Π admit a definition by an equation of the form $d\sigma = 0$, where σ a function, i.e., that one may define the isotropic direction field in question by a gradient.

If l_α is a gradient then one has:

(12-1) $$\nabla_\beta l_\alpha - \nabla_\alpha l_\beta = 0,$$

and, as a result:

$$l^\beta(\nabla_\beta l_\alpha - \nabla_\alpha l_\beta) = 0,$$

i.e., since l_α has null length:

(12-2) $$l^\beta \nabla_\beta l_\alpha = 0.$$

Therefore, in order for a field to be of integrable type, the trajectories of one of its isotropic direction fields must also be null-length geodesics.

13. Permanence of a singular field of integrable type. Consider an electromagnetic field F of integrable type *that satisfies the vacuum MAXWELL equations,* and suppose that there exists a *space-oriented* hypersurface Σ on which F is singular.

We say that F is of integrable type, i.e., that there exists an isotropic vector field $\vec{l}$ that is a gradient, and which satisfies:

(13-1) $$l_\alpha F^{\alpha\beta} = a l^\beta.$$

One then also has:

(13-2) $$l_\alpha ({}^*F)^{\alpha\beta} = b l^\beta.$$

The form F is singular on Σ, and, as a result, one necessarily has $a = b = 0$ on Σ.

Upon deriving (13-1), we obtain:

$$\nabla_\beta (a l^\beta) = \nabla_\beta l_\alpha F^{\alpha\beta} + l_\alpha \nabla_\beta F^{\alpha\beta}.$$

From (12-1) and MAXWELL's equations, it results that the right-hand side is zero. Therefore:

(13-3) $$\nabla_\beta (a l^\beta) = l^\beta \partial_\beta a + a \nabla_\beta l^\beta = 0,$$

and similarly,

(13-4) $$\nabla_\beta (b l^\beta) = l^\beta \partial_\beta b + b \nabla_\beta l^\beta = 0.$$

Since a and b are null on Σ, it results from (13-3) and (13-4) that $a = b = 0$ outside of Σ. As a result, the electromagnetic field is singular outside of Σ. We assert ([3]):

THEOREM. – *If an electromagnetic field of integrable type is singular on a space-oriented hypersurface* Σ *and satisfies the vacuum MAXWELL equations then it singular outside of* Σ.

14. Study of the singular field of integrable type.

a) If we are given a completely integrable singular field on a neighborhood U then there exists a function σ on this neighborhood, which is defined up to a transformation $\sigma \to f(\sigma)$, where f is an arbitrary function, such that Π is defined by an equation $d\sigma = 0$. It results from this that:

(14-1) $$d\sigma \wedge \mathrm{F} = 0 \qquad d\sigma \wedge (*\mathrm{F}) = 0.$$

One calls the function σ the *phase function* for the field in question. The relations (14-1) characterize the singular fields of integrable type. The manifolds σ = const. are the characteristic manifolds of the MAXWELL equations.

b) Suppose we are given a singular field that we refer to an adapted frame. If $\lambda = \theta^0 - \theta^1$ then the field Π may be defined by the equation $\lambda = 0$. From the method of FROBENIUS, in order for a field to be completely integrable it is necessary and sufficient that:

(14-2) $$\lambda \wedge d\lambda = 0.$$

Condition (14-2) is equivalent to the annihilation of the four forms:

(14-3) $$\theta^\rho \wedge \lambda \wedge d\lambda = 0 \qquad (\rho = 0, 1, 2, 3).$$

For $\rho = 0$ and 1, one obtains the unique condition:

$$\theta^0 \wedge \theta^1 \wedge d\lambda = 0.$$

For $\rho = 2$ or 3, one obtains two conditions that found to be satisfied identically by virtue of the MAXWELL equations. By means of (7-3), taken with $Z = 0$, they are:

$$F = Y\,\theta^2 \wedge \lambda \qquad *F = -\,Y\,\theta^3 \wedge \lambda.$$

Since $dF = d{*F} = 0$, differentiation produces:

$$d(Y\theta^2) \wedge \lambda - Y\theta^2 \wedge d\lambda = 0 \qquad d(Y\theta^3) \wedge \lambda - Y\theta^3 \wedge d\lambda = 0.$$

Upon taking the exterior product with λ, one sees that (14-3) is satisfied for $\rho = 2$ and 3. Doing this for a pure electromagnetic radiation field of integrable type introduces only one supplementary scalar condition.

[3] A slightly more general result has been established by L. MARIOT.

If F is *singular of integrable type,* then

$$\nabla_\alpha l_\beta - \nabla_\beta l_\alpha = 0.$$

From (11-8) and (11-9), by term-wise addition, we obtain the differential relation:

(14-3) $$2l^\rho \nabla_\rho F_{\alpha\beta} + (\nabla_\rho l^\rho)\, F_{\alpha\beta} = 0,$$

a relation that is analogous to (9-11) or (9-12).

15. Application. Suppose we are given a non-constant function σ on U and an electromagnetic field that satisfies the vacuum MAXWELL equations. Consider the 2-form defined by:

(15-1) $$H = f(\sigma)\, F,$$

where f is an arbitrary non-constant function. One obviously has:

(15-2) $$*H = f(\sigma)(*F).$$

By exterior differentiation, we obtain:

$$dH = f'(\sigma) d\sigma \wedge F \qquad d(*H) = f'(\sigma) d\sigma \wedge (*F).$$

In order for H to satisfy the vacuum MAXWELL equations, it is necessary and sufficient that:

$$d\sigma \wedge F = 0 \qquad d\sigma \wedge *F = 0,$$

i.e., that F define a singular field of integrable type that admits the phase function σ.

THEOREM. – *If we are given a non-constant function σ and an electromagnetic field F that satisfies the vacuum MAXWELL equations then in order for the field:*

$$H = f(\mathrm{s})\, F,$$

(where f is an arbitrary non-constant function) to satisfy the same equations, it is necessary and sufficient that F be a singular field of integrable type that admits σ as a phase function.

Note that H then satisfies the vacuum MAXWELL equations for any f.

II. Gravitational wavefronts in general relativity.

16. Expressions for the curvature tensor and the Ricci tensor. In this section, we propose to study the notion of gravitational wavefront in general relativity. The curvature tensor of the Riemannian manifold V_4 will play an essential role in the course of this study, and we must analyze how the second derivatives of the gravitational potentials with respect to the local coordinates which enter into the expression of this tensor. Consider a riemannian manifold V_{m+1} of arbitrary signature. In a neighborhood U on which local coordinates (x^α) are defined, let $\Gamma^\alpha_{\beta\gamma}$ be the coefficients of the Riemannian connection of V_{m+1} relative to the local coordinates. By definition, the curvature tensor of V_{m+1} is the tensor with components:

(16-1) $$R^\alpha_{\beta,\lambda\mu} = \partial_\lambda \Gamma^\alpha_{\beta\mu} - \partial_\mu \Gamma^\alpha_{\beta\lambda} + \Gamma^\alpha_{\rho\lambda}\Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\beta\lambda}.$$

Upon expressing the $\Gamma^\alpha_{\beta\gamma}$ with the aid of the CHRISTOFFEL symbols, we obtain:

$$R^\alpha_{\beta,\lambda\mu} = g^{\alpha\rho}(\partial_\lambda[\beta\mu,\rho] - \partial_\mu[\mu\lambda,\rho]) + K^\alpha_{\beta,\lambda\mu},$$

where the $K^\alpha_{\beta,\lambda\mu}$ depend only on the first derivatives of the potentials $g_{\rho\sigma}$ and the potentials themselves. From this, one deduces:

$$R_{\alpha\beta,\lambda\mu} = \partial_\lambda[\beta\mu,\alpha] - \partial_\mu[\beta\lambda,\alpha] + K_{\alpha\beta,\lambda\mu},$$

and, upon developing the CHRISTOFFEL symbols:

$$R_{\alpha\beta,\lambda\mu} = \frac{1}{2}(\partial_{\beta\lambda} g_{\alpha\mu} + \partial_{\lambda\mu} g_{\alpha\beta} - \partial_{\alpha\lambda} g_{\beta\mu} - \partial_{\beta\mu} g_{\alpha\lambda} - \partial_{\lambda\mu} g_{\alpha\beta} + \partial_{\alpha\mu} g_{\beta\lambda}) + K_{\alpha\beta,\lambda\mu}.$$

One thus obtains:

(16-2) $$R_{\alpha\beta,\lambda\mu} = \frac{1}{2}(\partial_{\beta\lambda} g_{\alpha\mu} + \partial_{\alpha\mu} g_{\beta\lambda} - \partial_{\beta\mu} g_{\alpha\lambda} - \partial_{\alpha\lambda} g_{\beta\mu}) + K_{\alpha\beta,\lambda\mu}.$$

The RICCI tensor on V_{m+1} is defined by:

(16-3) $$R_{\alpha\beta} = R^\sigma_{\beta,\sigma\alpha} = g^{\rho\sigma} R_{\rho\beta,\sigma\alpha}.$$

By contracting the indices α and λ in (16-2) and changing the name of the indices, this becomes:

(16-4) $$R_{\alpha\beta} = \frac{1}{2} g\rho\sigma(\partial_{\beta\sigma} g_{\alpha\rho} + \partial_{\alpha\rho} g_{\beta\sigma} - \partial_{\alpha\beta} g_{\rho\sigma} - \partial_{\rho\sigma} g_{\alpha\beta}) + K_{\alpha\beta.}$$

17. **Discontinuities of the second derivatives of the potentials.** Suppose we have a neighborhood of the "spacetime" manifold V_4 with a system of local coordinates (x^α). Since the gravitational tensor is supposed to be (C^1, piecewise C^3), the second derivatives of the gravitational potentials $g_{\alpha\beta}$ with respect to the local coordinates may admit certain discontinuities upon crossing certain hypersurfaces S, and it is those discontinuities that we proceed to study.

Therefore, let V_{m+1} be a differentiable manifold of class (C^2, piecewise C^4) endowed with a Riemannian metric of hyperbolic normal type and class (C^1, piecewise C^3). If a neighborhood U of V_{m+1} is referred to a system of local coordinates (x^α) then let $f(x^\alpha) = 0$ be the local equation of a hypersurface S that produces discontinuities when one crosses it. From the HADAMARD conditions on the propagation of waves, it results that there exists a system of local quantities $a_{\alpha\beta}$ at the points of S, such that the discontinuities $[\partial_{\lambda\mu} g_{\alpha\beta}]$ may be expressed by the formula:

(17-1) $$[\partial_{\lambda\mu} g_{\alpha\beta}] = a_{\alpha\beta} \partial_\lambda f \partial_\mu f .$$

namely, upon designating the gradient of f by l_λ:

(17-2) $$[\partial_{\lambda\mu} g_{\alpha\beta}] = a_{\alpha\beta} l_\lambda l_\mu .$$

We study how the system of $a_{\alpha\beta}$ transforms under a change of local coordinates. If $x^\alpha = x^\alpha(x^{\sigma'})$ defines this change of coordinates, one has:

$$g_{\sigma'\tau'} = A^\alpha_{\sigma'} A^\beta_\tau g_{\alpha\beta} \qquad \left(A^\alpha_{\sigma'} = \frac{\partial x^\alpha}{\partial x^{\sigma'}} \right).$$

By derivation, this becomes:

$$\partial_{\nu'} g_{\sigma'\tau'} = A^\alpha_{\sigma'} A^\beta_{\tau'} A^\lambda_{\nu'} \partial_\lambda g_{\alpha\beta} + (\partial_{\nu'} A^\alpha_{\sigma'} \cdot A^\beta_{\tau'} + A^\beta_{\tau'} \partial_{\nu'} A^\beta_{\tau'}) g_{\alpha\beta} .$$

The third derivatives $\partial_{\nu'\rho'} A^\alpha_{\sigma'}$ may be discontinuous upon crossing S, while the discontinuities in the second derivatives of the potentials of the metric upon crossing S are related by the formula:

$$[\partial_{\nu'\rho'} g_{\sigma'\tau'}] = A^\alpha_{\sigma'} A^\beta_{\tau'} A^\lambda_{\nu'} A^\mu_{\rho'} [\partial_{\lambda\mu} g_{\alpha\beta}] + [\partial_{\nu'\rho'} A^\alpha_{\sigma'}] A^\beta_{\tau'} + A^\alpha_{\sigma'} [\partial_{\nu'\rho'} A^\beta_{\tau'}]) g_{\alpha\beta} .$$

Conforming to the HADAMARD conditions, if we set:

(17-3) $$[\partial_{\nu'\rho'} A^\alpha_{\sigma'}] = t_\alpha l_{\nu'} l_{\rho'} l_{\sigma'}$$

and:

(17-4) $$t_\beta = t^\alpha g_{\alpha\beta}$$

then we obtain:

(17-5) $$a_{\sigma'\tau'} = A^\alpha_{\sigma'} A^\beta_{\tau'} (a_{\alpha\beta} + t_\beta l_\alpha + t_\alpha l_\beta).$$

Therefore, if the coordinate change is of class C^3 in a neighborhood of S then the $a_{\alpha\beta}$ transform according to the tensor law:

(17-6)
$$a_{\sigma'\tau'} = A^{\alpha}_{\sigma'} A^{\beta}_{\nu'} a_{\alpha\beta}.$$

If the coordinate transformation in question is tangent to the identity transformation along S and admits discontinuous third derivatives when one crosses S then the $a_{\alpha\beta}$ are subject to the transformation:

(17-7)
$$a_{\alpha\beta} \to a_{\alpha\beta} + t_{\alpha} l_{\beta} + t_{\beta} l_{\alpha}.$$

Formula (17-5) may be considered as the result of the composition of (17-7) and (17-6).

Taking formula (17-2) into account, one obtains, upon starting from (16-2):

(17-8)
$$[R_{\alpha\beta,\lambda\mu}] = \frac{1}{2}(a_{\alpha\mu} l_{\beta} l_{\lambda} + a_{\beta\lambda} l_{\alpha} l_{\mu} - a_{\alpha\lambda} l_{\beta} l_{\mu} - a_{\alpha\lambda} l_{\beta} l_{\mu}).$$

Since the $R_{\alpha\beta,\lambda\mu}$ define a tensor, the same is true for the $[R_{\alpha\beta,\lambda\mu}]$, and formula (17-8) is necessarily invariant under the transformation (17-7), as one easily verifies.

As far as the RICCI tensor is concerned, one has, from (16-4):

(17-9)
$$[R_{\alpha\beta}] = \frac{1}{2} g^{\alpha\beta} (a_{\alpha\rho} l_{\beta} l_{\sigma} + a_{\beta\sigma} l_{\alpha} l_{\rho} - a_{\rho\sigma} l_{\alpha} l_{\beta} - a_{\alpha\beta} l_{\rho} l_{\sigma}).$$

18. Characteristic manifolds of the Einstein equations.

a) Suppose that the metric of the manifold V_{m+1} satisfies generalized "EINSTEIN equations" of the form:

(18-1)
$$S_{\alpha\beta} = \chi T_{\alpha\beta},$$

where $T_{\alpha\beta}$ is a given symmetric tensor – *which is supposed to be continuous in the domain in question* – χ is a constant, and $S_{\alpha\beta}$ denotes the tensor:

(18-2)
$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R.$$

From (18-2), one deduces, by contraction:

$$S = R - \frac{m+1}{2} R = -\frac{m-1}{2} R.$$

When m ≠ 1, this implies:

(18-3)
$$R_{\alpha\beta} = S_{\alpha\beta} - \frac{1}{m-1} g_{\alpha\beta} S.$$

From the continuity of $T_{\alpha\beta}$, one deduces from (18-1) that $[S_{\alpha\beta}] = 0$, and, as a result, by virtue of (18-3) and the continuity of the metric:

$$[R_{\alpha\beta}] = 0. \tag{18-4}$$

b) Choose a system of local coordinates such that the hypersurface *S* that is responsible for the discontinuities that we are studying has the local equation $x^0 = 0$; in this case, l_α admits the components:

$$l_0 = 1 \qquad l_u = 0 \qquad (u, v, \text{etc.} \ldots = 1, 2, \ldots, m). \tag{18-5}$$

The only second derivatives that may be discontinuous upon crossing *S* are the $\partial_{00} g_{\alpha\beta}$, and one has:

$$[\partial_{00} g_{uv}] = a_{uv} \qquad [\partial_{00} g_{0\alpha}] = a_{0\alpha},$$

where the a_{uv} are invariant under the transformation (17-7) and the $a_{0\alpha}$ transform according to:

$$a_{0\alpha} \to a_{0\alpha} + t_\alpha + t_0\, l_\alpha.$$

The choice of t_α or of t^α permits us to annihilate the discontinuities of $\partial_{00} g_{0\alpha}$ or make them appear. Moreover, from (17-9), one obtains:

$$[R_{uv}] \equiv -\frac{1}{2} g^{00} a_{uv} = 0 \tag{18-6}$$

$$[R_{0u}] \equiv \frac{1}{2} g^{0v} a_{uv} = 0 \tag{18-7}$$

$$[R_{00}] \equiv -\frac{1}{2} g^{uv} a_{uv} = 0, \tag{18-8}$$

and the left-hand sides of (18-6), (18-7), and (18-8) do not involve the $[\partial_{00} g_{0\alpha}]$. We say that the derivatives $\partial_{00} g_{uv}$ are *significant* relative to the hypersurface, and that the $(m+1)$ derivatives $\partial_{00} g_{0\alpha}$ are *insignificant.* It is the case where the significant derivatives present discontinuities that we now proceed to examine.

If all of the discontinuities of the significant derivatives are non-null, then there exists an $a_{uv} \neq 0$, and from (18-6) it results that

$$g^{00} = 0. \tag{18-9}$$

The vector l_α then admits the components:

$$l_0 = 0 \qquad l_u = g_{0u},$$

and, from (18-7), it satisfies the relations:

(18-10) $$a_{uv}\, l^v = 0.$$
Moreover, from (18-8):
(18-11) $$a \equiv g^{\alpha\beta} a_{\alpha\beta} = 2g^{0u} a_{0u} = 2a_{0u}\, l^u.$$

In arbitrary local coordinates, (18-9) may be written:

$$g^{\alpha\beta} l_\alpha l_\beta = 0,$$

and the gradient of f has null length. Therefore, S is necessarily a solution of the first order partial differential equation:

$$\Delta_1 f \equiv g^{\alpha\beta} \partial_\alpha f \partial_\beta = 0,$$

i.e., the general relativistic analog of (9-4).

Therefore, in this case (m = 3), the *gravitational wavefronts* – or characteristic manifolds – of the EINSTEIN equations are the hypersurfaces that are tangent to the elementary cone at each of their points. The *gravitational rays,* which are trajectories of the vector field defined by $\vec{l}$ on such a hypersurface are always null-length geodesics.

Recall the general case of a manifold V_{m+1}. Equations (18-10) and (18-11) may be translated into arbitrary local coordinates by the relation:

(18-12) $$a_{\alpha\beta} l^\beta = \frac{a}{2} l_\alpha \qquad (a = g^{\alpha\beta} a_{\alpha\beta}).$$

In particular, one sees that the null-length vector l^α is a proper vector of the matrix ($a_{\alpha\beta}$). One immediately verifies that (18-12) is invariant under the transformation (17-7).

Conversely, if l^α has null length and satisfies (18-12) then one has $[R_{\alpha\beta}] = 0$.

19. Lemma on the manifolds that admit a group of isometries. Five-dimensional case. It is known that the consideration of a five-dimensional manifold endowed with a one-parameter group of isometries permits the geometric unification of the gravitational field and the electromagnetic field (in the absence of induction). More generally, we shall review the formulas that relate to a manifold V_{m+1} that admits a one-parameter group of isometries.

a) Consider a manifold V_{m+1} of dimension (m+1) that satisfies the same differentiability hypotheses as the spacetime manifold, i.e., (C^2, piecewise C^4). We suppose that a Riemannian metric $d\sigma^2$ is defined on this manifold, which is of hyperbolic normal type, with 1 positive square and m negative ones, and class (C^1, piecewise C^3). Throughout this section, we suppose that Greek indices take the values 0, 1, …, m and Latin indices take the values 1, 2, …, m. In local coordinates, the metric may be written:

(19-1) $$d\sigma^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta,$$

and in a neighborhood they may be decomposed into the algebraic sum of squares:

(19-2) $$d\sigma^2 = (\theta^m)^2 - (\theta^0)^2 - (\theta^1)^2 - \dots (\theta^{m+1})^2,$$

where the θ^α are linearly independent local PFAFF forms. By duality, the neighborhood of V_{m+1} in question is found to be endowed with an orthonormal frame.

We suppose that V_{m+1} admits a connected global one-parameter group of isometries with trajectories that are oriented so that $d\sigma^2 < 0$, leaving no point of V_{m+1} invariant, and enjoying the following property: upon passage to the quotient under the equivalence relation defined by the group of isometries, one obtains a differentiable manifold V_m of class (C^2, piecewise C^4); the points z of V_m may be identified with the different trajectories of the group in V_{m+1}.

Let $\vec{\xi}$ be the infinitesimal generator of the group of isometries; since no point of V_{m+1} is invariant, $\vec{\xi}$ is non-null. It satisfies the KILLING equations:

(19-3) $$L(\vec{\xi})g_{\alpha\beta} = \nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta = 0.$$

Let (x^i) be an arbitrary system of local coordinates on V_m . A point x of V_{m+1} may be represented by its trajectory $z(x)$, and, on that trajectory, by a coordinate $(x^{0'})$. In the local coordinates $(x^{\lambda'})$ the trajectories of the vector field $\vec{\xi}$ are the lines $x^{i'}$ = const., and one has:

$$\xi^{i'} = 0 \qquad\qquad \xi^{0'} \neq 0.$$

We perform the change of local coordinates defined by:

$$x^i = x^{i'} \qquad\qquad x^0 = f(x^{0'}, x^{j'}).$$

One may choose f in such a manner that the new components are:

$$\xi^0 = A_0^0, \qquad \xi^{0'} = \frac{\partial f}{\partial x^{0'}}\xi^{0'} = 1.$$

It suffices to take:

$$\frac{\partial f}{\partial x^{0'}} = \frac{1}{\xi^{0'}} \qquad\qquad (\xi^{0'} \neq 0),$$

and the function f is found to be defined up to an additive function of the coordinates $(x^{j'})$. In the system of coordinates thus defined (x^λ):

$$\xi^i = 0 \qquad\qquad \xi^0 = 1,$$

and x^λ is called *adapted to the group of isometries* in question. In this system, $\xi^\alpha = g_{\alpha 0}$, and, as a result, $\nabla_\beta \xi_\alpha = [\beta 0, \alpha]$. As a result, the KILLING equations (19-3) translate into:

(19-4) $$\partial_0 \gamma_{\alpha\beta} = 0.$$

Systems of local coordinates adapted to the group of isometries are defined up to a change:

(19-5) $$x^{i'} = \psi^{i'}(x^j) \qquad x^{0'} = x^0 + \psi(x^j),$$

where ψ is an arbitrary function of the x^j.

To each point x of a neighborhood V_{m+1}, there is an associated orthonormal frame whose first vector is tangent at x to the trajectory passing through this point. Such a frame is called *adapted to the group of isometries.* Relative to the adapted frame, we have:

(19-6) $$d\sigma^2 = -(\theta^0)^2 + (\theta^m)^2 - (\theta^1)^2 - \ldots (\theta^{m-1})^2,$$

and the θ^i are annihilated along the trajectories. If (x^λ) is an adapted system of local coordinates, then the θ^i are PFAFF forms with respect to the dx^j and (19-6) is none other than the decomposition into squares of $d\sigma^2$, with the variable dx^0 playing the role of directrix variable. From this, one deduces:

(19-7) $$d\sigma^2 = -(\theta^0)^2 + ds^2,$$

with

(19-8) $$\theta^0 = \frac{1}{\sqrt{|\gamma_{00}|}}(\gamma_{00}dx^0 + \gamma_{0i}dx^i)$$

and

(19-9) $$ds^2 = (\theta^m)^2 - (\theta^1)^2 - \cdots - (\theta^{m-1})^2 = \left(\gamma_{ij} - \frac{\gamma_{0i}\gamma_{0j}}{\gamma_{00}}\right)dx^i dx^j.$$

From (19-9), it results that ds^2 is independent of the chosen system of adapted local coordinates, and, in this system, it is independent of x^0. The quadratic form ds^2 determined a Riemannian metric of hyperbolic type on V_m, which is called the *quotient metric,* with coefficients:

(19-10) $$g_{ij} = \gamma_{ij} - \frac{\gamma_{0i}\gamma_{0j}}{\gamma_{00}}.$$

b) The square of the vector $\vec{\xi}$ is strictly negative, and we designate it by $-\xi^2$. In adapted coordinates $\gamma_{00} = -\xi^2$ and ξ defines a *scalar* field on V_m.

Now consider the vector φ_α of V_{m+1} that is collinear to ξ_α, and is defined by:

(19-11) $$\beta\varphi_i = \frac{\xi_\alpha}{-\xi^2},$$

where β designates a suitable constant. In adapted coordinates:

(19-12) $$\beta\varphi_i = \frac{\gamma_{0i}}{\gamma_{00}} \qquad \beta\varphi_0 = 1.$$

Consider $F_{\alpha\beta}$, which is the rotation of φ_α. From (19-12):

$$F_{0\alpha} = \partial_0\varphi_\alpha - \partial_\alpha\varphi_0 = 0.$$

On the other hand, if one performs the change of adapted local coordinates defined by $x^{i'} = x^i,\ x^{0'} = x^0 + \psi(x^i)$:

(19-13) $$\varphi_i = \varphi_{i'} + \frac{1}{\beta}\frac{\partial\psi}{\partial x^i},$$

and, as a result, $F_{ij} = F_{i'j'}$. Therefore the F_{ij} define *an anti-symmetric tensor on* V_m .

It is easy to see the geometric significance of the vanishing of F_{ij} . If it is possible to find adapted local coordinates for a neighborhood U of V_m , such that the trajectories corresponding to U are *orthogonal trajectories* of the hypersurfaces x^0 = const. then one has $\xi_i = \gamma_{0i} = 0$, $\varphi_i = 0$ in this system, and, as a result $F_{ij} = 0$. Conversely, if $F_{ij} = 0$ then the tensor $F_{\lambda\mu}$ of V_{m+1} is null, and ($\beta_{j0} = 1$, $\beta\varphi_i$) locally defines a gradient field. There thus exists a function $\psi(\xi^i)$ in a neighborhood U of V_m such that:

$$\beta\varphi_\alpha = \partial_\alpha[x^0 + \psi(x^i)].$$

Therefore, $F_{ij} = 0$ expresses the idea that there exist adapted local coordinates such that the corresponding trajectories of the group of isometries on a neighborhood U are orthogonal trajectories of the hypersurface x^0 = const.

c) We have thus defined a metric (19-10) on V_m, a scalar ξ, and an anti-symmetric tensor F_{ij}, after starting with the metric on V_{m+1} and the group of isometries.

Refer V_{m+1} to an adapted orthonormal frame. From (19-9), the manifold V_m, which is considered to be a Riemannian manifold for the quotient metric, is therefore found to be referred to the orthonormal frame. One may express the components $R_{\alpha\beta,\gamma\delta}$ of the curvature tensor of V_{m+1} as a function of the components of the curvature tensor for V_m of ξ and of the components of the anti-symmetric tensor F.

Indeed, one proves the following formulas ([4]):

(19-14) $$R_{ij,kl} = \overset{*}{R}_{ij,kl} + \frac{\beta^2\xi^2}{4}(F_{ik}F_{ji} - F_{il}F_{jk}) + \frac{\beta^1\xi^2}{2}F_{ij}F_{kl}$$

(19-15) $$R_{ij,k0} = \frac{\beta}{2}(\xi\overset{*}{\nabla}_k F_{ij} + 2\partial_k\xi F_{ij} - \partial_i\xi F_{jk} + \partial_j\xi F_{ik})$$

---

[4] See LICHNEROWICZ, *Théories relativistes de la gravitation et de l'electromagnetisme*, pp. 119; the formulas have been transformed according to the signature of the metric and the orientation of the trajectories.

(19-16) $$R_{i0,k0} = \frac{1}{\xi}\overset{*}{\nabla}_k(\partial_i\xi) + \frac{\beta^2\xi^2}{4}F_{ir}F_k^{\ r},$$

where the elements that relate to the metric ds^2 have been given a *.

d) We place ourselves in the five-dimensional case (m = 4). For the theories of JORDAN-THIRY and KALUZA-KLEIN, the preceding hypotheses are satisfied, and the anti-symmetric tensor, F_{ij}, may be interpreted as the electromagnetic field.

From the preceding formulas, it results that upon crossing a hypersurface S the discontinuities of the components of the curvature tensor of V_5 may be written:

(19-17) $$[R_{ij,kl}] = [\overset{*}{R}_{ij,kl}]$$

(19-18) $$[R_{ij,k0}] = \frac{\beta}{2}\xi[\overset{*}{\nabla}_k F_{ij}]$$

(19-19) $$[R_{i0,k0}] = \frac{1}{\xi}[\overset{*}{\nabla}_k(\partial_i\xi)].$$

(19-18) show that the study of the discontinuities of the first derivatives $[\overset{*}{\nabla}_k F_{ij}]$ of the electromagnetic field are equivalent to those of the discontinuities of the components $R_{ij,k0}$ of the curvature tensor of V_5.

If we wish to analyze the structure of gravitational waves in spacetime, we are therefore led to study the discontinuities in the components of the curvature tensor of the metric of general relativity.

20. Formulas that relate to the discontinuities of the curvature tensor for V_{m+1}. Let us return to the case of a Riemannian manifold V_{m+1} that is endowed with a Riemannian metric of hyperbolic normal type with one positive square an m negative squares.

a) From formula (17-8), it is possible to deduce an interesting relation between $l_\gamma = \partial_\gamma f$ and the tensor $[R_{\alpha\beta,\lambda\mu}]$ that expresses the discontinuity in the curvature tensor upon crossing the hypersurface S that has the local equation $f(x^\alpha) = 0$. One has:

(20-1) $$l_\gamma[R_{\alpha\beta,\lambda\mu}] = \tfrac{1}{2}(a_{\alpha\mu}\,l_\beta\,l_\gamma - a_{\beta\mu}\,l_\gamma\,l_\alpha)\,l_\lambda + \tfrac{1}{2}(a_{\alpha\mu}\,l_\beta\,l_\gamma - a_{\alpha\mu}\,l_\beta\,l_\gamma)\,l_\mu\,.$$

By cyclic permutation of the indices α, β, γ one obtains:

(20-2) $$l_\alpha[R_{\beta\gamma,\lambda\mu}] = \tfrac{1}{2}(a_{\beta\mu}\,l_\gamma\,l_\alpha - a_{\gamma\mu}\,l_\alpha\,l_\beta)\,l_\lambda + \tfrac{1}{2}(a_{\mu\lambda}\,l_\alpha\,l_\beta - a_{\beta\lambda}\,l_\gamma\,l_\alpha)\,l_\mu$$

and:

(20-3) $$l_\beta[R_{\gamma\alpha,\lambda\mu}] = \tfrac{1}{2}(a_{\gamma\mu}\,l_\alpha\,l_\beta - a_{\alpha\mu}\,l_\beta\,l_\gamma)\,l_\lambda + \tfrac{1}{2}(a_{\alpha\lambda}\,l_\beta\,l_\gamma - a_{\gamma\lambda}\,l_\alpha\,l_\beta)\,l_\mu\,.$$

By adding (20-1), (20-2), and (20-3), we obtain the relations:

$$l_\alpha[R_{\beta\gamma,\lambda\mu}] + l_\beta[R_{\gamma\alpha,\lambda\mu}] + l_\gamma[R_{\alpha\beta,\lambda\mu}] = 0. \tag{20-4}$$

b) Suppose, moreover, that the metric on V_{n+1} satisfies the generalized "Einstein equations" with a continuous right-hand side.

Upon crossing the hypersurface:

$$[R_{\alpha\beta}] = 0, \tag{20-5}$$

and from the analysis that was done in sec. 18, the vector l_α satisfies relation (18-12).

From (17-18), one deduces:

$$l^\alpha[R_{\alpha\beta,\lambda\mu}] = \tfrac{1}{2}(a_{\alpha\mu}\, l^\alpha l_\beta l_\lambda - a_{\alpha\lambda}\, l^\alpha l_\beta l_\mu) + \tfrac{1}{2}(a_{\beta\lambda}\, l_\mu - a_{\beta\lambda}\, l_\lambda)\, l_\alpha l^\alpha,$$

and since l^α has null length, the second term of the right-hand side is zero. Moreover, from (18-12):

$$a_{\alpha\mu} l^\alpha l_\lambda - a_{\alpha\lambda} l^\alpha l_\mu = \frac{a}{2}(l_\lambda l_\mu - l_\mu l_\lambda) = 0.$$

Thus, (20-5) entails that:

$$l_\alpha[R_{\alpha\beta,\lambda\mu}] = 0. \tag{20-6}$$

c) One may also establish relations (20-4) and (20-6) directly in the following manner: since the coefficients Γ of the Riemannian connection in local coordinates are continuous upon crossing *S*, it results from the HADAMARD conditions that there exists a system of quantities $u^\alpha_{\beta\mu}$ such that:

$$[\partial_\lambda \Gamma^\alpha_{\beta\mu}] = u^\alpha_{\beta\mu}\, l_\lambda \qquad (l_\lambda = \partial_\lambda f).$$

From (16-1), it results that:

$$[R^\alpha{}_{\beta,\lambda\mu}] = u^\alpha_{\beta\mu}\, l_\lambda - u^\alpha_{\beta\lambda}\, l_\mu\,. \tag{20-7}$$

By multiplying with l_ν and cyclically permuting λ, μ, ν it follows that:

$$l_\lambda\,[R^\alpha{}_{\beta,\mu\nu}] + l_\mu\,[R^\alpha{}_{\beta,\nu\mu}] + l_\nu\,[R^\alpha{}_{\beta,\lambda\mu}] = 0, \tag{20-8}$$

which is equivalent to (20-4).

If the relation (20-7) is satisfied then one deduces from (20-7) that:

$$u^\rho_{\beta\mu}\, l_\rho - u^\rho_{\beta\rho}\, l_\mu = 0.$$

After multiplying (20-7) by l_α, it thus follows that:

$$l_\alpha[R^\alpha{}_{\beta,\lambda\mu}] = u^\rho_{\beta\rho}\, l_\lambda l_\mu - u^\rho_{\beta\rho}\, l_\lambda l_\mu = 0,$$

i.e., (20-6). Introduce the curvature forms:

$$\Omega^{\alpha}_{\beta} = \tfrac{1}{2} R^{\alpha}{}_{\beta,\lambda\mu} \theta^{\lambda} \wedge \theta^{\mu} .$$

We associate 2-forms that are defined on S to the discontinuities of the curvature tensor:

$$[\Omega^{\alpha}_{\beta}] = \tfrac{1}{2} [\mathrm{R}^{\alpha}{}_{\beta,\lambda\mu}] \theta^{\lambda} \wedge \theta^{\mu} .$$

Relations (20-6) and (20-8) express that *all of the local forms are singular* $[\Omega^{\alpha}_{\beta}]$ *since the – necessarily isotropic – vector* l_α *is a common proper isotropic vector.*

21. General case of a tensor that admits the symmetry type of a curvature tensor. Consider a tensor $H_{\alpha\beta,\lambda\mu}$ ($\neq 0$) at a point x of a Riemannian manifold V_{n+1} that admits a metric of hyperbolic normal type, and which enjoys the same symmetry properties as the curvature tensor:

(21-1) $$H_{\alpha\beta,\lambda\mu} = - H_{\beta\alpha,\lambda\mu} = - H_{\alpha\beta,\mu\lambda} \qquad H_{\alpha\beta,\lambda\mu} = H_{\lambda\mu,\alpha\beta} .$$

Suppose that there exists a vector *la* that satisfies the relations:

(21-2) $$l_\alpha H_{\beta\gamma,\lambda\mu} + l_\beta H_{\gamma\alpha,\lambda\mu} + l_\gamma H_{\alpha\beta,\lambda\mu} = 0$$

and:

(21-3) $$l^\alpha H_{\alpha\beta,\lambda\mu} = 0.$$

If we set:

(21-4) $$\Pi_{\lambda\mu} = \tfrac{1}{2} H_{\alpha\beta,\lambda\mu}\, \theta^\alpha \wedge \theta^\beta$$

then relations (21-2) and (21-3) express that the forms $\Pi_{\lambda\mu}$ are all singular and admit the vector l_α, which, from sec. 7, is necessarily isotropic, as a common isotropic proper vector.

Therefore, *the vector* l_α *that we envision is necessarily isotropic.*

We propose to study the structure of the contracted tensor:

(21-5) $$H_{\alpha\beta} = g^{\rho\sigma} H_{\alpha\rho,\beta\sigma} .$$

Upon contracting the indices α and λ in (21-2), and on account of the symmetry properties, it follows that:

(21-6) $$l^\alpha \mathrm{H}_{\alpha\mu,\beta\gamma} + l_\gamma \mathrm{H}_{\beta\mu} - l_\beta \mathrm{H}_{\gamma\mu} = 0,$$

which is just a consequence of (21-2).

Upon accounting for (21-3) in (21-6), it follows that:

(21-7) $$l_\alpha H_{\beta\mu} - l_\beta H_{\gamma\mu} = 0.$$

If v^μ is an arbitrary vector at x:

$$l_\gamma H_{\beta\mu} v^\mu - l_\beta H_{\gamma\mu} v^\mu = 0.$$

It then results that:

$$H_{\beta\mu} v^\mu = \lambda(\mathbf{v})\, l_\beta,$$

in which $\lambda(\mathbf{v})$ is a linear form in $\mathbf{v}$. There thus exists a scalar τ such that:

$$H_{\alpha\beta} = \tau\, l_\alpha l_\beta . \tag{21-8}$$

b) Conversely, suppose that the tensor $H_{\alpha\beta,\lambda\mu}$ admits the properties (21-1), and the vector l_α satisfies the relations (21-2) and (21-8). We have seen that (21-2) entails (21-6). Taking (21-8) into account, one thus obtains:

$$l^\alpha H_{\alpha\mu,\beta\gamma} = 0,$$

i.e., (21-3), and l^α is necessarily isotropic. We state:

THEOREM. – *If one is given a tensor $H_{\alpha\mu,\beta\gamma}$ at a point x of a Riemannian manifold V_{m+1} that admits a hyperbolic normal metric, and this tensor enjoys the symmetry properties* (21-1), *as well as a vector l_α for which these elements are coupled by relations* (21-2), (21-3), *and* (21-8) *then it is necessary and sufficient that either the relations* (21-2) *and* (21-3) *are satisfied or the relations* (21-2) *and* (21-8); ***l*** *is then isotropic.*

c) If the indices λ and μ are fixed then the 2-form $\Pi_{\lambda\mu}$ at x is singular, with a fundamental isotropic vector ***l***. If we introduce the basis $\varphi^{(i)}$ of singular 2-forms with fundamental vector ***l*** that was defined in sec. 7 then it follows that:

$$H_{\alpha\mu,\beta\gamma} = \sum_i a_{i(\lambda\mu)} \varphi^{(i)}_{\alpha\beta} \qquad (i = 1, \dots, (m-1)).$$

For a fixed i the $a_{i(\lambda\mu)}$ also define a singular 2-form with fundamental vector l and:

$$a_{i(\lambda\mu)} = \sum_j a_{ij} \varphi^{(j)}_{\lambda\mu} .$$

From this, it results that:

$$H_{\alpha\beta,\lambda\mu} = \sum_{i,j} a_{ij} \varphi^{(i)}_{\alpha\beta} \varphi^{(j)}_{\lambda\mu} ,$$

and from the symmetry property $H_{\alpha\beta,\lambda\mu} = H_{\lambda\mu,\alpha\beta}$, it follows that $a_{ij} = a_{ij}$. Therefore:

$$H_{\alpha\beta,\lambda\mu} = \sum_{i,j} a_{ij} (l_\alpha n^{(i)}_\beta - l_\beta n^{(i)}_\alpha)(l_\lambda n^{(j)}_\mu - l_\mu n^{(j)}_\lambda) . \tag{21-9}$$

By contraction, one obtains:

(21-10) $$H_{\alpha\beta} = -\sum_i (a_{ii})\, l_\alpha l_\beta .$$

When $H_{\alpha\beta,\lambda\mu}$ and l^β are given the a_{ij} may not depend on the $\mathbf{n}^{(i)}$. The forms $\varphi^{(i)}$, and consequently the a_{ij}, are invariants of the transformation $\mathbf{n}^{(i)}$ τ $\mathbf{n}^{(i)} + k^{(i)}\mathbf{l}$: the a_{ij} are the components of a symmetric tensor under a rotation of the system of $\mathbf{n}^{(i)}$ in the $(m-1)$-plane that they determine. In particular, when H is given it is possible to choose $\mathbf{n}^{(i)}$ to be a system of proper vectors of the matrix (a_{ij}) with respect to the unit matrix, and to thus annul a_{ij} for $i \neq j$.

If we introduce the symmetric quantities:

(21-11) $$b_{\alpha\lambda} = \sum a_{ij} n_\alpha^{(i)} n_\lambda^{(j)}$$

then, according to (21-9) it follows that:

(21-12) $$H_{\alpha\beta,\lambda\mu} = b_{\alpha\lambda}\, l_\beta l_\mu + b_{\beta\mu}\, l_\alpha l_\lambda - b_{\alpha\mu}\, l_\beta l_\lambda - b_{\beta\lambda}\, l_\alpha l_\mu$$

and one obviously has:

(21-13) $$b_{\alpha\lambda}\, l^\lambda = 0.$$

Therefore, there exist quantities $b_{\alpha\lambda}$ that satisfy (21-13) and are such that the tensor $H_{\alpha\beta,\lambda\mu}$ admits the expression (21-12). We look for the sort of transformation up to which these quantities are defined: To that effect, we multiply (21-12) by v^β, in which v^β denotes an arbitrary vector. If the tensor H is null then it follows that:

$$\{(v^\beta l_\beta)\, b_{\alpha\lambda} - v^\beta b_{\beta\lambda}\ l_\alpha\}\ l_\mu - \{(v^\beta l_\beta)\, b_{\alpha\mu} - v^\beta b_{\beta\mu}\, l_\alpha\} l_\lambda = 0.$$

From this, it results that there exist quantities t_α such that:

$$(v^\beta l_\beta)\, b_{\alpha\lambda} - v^\beta b_{\beta\lambda}\, l_\alpha = t_\alpha l_\lambda (v^\beta l_\beta).$$

Therefore, the $b_{\alpha\lambda}$ necessarily have the form:

$$b_{\alpha\lambda} = t_\alpha l_\lambda + t_\lambda l_\alpha = t_\alpha l_\lambda + t_\lambda l_\alpha + (u_\lambda - t_\lambda)\, l_\alpha .$$

From the symmetry of the $b_{\alpha\lambda}$, $u_\lambda - t_\lambda$ is collinear to l_λ and may annulled by modifying the t_α. When we take (21-13) into account one sees that the $b_{\alpha\lambda}$ that satisfy (21-12), (21-13) are defined up to a transformation of the form:

(21-14) $$b_{\alpha\lambda} \to b_{\alpha\lambda} + t_\alpha l_\lambda + t_\lambda l_\alpha \qquad \text{(with } t_\alpha l^\alpha = 0\text{)}.$$

One will note that by virtue of (21-14) the scalar:

$$e = b_{\alpha\lambda} b^{\alpha\lambda}$$

depends only on the tensor H and the choice of vector $\boldsymbol{l}$. Moreover, if the $b_{\alpha\lambda}$ are defined by starting with (21-11), one has:

$$e = \sum_{i,j} (a_{ij})^2 . \tag{21-15}$$

Therefore, e is strictly positive and is annulled only if $H = 0$.

d) We place ourselves in the case of *general relativity*, for which $m = 3$. Let $\mathbf{e}_0$ be a normal vector that is oriented so that $ds^2 > 0$, $\mathbf{e}_1$, a normal vector that is orthogonal to $\mathbf{e}_0$ and such that one may take:

$$\boldsymbol{l} = \mathbf{e}_0 + \mathbf{e}_1.$$

By starting with $\mathbf{e}_0$, $\mathbf{e}_1$, we construct an orthonormal frame $(\mathbf{e}_\lambda)$ at the point x of V_4. Here, we shall denote indices that take the values 1, 2, 3 by u, v, …, and indices that take the values 2, 3 by A, B, … For this frame, relations (21-3) translate into:

$$H_{0\beta,\lambda\mu} + H_{1\beta,\lambda\mu} = 0. \tag{21-16}$$

We set $\alpha = A$, $\beta = B$, $\gamma = 0$ in (21-12). It follows that:

$$H_{\mathrm{AB},\lambda\mu} = 0. \tag{21-17}$$

If one sets $\alpha = 1$, $\beta = B$, $\gamma = 0$ in the same formula then one has:

$$H_{0\mathrm{B},\lambda\mu} + H_{1\mathrm{B},\lambda\mu} = 0 \tag{21-18}$$

and the set of relations (21-17), (21-18) is equivalent to (21-2). As a result, if $H_{\alpha\beta,\lambda\mu}$ and l_α are isotropic and related by relations (21-3) and (21-8) then the relations (21-2) only amount to (21-17), namely:

$$H_{23,\lambda\mu} = 0.$$

We propose to establish that for $m = 3$, (21-2) is a consequence of (21-3) and (21-8), in which l_α is assumed to be isotropic. From (21-16), $H_{23,10} = 0$. Moreover, from the same relation:

$$H_{A0,B0} = -H_{A0,B1} = H_{A1,B1} .$$

In order to bring (21-8) into the picture, we remark that in an orthonormal frame, because of its signature, one has:

$$H_{\alpha\beta} = H_{\alpha 0,\beta 0} - H_{\alpha u,\beta u}.$$

From (21-8), one thus deduces:

(21-19)
$$\begin{cases} H_{AB} = H_{A0,B0} - H_{A1,B1} - H_{AC,BC} = 0, \\ H_{1B} = -H_{1A,BA} = 0, \\ H_{0B} = -H_{0A,BA} = 0. \end{cases}$$

If we give A, B, C the values 2, 3, in (21-19) then one has:

$$H_{23,23} = 0 \qquad H_{23,31} = H_{23,12}, \qquad H_{23,20} = H_{23,30} = 0;$$

i.e., (21-17); (21-2) is thus established. We thus state:

THEOREM. – *Suppose we are given a tensor $H_{\alpha\beta,\lambda\mu}$ at a point x of a Riemannian manifold V_4 that admits a metric of hyperbolic normal type, a tensor that enjoys the symmetry properties* (21-1), *and an isotropic vector l^α. If these elements satisfy* (21-3) *and* (21-8) *then they also satisfy* (21-2).

e) The tensor $H_{\alpha\beta,\lambda\mu}$ may be identified with a symmetric tensor that is constructed over the space of bivectors at x. This space admits the metric that is defined by the tensor:

(21-20)
$$\gamma_{\alpha\beta,\lambda\mu} = g_{\alpha\lambda}\, g_{\beta\mu} - g_{\alpha\lambda}\, g_{\beta\mu} .$$

In the case of the manifold V_4, of general relativity the space of bivectors in question is six-dimensional. If $\mathbf{e}_\lambda$ is an orthonormal frame at x in V_4 then we agree to set:

(21-21)
$$\begin{cases} \mathbf{e}_2 \wedge \mathbf{e}_3 = \tau_1 \\ \mathbf{e}_3 \wedge \mathbf{e}_1 = \tau_2 \\ \mathbf{e}_1 \wedge \mathbf{e}_2 = \tau_3 \end{cases} \qquad \begin{cases} \mathbf{e}_1 \wedge \mathbf{e}_0 = \tau_4 \\ \mathbf{e}_2 \wedge \mathbf{e}_0 = \tau_5 \\ \mathbf{e}_3 \wedge \mathbf{e}_0 = \tau_6 . \end{cases}$$

The τ_J (I, J = 1, …, 6) define a basis for the space envisioned, which is orthonormal for the metric determined by (21-20). In this basis, the metric tensor (21-20) admits for its only non-null components:

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = 1 \qquad\qquad \gamma_{44} = \gamma_{55} = \gamma_{66} = -1.$$

This gives us the signature of that metric. We propose to study the *matrix representative* of the tensor $H_{\alpha\beta,\lambda\mu}$ in the basis thus introduced. It is a symmetric matrix 6×6, which we denote by (H_{IJ}).

For $e_2 = n^{(1)}$, $e_3 = n^{(2)}$ the two forms $\varphi^{(i)}$ have the components:

$$\begin{cases} \varphi^{(1)}_{23} = 0 & \varphi^{(1)}_{10} = 0 \\ \varphi^{(1)}_{31} = 0 & \varphi^{(1)}_{20} = 1 \\ \varphi^{(1)}_{12} = 1 & \varphi^{(1)}_{30} = 0 \end{cases} \qquad \begin{cases} \varphi^{(2)}_{23} = 0 & \varphi^{(2)}_{10} = 0 \\ \varphi^{(2)}_{31} = -1 & \varphi^{(2)}_{20} = 0 \\ \varphi^{(2)}_{12} = 0 & \varphi^{(2)}_{30} = 1. \end{cases}$$

From (21-9), the matrix (H_{IJ}) has the resulting form:

(21-22) $$(H_{IJ}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & -a_{12} & 0 & -a_{12} & -a_{22} \\ 0 & -a_{12} & a_{11} & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_{12} & a_{11} & 0 & a_{11} & a_{12} \\ 0 & -a_{22} & a_{12} & 0 & a_{12} & a_{22} \end{pmatrix}$$

$H_{\alpha\beta}$ satisfies (21-8) with:

(21-23) $$\tau = -(a_{11} + a_{22}).$$

22. – **Matrix of the curvature discontinuity tensor in general relativity.**

a) We now study the matrix ($[R_{IJ}]$) ($I, J = 1, \ldots, 6$) that represents the tensor $[R_{\alpha\beta,\lambda\mu}]$ at a point x of a hypersurface S at which the curvature tensor $R_{\alpha\beta,\lambda\mu}$ of the spacetime manifold V_4 is discontinuous, relative to an orthonormal frame such $\boldsymbol{l} = \mathbf{e}_0 + \mathbf{e}_1$.

From the results of the preceding paragraph, it is easy to deduce its form since $[R_{\alpha\beta,\lambda\mu}]$ satisfies the relations (21-2) and (21-3). The contracted tensor $[R_{\alpha\beta}]$ is assumed to be null here; it then results from (21-23) that:

$$a_{11} + a_{22} = 0.$$

Set:

$$a_{11} = -a_{22} = \sigma \qquad a_{12} = \rho.$$

One thus obtains the following form ([5]) for the matrix ($[R_{IJ}]$):

(22-1) $$([R_{IJ}]) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sigma & -\rho & 0 & -\rho & \sigma \\ 0 & -\rho & \sigma & 0 & \sigma & \rho \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho & \sigma & 0 & \sigma & \rho \\ 0 & \sigma & \rho & 0 & \rho & -\sigma \end{pmatrix}.$$

When one is given the tensor $[R_{\alpha\beta,\lambda\mu}]$ one may naturally make $\rho = 0$ by a choice of convenient choice of vectors $\mathbf{e}_2$ and $\mathbf{e}_3$. According to (21-9), (22-1) may be translated into:

(22-2) $$[R_{IJ}] = \sigma(\varphi_{\mathrm{I}}^{(1)}\varphi_{\mathrm{J}}^{(1)} - \varphi_{\mathrm{I}}^{(1)}\varphi_{\mathrm{J}}^{(2)}) + \rho(\varphi_{\mathrm{I}}^{(1)}\varphi_{\mathrm{J}}^{(1)} + \varphi_{\mathrm{J}}^{(1)}\varphi_{\mathrm{I}}^{(2)}) .$$

[5] This form was obtained by PIRANI [2] by using the local coordinates that were introduced in sec. 8.

b) We have seen (cf. (17-8)) that:

(22-3) $$[R_{\alpha\beta,\lambda\mu}] = \tfrac{1}{2}\,(a_{\alpha\mu}\, l_\beta l_\lambda + a_{\alpha\mu}\, l_\beta l_\lambda - a_{\alpha\lambda}\, l_\beta l_\mu - a_{\beta\mu}\, l_\alpha l_\lambda)\ ,$$

in which the $a_{\alpha\lambda}$ are restricted only by condition (18-12), namely:

(22-4) $$a_{\alpha\lambda}\, l^\lambda = \frac{a}{2}\, l_\alpha,$$

and are defined up to the transformation:

(22-5) $$a_{\alpha\lambda} \to a_{\alpha\lambda} + t_\alpha l_\lambda + t_\alpha l_\lambda\ .$$

a transforms according to:

$$a \to a + 2t_\alpha l^\alpha,$$

and it is possible to choose t_α in such a fashion as to annul a. Suppose that this is the case; for these special $a_{\alpha\lambda}$ the transformation (22-5) is restricted by the condition $t_\alpha l^\alpha = 0$; if we then set:

$$b_{\alpha\lambda} = -\tfrac{1}{2}\, a_{\alpha\lambda}$$

then (22-3) take the form (21-12), and, from (22-4), the $b_{\alpha\lambda}$ satisfy (21-13) precisely. It then results, in particular, that *any tensor* at the point x of V_4 satisfies the symmetry properties (21-1) and the relations (21-2), and (21-3) might be the discontinuity tensor at x for the curvature tensor upon crossing the 3-plane that is tangent to the elementary cone at x along $\boldsymbol{l}$.

One immediately verifies that the scalar ([6]):

$$a_{\alpha\lambda}\, a^{\alpha\lambda} - \tfrac{1}{2}\, a^2$$

is invariant under the general transformation (22-5). As a result, one has for the tensor $[R_{\alpha\beta,\lambda\mu}]$:

$$e = \tfrac{1}{4}\left(a_{\alpha\lambda} a^{\alpha\lambda} - \tfrac{1}{2} a^2\right) = b_{\alpha\lambda}\, b^{\alpha\lambda} > 0.$$

23. – Differential relations for the discontinuities of the curvature tensor. We return to a manifold V_{m+1} that is endowed a metric of hyperbolic normal type that satisfies the Einstein equations with *null right-hand side*. If the curvature of V_{m+1} is discontinuous upon crossing a hypersurface S then the tensor $[R_{\alpha\beta,\lambda\mu}]$ satisfies an interesting differential relation on S that is analogous to (9-11).

[6] The introduction of this quantity is due to STELLMACHER.

Let $f(x^\alpha) = 0$ be the local equation for S, where f is assumed to be of class C^2. For $l_\alpha = \partial_\alpha f$, one has:

(23-1) $$\nabla_\alpha l_\beta - \nabla_\beta l_\alpha = 0.$$

a) For a neighborhood U, we adopt local coordinates and notations that are identical to the ones in sec. 9. In particular, we have $\nabla_0 l^0 = 0$ in these coordinates. The tensor $[R_{\alpha\beta,\lambda\mu}]$ satisfies the relations:

(23-2) $$l_\rho [R_{\alpha\beta,\lambda\mu}] + l_\lambda [R_{\alpha\beta,\mu\rho}] + l_\mu [R_{\alpha\beta,\rho\lambda}] = 0$$

and:

(23-3) $$l_\rho [R_{\alpha\beta,}{}^\rho{}_\mu] = 0$$

on S. In these local coordinates, (23-2) and (23-3) can be written:

(23-4) $$[R_{\alpha\beta,uv}] = 0$$

(23-5) $$[R_{\alpha\beta,}{}^0{}_\mu] = 0,$$

respectively.

b) Since the relation (23-2) is satisfied on S, by differentiating on this hypersurface, it follows that:

$$\nabla_u (l_\rho [R_{\alpha\beta,\lambda\mu}]) + \nabla_u (l_\lambda [R_{\alpha\beta,\mu}{}^\rho]) + \nabla_u (l_\mu [R_{\alpha\beta,}{}^\rho{}_\lambda]) = 0.$$

If we set ρ equal to u and sum then one obtains a relation that, since $\nabla_0 l^0 = 0$, may be written:

(23-6) $$l^\rho \nabla_\rho [R_{\alpha\beta,\lambda\mu}] + (\nabla_\rho l^\rho)[R_{\alpha\beta,\lambda\mu}] + Q_{\alpha\beta,\lambda\mu} = 0,$$

in which we have set:

(23-7) $$Q_{\alpha\beta,\lambda\mu} = \nabla_\rho l_\lambda [R_{\alpha\beta,\mu}{}^\rho] + \nabla_\rho l_\mu [R_{\alpha\beta,}{}^\rho{}_\lambda] + l_\lambda [\nabla_u R_{\alpha\beta,\lambda}{}^u] + l_\mu [\nabla_u R_{\alpha\beta,}{}^u{}_\lambda].$$

Since $R_{\alpha\beta} = 0$, one knows that:

$$\nabla_\rho [R_{\alpha\beta,}{}^\rho{}_\lambda] = 0.$$

As a result, $Q_{\alpha\beta,\lambda\mu}$ may be put into the form:

$$Q_{\alpha\beta,\lambda\mu} = [\nabla_\lambda l_\rho R_{\alpha\beta,\mu}{}^\rho + \nabla_\mu l_\rho R_{\alpha\beta,}{}^\rho{}_\lambda] - l_\lambda [\nabla_0 R_{\alpha\beta,\lambda}{}^0] - l_\mu [\nabla_0 R_{\alpha\beta,}{}^0{}_\lambda].$$

Now:

$$\nabla_\lambda l_\rho \cdot R_{\alpha\beta,\mu}{}^\rho + \nabla_\mu l_\rho \cdot R_{\alpha\beta,}{}^\rho{}_\lambda = \nabla_\lambda (l_\rho R_{\alpha\beta,\mu}{}^\rho) + \nabla_\mu(l_\rho R_{\alpha\beta,}{}^\rho{}_\lambda) - l_\rho(\nabla_\lambda R_{\alpha\beta,\mu}{}^\rho + \nabla_\mu R_{\alpha\beta,}{}^\rho{}_\lambda).$$

From the BIANCHI identity, one thus deduces that:

$$Q_{\alpha\beta,\lambda\mu} = l^\rho \nabla_\rho [R_{\alpha\beta,\lambda\mu}] + [\nabla_\lambda (l_\rho R_{\alpha\beta,\mu}{}^\rho) + \nabla_\mu (l_\rho R_{\alpha\beta,}{}^\rho{}_\lambda)] - l_\lambda [\nabla_0 R_{\alpha\beta,\mu}{}^0] - l_\mu [\nabla_0 R_{\alpha\beta,}{}^0{}_\lambda].$$

If we set $\lambda = u$ in the second term on the right-hand side then we have:

$$[\nabla_u (l_\rho R_{\alpha\beta,\mu}{}^\rho)] = \nabla_u (l_\rho [R_{\alpha\beta,\mu}{}^\rho]) = 0.$$

As a result:

$$Q_{\alpha\beta,uv} = l^\rho \nabla_\rho [R_{\alpha\beta,uv}].$$

Similarly, when $\lambda = 0$ and $\mu = u$:

$$[\nabla_0 (l_\rho R_{\alpha\beta,u}{}^\rho)] = [\nabla_0 l^\rho R_{\alpha\beta,u\rho} + \nabla_\rho l_0 R_{\alpha\beta,u}{}^\rho] = [\nabla_0 R_{\alpha\beta,u}{}^0]$$

since $\nabla_0 l^0 = 0$. One thus obtains:

$$Q_{\alpha\beta,\lambda\mu} = l^\rho \nabla_\rho [R_{\alpha\beta,\lambda\mu}].$$

If we substitute this into (23-6) then one sees that the tensor $[R_{\alpha\beta,\lambda\mu}]$ satisfies the differential relation ([7]):

$$2l^\rho \nabla_\rho [R_{\alpha\beta,\lambda\mu}] + (l^\rho \nabla_\rho)[R_{\alpha\beta,\lambda\mu}] = 0 \tag{23-8}$$

on S, which entails consequences that are analogous to (9-11) as far as $[R_{\alpha\beta,\lambda\mu}]$ is concerned: if the tensor $[R_{\alpha\beta,\lambda\mu}]$ is annulled at a point x of S then it is annulled all along the isotropic geodesic that issues from x and is situated on S.

Let Σ be spatially oriented hypersurface that cuts S ($x^0 = 0$) along a 2-surface U. We choose CAUCHY data $(g_{\alpha\beta})_\Sigma$, $(\partial_\lambda g_{\alpha\beta})_\Sigma$ on Σ such that the second derivatives experience discontinuities $[\partial_{00} g_{\alpha\beta}]_4 = (a_{\alpha\beta})_4$ when crossing Σ, in which the $(a_{\alpha\beta})_U$ are restricted only by the condition (18-12):

$$\left\{\left(a_{\alpha\beta} - \frac{a}{2} g_{\alpha\beta}\right) l^\beta\right\}_U = 0.$$

This amounts to being given a tensor $[R_{\alpha\beta,\lambda\mu}]_U$ at the points of U that admits $(l^\rho)_U$ for a fundamental vector and whose contracted tensor is null. A solution to the EINSTEIN equation $R_{\alpha\beta} = 0$ corresponds to the CAUCHY data in question such that the curvature tensor experiences a discontinuity $[R_{\alpha\beta,\lambda\mu}]$ upon crossing S. The tensor $[R_{\alpha\beta,\lambda\mu}]$ is necessarily the solution to (23-8) that corresponds to the initial data $[R_{\alpha\beta,\lambda\mu}]_U$.

c) We have seen that the tensor $[R_{\alpha\beta,\lambda\mu}]$ may be written:

$$[R_{\alpha\beta,\lambda\mu}] = b_{\alpha\lambda} l_\beta l_\mu + b_{\beta\mu} l_\alpha l_\lambda - b_{\beta\mu} l_\alpha l_\lambda - b_{\beta\lambda} l_\alpha l_\mu$$

in which the $b_{\alpha\lambda}$ satisfy:

$$b_{\alpha\lambda} l^\lambda = 0.$$

Set:

$$c_{\alpha\lambda} = 2l^\rho \nabla_\rho b_{\alpha\lambda} + (\nabla_\rho l^\rho)\, b_{\alpha\lambda}.$$

[7] This relation was established independently by A. TRAUTMAN [1].

If one takes into account that $l^\rho \nabla_\rho l_\alpha = 0$ then one sees that (23-8) may be written:

$$c_{\alpha\lambda} l_\beta l_\mu + c_{\beta\mu} l_\alpha l_\lambda - c_{\alpha\mu} l_\beta l_\lambda - c_{\beta\lambda} l_\alpha l_\mu = 0. \tag{23-9}$$

If one takes the product (23-9) with $b^{\alpha\lambda}$ then one obtains:

$$b^{\alpha\lambda} c_{\alpha\lambda} = 0,$$

that is:

$$2l^\rho b^{\alpha\beta} \nabla_\rho b_{\alpha\beta} + (\nabla_\rho l^\rho)\, b^{\alpha\beta} b_{\alpha\beta} = 0. \tag{23-10}$$

Therefore, the scalar:

$$e = b^{\alpha\beta} b_{\alpha\beta} = \frac{1}{4}\left(a^{\alpha\beta} a_{\alpha\beta} - \frac{1}{2} a^2 \right) > 0$$

satisfies the conservation identity:

$$\nabla_\rho (e\, l^\rho) = 0, \tag{23-11}$$

a relation that is similar to (10-1). We introduce the fourth order tensor:

$$\tau_{\alpha\beta,\lambda\mu} = e\, l_\alpha l_\beta l_\lambda l_\mu . \tag{23-12}$$

If $\boldsymbol{l}$ is subjected to the transformation $l_\alpha \to \lambda l_\alpha$ then $b_{\alpha\lambda} \to \lambda^{-2} b_{\alpha\lambda}$ and $e \to \lambda^{-4} e$. The tensor τ thus depends only upon the tensor $[R_{\alpha\beta,\lambda\mu}]$, and, since $l^\rho \nabla_\rho l_\alpha = 0$, (23-11) expresses the idea that τ is conservative:

$$\nabla_\alpha \tau^\alpha{}_{\beta,\lambda\mu} = 0. \tag{23-13}$$

Furthermore, from expression for $[R_{\alpha\beta,\lambda\mu}]$ in terms of $b_{\alpha\lambda}$ one immediately verifies that:

$$\tau_{\alpha\beta,\lambda\mu} = \tfrac{1}{2} \{ [R^\rho{}_{\beta,}{}^\sigma{}_\lambda][R_{\rho\beta,\sigma\mu}] + [R^\rho{}_{\alpha,}{}^\sigma{}_\mu][R_{\rho\beta,\sigma\lambda}] \}. \tag{23-14}$$

23 bis. – **Case in which there exists an electromagnetic field.** In the case where there exists an electromagnetic field in V_4 that satisfies the vacuum MAXWELL equations and relates to the gravitational field through the EINSTEIN equations:

$$R_{\alpha\beta} = \chi \tau_{\alpha\beta} \qquad (\tau_{\alpha\beta} = \text{MAXWELL tensor}), \tag{23 bis-1}$$

suppose that the derived tensors of the electromagnetic field and the curvature tensor are discontinuous upon crossing the hypersurface S.

a) With the same hypotheses and notations as in sec. 9, formula (9-11) may be modified: here, the RICCI identity gives:

$$[\nabla_\rho \nabla_\sigma F_{\beta\gamma} - \nabla_\sigma \nabla_\rho F_{\beta\gamma}] = -[R^\lambda{}_{\beta,\rho\sigma}]\, F_{\lambda\gamma} - [R^\lambda{}_{\gamma,\rho\sigma}]\, F_{\beta\lambda} .$$

From this, it easily results that:

$$[l_\beta \nabla_\rho \nabla_\sigma F_\gamma{}^\beta + l_\gamma \nabla_\sigma \nabla_\rho F^\rho{}_\beta] = - l_\lambda F^{\lambda\rho}[R_{\rho\sigma,\beta\gamma}]$$

and:

$$[l_\rho \nabla_\beta \nabla_\sigma F_\gamma{}^\rho + l_\rho \nabla_\gamma \nabla_\sigma F^\rho{}_\beta] = - l^\rho \nabla_\rho [\nabla_\sigma F_{\beta\gamma}] + l_\lambda F^{\lambda\rho}[R_{\rho\sigma,\beta\gamma}].$$

By the same argument as in sec. 9, one then deduces:

(23 bis-2) $$2l^\rho \nabla_\rho [\nabla_\sigma F_{\beta\gamma}] + (l_\rho \nabla^\rho)[\nabla_\sigma F_{\beta\gamma}] = 2l_\lambda F^{\lambda\rho}[R_{\rho\sigma,\beta\gamma}].$$

In (23 bis-2), if we set:

(23 bis-3) $$[\nabla_\sigma F_{\beta\gamma}] = \varphi_{\beta\gamma} l_\sigma = (l_\beta b_\gamma - l_\gamma b_\beta)\, l_\sigma$$

and:

(23 bis-4) $$[R_{\rho\sigma,\beta\gamma}] = b_{\rho\beta} l_\sigma l_\gamma + b_{\sigma\gamma}\, l_\rho l_\beta - b_{\rho\gamma} l_\sigma l_\beta - b_{\sigma\beta} l_\rho l_\gamma \qquad (b_\beta{}^\beta = 0)$$

in (23 bis-2) then it follows that:

(23 bis-5) $$2l^\rho \nabla_\rho (l_\beta b_\gamma - l_\gamma b_\beta) + (l_\rho \nabla^\rho)(l_\beta b_\gamma - l_\gamma b_\beta) = 2l_\lambda F^{\lambda\rho}(b_{\mu\beta} l_\gamma - b_{\rho\gamma} l_\beta).$$

Set:

(23 bis-6) $$e_{em} = - b^\beta b_\beta > 0 \qquad\qquad e_\sigma = b^{\alpha\lambda} b_{\alpha\lambda} > 0.$$

If we multiply (23 bis-5) by b^β then it follows that:

(23 bis-7) $$\nabla_\rho (e_{em}\, l^\rho) = 2l_\lambda F^{\lambda\rho} b^\sigma b_{\rho\sigma}.$$

b) Now recall the argument of sec. 23, while noting that in the present case one has (23 bis-1) instead of $R_{\alpha\beta} = 0$. It follows that:

$$[\nabla R_{\alpha\beta,\mu}{}^\rho] = -\chi\, [\nabla_\alpha \tau_{\beta\mu} - \nabla_\beta \tau_{\alpha\mu}].$$

As a result, instead of the formula (23-8) one now substitutes:

(23 bis-8) $$2l^\rho \nabla_\rho [R_{\alpha\beta,\lambda\mu}] + (l_\rho \nabla^\rho)[R_{\alpha\beta,\lambda\mu}]$$
$$= -\chi\, l_\lambda [\nabla_\alpha \tau_{\beta\mu} - \nabla_\beta \tau_{\alpha\mu}] - \chi\, l_\mu [\nabla_\alpha \tau_{\beta\lambda} - \nabla_\beta \tau_{\alpha\lambda}].$$

Now:

$$[\nabla_\beta \tau_{\alpha\lambda}] = \tfrac{1}{2}\, g_{\alpha\lambda} F^{\rho\sigma}[\nabla_\beta F_{\rho\sigma}] - F_\lambda{}^\rho [\nabla_\beta F_{\alpha\rho}] - F_{\alpha\rho}[\nabla_\beta F_\lambda{}^\rho],$$

namely:

(23 bis-9) $$[\nabla_\beta \tau_{\alpha\lambda}] = \tfrac{1}{2}\, g_{\alpha\lambda} F^{\rho\sigma} \varphi_{\rho\sigma} l_\beta - F_\lambda{}^\rho \varphi_{\alpha\rho} l_\alpha - F_{\alpha\rho} \varphi_\lambda{}^\rho l_\beta.$$

If we substitute the expressions for $[R_{\alpha\beta,\lambda\mu}]$ and $[\nabla_\beta \tau_{\alpha\lambda}]$ in (23 bis-8) and multiply by $b^{\alpha\lambda}$ then we obtain:

(23 bis-10) $$\nabla_\rho (e_g\, l^\rho) = - 2\chi\, l_\lambda F^{\lambda\rho} b^\sigma b_{\rho\sigma}.$$

From (23 bis-7) and (23 bis-10), one concludes:

(23 bis-11) $$\nabla_\rho\{e_g + e_{em}\} = 0,$$

which can be interpreted as the conservation of the total "energy of the discontinuity" relative to the gravitational field and the electromagnetic field. The fourth order tensor that is obtained by combining (23-14) and (10-2) is thus conservative.

III. Gravitational radiation in general relativity.

24. **Notion of pure gravitational radiation.**

a) From the study that was made in part II, it results that in general relativity one must focus on metrics for which there exists a vector l_α such that curvature tensor $R_{\alpha\beta,\lambda\mu}$ satisfies the relations:

(24-1) $$l_\alpha R_{\beta\gamma,\lambda\mu} + l_\beta R_{\gamma\alpha,\lambda\mu} + l_\gamma R_{\alpha\beta,\lambda\mu} = 0$$

and:

(24-2) $$l^\alpha R_{\alpha\beta,\lambda\mu} = 0.$$

If $R_{\alpha\beta,\lambda\mu}$ is not identically null then l_α *is necessarily isotropic.* The RICCI tensor of the metric then has the form:

(24-3) $$R_{\alpha\beta} = \tau\, l_\alpha l_\beta.$$

If this is true at a point x of the manifold V_4 then we say that the metric corresponds to a state of *pure total radiation* at this point. If, in addition, the RICCI tensor $R_{\alpha\beta}$ is null then we say that we have a state of *pure gravitational radiation*.

We shall ultimately come upon an example in which this may possibly be the case at all of the points of a four-dimensional domain of the manifold V_4. In such a domain, l_α defines a field of generatrices of elementary cones and, as a result, a field of 3-planes that are tangent to these cones along the generatrices. If this field of planes is completely integrable then we say that *the radiation in question is of integrable type.*

b) Suppose that there exists a vector field l_α on the manifold V_{m+1} such that (24-1) and (24-2) are satisfied. From the BIANCHI identity:

$$\nabla_\rho R_{\alpha\beta,\lambda\mu} + \nabla_\alpha R_{\beta\rho,\lambda\mu} + \nabla_\beta R_{\rho\alpha,\lambda\mu} = 0,$$

if one takes the contracted product with l^ρ then, on account of (24-2), one deduces:

(24-4) $$l^\rho \nabla_\rho R_{\alpha\beta,\lambda\mu} - \nabla_\alpha l_\rho R_\beta{}^\rho{}_{,\lambda\mu} - \nabla_\beta l_\rho R^\rho{}_{\alpha,\lambda\mu} = 0,$$

a relation that plays the same role here that (11-8) or $L(\boldsymbol{l})F = 0$ plays in the case of electromagnetic radiation.

Suppose that $R_{\alpha\beta} = 0$. By contracted differentiation of:

$$l_\rho R_{\alpha\beta,\lambda\mu} + l_\alpha R_{\beta\rho,\lambda\mu} + l_\beta R_{\rho\alpha,\lambda\mu} = 0,$$

one obtains, since $\nabla_\rho \mathrm{R}_\alpha{}^\rho{}_{,\lambda\mu} = 0$:

(24-5) $$\nabla_\rho(l^\rho R_{\alpha\beta,\lambda\mu}) + \nabla_\rho l_\alpha R_\beta{}^\rho{}_{,\lambda\mu} + \nabla_\rho l_\beta R^\rho{}_{\alpha,\lambda\mu} = 0,$$

which plays the same role here as (11-9).

Suppose, in addition, that the field is *of integrable type*: $\nabla_\alpha l_\rho = \nabla_\rho l_\alpha$. By adding (24-4) and (24-5) one obtains the relations:

(24-6) $$2\, l^\rho \nabla_\rho R_{\alpha\beta,\lambda\mu} + (\nabla_\rho l^\rho) R_{\alpha\beta,\lambda\mu} = 0$$

which is similar to (14-4).

c) If the RICCI tensor $R_{\alpha\beta}$ of V_{m+1} is $\neq 0$ then it is clear that the trajectories of the vector field that is defined by l_α are isotropic geodesics. Indeed, one necessarily $R = 0$, and, as a result:

$$S_{\alpha\beta} = \tau\, l_\alpha l_\beta \qquad (\tau \neq 0).$$

From the conservation identities, one deduces:

$$\nabla_\alpha S^\alpha{}_\beta = \nabla_\alpha(\tau\, l_\alpha)\, l_\beta + \tau\, l^\alpha \nabla_\alpha l_\beta = 0.$$

Therefore, for $\tau \neq 0$, $l^\alpha \nabla_\alpha l_\beta$ is precisely collinear to l_β.

It is easy to extend this result to the general case. To that effect, we use (24-4), and introduce the tensor that is defined by:

(24-7) $$P_{\alpha\beta,\lambda\mu} = l^\rho \nabla_\rho R_{\alpha\beta,\lambda\mu},$$

so it has the same symmetry type as the curvature tensor. From (24-4), one deduces that the tensor $P_{\alpha\beta,\lambda\mu}$ satisfies the relations:

(24-8) $$l_\nu P_{\alpha\beta,\lambda\mu} + l_\lambda P_{\alpha\beta,\mu\nu} + l_\mu P_{\alpha\beta,\nu\lambda} = 0$$

and:

(24-9) $$l^\lambda P_{\alpha\beta,\lambda\nu} = 0.$$

On the other hand, differentiate (24-1) and take the contracted product with l_ρ. If we set $l^\rho \nabla_\rho l_\alpha = u_\alpha$ then one obtains:

$$u_\alpha R_{\alpha\beta,\lambda\mu} + u_\beta R_{\gamma\alpha,\lambda\mu} + u_\gamma R_{\alpha\beta,\lambda\mu} + l_\alpha P_{\beta\gamma,\lambda\mu} + l_\beta P_{\gamma\alpha,\lambda\mu} + l_\gamma P_{\alpha\beta,\lambda\mu} = 0.$$

If we proceed as we did with (24-2) then we have:

$$u^\alpha R_{\alpha\beta,\lambda\mu} + l^\alpha P_{\alpha\beta,\lambda\mu} = 0.$$

From (24-8) and (24-9), one thus deduces that the vector u_α and the curvature tensor also satisfy the relations (24-1) and (24-2). Since the curvature tensor is not identically null, u_α is necessarily isotropic; however, it is orthogonal to l_α. Hence, it may not be collinear. Therefore:

THEOREM. – *If a vector field la exists that satisfies relations* (24-1) *and* (24-2) *on a Riemannian manifold* V_{m+1} *whose metric of normal hyperbolic type has non-null curvature then the trajectories of that vector field are null-length geodesics of that metric.*

25. The conservation identity. We propose to establish a conservation identity for *pure gravitational radiation* that is analogous to (11-11). To that effect, we adapt the method that was indicated in sec. 11*d* to the gravitational case.

By virtue of (24-1) and (24-2), we may set:

$$R_{\alpha\beta,\lambda\mu} = b_{\alpha\lambda}\, l_\beta l_\mu + b_{\beta\lambda}\, l_\alpha l_\lambda - b_{\alpha\mu}\, l_\beta l_\lambda - b_{\beta\lambda}\, l_\alpha l_\mu$$

in which the $b_{\alpha\lambda}$ satisfy:

$$b_{\alpha\lambda}\, l^\lambda = 0$$

and l is restricted by:

(25-1) $$l^\rho\, \nabla_\rho l_\alpha = 0.$$

Set $b_{\alpha\lambda} = l^\rho\, \nabla_\rho b_{\alpha\lambda}$. If we substitute the preceding expression for $R_{\alpha\beta,\lambda\mu}$ in (24-4) then we obtain:

$$d_{\alpha\lambda}\, l_\beta l_\mu + d_{\beta\lambda}\, l_\alpha l_\lambda - d_{\alpha\mu}\, l_\beta l_\lambda - d_{\beta\lambda}\, l_\alpha l_\mu - \nabla_\alpha l_\rho (b^\rho{}_\mu\, l_\beta l_\lambda - b^\rho{}_\lambda\, l_\beta l_\mu)$$
$$- \nabla_\beta l_\rho (b^\rho{}_\lambda\, l_\alpha l_\mu - b^\rho{}_\mu\, l_\alpha l_\lambda) = 0.$$

Upon multiplying by $b^{\alpha\lambda}$ one obtains the relations:

(25-2) $$l_\rho b^{\alpha\beta}\, \nabla_\rho b_{\alpha\beta} + \nabla_\sigma l_\rho b^\rho{}_\lambda\, b^{\sigma\lambda} = 0.$$

Similarly, if we substitute the expression for $R_{\alpha\beta,\lambda\mu}$ into (24-5) then one has:

$$(d_{\alpha\lambda} + (l_\rho \nabla^\rho)\, b_{\alpha\lambda})\, l_\beta l_\mu + (d_{\beta\mu} + (l_\rho \nabla^\rho)\, b_{\beta\mu})\, l_\mu\, l_\lambda - (d_{\alpha\mu} + (l_\rho \nabla^\rho)\, b_{\alpha\mu})\, l_\beta l_\lambda$$
$$- (d_{\beta\lambda} + (l_\rho \nabla^\rho)\, b_{\beta\lambda})\, l_\alpha\, l_\mu + \nabla_\rho l_\alpha (b^\rho{}_\mu\, l_\beta l_\lambda - b^\rho{}_\lambda\, l_\beta l_\mu) + \nabla_\rho l_\beta (b^\rho{}_\lambda\, l_\beta l_\mu - b^\rho{}_\mu\, l_\beta l_\lambda) = 0.$$

After multiplying this by $b^{\alpha\lambda}$ one obtains:

(25-3) $$l^\rho b^{\alpha\beta} \nabla_\rho b_{\alpha\beta} + (l_\rho \nabla^\rho)\, b^{\alpha\beta} b_{\alpha\beta} - \nabla_\rho l_\sigma b^\rho{}_\lambda\, b^{\sigma\lambda} = 0.$$

Adding corresponding sides of equations (25-1) and (25-2) gives:

$$2\, l^\rho b^{\alpha\beta} \nabla_\rho b_{\alpha\beta} + (l_\rho \nabla^\rho)\, b^{\alpha\beta} b_{\alpha\beta} = 0$$

namely:

(25-4) $$\nabla_\rho (e\, l^\rho) = 0 \quad \text{with} \quad e = b^{\alpha\beta} b_{\alpha\beta} > 0.$$

If we introduce the fourth order tensor:

$$\tau_{\alpha\beta\lambda\mu} = e\, l_\alpha l_\beta l_\lambda l_\mu\,,$$

which depends only upon the curvature tensor, one sees, on account of (25-1), (25-4), that τ is conservative.

26. The Bel tensor.

1) Consider a tensor $K_{\alpha\beta,\gamma\delta}$ at the point x of the Riemannian manifold V_4 that satisfies the symmetry properties (21-1) of the curvature tensor, is such that its contracted tensor satisfies:

(26-1) $$K_{\alpha\beta} = \lambda\, g_{\alpha\beta}.$$

We associate it with the tensor:

(26-2) $$K^*_{\alpha\beta,\gamma\delta} = \frac{1}{2}\,\eta_{\alpha\beta\gamma\delta}\,K^{\rho\sigma}{}_{,\gamma\delta}.$$

It is easy to show that under the hypothesis (26-1) the tensor (26-2) also enjoys the symmetry properties (21-1), i.e., it satisfies:

(26-3) $$K^*_{\alpha\beta,\gamma\delta} = K^*_{\gamma\delta,\alpha\beta}.$$

Indeed, if S denotes the summation after cyclic permutation of the three indices α, β, γ, which are assumed to be distinct, then we obtain:

$$S\,K^*_{\alpha\beta,\gamma\delta} = \frac{1}{2}\,S\eta_{\alpha\beta\gamma\delta}\,K^{\rho\sigma}{}_{,\gamma\delta} = \eta_{\alpha\beta\gamma\delta}\,K^{\sigma}{}_{\delta}.$$

One thus sees that for any $\alpha, \beta, \gamma, \delta$ one has:

(26-4) $$K^*_{\alpha\beta,\gamma\delta} + K^*_{\beta\gamma,\alpha\delta} + K^*_{\gamma\alpha,\beta\delta} = \lambda\,\eta_{\alpha\beta\gamma\delta}.$$

(26-4) may also be written:

$$K^*_{\beta\alpha,\delta\gamma} + K^*_{\alpha\delta,\beta\gamma} + K^*_{\delta\beta,\alpha\gamma} = \lambda\,\eta_{\alpha\beta\gamma\delta}.$$

Adding both sides of these last two equations gives:

(26-5) $$K^*_{\alpha\beta,\gamma\delta} = L_{\alpha\beta,\gamma\delta} + L_{\alpha\delta,\gamma\beta} + \lambda\,\eta_{\alpha\beta\gamma\delta},$$

in which one has set:

$$2L_{\alpha\beta,\gamma\delta} = K^*_{\alpha\beta,\gamma\delta} + K^*_{\gamma\delta,\alpha\beta}.$$

By exchanging α and γ, β, and δ in (26-5), one sees that (26-3) is satisfied.

Now consider the tensor:

$$K^{**}_{\alpha\beta,\gamma\delta} = \tfrac{1}{4}\,\eta^*_{\alpha\beta\lambda\mu}\,\eta_{\gamma\delta\rho\sigma}\,K^{\lambda\mu,\rho\sigma} = \tfrac{1}{2}\,\eta_{\alpha\beta\lambda\mu}\,K^*_{\gamma\delta,}{}^{\lambda\mu}.$$

From (26-3), it results that:

$$K^{**}_{\alpha\beta,\gamma\delta} = \tfrac{1}{4}\,\eta_{\alpha\beta\lambda\mu}\,\eta^{\lambda\mu\rho\sigma}K_{\gamma\delta,\rho\sigma} = -\tfrac{1}{2}\,\varepsilon^{\rho\sigma}_{\alpha\beta}\,K_{\rho\sigma,\gamma\delta},$$

that is:

$$K^{**}_{\alpha\beta,\gamma\delta} = -K_{\alpha\beta,\gamma\delta}. \tag{26-6}$$

One may derive a formula from relation (26-6) that will be important in what follows. To the end, we form the quantity:

$$K^{**\alpha\beta,\gamma\lambda}\;K^{**}_{\alpha\beta,\gamma\mu} = \tfrac{1}{16}\,\eta^{\alpha\beta\rho\sigma}\,\eta^{\gamma\lambda\tau u}\,\eta_{\alpha\beta\phi\psi}\,\eta_{\gamma\mu\chi\pi}\,K_{\rho\sigma,\tau u}\,K^{\gamma\psi,\chi\pi}.$$

Upon introducing the KRONECKER indicators, one obtains:

$$K^{**\alpha\beta,\gamma\lambda}\;K^{**}_{\alpha\beta,\gamma\mu} = \tfrac{1}{8}\,\varepsilon^{\rho\sigma}_{\phi\psi}\,\varepsilon^{\lambda\tau u}_{\mu\chi\pi}\,K_{\rho\sigma,\tau u}\,K^{\phi\psi,\chi\pi} = \tfrac{1}{4}\,\varepsilon^{\lambda\alpha\beta}_{\mu\gamma\delta}\,K_{\rho\sigma,\alpha\beta}K^{\rho\sigma,\gamma\delta},$$

namely:

$$K^{**\alpha\beta,\gamma\lambda}\;K^{**}_{\alpha\beta,\gamma\mu} = \tfrac{1}{2}\,\delta^{\lambda}_{\mu}\,K^{\rho\sigma,\alpha\beta}K_{\rho\sigma,\alpha\beta} - K^{\rho\sigma,\lambda\delta}K_{\rho\sigma,\mu\delta}.$$

One thus obtains:

$$K^{**\alpha\beta,\gamma\lambda}\;K^{**}_{\alpha\beta,\gamma\mu} + K^{\alpha\beta,\gamma\lambda}\,K_{\alpha\beta,\gamma\mu} = \tfrac{1}{2}\,\delta^{\lambda}_{\mu}\,K^{\alpha\beta,\gamma\delta}\,K_{\alpha\beta,\gamma\delta}. \tag{26-7}$$

From (26-6), it then follows that:

$$K^{\alpha\beta,\gamma\lambda}\,K_{\alpha\beta,\gamma\mu} = \tfrac{1}{4}\,\delta^{\lambda}_{\mu}\,K^{\alpha\beta,\gamma\delta}K_{\alpha\beta,\gamma\delta}. \tag{26-8}$$

2) If we are given a *unitary* vector **u** then we can associate the tensor $K_{\alpha\beta,\gamma\delta}$ with two symmetric tensor:

$$E_{\alpha\beta}(\mathbf{u}) = K_{\alpha\rho,\beta\sigma}\,u^{\rho}u^{\sigma}, \qquad H_{\alpha\beta}(\mathbf{u}) = -\,K^{*}_{\alpha\rho,\beta\sigma}\,u^{\rho}u^{\sigma}, \tag{26-9}$$

which obviously satisfy:

$$E_{\alpha\beta}u^{\beta} = 0 \qquad H_{\alpha\beta}u^{\beta} = 0.$$

The data of these two tensors completely determines the tensor $K_{\alpha\beta,\gamma\delta}$, which is assumed to satisfy (26-1). Indeed, we adopt an orthonormal frame such that $\mathbf{e}_0 = \mathbf{u}$; E and H are spatial tensors with the components:

$$E_{rs} = K_{r0,s0} \qquad H_{rs} = -K^{*}_{r0,s0} \qquad (r, s, \ldots, = 1, 2, 3)$$

and one sees that the components $K_{rs,tu}$ are given by the tensor E, while the components $K_{rs,t0}$ by the tensor H. In particular, we evaluate:

$$A = \tfrac{1}{8}\,K^{\alpha\beta,\gamma\delta}K_{\alpha\beta,\gamma\delta}. \tag{26-10}$$

From this, it follows that:

$$A = \tfrac{1}{2} K^{r0,s0} K_{r0,s0} + \tfrac{1}{2} K^{rs,t0} K_{rs,t0} + \tfrac{1}{8} K^{rs,tu} K_{rs,tu}.$$

Now:

$$K^{r0,s0} K_{r0,s0} = K^{rs} K_{rs} .$$

On the other hand:

$$K^{rs,t0} K_{rs,t0} = \eta^{rsv0} \eta_{rsv0} K^{*}_{u0,}{}^{t0} K^{*v0}{}_{,t0} = -\delta^u_v K^{*}_{u0,}{}^{t0} K^{*v0}{}_{,t0} = -2H^{rs} H_{rs} ,$$

and finally, we have:

$$K^{rs,tu} K_{rs,tu} = K^{**rs,tu} K^{**}_{rs,tu} = \eta^{rsv0} \eta^{tuv0} \eta_{rsp0} \eta_{tuq0} K_{v0,w0} K^{p0,q0},$$

namely:

$$K^{rs,tu} K_{rs,tu} = 4K^{v0,w0} K_{v0,w0} = 4E^{rs} E_{rs} .$$

3) The fourth order tensor that appeared in sec. 23, and the analogous tensor for a pure gravitational radiation led Bel [2] to associate with any metric that satisfies:

(26-12) $$R_{ab} - \lambda g_{ab} = 0$$

a fourth order tensor:

(26-13) $$B_{\alpha\beta,\lambda\mu} = \tfrac{1}{2} (R^{\rho,}{}_{\alpha,}{}^{\sigma}{}_{\lambda} R_{\rho\beta,\sigma\mu} + R^{\rho,}{}_{\alpha,}{}^{\sigma}{}_{\mu} R_{\rho\beta,\sigma\lambda}).$$

This tensor is symmetric in α, β, as well as in λ, μ, and is also symmetric in the pairs α,β and λ,μ; we evaluate its contracted covariant derivative. From:

$$\nabla_\alpha R^{\alpha}{}_{\beta,\gamma\delta} = 0,$$

which is a consequence of (26-12), it results that:

$$2\nabla_\alpha B^{\alpha}{}_{\beta,\gamma\delta} = R^{\alpha\beta,\gamma}{}_{\delta} \nabla_\alpha R^{\alpha}{}_{\beta,\gamma\delta} + R^{\rho\alpha,\sigma}{}_{\mu} \nabla_\alpha R_{\rho\beta,\sigma\lambda} .$$

Hence:

$$4\nabla_\alpha B^{\alpha}{}_{\beta,\lambda\mu} = R^{\rho\alpha,\sigma}{}_{\lambda} (\nabla_\alpha R_{\rho\beta,\sigma\mu} - \nabla_\alpha R_{\rho\beta,\sigma\mu}) + R^{\rho\alpha,\sigma}{}_{\mu} (\nabla_\alpha R_{\rho\beta,\sigma\lambda} - \nabla_\alpha R_{\rho\beta,\sigma\lambda}).$$

Therefore, by virtue of the BIANCHI identity:

$$4\nabla_\alpha B^{\alpha}{}_{\beta,\lambda\mu} = R^{\rho\alpha,\sigma}{}_{\lambda} \nabla_\alpha R_{\rho\beta,\sigma\mu} + R^{\rho\alpha,\sigma}{}_{\mu} \nabla_\alpha R_{\rho\beta,\sigma\lambda} .$$

Hence:

$$4\nabla_\alpha B^{\alpha}{}_{\beta,\lambda\mu} = \nabla_\beta (R^{\rho\alpha,\sigma}{}_{\lambda} R_{\rho\alpha,\sigma\mu}).$$

From (26-8), it thus results that:

$$\nabla_\alpha B^{\alpha}{}_{\beta,\lambda\mu} = \tfrac{1}{2} \nabla_\beta (A g_{\lambda\mu}) \qquad \text{(with } A = \tfrac{1}{8} R^{\alpha\beta,\gamma\delta} R_{\alpha\beta,\gamma\delta}\text{)}.$$

Therefore, the tensor:

(26-14) $$T_{\alpha\beta,\lambda\mu} = B_{\alpha\beta,\lambda\mu} - \tfrac{1}{2} A g_{\alpha\beta} g_{\lambda\mu} ,$$

which exhibits the same symmetry properties as B, satisfies the conservation identity:

(26-15) $$\nabla_\alpha T^\alpha{}_{\beta,\lambda\mu} = 0.$$

For pure gravitational radiation ($R_{\alpha\rho} = 0$), $T_{\alpha\beta,\lambda\mu}$ reduces to the tensor $\tau_{\alpha\beta\lambda\mu}$ of sec. 25. If **u** is a unitary vector then we study the quantity:

$$T(\mathbf{u}) = T_{\alpha\beta,\lambda\mu}\, u^\alpha u^\beta u^\lambda u^\mu.$$

In an orthonormal frame such that $\mathbf{e}_0 = \mathbf{u}$:

$$T(\mathbf{u}) = T_{00,00} = B_{00,00} - \tfrac{1}{2} A = R^\rho{}_{0,}{}^\sigma{}_0\, R_{\rho 0,\sigma 0} - \tfrac{1}{2} A = E^{rs}(\mathbf{u}) E_{rs}(\mathbf{u}) - \tfrac{1}{2} A,$$

namely:

$$T(\mathbf{u}) = \tfrac{1}{2} \mid E^{rs}(\mathbf{u}) E_{rs}(\mathbf{u}) + H^{rs}(\mathbf{u}) H_{rs}(\mathbf{u}) \mid > 0.$$

Therefore, $T(\mathbf{u})$ is strictly positive, and it is null only if the curvature tensor is null. The tensor $T_{\alpha\beta,\lambda\mu}$ therefore seems to generalize the MAXWELL tensor in this case.

27. Construction of an example of radiation.

a) Suppose we have a neighborhood U of V_{m+1} that is endowed with the metric:

$$ds^2 = g_{00}(x^i)(dx^0)^2 + ds^{*2} \qquad (g_{00} = -\xi^2 < 0;\ i = 1, \ldots, m)$$

in which ds^{*2} is a quadratic form in the variables (x^i). The metric ds^2 admits a one-parameter group of isometries, $x^0 \to x^0 + \text{const.}$, whose generator $\boldsymbol{\xi}$ has the components ($\xi^0 = 1$, $\xi^1 = 0$). The quotient metric is ds^{*2}, and the anti-symmetric F_{ij} of sec. 19 is null. If $\Gamma^\alpha_{\beta\gamma}$ ($\alpha, \ldots, = 0, 1, \ldots, m$) and $\overset{*}{\Gamma}{}^i_{jk}$ are the coefficients of the Riemannian connection on V_{m+1} and V_m relative to the coordinates envisioned:

$$\overset{*}{\Gamma}{}^i_{jk} = \Gamma^i_{jk}\,.$$

With the notations of sec. 19, the antisymmetric tensor F_{ij} is null, and if one refers U to an adapted orthonormal frame then formulas (19-14), (19-15), and (19-16) become:

$$\begin{aligned}
&\mathrm{R}_{\underline{i}\underline{j},\underline{k}\underline{l}} = \overset{*}{\mathrm{R}}_{\underline{i}\underline{j},\underline{k}\underline{l}}\\
&\mathrm{R}_{\underline{i}\underline{j},\underline{k}\underline{0}} = 0\\
&\mathrm{R}_{\underline{i}\underline{0},\underline{k}\underline{0}} = \overline{\xi}^1\, \overset{*}{\nabla}_{\underline{k}} (\partial_{\underline{i}}\, \xi)\,.
\end{aligned}$$

We shall use the preceding formulas in order to construct an example of radiation.

b) Let the numerical space $\mathbb{R}^4$ be referred to the coordinates:

$$t = x^0 \qquad x = x^1 \qquad y = x^2 \qquad z = x^3 \qquad (\alpha \ldots = 0, 1, 2, 3),$$

set:

(27-1) $$u = t - x = x^0 - x^1,$$

and endow a domain U with the metric that is defined by:

(27-2) $$ds^2 = e^{2\varphi}(dt^2 - dx^2) - (x^2\, dy^2 + h^2\, dz^2) = g_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

in which φ, $\xi > 0$, $\eta > 0$ are three functions of the variable u. If we are given a function $f(u)$ then we denote its derivative with respect to u by $f'(u)$; one then has:

$$\partial_0 f = f' \qquad \partial_1 f = -f' \qquad \partial_A f = 0 \qquad (A, \ldots, = 2, 3)$$

and:

$$\partial_{00} f = f'' \qquad \partial_{01} f = -f'' \qquad \partial_{11} f = f'' \qquad \partial_{A\alpha} f = 0.$$

The metric (27-2) admits two one-parameter groups of isometries that are defined by the two generators:

$$\xi^0 = \xi^1 = \xi^3 = 0 \qquad \xi^2 = 1$$

and:

$$\eta^0 = \eta^1 = \eta^2 = 0 \qquad \eta^3 = 1,$$

which both satisfy the hypotheses of the preceding section. The numbers ξ and η are the scalar numbers that are associated with the two generators.

Consider the decomposition of ds^2 into squares:

$$ds^2 = (\theta^{\underline{0}})^2 - (\theta^{\underline{1}})^2 - (\theta^{\underline{2}})^2 - (\theta^{\underline{3}})^2$$

with:

$$\theta^{\underline{0}} = e^\varphi dx^0 \qquad \theta^{\underline{1}} = e^\varphi dx^1 \qquad \theta^{\underline{2}} = \xi\, dx^2 \qquad \theta^{\underline{3}} = \eta\, dx^3.$$

We thus define orthonormal frames that are adapted to both groups of isometries. The matrix $(A^{\underline{\alpha}}_\beta)$ for the transition from the coordinates (x^β) to the orthonormal frame is diagonal, and it has the elements:

(27-4) $$A^{\underline{0}}_0 = e^\varphi \qquad A^{\underline{1}}_1 = e^\varphi \qquad A^{\underline{2}}_2 = \xi \qquad A^{\underline{3}}_3 = \eta,$$

and the only non-null elements of the inverse matrix are:

(27-5) $$A^0_{\underline{0}} = e^{-\varphi} \qquad A^1_{\underline{1}} = e^{-\varphi} \qquad A^2_{\underline{2}} = \xi^{-1} \qquad A^3_{\underline{3}} = \eta^{-1}.$$

c) We propose to compute the *components of the curvature tensor* of (27-2) with respect to the orthonormal frame described.

From preliminary formulas, it results that if exactly one of the indices $\alpha, \beta, \lambda, \mu$ is equal to the values 2 or the value 3 then $R_{\underline{\alpha}\underline{\beta},\underline{\lambda}\underline{\mu}} = 0$.

One thus obtains:

$$\begin{array}{lllll}
R_{\underline{2}\underline{3},\underline{3}\underline{1}} = 0 & R_{\underline{2}\underline{3},\underline{1}\underline{2}} = 0 & R_{\underline{2}\underline{3},\underline{1}\underline{0}} = 0 & R_{\underline{2}\underline{3},\underline{2}\underline{0}} = 0 & R_{\underline{2}\underline{3},\underline{3}\underline{0}} = 0 \\
 & R_{\underline{3}\underline{1},\underline{1}\underline{2}} = 0 & R_{\underline{3}\underline{1},\underline{1}\underline{0}} = 0 & R_{\underline{3}\underline{1},\underline{2}\underline{0}} = 0 & \\
 & & R_{\underline{1}\underline{2},\underline{1}\underline{0}} = 0 & & R_{\underline{1}\underline{2},\underline{3}\underline{0}} = 0 \\
 & & & R_{\underline{1}\underline{0},\underline{2}\underline{0}} = 0 & R_{\underline{1}\underline{0},\underline{3}\underline{0}} = 0 \\
 & & & & R_{\underline{2}\underline{0},\underline{3}\underline{0}} = 0.
\end{array}$$

With the notations of sec 22, the components of the curvature tensor may be arranged into the matrix:

$$(27\text{-}6) \qquad (R_{IJ}) = \begin{pmatrix} R_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & 0 & R_{35} & 0 \\ 0 & 0 & 0 & R_{44} & 0 & 0 \\ 0 & 0 & R_{35} & 0 & R_{55} & 0 \\ 0 & R_{26} & 0 & 0 & 0 & R_{66} \end{pmatrix}.$$

Now we must evaluate the various elements that appear in (27-6).

First of all, the isometry, $x^2 \to x^2$ + const., gives us, upon passing to the quotient, the metric:

$$ds^{*2} = e^{2\varphi}(dt^2 - dx^2) - \eta^2 dz^2,$$

which further admits the isometry, $x^3 \to x^3$ + const., which leads us to the new metric:

$$ds^{**2} = e^{2\varphi(u)}(dt^2 - dx^2) - \eta^2 dz^2 = e^{2\varphi(u)}\, du(t + x),$$

which is obviously Euclidian. From the preliminary formulas, and using an obvious notation, it results that:

$$R_{\underline{1}\underline{0},\underline{1}\underline{0}} = \overset{*}{R}_{\underline{1}\underline{0},\underline{1}\underline{0}} = \overset{**}{R}_{\underline{1}\underline{0},\underline{1}\underline{0}} .$$

One thus has:

$$(27\text{-}7) \qquad R_{\underline{1}\underline{0},\underline{1}\underline{0}} = 0.$$

We now evaluate $R_{\underline{2}\underline{3},\underline{2}\underline{3}}$. From (26-5), it results that:

$$R_{\underline{2}\underline{3},\underline{2}\underline{3}} = \xi^{1} \overset{*}{\nabla}_{\underline{3}}(\partial_{\underline{3}}\xi) .$$

From (27-4), it follows that:

$$R_{23,23} = \xi \overset{*}{\nabla}_3(\partial_3\xi) .$$

Since x depends only on u, and recalling the preliminary formulas, one has:

$$\overset{*}{\nabla}_3(\partial_3\xi) = -(\Gamma^0_{33} - \Gamma^1_{33})\xi' .$$

Now:

$$\Gamma^0_{33} = g^{00}[33,\,0] = -\tfrac{1}{2}\,g^{00}g'_{33} \qquad \Gamma^1_{33} = g^{11}[33,\,1] = -\tfrac{1}{2}\,g^{00}g'_{33} .$$

From this, it results that $R_{23,23} = 0$, and, as a result:

(27-8) $$R_{\underline{2}\underline{3},\underline{2}\underline{3}} = 0.$$

d) One likewise has:

$$R_{\underline{1}\underline{2},\underline{1}\underline{2}} = \xi^{-1}\overset{*}{\nabla}_{\underline{1}}(\partial_{\underline{1}}\xi) ,$$

namely, from (27-4):

$$R_{12,12} = \xi\overset{*}{\nabla}_1(\partial_1\xi) .$$

By specifying the covariant derivative, one gets:

$$R_{12,12} = \xi[\xi'' - (\Gamma^0_{11} - \Gamma^1_{11})\xi'] .$$

Now:

$$\Gamma^0_{11} = g^{00}[11,\,0] = \tfrac{1}{2}\,g^{00}g'_{00} = \varphi' \qquad \Gamma^1_{11} = g^{11}[11,\,0] = -\tfrac{1}{2}\,g^{00}g'_{00} = -\varphi' .$$

From this, one deduces that:

$$R_{12,12} = \xi(\xi'' - 2\varphi'\xi') ,$$

and, from (27-5):

(27-9) $$R_{\underline{1}\underline{2},\underline{1}\underline{2}} = e^{-2\varphi}\xi^{-1}(\xi'' - 2\varphi'\xi') .$$

On the other hand:

$$R_{\underline{1}\underline{2},\underline{2}\underline{0}} = -\xi^{-1}\overset{*}{\nabla}_{\underline{1}}(\partial_{\underline{0}}\xi) ,$$

i.e.:

(27-10) $$R_{12,20} = -\xi\nabla_1(\partial_0\xi) = \xi[\xi'' - (\Gamma^0_{11} - \Gamma^1_{11})\xi'] .$$

Now:

$$\Gamma^0_{10} = g^{00}[10,\,0] = -\tfrac{1}{2}\,g^{00}g'_{00} = -\varphi' \qquad \Gamma^1_{10} = g^{11}[10,\,0] = \tfrac{1}{2}\,g^{00}g'_{00} = \varphi' .$$

One thus obtains:

(27-11) $$R_{\underline{1}\underline{2},\underline{2}\underline{0}} = R_{\underline{2}\underline{0},\underline{2}\underline{0}} = e^{-2\varphi}\xi^{-1}(\xi'' - 2\varphi'\xi') .$$

As far as the matrix (27-6) is concerned, we have thus established:

$$R_{11} = R_{44} = 0$$

and:

(27-12) $$R_{11} = R_{44} = R_{55} = e^{-2\varphi}\xi^{-1}(\xi'' - 2\varphi'\xi').$$

If we consider the other group of isometries then one likewise establishes:

(27-13) $$R_{22} = -R_{26} = R_{66} = e^{-2\varphi}\eta^{-1}(\eta'' - 2\varphi'\eta').$$

If we set:

(27-14) $$a = e^{-2\varphi}\xi^{-1}(\xi'' - 2\varphi'\xi') \qquad b = e^{-2\varphi}\eta^{-1}(\eta'' - 2\varphi'\eta')$$

then the matrix (R_{IJ}) takes the form (21-27). From this, it results that the curvature tensor $R_{\alpha\beta,\gamma\delta}$ of (27-2) and the vector l_α whose components are:

$$l_0 = 1,\ \ l_1 = -1 \qquad l_A = 0,$$

which is the gradient of u, satisfy the relations (24-1), (24-2). As a result, there exists a scalar τ such that:

(27-15) $$R_{\alpha\beta} = \tau\, l_\alpha l_\beta.$$

Therefore, at the various points of the domain U where it is regular, the metric (27-2) represents a *pure total radiation of integrable type*. In order to be dealing with a pure gravitational radiation, it is necessary and sufficient that $a + b = 0$, i.e., that ξ and η satisfy the relation:

(27-16) $$\eta^{-1}(\eta'' - 2\varphi'\eta') + \xi^{-1}(\xi'' - 2\varphi'\xi') = 0.$$

This state will be non-trivial if the curvature tensor in question is non-null, i.e., if the relation:

(27-17) $$\eta^{-1}(\eta'' - 2\varphi'\eta') - \xi^{-1}(\xi'' - 2\varphi'\xi') = 0$$

is not also satisfied. Set:

$$\xi\eta = e^{2\alpha} \qquad \eta / \xi = e^{2\beta}.$$

The relation (27-16) takes the form:

(27-18) $$\alpha'' + \alpha'^2 + \beta'^2 - 2\alpha'\varphi' = 0,$$

and the relation (27-17) takes the form:

(27-19) $$\beta'' + 2\alpha'\beta' - 2\beta'\varphi' = 0.$$

e) We remark that in the case where the metric (27-2) is Euclidian, one has:

$$2\varphi' = \frac{\xi''}{\xi'} = \frac{\eta''}{\eta'}.$$

By integration, one deduces that there exist two constants C_1 and C_2 such that:

$$C_1\xi + C_2\eta = \text{const.}$$

If ξ and η are themselves constants then φ may be chosen arbitrarily. If this is not the case then:

$$e^{2\varphi} = C_3\xi'.$$

28. – Rosen form for the metric. A ds^2 of the type (27-2) was introduced in 1937 by ROSEN ([8]) in a different optical context. However, it is possible to reduce (27-2) to the form that was indicated by ROSEN. Indeed, the expression (27-2) for our ds^2 admits an arbitrary change $u \to f(u)$, it is possible to use this fact to simplify the form of our metric.

a) If $\xi\eta$ = const., that is, α = const., then it results from (27-18) that in the gravitational case ($R_{\alpha\beta} = 0$) one then has η / ξ = const., namely, ξ = const., η = const., and the ds^2 envisioned is necessarily Euclidian. We thus choose our variable u to be the one defined by:

$$\xi\eta = u^2,$$

and set:

$$v = t + x.$$

Consider the subset $(\mathbb{R}^4)^+$ of $\mathbb{R}^4$ that is defined by:

$$u = t - x > 0.$$

On this subset, the metric that is defined by:

$$ds^2 = e^{2\varphi}(dt^2 - dx^2) - u^2(e^{-2\beta}dy^2 + e^{2\beta}dz^2) \tag{28-1}$$

is regular if the functions $\beta(u)$ and $\varphi(u)$ are of class (C^1, piecewise C^3) for $u > 0$. We then have:

$$\alpha = \log u \qquad \alpha' = \frac{1}{u} \qquad \alpha'' = -\frac{1}{u^2}.$$

In order for (28-1) to satisfy the equation $R_{\alpha\beta} = 0$, it is necessary and sufficient that:

$$2\varphi' = u\beta'^2. \tag{28-2}$$

We may arbitrarily choose $\beta(u)$ to be a function of class (C^1, piecewise C^3) for $u > 0$, and φ will be found by a quadrature from starting with relation (28-2). The metric that is obtained on $(\mathbb{R}^4)^+$ has a non-null curvature if the relation:

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[8] ROSEN, Phys. Z. Sovjet Union, 12, pp. 366 (1937); EINSTEIN and ROSEN J., Franklin Inst., 223, 43, (1937).

(28-3) $$u\beta'' + 2\beta' - u^2\beta'^3 = 0$$

is not satisfied by β. In particular, the curvature tensor is non-null at a point where $\beta' = \beta'' = 0$.

We have thus obtained the metric that was indicated by ROSEN, but it is valid only on $(\mathbb{R}^4)^+$, which seems to limit its interest.

b) If we adopt other coordinates then it is easy to endow $\mathbb{R}^4$ with an everywhere regular metric that is Euclidian in some domains and coincides with the ROSEN metric in other domains.

One may reduce (28-1) to the Galilean form at a point where $\beta'(u) = 0$, namely:

(28-4) $$ds^2 = e^{2\varphi} du\, dv - u^2(e^{-2\beta} dy^2 + e^{2\beta} dz^2).$$

To that effect, we adopt the coordinates u, v and:

(28-5) $$\bar{y} = u\, e^{-\beta} y \qquad \bar{z} = u\, e^{\beta} z.$$

By differentiation, it follows that:

$$u\, e^{-\beta} dy = \bar{y} - \frac{\bar{y}}{u} du \qquad\qquad u\, e^{\beta} dz = \bar{z} - \frac{\bar{z}}{u} du\,.$$

From this, one deduces that:

$$u^2(e^{-2\beta} dy^2 + e^{2\beta} dz^2) = d\bar{y}^2 + d\bar{z}^2 - 2\frac{\bar{y}d\bar{y} + \bar{z}d\bar{z}}{u} du + \frac{\bar{y}^2 + \bar{z}^2}{u^2} du^2$$

and:

$$ds^2 = du\left(e^{2\varphi} dv + 2\frac{\bar{y}d\bar{y} + \bar{z}d\bar{z}}{u} + \frac{\bar{y}^2 + \bar{z}^2}{u^2} du\right) - (d\bar{y}^2 + d\bar{z}^2)\,.$$

By performing a new change of variables:

(28-6) $$\bar{u} = u \qquad\qquad \bar{v} = e^{2\varphi} + \frac{\bar{y}^2 + \bar{z}^2}{u^2}$$

and after taking (28-2) into account, one obtains at the point considered:

$$ds^2 = d\bar{u}d\bar{v} - (d\bar{y}^2 + d\bar{z}^2)\,,$$

and it suffices to set $\bar{u} = \bar{t} - \bar{x}$, $\bar{v} = \bar{t} + \bar{x}$ in order to reduce it to the Galilean form.

Having said this, suppose that β is an arbitrary function of class (C^1, piecewise C^3), and perform the transformation ([9]) on (28-4) that is defined by:

$$\bar{u} = u, \quad \bar{v} = e^{2\varphi} v + \frac{\bar{y}^2 + \bar{z}^2}{u}, \quad \bar{y} = u\, e^{-\beta} y, \qquad \bar{z} = u\, e^{\beta} y \,. \tag{28-7}$$

This transformation defines a bijective map with a non-null Jacobian from $(\mathbb{R}^4)^+$ to itself. It follows that:

$$u\, e^{-\beta} dy = d\bar{y} - \frac{\bar{y}}{u} du + \beta' \bar{y} du \,, \qquad u\, e^{\beta} dz = d\bar{z} - \frac{\bar{y}}{u} du + \beta' \bar{z} du \,.$$

As a result:

$$u^2(e^{-2\beta} dy^2 + e^{2\beta} dz^2) = d\bar{z}^2 + d\bar{z}^2 - 2\frac{\bar{y}d\bar{y} + \bar{z}d\bar{z}}{u} du + \frac{\bar{y}^2 + \bar{z}^2}{u^2} du^2$$
$$+2\beta'\left(\bar{y}d\bar{y} - \bar{z}d\bar{z} - \frac{\bar{y}^2 - \bar{z}^2}{u} du \right) du + \beta'^2 (\bar{y}^2 + \bar{z}^2) du^2 \,.$$

On the other hand:

$$d\bar{v} = e^{2\varphi} dv + 2\frac{\bar{y}d\bar{y} + \bar{z}d\bar{z}}{u} - \frac{\bar{y}^2 + \bar{z}^2}{u^2} du + \beta'^2 [uv - (\bar{y}^2 + \bar{z}^2)] du \,.$$

From this, one deduces that:

$$ds^2 = du d\bar{v} - (d\bar{y}^2 + d\bar{z}^2) - 2\beta'\left(\bar{y}d\bar{y} - \bar{z}d\bar{z} - \frac{\bar{y}^2 - \bar{z}^2}{u} du \right) du + \beta'^2 u\bar{v} du^2 \,. \tag{28-8}$$

If u_0 and u_1 are *two positive numbers* then choose β' to be a function of class (C^1, piecewise C^3) that is defined on $u_0 \leq u \leq u_1$, and is such that:

$$\beta'(u_0) = \beta'(u_1) = 0 \qquad\qquad \beta''(u_0) = \beta''(u_1) = 0.$$

Consider the numerical space $\mathbb{R}^4$ to be the set of all $(u, \bar{v}, \bar{y}, \bar{z})$ and give it the metric that is defined in the following manner:

for $u \leq u_0$	$ds^2 = du d\bar{v} - (d\bar{y}^2 + d\bar{z}^2)$
for $u_0 \leq u \leq u_1$	ds^2 is given by (28-8)
for $u \geq u_1$	$ds^2 = du d\bar{v} - (d\bar{y}^2 + d\bar{z}^2)$.

The metric that is thus defined on $\mathbb{R}^4$ satisfies the equations $R_{\alpha\beta} = 0$, and is likewise of class C^2 everywhere. It is non-Euclidian for $u_0 < u < u_1$ when β' does not satisfy (28-3).

[9] This transformation and the argument that follows are due to BONDI, "Nature," t. 179, 1072, (1957).

In that region, it represents a *state of pure gravitational radiation*, and it is of integrable type; of course, it is non-Euclidian in the domain at spatial infinity.

IV. Deviation formulas.

29. – **Geodesic deviation formula.** In order to study the physical effects of either the existence of a gravitational wave front with discontinuities in the curvature tensor or radiative state, we shall use the formulas that are called "the equations of geodesic deviation." Consider the spatio-temporal trajectories of a family of test particles in a domain Δ of V. Each of these trajectories is a geodesic of V_4 that is time-oriented. We propose to study the "relative acceleration" (with respect to s) of two infinitely close particles at points of V_4 that define a vector that is orthogonal to the trajectories.

a) The following considerations are naturally local. We take a congruence of time-oriented curves in a neighborhood of V_4 and denote the unitary tangent vector by **u**. We extract a one-parameter family of curves Γ_t from this congruence that generates a surface S of dimension 2. We use the given parameter s on each curve of Γ_t, starting from a suitable origin. The surface S is thus parameterized by two parameters s and t, and $x(s, t)$, where $x \in S$, is assumed to be twice continuously differentiable with respect to (s, t). The unitary vector **u** that is tangent to Γ_t at x is the vector $\partial x / \partial s$, and we denote the vector $\partial x / \partial t$ by **v**.

If the neighborhood envisioned is referred to local coordinates (x^α) then the vectors u and v have the components:

$$u^\alpha = \frac{\partial x^\alpha}{\partial s} \qquad v^\alpha = \frac{\partial x^\alpha}{\partial t}$$

and:

(29-1)
$$\frac{\partial u^\alpha}{\partial t} = \frac{\partial v^\alpha}{\partial s},$$

respectively.

In terms of absolute differentiation, it follows that:

$$\frac{\nabla u^\alpha}{dt} = \frac{\partial u^\alpha}{\partial t} + \Gamma^\alpha_{\beta\gamma} u^\beta \frac{\partial x^\gamma}{\partial t} = \frac{\partial v^\alpha}{\partial s} + \Gamma^\alpha_{\beta\gamma} u^\beta v^\gamma .$$

One thus has:

(29-2)
$$\frac{\nabla u^\alpha}{dt} = \frac{\nabla v^\alpha}{ds} .$$

Having said this, we apply the RICCI identity to **u**:

(29-3)
$$\nabla_\lambda \nabla_\mu u^\alpha - \nabla_\mu \nabla_\lambda u^\alpha = R^\alpha{}_{\rho,\lambda\mu}\, u^\rho .$$

We take the product of the left-hand side with $v^\lambda u^\mu$ and get:

$$v^\lambda u^\mu (\nabla_\lambda \nabla_\mu u^\alpha - \nabla_\mu \nabla_\lambda u^\alpha) = u^\mu \frac{\nabla}{dt} (\nabla_\mu u^\alpha) - v^\lambda \frac{\nabla}{dt} (\nabla_\lambda u^\alpha)$$

namely:

$$v^\lambda u^\mu(\nabla_\lambda \nabla_\mu u^\alpha - \nabla_\mu \nabla_\lambda u^\alpha) = \frac{\nabla}{dt}\frac{\nabla u^\alpha}{ds} - \frac{\nabla}{ds}\frac{\nabla u^\alpha}{dt} - \frac{\nabla u^\alpha}{dt}\nabla_\mu u^\alpha + \frac{\nabla v^\lambda}{dt}\nabla_\lambda u^\alpha.$$

From (29-2), one obtains:

$$v^\lambda u^\mu(\nabla_\lambda \nabla_\mu u^\alpha - \nabla_\mu \nabla_\lambda u^\alpha) = \frac{\nabla}{dt}\frac{\nabla u^\alpha}{ds} - \frac{\nabla^2 v^\alpha}{ds^2}.$$

From (29-3), one thus derives the formula:

(29-4) $$\frac{\nabla^2 v^\alpha}{ds^2} + R^\alpha{}_{\rho,\lambda\mu}\, u^\rho v^\lambda u^\mu = \frac{\nabla}{dt}\frac{\nabla u^\alpha}{ds}.$$

b) Suppose that the curves Γ_t envisioned are geodesics of V_4. One then has:

(29-5) $$\frac{\nabla u^\alpha}{ds} = 0.$$

Since the vector u^α is unitary:

$$u^\alpha \frac{\nabla u^\alpha}{dt} = u^\alpha \frac{\nabla u^\alpha}{ds} = 0,$$

and, from (29-5):

$$\frac{\nabla}{ds}(u_\alpha v^\alpha) = \frac{\nabla u^\alpha}{ds} v^\alpha + u^\alpha \frac{\nabla u^\alpha}{ds} = 0.$$

Therefore, one each curve Γ_t one has:

$$u_\alpha v^\alpha = \text{const.}$$

Let C be an orthogonal trajectory on S of the geodesics Γ_t, and adopt a point of C to be the origin of the arc s on each Γ. **n** and **v** are orthogonal at the points of C ($s = 0$); as a result, $u_\alpha v^\alpha = 0$ on any S, and **v** is orthogonal to the geodesics.

From (29-4), it results that:

(29-6) $$\frac{\nabla^2}{ds^2} v^\alpha + R^\alpha{}_{\rho,\lambda\mu}\, u^\rho v^\lambda u^\mu = 0.$$

We give this formula the name of "the geodesic deviation formula." Since the corresponding points on two infinitely close geodesics are the ones that are defined by the direction of v^α that is orthogonal to the geodesics, they give us what one may call the relative acceleration, with respect to s, of the two test particles that describe these geodesics.

30. – Applications.

a) Consider the spatio-temporal trajectories of the test particles envisioned in a neighborhood of the point *x*, and assume that these particles are subject to only the gravitational field. If Γ is the trajectory that issues from *x* then we introduce an orthonormal frame $(\mathbf{e}_\alpha)$ such that the first vector $\mathbf{e}_0$ coincides with the unitary vector that is tangent to Γ at *x*. At the point *x* the vector **u** admits the components:

$$u^0 = 0 \qquad u^u = 0 \qquad (u = 1, 2, 3), \tag{30-1}$$

and since **v** is orthogonal to **u** it admits the components:

$$v^0 = 0 \qquad v^u. \tag{30-2}$$

We parallel transport the initial frame along Γ and endow a neighborhood of Γ with a family of orthonormal frames that is a C^2 extension of the frames that are thus attached to the various points of Γ. If $\gamma^\alpha{}_{\beta\gamma}$ are the coefficients of the Riemannian connection of V_4 relative to that family of frames then one has on Γ:

$$\gamma^\alpha{}_{\beta\gamma}\, u^\gamma = 0. \tag{30-3}$$

We evaluate the components of $\dfrac{\nabla^2 v^\alpha}{ds^2}$ at the points of Γ. At these points, one obtains:

$$\frac{\nabla v^\alpha}{ds} = \frac{dv^\alpha}{ds} + \gamma^\alpha{}_{\beta\gamma}\, u^\gamma v^\beta,$$

and, by derivation:

$$\frac{\nabla^2 v^\alpha}{ds^2} = \frac{d^2 v^\alpha}{ds^2} + \frac{d}{ds}(\gamma^\alpha{}_{\beta\gamma} u^\gamma v^\beta) + \gamma^\alpha{}_{\beta\gamma} u^\gamma v^\beta\, \frac{\nabla^2 v^\beta}{ds^2}. \tag{30-4}$$

From (30-3), one thus has on Γ:

$$\frac{\nabla^2 v^\alpha}{ds^2} = \frac{d^2 v^\alpha}{ds^2}$$

and the equation (29-6) takes the form:

$$\frac{d^2 v^\alpha}{ds^2} + R^\alpha{}_{\rho,\lambda\mu}\, u^\rho v^\lambda u^\mu = 0. \tag{30-5}$$

b) The trajectories of test particles that are subject to only the gravitational field are time-oriented geodesics in V_4. From this, it results that $\mathbf{e}_0 = \mathbf{u}$ at any point of *G*, and, as a result, $v^0 = 0$ on Γ.

At the chosen point *x* (30-5) may be written for $\alpha = u$:

(30-6) $$\frac{d^2 v^u}{ds^2} + R^u{}_{0,v0}\, v^v = 0,$$

and, from the preceding remark, we obtain only an identity when $\alpha = 0$. Suppose that there exists a gravitational wave front on V_4 that passes through x and has discontinuities in its curvature tensor. From (30.6), it results:

(30-7) $$\left[\frac{d^2 v^u}{ds^2}\right] = -\,[R^u{}_{0,v0}]\, v^v,$$

namely:

(30-8) $$\left[\frac{d^2 v^u}{ds^2}\right] = K^u{}_v\, v^v,$$

if we set:

$$K^u{}_v = -\,[R^u{}_{0,v0}].$$

If the vector $\mathbf{e}_0$ is fixed at x then we have to choose the vector $\mathbf{e}_u$ of the frame at that point in such a fashion that the discontinuity matrix of the curvature tensor takes the form (22.1). The matrix $(K^u{}_v)$ may then be written (with $\rho = 0$):

(30-9) $$(K^u{}_v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & -\sigma \end{pmatrix}$$

and (30-10) may be specified by the relations:

(30-10) $$\left[\frac{d^2 v^1}{ds^2}\right] = 0 \qquad \left[\frac{d^2 v^2}{ds^2}\right] = \sigma v^2 \qquad \left[\frac{d^2 v^3}{ds^2}\right] = -\sigma v^2,$$

which gives us the components of the discontinuity of the relative acceleration. The component along $\mathbf{e}_1$, i.e., in the spatial direction of the wave propagation, is always null. Since the discontinuity envisioned depends linearly on the vector $\mathbf{v}$ at x, we may study the general case as a superposition of the following two particular cases:

1) If $\mathbf{v}$ is collinear with $\mathbf{e}_1$ then one has $v^2 = v^3 = 0$, and the discontinuity in the relative acceleration is null.

2) If $\mathbf{v}$ is in the 2-plane $(\mathbf{e}_2, \mathbf{e}_3)$, i.e., it is orthogonal to the spatial direction of wave propagation:

(30-11) $$v^2 = v\cos\vartheta \qquad v^3 = v\sin\vartheta,$$

in which ϑ is the angle between $\mathbf{e}_2$ and $\mathbf{v}$. One then has:

(30-12) $$\left[\frac{d^2 v^2}{ds^2}\right] = \sigma v\cos\vartheta, \quad \left[\frac{d^2 v^2}{ds^2}\right] = -\sigma v\sin\vartheta,$$

and the discontinuity is carried by the direction in the ($\mathbf{e}_2$, $\mathbf{e}_3$) that is defined by starting with $\mathbf{e}_3$ and the angle $-\vartheta$.

One has exhibited the transversal character of the gravitational wave and located the discontinuity of the relative acceleration that is produced by the wave front.

In a domain of V_4 where the metric represents a state of *pure gravitational radiation*, one sees from (30-6) that the results are identical to the ones that relate to the relative acceleration itself.

31. Deviation of the trajectories of charged particles in the presence of an electromagnetic field. We suppose that there exists an electromagnetic field $F_{\alpha\beta}$ in a domain Δ of V_4, and we propose to study, by analogy with the geodesic case, the deviation of the trajectory of a charged test particle that is subject to both the gravitational electromagnetic fields. The spatio-temporal trajectories of the charged particles satisfy the differential system that is expressed in local coordinates by:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = kF^\alpha{}_\beta\frac{dx^\beta}{ds}, \tag{31-1}$$

in which the constant k characterizes the ratio of the charge to the mass for the particle in question.

a) We assume that we have a cloud of charged particles in Δ for which the ration k is the same, and we isolate a one-parameter family of trajectories Γ_t from the congruence of their trajectories in Δ. With the same hypotheses and notations that were made in sec. 29, the differential system (31-1) may be written:

$$\frac{\nabla u^\alpha}{ds} = kF^\alpha{}_\beta u^\beta. \tag{31-2}$$

On the other hand, one has (29-4), namely:

$$\frac{\nabla^2 v^\alpha}{ds^2} + R^\alpha{}_{\rho,\lambda\mu}u^\rho v^\lambda u^\mu = \frac{\nabla}{dt}\frac{\nabla u^\alpha}{ds}. \tag{31-3}$$

From (31-1), it results that:

$$\frac{\nabla}{dt}\frac{\nabla u^\alpha}{ds} = k\left(\nabla_\rho F^\alpha{}_\beta u^\beta v^\rho + F^\alpha{}_\beta\frac{\nabla v^\beta}{ds}\right).$$

Therefore, the vector v^α satisfies the differential system:

(31-4) $$\frac{\nabla^2 v^\alpha}{ds^2} + R^\alpha{}_{\rho,\lambda\mu} u^\rho v^\lambda u^\mu = k\left(\nabla_\rho F^\alpha{}_\beta u^\beta v^\rho + F^\alpha{}_\beta \frac{\nabla v^\beta}{ds}\right).$$

Let C be an orthogonal trajectory of the curves Γ_t envisioned one the surface that is generated. We adopt a point of C to be the origin of the arc on each Γ_t. **u** and **v** are orthogonal to the points of C (s = 0), but the same thing is not true outside of C since:

$$\frac{d}{dt}(u_\alpha v^a) = \frac{\nabla u^\alpha}{ds} v^\alpha = k\, F_{\alpha\beta} v^\alpha u_\beta,$$

is not identically null.

Formula (31-4) provides the relative acceleration, with respect to s, of two infinitely close test particles, such that the corresponding points on the two spatio-temporal trajectories are the ones with the same s, when measured from C.

b) If x is a definite point of one of the preceding spatio-temporal trajectories Γ then we introduce an orthonormal frame ($\mathbf{e}_\alpha$) at this point such that the first vector $\mathbf{e}_0$ coincides with the unitary vector **u** that is tangent to Γ at x. From a preceding remark, we may assume that **v** is *orthogonal to* **u** at this point x. Therefore, the vectors **u**, **v**, admit the components (30-1) and (30-2) at x.

If we adopt a family of frames in a neighborhood of G that are identical to the ones that were defined in sec. 30 then (31-4) takes the form:

(31-5) $$\frac{d^2 v^\alpha}{ds^2} + R^\alpha{}_{\rho,\lambda\mu} u^\rho v^\lambda u^\mu = k\left(\nabla_\rho F^\alpha{}_\beta u^\beta v^\rho + F^\alpha{}_\beta \frac{dv^\beta}{ds}\right).$$

Suppose that there exists a wave front S on V_4 that is both gravitational and electromagnetic and passes through x; there are discontinuities in the curvature tensor and the derived tensor of the electromagnetic field upon crossing S.

We naturally suppose that the metric satisfies the EINSTEIN equations with a continuous right-hand side, and that $F_{\alpha\beta}$ satisfies the MAXWELL equations with a continuous current vector. From (31-5), it results that at the point x in question:

(31-6) $$\left[\frac{d^2 v^\alpha}{ds^2}\right] + [R^\alpha{}_{\rho,\lambda\mu}]\, u^\rho v^\lambda u^\mu = k[\nabla_\rho F^\alpha{}_\beta]\, u^\beta v^\rho.$$

For a = 0, if we keep in mind the choice of frame at x, (31-6) gives:

(31-7) $$\left[\frac{d^2 v^0}{ds^2}\right] = 0.$$

For $\alpha = u$, keeping in mind the expressions (30-1) and (30-2) for the components of **u** and **v**:

(31-8)
$$\left[\frac{d^2 v^\alpha}{ds^2}\right] = K^u{}_v v^v + k\,[\nabla_v F^u{}_0]\, v^v,$$

in which the notations are identical to the ones in sec. 30.

If the vector $\mathbf{e}_0$ is fixed at x and the vector $\boldsymbol{l}$ is determined by S then we choose $\mathbf{e}_1$ in such a way that one may take $\boldsymbol{l} = \mathbf{e}_0 + \mathbf{e}_1$. For a convenient choice of $\mathbf{e}_2$ in the 2-plane ($\mathbf{e}_2$, $\mathbf{e}_3$), the matrix $K^u{}_v$ has the form (30-9). As for the matrix $(\nabla_v F^u{}_0)$, from the considerations of sec. 9, all of its elements are null for $v \neq 1$, and, from (7, 3):

$$[\nabla_1 F^1{}_0] = 0.$$

We set:

$$[\nabla_1 F^2{}_0] = \mu \qquad\qquad [\nabla_1 F^3{}_0] = \nu$$

and (31-8) may be specified by the relations:

$$\left[\frac{d^2 v^1}{ds^2}\right] = 0, \quad \left[\frac{d^2 v^2}{ds^2}\right] = \sigma v^2 + k\mu\, v^1, \qquad \left[\frac{d^2 v^2}{ds^2}\right] = -\sigma v^3 + k\nu\, v^1,$$

which gives the components of the discontinuity in the relative acceleration at x.

The component of this discontinuity along $\mathbf{e}_1$ is always null. If $\mathbf{v}$ is collinear with $\mathbf{e}_1$ then the discontinuity is of purely electromagnetic origin. If $\mathbf{v}$ is orthogonal to $\mathbf{e}_1$ then the discontinuity is of purely gravitational origin.

V. Remarks on the penta-dimensional case

32. Components of the curvature tensor, the Ricci tensor and the Einstein tensor. The preceding results lead us to study the wave fronts and discontinuities of the curvature tensor in the context of the penta-dimensional theories of KALUZA-KLEIN and the theory of JORDAN-THIRY.

We consider a Riemannian manifold V_5 that satisfies the hypotheses of sec. 19, and we recall first the expressions for the components $R_{\alpha\beta,\lambda\mu}$ (α, …, any Greek index = 0, 1, 2, 3, 4) of the curvature tensor in an adapted orthonormal frame. If $i, j, \ldots = 1, 2, 3, 4$ then one obtains:

$$R_{\underline{ij},\underline{kl}} = \overset{*}{R}_{\underline{ij},\underline{kl}} + \frac{\beta^2\xi^2}{4}(F_{\underline{ik}}F_{\underline{jl}} - F_{\underline{il}}F_{\underline{jk}}) + \frac{\beta^2\xi^2}{2}F_{\underline{ij}}F_{\underline{kl}} \tag{32-1}$$

$$R_{\underline{ij},\underline{k0}} = \frac{\beta}{2}(\xi\overset{*}{\nabla}_{\underline{k}} F_{\underline{ij}} + 2\partial_{\underline{k}}\xi F_{\underline{ij}} - \partial_{\underline{i}}\xi F_{\underline{jk}} + \partial_{\underline{j}}\xi F_{\underline{ik}}) \tag{32-2}$$

$$R_{\underline{i0},\underline{k0}} = \xi^{-1}\overset{*}{\nabla}_{\underline{k}}(\partial_{\underline{i}}\xi) + \frac{\beta^2\xi^2}{4}F_{\underline{ir}}F_{\underline{k}}{}^{\underline{r}}, \tag{32-3}$$

in which F_{ij} is identified with the electromagnetic tensor, b is a constant, and the elements that are marked with a * are defined relative to the quotient metric.

From (32-1), one deduces by contraction:

$$R_{\underline{ij},\underline{k}}{}^{\underline{r}} = \overset{*}{R}_{\underline{ik}} + \frac{3}{4}\beta^2\xi^2F_{\underline{ir}}F_{\underline{j}}{}^{\underline{r}}. \tag{32-4}$$

From (32-2), it follows:

$$R_{\underline{j0}} = \frac{\beta}{2}(\xi\nabla_{\underline{r}}F^{\underline{r}}{}_{\underline{j}} + 3\partial_{\underline{r}}\xi F^{\underline{r}}{}_{\underline{j}}). \tag{32-5}$$

By starting with:

$$R_{\underline{ik}} = R_{\underline{ir},\underline{k}}{}^{\underline{r}} + R_{\underline{i0},\underline{k}}{}^{\underline{0}}$$

and from (32-4) and (32-5) one obtains the following expressions for the components of the RICCI tensor:

$$R_{\underline{ik}} = \overset{*}{R}_{\underline{ik}} + \frac{\beta^2}{2}\xi^2F_{\underline{ir}}F_{\underline{k}}{}^{\underline{r}} - \xi^{-1}\overset{*}{\nabla}_{\underline{k}}(\partial_{\underline{i}}\xi) \tag{32-6}$$

$$R_{\underline{i0}} = \frac{\beta}{2}\xi^{-2}\overset{*}{\nabla}_{\underline{r}}(\xi^3F^{\underline{r}}{}_{\underline{i}}) \tag{32-7}$$

$$R_{\underline{00}} = \xi^{-1}\Delta^*\xi + \frac{\beta^2\xi^2}{4}F_{\underline{rs}}F^{\underline{rs}}. \tag{32-8}$$

Finally, if we evaluate the components of the EINSTEIN tensor in V_5:

$$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\,\gamma_{\alpha\beta} R.$$

From (32-6), it results:

$$R_{\underline{i}}^{\ \underline{i}} = R^* + \frac{\beta^2\xi^2}{2} F_{\underline{rs}} F^{\underline{rs}} - \xi^{-1}\Delta^*\xi$$

and, from (32-8):

$$R_{\underline{0}}^{\ \underline{0}} = -\xi^{-1}\Delta^*\xi - \frac{\beta^2\xi^2}{2} F_{\underline{rs}} F^{\underline{rs}}\,.$$

Therefore, the Riemannian scalar curvature R has the values:

(32-9) $$R = R^* + \frac{\beta^2\xi^2}{4} F_{\underline{rs}} F^{\underline{rs}} - 2\xi^{-1}\Delta^*\xi\,.$$

One thus obtains the following components of for the EINSTEIN tensor:

(32-10) $$S_{\underline{ik}} = \overset{*}{S}_{\underline{ik}} - \frac{\beta^2\xi^2}{2}\left(\tfrac{1}{4} g_{\underline{ik}} F_{\underline{rs}} F^{\underline{rs}} - F_{\underline{ir}} F_{\underline{k}}^{\ \underline{r}}\right) - \xi^{-1}(\overset{*}{\nabla}_{\underline{k}}\,\partial_{\underline{i}}\,\xi - g_{\underline{ik}}\Delta^*\xi)$$

(32-11) $$S_{\underline{i0}} = \frac{\beta}{2}\xi^{-2}\,\overset{*}{\nabla}_{\underline{r}}\,(\xi^3 F^{\underline{r}}_{\ \ \underline{i}})$$

(32-12) $$S_{\underline{00}} = \tfrac{1}{2} R^* + \frac{3}{8}\beta^2\xi^2 F_{\underline{rs}} F^{\underline{rs}}\,.$$

The field equations for the JORDAN-THIRY may be written:

(32-13) $$S_{\alpha\beta} = \Theta_{\alpha\beta},$$

in which the tensor $\Theta_{\alpha\beta}$ on the right-hand side describes the field sources. In a domain with no sources this tensor is null (exterior unitary case).

33. Field equations in the Kaluza-Klein. In the KALUZA-KLEIN theory $\xi = 1$, and the preceding formulas that relate to the curvature tensor take the form:

(33-1) $$R_{\underline{ij},\underline{kl}} = \overset{*}{R}_{\underline{ij},\underline{kl}} + \frac{\beta^2}{4}(F_{\underline{ik}} F_{\underline{jl}} - F_{\underline{il}} F_{\underline{jk}}) + \frac{\beta^2}{2} F_{\underline{ij}} F_{\underline{kl}}$$

(33-2) $$R_{\underline{ij},\underline{k0}} = \frac{\beta}{2}\overset{*}{\nabla}_{\underline{k}}\, F_{\underline{ij}}$$

(33-3) $$R_{\underline{i0},\underline{k0}} = \frac{\beta^2}{4} F_{\underline{ir}} F_{\underline{k}}^{\ \underline{r}}\,.$$

The ones that relate to the RICCI tensor may be written:

(33-4) $$R_{\underline{ik}} = \overset{*}{R}_{\underline{ik}} + \frac{\beta^2}{2} F_{\underline{ir}} F_{\underline{k}}{}^{\underline{r}}$$

(33-5) $$R_{\underline{i0}} = \frac{\beta}{2} \overset{*}{\nabla}_{\underline{r}} F^{\underline{r}}{}_{\underline{i}}$$

(33-6) $$R_{\underline{00}} = \frac{\beta^2}{4} F_{\underline{rs}} F^{\underline{rs}}$$

and the ones that relate to the EINSTEIN tensor:

(33-7) $$S_{\underline{ik}} = \overset{*}{S}_{\underline{ik}} - \frac{\beta^2}{2}\left(\tfrac{1}{4} g_{\underline{ik}} F_{\underline{rs}} F^{\underline{rs}}\right)$$

(33-8) $$S_{\underline{i0}} = \frac{\beta}{2} \overset{*}{\nabla}_{\underline{r}} F^{\underline{r}}{}_{\underline{i}}$$

(33-9) $$S_{\underline{00}} = \tfrac{1}{2} \overset{*}{R} + \frac{3}{8} \beta^2 F_{\underline{rs}} F^{\underline{rs}}.$$

The fourteen *field equations* may then be written:

(33-10) $$S_{\underline{ik}} \equiv R_{\underline{ik}} - \tfrac{1}{2} g_{\underline{ik}} R = \Theta_{\underline{ik}}$$

and:

(33-11) $$S_{\underline{i0}} \equiv R_{\underline{i0}} = \Theta_{\underline{i0}},$$

in which the Θ describe field sources. From (33-10), one deduces by contraction:

$$g^{\underline{ik}} R_{\underline{ik}} - 2R = \Theta \qquad (\Theta = g^{\underline{ik}} \Theta_{\underline{ik}}),$$

namely:

$$R + R_{00} - 2R = \Theta.$$

From this, it results that:

$$R = R_{00} - \Theta$$

and (33-10) may be put into the form:

(33-12) $$R_{\underline{ik}} = \frac{\beta^2}{8} g_{\underline{ik}} F_{\underline{rs}} F^{\underline{rs}} + \left(\Theta_{\underline{ik}} - \tfrac{1}{2} g_{\underline{ik}} \Theta\right).$$

34. Discontinuities in the curvature tensor of V_5. The metric $\gamma_{\alpha\beta}$ of V_5 and the infinitesimal generator $\boldsymbol{\xi}$ of the isometry group are assumed to be of class (C^1, piecewise C^3). It then results that the quotient metric and the scalar ξ are also (C^1, piecewise C^3), whereas the tensor F_{ij} is continuous, and has discontinuous first derivatives.

We put ourselves in a domain of V_5 where the right-hand sides of the field equations are continuous, and we study the discontinuities in the curvature tensor of V_5 upon crossing a hypersurface S that is generated by the trajectories of the isometry group. In local adapted coordinates, any vector $\boldsymbol{l}$ that is collinear with the gradient of f satisfies $l_0 = 0$, i.e., it is orthogonal to the vector that is tangent to the trajectory of the isometry group. In an adapted orthonormal frame $l_0 = 0$.

Under the hypotheses that we made in the JORDAN-THIRY theory, by virtue of (18-4), when one crosses S, one has:

(34-1) $$[R_{\alpha\beta}] = 0.$$

By virtue of (33-12), (33-11), and (33-6), this relation is further verified under the same hypotheses in the KALUZA-KLEIN theory.

From the general study that was made in sec. 20, $\boldsymbol{l}$ has null length, and the tensor $[R_{\alpha\beta,\lambda\mu}]$ satisfies the relations:

(34-2) $$l_\alpha[R_{\beta\gamma,\lambda\mu}] + l_\beta[R_{\gamma\beta,\lambda\mu}] + l_\gamma[R_{\alpha\beta,\lambda\mu}] = 0$$

(34-3) $$l^\alpha[R_{\alpha\beta,\lambda\mu}] = 0,$$

at a point x of S.

At this point, we consider the normed vector $\mathbf{e}_0$ that is collinear with $\boldsymbol{\xi}$, and give ourselves a unitary vector $\mathbf{e}_4$ ($\mathbf{e}_4^2 = 1$) that is restricted only to be orthogonal to $\mathbf{e}_4$. The vectors $\mathbf{e}_0$ and $\boldsymbol{l}$ define a 2-plane that is orthogonal to $\mathbf{e}$. In this 2-plane, let $\mathbf{e}_1$ be a normal vector that is orthogonal to $\mathbf{e}_4$ such that one may take:

(34-3) $$\boldsymbol{l} = \mathbf{e}_4 + \mathbf{e}_1.$$

One may complete this frame with two vectors $\mathbf{e}_2$, $\mathbf{e}_3$ in the 2-plane that is orthogonal to the 3-plane that is defined by ($\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_4$) in such a fashion as to obtain an adapted orthonormal frame ($\mathbf{e}_a$) at x that satisfies (34-4) (?). We shall reason relative to such a frame in what follows.

35. Representative matrix of a tensor $H_{\alpha\beta,\lambda\mu}$ at x. Let $H_{\alpha\beta,\lambda\mu} \neq 0$ be a tensor that enjoys the same symmetry properties as the curvature tensor and satisfies (21-2) and (21-3) for a vector $\boldsymbol{l}$ that is *orthogonal to* $\mathbf{e}_0$. This vector necessarily has null length, and we may adopt the frame ($\mathbf{e}_a$) at x that was introduced in the preceding section.

(21-2) and (21-3) then translate into the relations:

(35-1) $$H_{\alpha\beta,\lambda\mu} = a_{00}\varphi^{(0)}_{\alpha\beta}\varphi^{(0)}_{\lambda\mu} + \sum_{\mathrm{A}} a_{0\mathrm{A}}(\varphi^{(0)}_{\alpha\beta}\varphi^{(\mathrm{A})}_{\lambda\mu} + \varphi^{(\mathrm{A})}_{\alpha\beta}\varphi^{(0)}_{\lambda\mu}) + \sum_{\mathrm{A,B}} a_{\mathrm{AB}}\varphi^{(\mathrm{A})}_{\alpha\beta}\varphi^{(\mathrm{B})}_{\lambda\mu}\,,$$

in which the tensors j are the exterior products of $\boldsymbol{l}$ with the vectors $\boldsymbol{n}^0 = \mathbf{e}_0$, $\boldsymbol{n}^{(1)} = \mathbf{e}_2$, $\boldsymbol{n}^{(2)} = \mathbf{e}_3$, and the indices A, B take the values 1, 2.

If we set:

$$\begin{cases} \mathbf{e}_2 \wedge \mathbf{e}_3 = \tau_1 & \mathbf{e}_1 \wedge \mathbf{e}_4 = \tau_4 & \mathbf{e}_1 \wedge \mathbf{e}_0 = \tau_7 \\ \mathbf{e}_3 \wedge \mathbf{e}_1 = \tau_2 & \mathbf{e}_2 \wedge \mathbf{e}_4 = \tau_5 & \mathbf{e}_2 \wedge \mathbf{e}_0 = \tau_8 \\ \mathbf{e}_1 \wedge \mathbf{e}_2 = \tau_3 & \mathbf{e}_3 \wedge \mathbf{e}_4 = \tau_6 & \mathbf{e}_3 \wedge \mathbf{e}_0 = \tau_9 \\ & & \mathbf{e}_4 \wedge \mathbf{e}_0 = \tau_{10} \end{cases}$$

then one is led, as in sec. 21, to represent $H_{\alpha\beta,\lambda\mu}$ by a symmetric matrix (H_{IJ}) ($I, J = 1, \ldots, 10$); because of the components of the tensor φ (35.1) leads to the matrix (H_{IJ}) having the form:

(35.2)

	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	0	0	0	0	0	0
2	0	a_{22}	$-a_{12}$	0	$-a_{12}$	$-a_{22}$	$-a_{02}$	0	0	a_{02}
3	0	$-a_{12}$	a_{11}	0	a_{11}	a_{12}	a_{01}	0	0	$-a_{01}$
4	0	0	0	0	0	0	0	0	0	0
5	0	a_{12}	a_{11}	0	a_{11}	a_{12}	a_{01}	0	0	$-a_{01}$
6	0	$-a_{22}$	a_{12}	0	a_{12}	a_{22}	a_{02}	0	0	$-a_{02}$
7	0	$-a_{02}$	a_{01}	0	a_{01}	a_{02}	a_{00}	0	0	$-a_{00}$
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	a_{02}	$-a_{01}$	0	$-a_{01}$	$-a_{02}$	a_{00}	0	0	a_{00}

One knows that it is necessary that:

$$H_{\alpha\beta} = \tau\, l_\alpha l_\beta .$$

By contracting (35.1), it follows that:

$$H_{\alpha\beta} = -(a_{00} + a_{11} + a_{22})\, l_\alpha l_\beta$$

namely:

$$\tau = -\,(a_{00} + a_{11} + a_{22}).$$

Therefore, in order that $H_{\alpha\beta} = 0$ it is necessary and sufficient that $a_{00} + a_{11} + a_{22} = 0$.

36. Discontinuity matrix for the curvature tensor in the penta-dimensional theories. In the case of the tensor $[R_{\alpha\beta,\lambda\mu}]$, one has $[R_{\alpha\beta}] = 0$, and, as a result, that $a_{00} + a_{11} + a_{22} = 0$. For the KALUZA-KLEIN theory, one obviously has $[R_{i0,k0}] = 0$, and, as a result, that $a_{00} = a_{11} + a_{22} = 0$. Therefore, the matrix $([R_{IJ}])$ has the form (35.2) so either $a_{00} + a_{11} + a_{22} = 0$ (JORDAN-THIRY theory) or $a_{00} = a_{11} + a_{22} = 0$ (KALUZA-KLEIN theory).

One easily recovers that reduced form by starting with the formulas:

$$[R_{\underline{ij},\underline{kl}}] = [\overset{*}{R}_{\underline{ij},\underline{kl}}], \qquad [R_{\underline{ij},\underline{k}\underline{0}}] = \frac{\beta}{2}\xi[\overset{*}{\nabla}_{\underline{k}}\, F_{\underline{ji}}], \qquad [R_{\underline{i}\,\underline{0},\underline{k}\,\underline{0}}] = \xi^{-1}[\overset{*}{\nabla}_{\underline{k}}(\partial_{\underline{i}}\xi)],$$

and from the prior results. We first note that, from the HADAMARD condition, there exists a scalar a_{00} at x such that:

$$\xi^{-1}\overset{*}{\nabla}_{\underline{k}}(\partial_{\underline{i}}\xi) = a_{00}\, l_{\underline{i}}\, l_{\underline{k}} . \tag{36.1}$$

If the adapted orthonormal frame is chosen in such a way that $\boldsymbol{l} = \mathbf{e}_4 + \mathbf{e}_1$ then it results from (36.1) that the submatrix of $([R_{IJ}])$ that corresponds to $7 \leq I, J \leq 10$ has the form that is described in (35.2). Moreover, from (32.6) one deduces that:

$$[\overset{*}{R}_{\underline{ik}}] = \xi^{-1}\overset{*}{\nabla}_{\underline{k}}(\partial_{\underline{i}}\xi) = a_{00}\, l_{\underline{i}}\, l_{\underline{k}} . \tag{36.2}$$

Therefore, $[\overset{*}{R}_{\underline{ij},\underline{kl}}]$, which satisfies:

$$l_{\underline{k}}[\overset{*}{R}_{\underline{ij},\underline{kl}}] + l_{\underline{i}}[\overset{*}{R}_{\underline{ik},\underline{kl}}] + l_{\underline{j}}[\overset{*}{R}_{\underline{ki},\underline{kl}}] = 0$$

and (36.2), represents the submatrix of $([R_{IJ}])$ that corresponds to $(1 \leq I, J \leq 6)$ in (35.2) with $a_{00} = -\,(a_{11} + a_{22})$. Finally, from:

$$\frac{\beta}{2}\xi[\overset{*}{\nabla}_{\underline{k}}\, F^{\underline{ij}}] = \varphi_{\underline{ij}}\, l_{\underline{k}}$$

in which $\varphi_{\underline{ij}}$ is singular, and the expression for this form when $\boldsymbol{l} = \mathbf{e}_4 + \mathbf{e}_1$, one deduces the submatrix that corresponds to $1 \leq I \leq 6$, $7 \leq J \leq 10$.

37. **Radiation in the Kaluza-Klein theory.**

1) We place ourselves in the context of the Kaluza-Klein theory and refer the manifold V_5 to an adapted orthonormal frame. To simply the notations, we suppress the index underline in what follows. Suppose that the spacetime metric ds^2 corresponds to a state of pure total radiation and that the electromagnetic field itself represents a state of electromagnetic radiation with the same isotropic fundamental vector. We introduce the isotropic vector on V_5 that is orthogonal to ξ, and project it onto V_4 along the preceding vector.

(37-1) (*a*) $S l_h R^*_{ij,kl} = 0$ (*b*) $l^i R^*_{ij,kl} = 0,$

in which S denotes the sum over all cyclic permutations of the indices h, i, j here, and, on the other hand:

(37-2) (*a*) $\mathrm{S}\, l_h F_{ij} = 0$ (*b*) $l^i F_{ij} = 0.$

For a singular 2-form F, the formula (33-1) leads us to study the tensor:

$$S\, l_h (F_{ik} F_{jl} - F_{il} F_{jk}).$$

From this, it follows that:

$$Sl_h(F_{ik} F_{jl} - F_{il} F_{jk}) = - Sl_h(F_{ik} F_{lj} + F_{il} F_{jk}).$$

Now, since F is a singular 2-form it is an exterior product, and:

$$F_{ik} F_{jl} + F_{il} F_{jk} + F_{ij} F_{kl} = 0.$$

Therefore:

$$Sl_h(F_{ik} F_{jl} - F_{il} F_{jk}) = (Sl_h F_{ij}) F_{kl}\,.$$

We therefore see that for a singular 2-form F:

(37-3) $Sl_h(F_{ik} F_{jl} - F_{il} F_{jk}) = 0\,,$ $l^i(F_{ik} F_{jl} - F_{il} F_{jk}) = 0.$

If we start with (37-3) then the hypotheses (37-1) and (37-2) thus entail:

(37-4) (*a*) $Sl_h R_{ij,kl} = 0$ (*b*) $l^i R_{ij,kl} = 0.$

On the other hand, from (33-3), one deduces:

$$l_h R_{i0,k0} - l_i R_{h0,k0} = \frac{\beta^2}{4}(l_h F_{ir} - l_h F_{hr}) F_k^{\ r}.$$

From $(37\text{-}2)_b$, one thus has:

(37-5) (*a*) $l_h R_{i0,k0} - l_i R_{0h,k0} = 0$ (*b*) $l^i R_{i0,k0} = 0.$

2) We introduce the tensor on V_5 that has the same symmetry type as the curvature tensor, and is defined, relative to an adapted orthonormal frame, by:

(37-6) $$P_{ij,kl} = R_{ij,kl}\,, \quad P_{ij,k0} = 0, \quad P_{i0,k0} = -R_{i0,k0},$$

in which the "–" sign appears for signature reasons. The tensor $P_{\alpha\beta}$, which is a contraction of $P_{\alpha\beta,\gamma\delta}$, verifies:

(37-7) $$P_{i0} = 0, \quad P_{00} = -R_{00}.$$

Since l_0 is null, relations (37-4) and (37-5) may be expressed by:

(37-8) $$(a) \quad S\, l_\alpha P_{\beta\gamma,\lambda\mu} = 0 \qquad (b) \quad l^\alpha P_{\alpha\beta,\lambda\mu} = 0,$$

in which the vector l_α is orthogonal to ξ_α.

3) Conversely, suppose that the tensor $P_{\beta\gamma,\lambda\mu}$ is defined by (37-6) in a adapted orthonormal frame is such that there exists a vector l_α that is orthogonal to ξ_α and satisfies the relations (37-8).

$P_{\alpha\beta}$ is proportional to $l_\alpha l_\beta$. Since $l_0 = 0$, one has $P_{00} = 0$, namely, $R_{00} = 0$, and, from (33-6):

(37-9) $$F_{rs} F^{rs} = 0.$$

Moreover, from $(37\text{-}5)_b$:

(37-10) $$l^i F_{ir} F_k^{\ r} = 0.$$

l^i is thus a proper vector with proper value 0 for the MAXWELL tensor of F and the 2-form F is singular. From (37-4), because of (37-3), one deduces that:

$$S l_h \overset{*}{R}_{ij,kl} = 0, \qquad l^i \overset{*}{R}_{ij,kl} = 0,$$

and the metric ds^2 defines a state of pure total radiation.

Suppose that the field equations (33-12) are satisfied in the absence of sources ($\Theta_{ik} = 0$). From (37-9), one deduces that:

$$R_{ik} = 0.$$

Since the 2-form F is singular:

$$\frac{\beta^2}{2} F_{ir} F_k^{\ r} = -\tau\, l_i\, l_k\,,$$

and, from (33-4):

$$\overset{*}{R}_{ik} = -\tau\, l_i\, l_k\,,$$

which explains the expression "state of pure total radiation."

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in which $l(\mathbf{v})$ is a linear form in **v**. From this, it results that $\overset{*}{\nabla}_k \partial_i \xi = \lambda_k l_i$, and by reason of symmetry:

(39-2) $$\overset{*}{\nabla}_k \partial_i \xi = \tau l_i l_k ,$$

in which τ is a scalar. Therefore, in order that the relations (39-1) are satisfied, it is necessary and sufficient that one have (39-2), where l_i is an *isotropic* vector. In particular:

(39-3) $$\Delta^* \xi = 0.$$

When the hypotheses (39-1) are satisfied we say that the scalar field ξ corresponds to a state of pure total radiation.

2) In context of the JORDAN-THIRY theory we suppose that the quotient metric ds^2 (which differs here from the spacetime metric $d\bar{s}^2 = \xi\, ds^2$), the form F and the scalar ξ satisfy the radiation conditions for the same isotropic fundamental vector. One has (39-2) in an adapted orthonormal frame:

(39-4) $$S l_h \overset{*}{R}_{ij,kl} = 0 \qquad l^i \overset{*}{R}_{ij,kl} = 0,$$

and:

(39-5) $$S l_h F_{ij} = 0 \qquad l^i F_{ij} = 0.$$

By virtue of (32-1) and (32-3), these relations entail:

$$S l_\alpha P_{\beta\gamma,\lambda\mu} = 0 \qquad l^\alpha P_{\alpha\beta,\lambda\mu} = 0,$$

in which $P_{\alpha\beta,\lambda\mu}$ is again defined by (38-5).

40. Construction of an example of pure radiation with an electromagnetic field. We propose to construct an example of pure total radiation that satisfies the KALUZA-KLEIN equations in the absence of sources. We must therefore obtain a gravitational field on the spacetime V_5 that satisfies (37-1) and (37-2) and a singular electromagnetic field that satisfies the EINSTEIN-MAXWELL equations of general relativity.

1) Consider a domain D of spacetime that is endowed with the metric (27-2), which may be written, with the following notations:

(40-1) $$ds^2 = e^{2\psi}[(dx^4)^2 - (dx^1)^2] - [\xi^2 (dx^2)^2 + \xi^2 (dx^3)^2] = g_{ij}\, dx^i\, dx^j,$$

in which ψ, $\xi > 0$, $\eta > 0$ are three functions of one variable:

$$u = x^4 - x^1.$$

Introduce the potential vector φ_i that is defined in (x^k) coordinates by:

$$\varphi_1 = \varphi_2 = \varphi_4 = 0 \qquad \varphi_3 = \varphi(u).$$

A 2-form F corresponds to the potential vector, and its only non-null components are:

$$F_{34} = -\partial_4\varphi = -\varphi' \qquad F_{31} = -\partial_1\varphi = \varphi'.$$

The vector l_i, which is the gradient of u, in which covariant components are:

$$l_1 = -1 \qquad l_2 = l_3 = 0 \qquad l_4 = 1$$

satisfies:

(40-2) $$S\, l_h F_{ij} = 0$$

in which S indices summation over cyclic permutations. This relation is indeed satisfied for $h = 4$, and either $i = 2$, $j = 3$, or $i = 3$, $j = 1$, or $i = 1$, $j = 2$. On the other hand, since l^i admits the contravariant components:

$$l^4 = l^1 = e^{-2\psi} \qquad l^2 = l^3 = 0$$

one has:

(40-3) $$l^i F_{ij} = 0.$$

Therefore, the form F is singular and admits the isotropic fundamental vector l_i.

2) We evaluate $\overset{*}{\nabla}_j\, l_i$, where $\overset{*}{\nabla}_j$ is the symbol for the covariant derivative in the metric (40-1). One first has:

$$\overset{*}{\nabla}_j\, l_i = -\overset{*}{\Gamma}{}^4_{i2} + \overset{*}{\Gamma}{}^1_{i2}.$$

Now:

$$\overset{*}{\Gamma}{}^4_{i2} = g^{44}[i2, 4] = \tfrac{1}{2}e^{-2\psi} g'_{i2} \qquad \overset{*}{\Gamma}{}^1_{i2} = g^{11}[i2, 1] = \tfrac{1}{2}e^{-2\psi} g'_{i2}.$$

One thus obtains:

$$\overset{*}{\nabla}_2\, l_i = \overset{*}{\nabla}_i\, l_2 = 0,$$

and, similarly:

$$\overset{*}{\nabla}_3\, l_i = \overset{*}{\nabla}_i\, l_3 = 0.$$

On the other hand:

$$\overset{*}{\nabla}_1\, l_1 = -\overset{*}{\Gamma}{}^4_{11} + \overset{*}{\Gamma}{}^1_{11}.$$

Now, from the results of sec. 27:

$$\overset{*}{\Gamma}{}^4_{11} = \psi' \qquad \overset{*}{\Gamma}{}^1_{11} = -\psi'.$$

From this, it results that:

$$\overset{*}{\nabla}_1 l_1 = -2\psi' ,$$

and likewise:

$$\overset{*}{\nabla}_4 l_1 = \overset{*}{\nabla}_1 l_4 = 2\psi' , \qquad \overset{*}{\nabla}_4 l_4 = -2\psi' .$$

We have thus established that:

(40-4) $$\overset{*}{\nabla}_j l_i = -2\psi' l_i l_j .$$

From relation (40-2), one deduces by derivation:

$$S l_h \overset{*}{\nabla}_k F_{ij} + S l_h \overset{*}{\nabla}_k \cdot F_{ij} = 0,$$

that is, by virtue of (40-4):

$$S l_h \overset{*}{\nabla}_k F_{ij} - 2\psi' l_h S\, l_k F_{ij} = 0,$$

i.e.:

(40-5) $$S l_h \overset{*}{\nabla}_k F_{ij} = 0.$$

Similarly, by deriving (40-3), one obtains:

$$l^i \overset{*}{\nabla}_k F_{ij} + \overset{*}{\nabla}_k l^i F_{ij} = 0.$$

That is, by virtue of (40-4):

(40-6) $$l_h \overset{*}{\nabla}_k F_{ij} = 0.$$

3) On the manifold V_5 of the Kaluza-Klein theory, consider the metric:

(40-7) $$d\sigma^2 = e^{2\psi}[(dx^4)^2 - (dx^1)^2] - [\xi^2(dx^2)^2 + \xi^2(dx^3)^2] - (dx^0 - \varphi\, dx^3)^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta.$$

The quotient metric under the isometry group $x^0 \to x^0$ + const. coincides with ds^2, and the relations (33-1), (33-2), (33-3) apply in adapted orthonormal frames with the constant β = 1. Since the F is singular, one has:

$$S l_{\underline{h}} (F_{\underline{i}\underline{k}} F_{\underline{j}\underline{l}} - F_{\underline{i}\underline{l}} F_{\underline{j}\underline{k}}) = 0, \qquad l^{\underline{i}} (F_{\underline{i}\underline{k}} F_{\underline{j}\underline{l}} - F_{\underline{i}\underline{l}} F_{\underline{j}\underline{k}}) = 0.$$

On the other hand, from the results of sec. 27:

$$S l_{\underline{h}} \overset{*}{R}_{\underline{i}\underline{j},\underline{k}\underline{l}} = 0 \qquad l^{\underline{i}} \overset{*}{R}_{\underline{i}\underline{j},\underline{k}\underline{l}} = 0.$$

Therefore, one has:

(40-8) $$S l_{\underline{h}} R_{\underline{i}\underline{j},\underline{k}\underline{l}} = 0 \qquad l^{\underline{i}} R_{\underline{i}\underline{j},\underline{k}\underline{l}} = 0.$$

The relations (40-5) and (40-6) express that:

(40-9) $$S l_{\underline{h}} \mathrm{R}_{\underline{i}\underline{j},\underline{k}\underline{0}} = 0 \qquad l^{\underline{i}} \mathrm{R}_{\underline{i}\underline{j},\underline{k}\underline{0}} = 0.$$

Finally, from the singular character of F and from sec. 37, no. 1, it results that:

(40-10) $$l_{\underline{h}} R_{\underline{i}\underline{0},\underline{k}\underline{0}} + l_{\underline{i}} R_{\underline{0}\underline{h},\underline{k}\underline{0}} = 0 \qquad l^{\underline{i}} R_{\underline{i}\underline{0},\underline{k}\underline{0}} = 0.$$

Therefore, the vector $l_{\underline{\alpha}}$, whose components are $l_{\underline{i}}$ is orthogonal to $\xi_{\underline{\alpha}}$ and satisfies:

(40-11) $$S l_\alpha R_{\beta\gamma,\lambda\mu} = 0 \qquad l^\alpha R_{\beta\gamma,\lambda\mu} = 0.$$

One knows that it then results that:

$$R_{\alpha\beta} = \tau\, l_\alpha l_\beta .$$

4) From (40-12), $R_{\underline{i}\underline{0}} = 0$ and the MAXWELL equations are satisfied by F. In order to verify the field equations (33-12) in the absence of source, namely $R_{\underline{i}\underline{k}} = 0$, from (40-12), it suffices for us to verify $R_{\underline{4}\underline{4}} = 0$, that is, from (33-4):

$$\overset{*}{R}_{44} = -\xi^{-1}(\xi'' - 2\psi'\xi') - \eta^{-1}(\eta'' - 2\psi'\eta') .$$

On the other hand:

$$F_{4r} F_4^{\ r} = F_{43} F_4^{\ 3} = -\eta^2 \varphi'^2 .$$

From this, it results that the functions ξ, η, η, φ must be coupled by the relation:

(40-13) $$\xi^{-1}(\xi'' - 2\psi'\xi') - \eta^{-1}(\eta'' - 2\psi'\eta') + \tfrac{1}{2}\eta^{-1}\varphi'^2 = 0.$$

If we limit ourselves to $u > 0$, and set $\xi\eta = u^2$, $\eta/\xi = e^{2\beta}$. (40-13) may then be put into the form:

(40-14) $$2\psi' = u\beta'^2 + \frac{1}{4u} e^{-2\beta} \varphi'^2 .$$

If β and φ are given functions for $u > 0$ then (40-10) allows us to determine ψ by quadrature in the same domain.

For the subset $(\mathbb{R}^4)^+$ $(u > 0)$ of spacetime $\mathbb{R}^4$ that is defined by x^i, we perform the change of variables (28-7) on the metric ds^2. Because of (40-14), one obtains the following expression for the metric on $(\mathbb{R}^4)^+$:

$$ds^2 = du\,d\bar{v} - (d\bar{y}^2 + d\bar{z}^2) - 2\beta'\left(\bar{y}\,d\bar{y} - \bar{z}\,d\bar{z} - \frac{\bar{y}^2 - \bar{z}^2}{u} du\right) du - \beta'^2 u\bar{v}\,du^2$$
$$- \left(\bar{v} - \frac{\bar{y}^2 + \bar{z}^2}{u} du\right) \frac{1}{4u} e^{-2\beta} \varphi'^2 du^2 .$$

Since u_0 is strictly positive, we choose β' and φ' to be functions of class C^2 that are *null for $u \leq u_0$*. One thus obtains a metric that is Euclidian for $u \leq u_0$ and non-Euclidian for $u > u_0$, and which, when combined with the form F that we introduced, defines a solution to the problem that we had posed.

VI. **A process of field quantization.**

A. THE ELECTROMAGNETIC FIELD.

41. Fourier transformation. In the quantum theory of fields the electromagnetic field $F_{\alpha\beta}$ is generally quantized by making recourse to the potential vector and the study of gauge transformations. We shall indicate a direct quantization process for the electromagnetic field in special relativity. This process, which is entirely linked with the notion of electromagnetic radiation, may be adapted, as we will see, to the quantization of the gravitational field.

In all of this section, V_4 is the spacetime of special relativity, which we assume to be referred to some orthonormal frame $(\mathbf{e}_\alpha)$ (α, or any Greek index = 0, 1, 2, 3). The metric on V_4, when referred to such a frame, will be denoted by:

(41-1) $$ds^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

in which the $\mathbf{x} \in V_4$ admits the coordinates (x^α). Consider an electromagnetic field F, which is assumed to satisfy the MAXWELL equations with an electric current vector that is identically null, i.e.:

(41-2) $$S\partial_\alpha F_{\beta\gamma} = 0 \qquad\qquad \partial_\alpha F^\alpha{}_\beta = 0,$$

in which S denotes the sum over all cyclic permutations of the three indices.

With these simple hypotheses, the current that is defined by F is a FOURIER transformation on spacetime, and one may write it, with an obvious meaining (by abuse of the usual notation amongst the school of physicists), as:

(41-3) $$F_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^4}\int U_{\alpha\beta}(\mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{x}}\, d\tau(\mathbf{p}),$$

in which $\mathbf{p}$ describes Minkowski space, and $d\tau(\mathbf{p})$ is the corresponding hypervolume element:

$$d\tau(\mathbf{p}) = dp^0 \wedge dp^1 \wedge dp^2 \wedge dp^3.$$

From (41-2), one deduces that:

$$S\, p_\alpha U_{\beta\gamma} = 0 \qquad\qquad p^\alpha U_{\alpha\beta} = 0.$$

As a result, $\mathbf{p}$ is different from zero only when it is isotropic. If V_4 is referred to a frame $(\mathbf{e}a)$ then we let $\mathbf{p}$ be an arbitrary vector of spacetime, and set:

$$\mathbf{p} = \boldsymbol{l} + \lambda\mathbf{e}_0,$$

in which the components of the *isotropic* vector:

$$\boldsymbol{l} = l^0 e_0 + l^u \mathbf{e}_u \qquad (u = 1, 2, 3)$$

satisfy:

$$(l^0)^2 = \sum_u (l^u)^2 .$$

From the expression for $d\tau(\mathbf{p})$, one has:

$$d\tau(\mathbf{p}) = \lambda^0 \, d\lambda \wedge d\Omega(\mathbf{e}),$$

in which:

$$d\Omega(\boldsymbol{l}) = \frac{dl^1 \wedge dl^2 \wedge dl^3}{l^0}$$

is the invariant volume element of the isotropic cone C. Upon introducing a DIRAC measure for the variable l one may put (41-3) into the form:

$$F_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^4} \int G_{\alpha\beta}(\boldsymbol{l}) \, e^{il\cdot x} \, d\Omega(\boldsymbol{l}).$$

42. Quantization conditions. We substitute a 2-form for the 2-form F with scalar values, which still has the same notation, but takes its values in a vector space of operators on a complex HILBERT space. We make the following hypotheses:

a) The values of F are Hermitian operators.

b) F verifies equations (41-2).

We denote the passage to an adjoint operator by an *. Formula (41-4) is again valid here, provided that $G_{\alpha\beta}$ is a tensor with values in M. In this formula, it results that:

$$F^*_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_C G^*_{\alpha\beta}(\boldsymbol{l}) \; e^{-il\cdot x} \, d\Omega(\boldsymbol{l}),$$

that is, after exchanging $\boldsymbol{l}$ with $-\boldsymbol{l}$:

$$F^*_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_C G^*_{\alpha\beta}(-\boldsymbol{l}) \, e^{\,il\cdot x} \, d\Omega(\boldsymbol{l}).$$

From the Hermitian character of the values of F, one thus deduces that:

(42-1) $$G^*_{\alpha\beta}(-\boldsymbol{l}) = G_{\alpha\beta}(\boldsymbol{l}).$$

It is possible to transform formula (41-4) in such a way as to reduce the domain of integration to the positive nappe C^+ of the isotropic cone. It thus follows that:

$$F_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{C^+} (G_{\alpha\beta}(\boldsymbol{l}) \, e^{il\cdot x} + G_{\alpha\beta}(-\boldsymbol{l}) e^{-il\cdot x}) \, d\Omega(\boldsymbol{l}),$$

namely, from (42-1):

$$F_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{C^+} (G_{\alpha\beta}(\boldsymbol{l})\, e^{i\boldsymbol{l}\cdot x} + G^*_{\alpha\beta}(-\boldsymbol{l})\, e^{-i\boldsymbol{l}\cdot x})\, d\Omega(\boldsymbol{l}). \tag{42-2}$$

The Hermitian character of the values of F is clear in the formula (42-2). The hypothesis b translates into the relations:

$$S\, l_\alpha G_{\beta\gamma} = 0 \qquad l^\alpha G_{\alpha\beta} = 0. \tag{42-3}$$

Suppose that $G_{\alpha\beta}$ is a 2-form with scalar values that satisfies (41-3). These relations express that the 2-form envisioned is singular. Let $\mathbf{n}^{(1)}(\boldsymbol{l})$ and $\mathbf{n}^{(2)}(\boldsymbol{l})$ denote two arbitrary normed orthogonal vectors in the 3-plane that is tangent to the isotropic cone along $\boldsymbol{l}$. These vectors define a spatially oriented 2-plane and a time-oriented orthogonal 2-plane that contains $\boldsymbol{l}$. If $\mathbf{e}_0$ is an arbitrary unitary vector in this 2-plane then one obtains an orthonormal frame $(\mathbf{e}_\alpha)$ such that:

$$\frac{1}{l^0}\boldsymbol{l} = \mathbf{e}_0 + \mathbf{e}_1, \quad \mathbf{e}_2 = \mathbf{n}^{(1)}, \qquad \mathbf{e}_2 = \mathbf{n}^{(2)}. \tag{42-4}$$

From the study in sec. 7 (in particular, see (7-3)) it results that for any $G_{\alpha\beta}$ that satisfies (42-3), it is necessary and sufficient that:

$$G_{\alpha\beta}(\boldsymbol{l}) = \sum_i a(i, \boldsymbol{l})(l_\alpha n^{(i)}_\beta - l_\beta n^{(i)}_\alpha), \tag{42-5}$$

in which the indices $i, j, \ldots$, take the values 1 and 2, and the $a(i, \boldsymbol{l})$ are scalars.

By introducing a linear form on M with scalar values one immediately sees that if $G_{\alpha\beta}$ is a 2-form with values in M that satisfies (42-3) then the formula (42-5) is again valid, with the condition that one take the $a(i, \boldsymbol{l})$ to be elements of M.

The quantization of the field F is accomplished by postulating the bracket conditions:

$$\begin{cases} [a(i,\boldsymbol{l}), a(j,\boldsymbol{l}')] = 0 \\ [a^*(i,\boldsymbol{l}), a(j,\boldsymbol{l}')] = \dfrac{\hbar}{i}\delta_{ij}\delta_\Omega(\boldsymbol{l},\boldsymbol{l}'), \end{cases} \tag{42-6}$$

in which $\boldsymbol{l}$ and $\boldsymbol{l}'$ are two vectors of C^+, δ_{ij} is the KRONECKER symbol, and δ_Ω is the DIRAC measure, relative to the isotropic cone that is given volume element $d\Omega$. One may thus obtain a more condensed formalism by introducing the indices A, B, which take the values +1 and –1, and by setting:

$$a^{A}(i, \boldsymbol{l}) = a(i, \boldsymbol{l}) \text{ for } A = 1, \qquad a^A(i, \boldsymbol{l}) = a^*(i, \boldsymbol{l}) \text{ for } A = -1.$$

With these notations, the quantization conditions (42-6) may be written:

(42-7) $$[a^*(i,\boldsymbol{l}), a(j,\boldsymbol{l}')] = \frac{\hbar}{i} A\delta_{AB}\delta_{ij}\delta_\Omega(\boldsymbol{l},\boldsymbol{l}') .$$

43. An auxiliary formula. In the course of calculating the commutation relations that field F must, by virtue of (42-7), satisfy, we will be led to evaluate the following tensor, which is defined by starting with $\boldsymbol{l}$, $\mathbf{n}^{(1)}(\boldsymbol{l})$, $\mathbf{n}^{(2)}(\boldsymbol{l})$:

(43-1) $$P_{\alpha\beta,\lambda\mu}(\boldsymbol{l}) = \sum \delta_{ij}(l_\alpha n_\beta^{(i)} - l_\beta n_\alpha^{(i)})(l_\lambda n_\mu^{(j)} - l_\mu n_\lambda^{(j)}) .$$

For this evaluation, we start with the orthonormal frame that was introduced in sec. 42 that satisfies (42-4); we set:

$$\mathbf{e}_0 = \mathbf{u}, \qquad \mathbf{e}_1 = \frac{1}{l^0}\boldsymbol{l} - \mathbf{u} = k\boldsymbol{l} - \mathbf{u} \qquad (k = 1 / l^0).$$

By starting with the components of the vectors in this orthonormal frame, the metric tensor $\eta_{\alpha\beta}$ of V_4 may be expressed by:

$$\eta_{\alpha\beta} = u_\alpha u_\beta - (kl_\alpha - u_\alpha)(kl_\beta - u_\beta) - n_\alpha^{(1)} n_\beta^{(1)} - n_\alpha^{(2)} n_\beta^{(2)} .$$

From this, one deduces that:

$$\sum_{i,j} \delta_{ij} n_\alpha^{(i)} n_\beta^{(j)} = n_\alpha^{(1)} n_\beta^{(1)} + n_\alpha^{(2)} n_\beta^{(2)} = k(l_\alpha u_\beta + l_\beta u_\alpha) - k^2 l_\alpha l_\beta - \eta_{\alpha\beta}.$$

If we develop the right-hand side of (43-1) then we obtain:

$$\begin{aligned} P_{\alpha\beta,\lambda\mu}(\boldsymbol{l}) = {} & l_\alpha l_\lambda\{k(l_\beta u_\mu + l_\mu u_\beta) - k^2 l_\alpha l_\beta - \eta_{\alpha\beta}\} + l_\beta l_\mu\{k(l_\alpha u_\lambda + l_\lambda u_\alpha) - k^2 l_\alpha l_\lambda - \eta_{\alpha\lambda}\} \\ & - l_\alpha l_\mu\{k(l_\beta u_\lambda + l_\lambda u_\beta) - k^2 l_\beta l_\lambda - \eta_{\alpha\beta}\} - l_\beta l_\lambda\{k(l_\alpha u_\mu + l_\mu u_\alpha) - k^2 l_\alpha l_\mu - \eta_{\alpha\mu}\} . \end{aligned}$$

After simplification, the following formula follows:

(43-2) $$\sum_{i,j} \delta_{ij}(l_\alpha n_\beta^{(i)} - l_\beta n_\alpha^{(i)})(l_\lambda n_\mu^{(j)} - l_\mu n_\lambda^{(j)}) = -(\eta_{\alpha\lambda} l_\beta l_\mu + \eta_{\beta\mu} l_\alpha l_\lambda - \eta_{\beta\mu} l_\alpha l_\lambda - \eta_{\beta\lambda} l_\alpha l_\mu).$$

44. The commutation relations. By virtue of (42-5), at the point $\mathbf{x}$ of V_4, $F_{\alpha\beta}(\mathbf{x})$ may be written:

(44-1) $$F_{\alpha\beta}(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum_{\mathrm{A},j} \int_{C^+} a^{\mathrm{A}}(i,\boldsymbol{l}) e^{i\mathrm{A}\boldsymbol{l}\cdot \boldsymbol{x}} (l_\alpha n_\beta^{(j)} - l_\beta n_\alpha^{(j)}) d\Omega(\boldsymbol{l}) .$$

Let $\mathbf{x}$ be another point of V_4 for which:

$$F_{\lambda\mu}(\mathbf{x}') = \frac{1}{(2\pi)^3}\sum_{\mathrm{B},j}\int_{C^+} a^{\mathrm{B}}(j,\boldsymbol{l}')e^{i\mathrm{B}\boldsymbol{l}'\bullet x'}(l'_\lambda n'^{(j)}_\mu - l'_\mu n'^{(j)}_\lambda)d\Omega(\boldsymbol{l}'),$$

in which the $\mathbf{n}'^{(j)}(\boldsymbol{l}')$ are normal orthogonal vectors that are tangent to C along $\boldsymbol{l}'$. We evaluate the bracket:

$$[F_{\alpha\beta}(\mathbf{x}),\ F_{\lambda\mu}(\mathbf{x}')] = \frac{1}{(2\pi)^6}\int_{C^+}\int_{C^+}[a^{-\mathrm{A}}(i,\boldsymbol{l}), a^{\mathrm{B}}(j,\boldsymbol{l}')]e^{i(-\mathrm{A}\boldsymbol{l}\bullet x+\mathrm{B}\boldsymbol{l}'\bullet \mathbf{x}')}$$
$$\times (l_\alpha n^{(i)}_\beta - l_\beta n^{(i)}_\alpha)(l'_\lambda n'^{(j)}_\mu - l'_\mu n'^{(j)}_\lambda)d\Omega(\boldsymbol{l})d\Omega(\boldsymbol{l}').$$

After integrating over the variable $\boldsymbol{l}'$, one obtains:

$$[F_{\alpha\beta}(\mathbf{x}), F_{\lambda\mu}(\mathbf{x}')]$$
$$= \frac{\hbar}{i}\frac{1}{2(2\pi)^3}\sum_{\mathrm{A}}\int_{C^+}\mathrm{A}e^{-i\mathrm{A}\boldsymbol{l}\bullet(x-\mathbf{x}')}\{\sum_{i,j}\delta_{ij}(l_\alpha n^{(i)}_\beta - l_\beta n^{(i)}_\alpha)(l_\lambda n^{(j)}_\mu - l_\mu n^{(j)}_\lambda)\}d\Omega(\boldsymbol{l}).$$

From the earlier formula (43-2), one thus deduces:

(44-2) $$[F_{\alpha\beta}(\mathbf{x}),\ F_{\lambda\mu}(\mathbf{x}')] = -\frac{\hbar}{i}\frac{1}{2(2\pi)^3}\sum_{\mathrm{A}}\int_{C^+}\mathrm{A}e^{-i\mathrm{A}\boldsymbol{l}\bullet(x-\mathbf{x}')}$$
$$\times (\eta_{\alpha\beta}l_\beta l_\mu + \eta_{\beta\mu}l_\alpha l_\mu - \eta_{\alpha\mu}l_\beta l_\lambda - \eta_{\alpha\lambda}l_\alpha l_\mu)d\Omega(\boldsymbol{l}).$$

We are thus led to introduce the invariant distribution that is defined by:

(44-3) $$D(\mathbf{x}) = \frac{1}{2(2\pi)^3}\sum_{\mathrm{A}}\int_{C^+}Ae^{-iA\boldsymbol{l}\bullet x}d\Omega(\boldsymbol{l}),$$

which is nothing but the JORDAN-PAULI "propagator." One thus immediately obtains:

$$\partial_\alpha D(\mathbf{x}) = -\frac{i}{2(2\pi)^3}\sum_{A}\int_{C^+}Ae^{-iA\boldsymbol{l}\bullet x}l_\alpha d\Omega(\boldsymbol{l})$$

and:

$$\partial_\alpha\partial_\beta D(\mathbf{x}) = -\frac{i}{2(2\pi)^3}\sum_{A}\int_{C^+}Ae^{-iA\boldsymbol{l}\bullet x}l_\alpha l_\beta d\Omega(\boldsymbol{l}).$$

As a result, formula (44-2) may be written:

(44-4) $$[F_{\alpha\beta}(\mathbf{x}),\ F_{\lambda\mu}(\mathbf{x}')] = \frac{\hbar}{i}(\eta_{\alpha\lambda}\partial_\beta\partial_\mu + \eta_{\beta\mu}\partial_\alpha\partial_\lambda - \eta_{\alpha\mu}\partial_\beta\partial_\lambda - \eta_{\beta\lambda}\partial_\alpha\partial_\mu)D(\mathbf{x}-\mathbf{x}'),$$

in which the derivatives are taken with respect to the variable **x**. Formula (44-4) may also be written by substituting the scalar product $\mathbf{e}_\alpha \cdot \mathbf{e}_\lambda$ of the frame vectors for $\eta_{\alpha\lambda}$ and introducing the derivatives with respect to the coordinates of the point $\mathbf{x}'$.

$$(44\text{-}5) \qquad [F_{\alpha\beta}(\mathbf{x}),\ F_{\lambda\mu}(\mathbf{x}')] = -\frac{\hbar}{i}\{(\mathbf{e}_\alpha\cdot\mathbf{e}_\lambda)\overset{x}{\partial}_\beta\overset{x'}{\partial}_\mu + (\mathbf{e}_\beta\cdot\mathbf{e}_\mu)\overset{x}{\partial}_\alpha\overset{x'}{\partial}_\lambda - (\mathbf{e}_\alpha\cdot\mathbf{e}_\mu)\overset{x}{\partial}_\beta\overset{x'}{\partial}_\lambda - (\mathbf{e}_\beta\cdot\mathbf{e}_\lambda)\overset{x}{\partial}_\alpha\overset{x'}{\partial}_\mu\}D(\mathbf{x}-\mathbf{x}').$$

In (44-4) or (44-5), we recover the classical commutation relation for the quantum theory of the electromagnetic field ([10]). As one immediately verifies, the bracket that one obtains is completely compatible with equations (41-2); if $K_{\alpha\beta,\lambda\mu}$ denotes the bracket:

$$S\partial_\gamma K_{\alpha\beta,\lambda\mu} = 0 \qquad \partial_\alpha K^\alpha{}_{\beta,\lambda\mu} = 0,$$

in which S indicates a summation over all cyclic permutations of the three indices α, β, γ. This quantization defines an irreducible unitary representation of the inhomogeneous LORENTZ group that may be characterized by its restriction to the "little group," in the sense of WIGNER (viz., the subgroup that leaves an isotropic vector invariant). As one knows, the representation thus obtained is characterized by a null mass and a spin that is equal to 1.

45. **Form of the commutation relation in an arbitrary frame**. Suppose that we have two arbitrary neighborhoods of points **x** and $\mathbf{x}'$ that are referred to two different frames $(\mathbf{e}_\alpha)$ and $(\mathbf{e}_{\lambda'})$. We thus refer the electromagnetic field at **x** to the frame $(\mathbf{e}_\alpha)$ and the electromagnetic field at $\mathbf{x}'$ to $(\mathbf{e}_{\lambda'})$. Upon multiplying by the frame transition matrices, formula (44-5) may be written, after suppressing the indices x and x' as no longer useful:

$$(45\text{-}1) \qquad [F_{\alpha\beta}(\mathbf{x}),\ F_{\lambda'\mu'}(\mathbf{x}')] = -\frac{\hbar}{i}\{(\mathbf{e}_\alpha\cdot\mathbf{e}_{\lambda'})\partial_\beta\partial_{\mu'} + (\mathbf{e}_\beta\cdot\mathbf{e}_{\mu'})\partial_\alpha\partial_{\lambda'} - (\mathbf{e}_\alpha\cdot\mathbf{e}_{\mu'})\partial_\beta\partial_{\lambda'} - (\mathbf{e}_\beta\cdot\mathbf{e}_{\lambda'})\partial_\alpha\partial_{\mu'}\}(\mathbf{x}-\mathbf{x}'),$$

in which the two sides represent a bitensor, an anti-symmetric tensor of order two, at **x**, and an anti-symmetric tensor of order two at $\mathbf{x}'$. The right-hand side clearly involves the bitensor that is defined by the products $(\mathbf{e}_\alpha\cdot\mathbf{e}_{\lambda'})$, and it is easy to obtain an interesting expression for it. Indeed, one has:

$$\mathbf{x}' - \mathbf{x} = x^{\lambda'}\mathbf{e}_{\lambda'} - x^\alpha\mathbf{e}_\alpha .$$

As a result, if s denotes the spatio-temporal interval that joins the point **x** to the point.

[10] For example, see G. WENTZEL, *Quantum Theory of Fields,* pp. 115.

$$s^2 = (\mathbf{x}' - \mathbf{x})^2 = x^{\lambda'} x^{\mu'} \mathbf{e}_{\lambda'} \cdot \mathbf{e}_{\mu'} + x^\alpha x^\beta \mathbf{e}_\alpha \cdot \mathbf{e}_\beta - 2x^\alpha x^{\lambda'} \mathbf{e}_\alpha \cdot \mathbf{e}_{\lambda'} .$$

By differentiation, one obtains:

$$\partial_\alpha \partial_{\lambda'} s^2 = -2\,\mathbf{e}_\alpha \cdot \mathbf{e}_{\lambda'} .$$

Therefore, if we introduce the bitensors at $\mathbf{x}$ and $\mathbf{x}'$ that are defined by:

(45-2) $$\theta_{\alpha\lambda'} = \partial_\alpha \partial_{\lambda'} s^2$$

then one obtains the following form for the commutation relation:

(45-3) $$[F_{\alpha\beta}(\mathbf{x}),\ F_{\lambda'\mu'}(\mathbf{x}')] = -\frac{\hbar}{i}\{\theta_{\alpha\lambda'}\nabla_\beta\nabla_{\mu'} + \theta_{\beta\mu'}\nabla_\alpha\nabla_{\lambda'} - \theta_{\alpha\mu'}\nabla_\beta\nabla_{\lambda'} - \theta_{\beta\lambda'}\nabla_\alpha\nabla_{\mu'}\}\,D(\mathbf{x} - \mathbf{x}').$$

In this form, the relation is valid in an arbitrary moving frame, and, in particular, if one refers spacetime to local curvilinear coordinates (x^α) in a neighborhood of $\mathbf{x}$, and $(x^{\lambda'})$ in a neighborhood of $\mathbf{x}'$.

B. THE GRAVITATIONAL FIELD.

46. The field equations. Consider a Riemannian spacetime V_4 and denote its curvature tensor by $R^\alpha{}_{\beta,\lambda\mu}$. It enjoys the following symmetry properties:

(46-1) $$R_{\alpha\beta,\lambda\mu} = -R_{\beta\alpha,\lambda\mu} = -R_{\alpha\beta,\mu\lambda} = R_{\lambda\mu,\alpha\beta},$$

and satisfies the identity:

(46-2) $$SR^\alpha{}_{\beta,\lambda\mu} = 0,$$

which is nothing but the integrability condition for the torsion in the case where it is zero. We assume that the spacetime in question V_4 satisfies the EINSTEIN conditions:

(46-3) $$R_{\alpha\beta} = 0.$$

The analogue of the MAXWELL equations, and the study made of the state of pure radiation lead us to adopt the *field equations:*

(46-4) $$(a)\ \ S\,\nabla_\alpha \mathrm{R}^\lambda{}_{\mu,\beta\gamma} = 0; \quad (b)\ \ \nabla_\alpha R^\alpha{}_{\beta,\lambda\mu} = 0,$$

in which ∇_α is the covariant derivative operator for the Riemannian connection. $(46\text{-}4)_a$ is nothing but the BIANCHI identity, an integrability condition for the equations that couple the curvature tensor to the connection. From $(46\text{-}4)_a$, one deduces, by contraction:

(46-5) $$\nabla_\alpha R^\alpha{}_{\mu,\beta\gamma} = \nabla_\beta R_{\mu\gamma} - \nabla_\gamma R_{\mu\beta},$$

and one sees that (46-3) entails that $(46\text{-}4)_b$.

2) We propose to show that, conversely, if the field equations (46-4) are satisfied then (46-3) may be considered to be a simple initial condition. More precisely, let Σ be a spatially oriented hypersurface in V_4 on which $R_{\alpha\beta}$ is annulled identically. If the equations (46-4) are satisfied then we shall establish that $R_{\alpha\beta}$ is also necessarily null outside of Σ.

Let $x^0 = 0$ be the local equation for Σ. Since this hypersurface is spatially oriented one has $g^{00} > 0$. From $(46\text{-}4)_b$ and (46-5), it results that:

$$\nabla_\beta R_{\mu\gamma} - \nabla_\gamma R_{\mu\beta} = 0, \tag{46-6}$$

and after contracting $(46\text{-}4)_b$ it follows that:

$$\nabla_\alpha R^\alpha{}_\lambda = 0,$$

which may also be written:

$$g^{\alpha\beta}\, \nabla_\alpha R_{\beta\lambda} = 0. \tag{46-7}$$

Let u, v, …, denote inidces that take the values 1, 2, 3. By expanding the left-hand side of (46-7), one has:

$$g^{00}\, \nabla_0 R_{0\lambda} + g^{0u}(\nabla_0 R_{u\lambda} + \nabla_u R_{0\lambda}) + g^{uv}\, \nabla_u R_{v\lambda} = 0,$$

namely, from (46-6):

$$g^{00}\, \nabla_0 R_{0\lambda} = -\,2g^{0u}\nabla_u R_{0\lambda} - g^{uv}\, \nabla_u R_{v\lambda}\,. \tag{46-8}$$

If we set $\beta = 0$ and $\gamma = 0$ in (46-6) and $\lambda = 0$ in (46-8) then one obtains the following linear homogeneous first order partial differential equation:

$$\begin{cases} \nabla_0 R_{uu} = \nabla_u R_{u0} \\ g^{00}\nabla_0 R_{00} = -2g^{0u}\nabla_u R_{00} - g^{uv}\nabla_u R_{v0} \end{cases}$$

in which $g^{00} \neq 0$. For an initial datum $R_{\alpha\beta}$ that is null on Σ, this system admits no solution but the null solution, which proves the property.

3) Suppose that V_4 is supported by a Minkowski space. If the gravitational field envisioned is weak then the metric tensor $g_{\alpha\beta}$ on V_4 may be written:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon\psi_{\alpha\beta}, \tag{46-9}$$

in which $\eta_{\alpha\beta}$ defines the Minkowski metric and ε is infinitesimal. If V_4 is referred to an orthonormal frame, relative to the metric $\eta_{\alpha\beta}$, then the corresponding coefficients of the Riemannian connection on V_4 are of order ε, as well as the components of the curvature tensor. If the indices are raised and lowered by means of the tensor $\eta_{\alpha\beta}$ then the principal part of the curvature tensor $H^\alpha{}_{\beta,\lambda\mu}$ satisfies the identity (46-1) and (46-2) and is restricted by equations that are deduced from (46-4) if we substitute ordinary derivatives for the covariant derivatives relative to $g_{\alpha\beta}$.

47. **Quantization conditions.** Therefore, take spacetime V_4 to be MINKOWSKI space, which we always assume to referred to orthonormal frames ($\mathbf{e}_\alpha$). We are thus led to describe the gravitational field by means of a tensor $H_{\alpha\beta,\lambda\mu}$ that satisfies the identities:

(47-1) $$H_{\alpha\beta,\lambda\mu} = - H_{\beta\alpha,\lambda\mu} = - H_{\alpha\beta,\mu\lambda} = H_{\lambda\mu,\alpha\beta},$$

and the identity:

(47-2) $$SH^{\lambda}{}_{\alpha,\beta\gamma} = 0,$$

where the indices are raised and lowered by means of the spacetime metric $\eta_{\alpha\beta}$. We take our field equations to be:

(47-3) $$(a)\quad S\partial_\alpha H^{\lambda}{}_{\alpha,\beta\gamma} = 0 \qquad (b)\qquad \partial_\alpha H^{\alpha}{}_{\beta,\lambda\mu} = 0,$$

and add the supplementary condition to these equations that:

(47-4) $$H_{\alpha\beta} = H^{\lambda}{}_{\alpha,\beta\gamma} = 0.$$

The same reasoning as in the preceding paragraph 2, in which one substitutes the tensor $H_{\alpha\beta,\lambda\mu}$ for $R_{\alpha\beta,\lambda\mu}$, shows that for a solution of (47-3) it suffices to verify this condition on a hypersurface Σ that is spatially oriented.

An ordinary tensor, such as H, may be identified with a multilinear form $H_{\alpha\beta,\lambda\mu} V^{\alpha}_{(1)} V^{\beta}_{(2)} V^{\lambda}_{(3)} V^{\mu}_{(4)}$ with scalar values. We replace it with a multilinear form that we denote with the same notation, but it takes its values in the vector space M and, naturally, it satisfies the conditions (47-1) and (47-2). In a manner that is analogous to the electromagnetic case, we assume that:

a) The values of H are Hermitian operators.
b) H satisfies equations (47-3).

Under some simple hypotheses that permit the introduction of the FOURIER transformation, and with notations that are analogous to the ones that were used in the electromagnetic case, H may be put into the form:

(47-5) $$H_{\alpha\beta,\lambda\mu}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{C^+} (K_{\alpha\beta,\lambda\mu}(\boldsymbol{l}) e^{i\boldsymbol{l}\cdot\boldsymbol{x}} + K^{*}_{\alpha\beta,\lambda\mu}(\boldsymbol{l}) e^{-i\boldsymbol{l}\cdot\boldsymbol{x}})\, d\Omega(\boldsymbol{l}),$$

in which $\mathbf{x} \in V_4$ and is a tensor with values in M. The hypotheses b translates into the relations:

(47-6) $$(a)\qquad S\, l_\alpha K_{\beta\gamma,\lambda\mu} = 0 \qquad\qquad (b)\qquad l^{\alpha} K_{\alpha\beta,\lambda\mu} = 0.$$

Suppose that $H_{\alpha\beta,\lambda\mu}$ is an ordinary tensor that enjoys the symmetry properties (47-1) and satisfies (47-3). The corresponding tensor $K_{\alpha\beta,\lambda\mu}(\boldsymbol{l})$ enjoys the same symmetry properties and, from (21-9), the solutions to (47-6) are given by:

$$K_{\alpha\beta,\lambda\mu}(\boldsymbol{l}) = \sum_{i,j} a(i,j,\boldsymbol{l})(l_\alpha n_\beta^{(i)} - l_\beta n_\alpha^{(i)})(l_\lambda n_\mu^{(j)} - l_\mu n_\lambda^{(j)}) \tag{47-7}$$

with:

$$a(i, j, \boldsymbol{l}) = a(j, i, \boldsymbol{l}),$$

by reason of symmetry.

By introducing a linear form with scalar values on M, one also sees that if $K_{\alpha\beta,\lambda\mu}$ is a tensor with values in that enjoys some specified symmetry properties and satisfies (47-6) then the formula (47-7) is always valid, on the condition that the $a(i, j, \boldsymbol{l})$ take their values in M.

The quantization of the field H is effected by postulating the bracket conditions:

$$\begin{cases} [a(i,j,\boldsymbol{l}), a(h,k,\boldsymbol{l}')] = 0 \\ [a^*(i,j,\boldsymbol{l}), a(h,k,\boldsymbol{l}')] = \dfrac{\hbar}{i}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh} - \delta_{ij}\delta_{hk})\delta_\Omega(\boldsymbol{l},\boldsymbol{l}') \end{cases} \tag{47-8}$$

in which $\boldsymbol{l}$ and $\boldsymbol{l}'$ are vectors in C^+. By means of the condensed formalism that was introduced in the electromagnetic case, one may translate (47-8) into:

$$[a^{-\mathrm{A}}(i,j,\boldsymbol{l}), a^{\mathrm{B}}(h,k,\boldsymbol{l}')] = \frac{\hbar}{i}\mathrm{A}_{\mathrm{AB}}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh} - \delta_{ij}\delta_{hk})\delta_\Omega(\boldsymbol{l},\boldsymbol{l}') . \tag{47-9}$$

48. The commutation relations. At the point $\mathbf{x}$ of V_4, by virtue of (47-5) and (47-7), $H_{\alpha\beta,\gamma\delta}(\mathbf{x})$ may be written:

$$H_{\alpha\beta,\gamma\delta}(\mathbf{x}) = \frac{1}{(2\pi)^3}\sum_{A,i,j}\int_{C^+} a^A(i,j,\boldsymbol{l})e^{iA\boldsymbol{l}\cdot x}(l_\alpha n_\beta^{(i)} - l_\beta n_\alpha^{(i)})(l_\gamma n_\delta^{(j)} - l_\delta n_\gamma^{(j)})\, d\Omega(\boldsymbol{l}) . \tag{48-1}$$

Let $\mathbf{x}'$ be another point of V_4 for which:

$$H_{\lambda\mu,\nu\rho}(\mathbf{x}') = \frac{1}{(2\pi)^3}\sum_{B,h,k}\int_{C^+} a^B(h,k,\boldsymbol{l}')e^{iB\boldsymbol{l}\cdot x}(l'_\lambda n'^{(h)}_\mu - l'_\mu n'^{(h)}_\lambda)(l'_\nu n'^{(k)}_\rho - l'_\rho n'^{(k)}_\nu)\, d\Omega(\boldsymbol{l}') ,$$

in which the $\mathbf{n}'^{(h)}(\boldsymbol{l}')$ are orthogonal vectors that are tangent to C along $\boldsymbol{l}'$.

From (47-9), one has:

$$[H_{\alpha\beta,\gamma\delta}(\mathbf{x}), H_{\lambda\mu,\nu\rho}(\mathbf{x}')]$$

$$= \frac{\hbar}{i}\frac{1}{(2\pi)^6}\sum_{\substack{A,B\\ i,j,k,h}}\int_{C^+}\int_{C^+} A\delta_{AB}(\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh}-\delta_{ij}\delta_{hk})\,\delta_\Omega(\boldsymbol{l},\boldsymbol{l}')\,e^{i(-A\boldsymbol{l}\cdot\mathbf{x}+B\boldsymbol{l}'\cdot\mathbf{x})}$$
$$\times\,(l_\alpha n_\beta^{(i)}-l_\beta n_\alpha^{(i)})(l_\gamma n_\delta^{(j)}-l_\delta n_\gamma^{(j)})(l'_\lambda n'^{(h)}_\mu-l'_\mu n'^{(h)}_\lambda)(l'_\nu n'^{(k)}_\rho-l'_\rho n'^{(k)}_\nu)\,d\Omega(\boldsymbol{l})\,d\Omega(\boldsymbol{l}')\,.$$

After integrating over the variable $\boldsymbol{l}$, one obtains:

$$[H_{\alpha\beta,\gamma\delta}(\mathbf{x}),H_{\lambda\mu,\nu\rho}(\mathbf{x}')]=\frac{\hbar}{i}\frac{1}{2(2\pi)^6}\sum_A\int_{C^+}Ae^{-i(A\boldsymbol{l}\cdot(\mathbf{x}-\mathbf{x})}\sum_{i,j,h,k}(\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh}-\delta_{ij}\delta_{hk})$$
$$\times(l_\alpha n_\beta^{(i)}-l_\beta n_\alpha^{(i)})(l_\gamma n_\delta^{(j)}-l_\delta n_\gamma^{(j)})(l_\lambda n_\mu^{(h)}-l_\mu n_\lambda^{(h)})(l_\nu n_\rho^{(k)}-l_\rho n_\nu^{(k)})\,d\Omega(\boldsymbol{l})\,.$$

From the auxiliary formula (43-2), it thus results that:

$$[H_{\alpha\beta,\gamma\delta}(\mathbf{x}),H_{\lambda\mu,\nu\rho}(\mathbf{x}')]=\frac{\hbar}{i}\frac{1}{2(2\pi)^3}\sum_A\int_{C^+}Ae^{-i(A\boldsymbol{l}\cdot(\mathbf{x}-\mathbf{x})}$$
$$\{(\eta_{\alpha\beta}l_\alpha l_\beta+\eta_{\beta\mu}l_\alpha l_\lambda-\eta_{\alpha\mu}l_\beta l_\lambda-\eta_{\beta\lambda}l_\alpha l_\mu)(\eta_{\gamma\nu}l_\delta l_\rho+\eta_{\delta\rho}l_\gamma l_\lambda-\eta_{\delta\mu}l_\beta l_\nu-\eta_{\delta\nu}l_\gamma l_\mu)$$
$$+(\eta_{\alpha\nu}l_\beta l_\rho+\eta_{\beta\mu}l_\alpha l_\nu-\eta_{\alpha\rho}l_\beta l_\nu-\eta_{\beta\nu}l_\alpha l_\rho)(\eta_{\gamma\lambda}l_\delta l_\mu+\eta_{\delta\mu}l_\gamma l_\lambda-\eta_{\gamma\mu}l_\delta l_\nu-\eta_{\delta\lambda}l_\gamma l_\mu)$$
$$+(\eta_{\alpha\gamma}l_\beta l_\delta+\eta_{\beta\delta}l_\alpha l_\gamma-\eta_{\alpha\delta}l_\beta l_\gamma-\eta_{\beta\gamma}l_\alpha l_\delta)(\eta_{\gamma\lambda}l_\mu l_\rho+\eta_{\mu\rho}l_\lambda l_\nu-\eta_{\lambda\rho}l_\mu l_\nu-\eta_{\mu\nu}l_\lambda l_\rho)\}\,d\Omega(\boldsymbol{l}')\,.$$

Upon introducing the JORDAN-PAULI propagator, one deduces that:

(48-2) $\qquad [H_{\alpha\beta,\gamma\delta}(\mathbf{x}),H_{\lambda\mu,\nu\rho}(\mathbf{x}')]=$

$$\frac{\hbar}{i}\{\,(\eta_{\alpha\lambda}\partial_\beta\partial_\mu+\eta_{\beta\mu}\partial_\alpha\partial_\lambda-\eta_{\alpha\mu}\partial_\beta\partial_\lambda-\eta_{\beta\lambda}\partial_\alpha\partial_\mu)\,(\eta_{\gamma\nu}\partial_\delta\partial_\rho+\eta_{\gamma\delta}\partial_\rho\partial_\nu-\eta_{\gamma\rho}\partial_\delta\partial_\nu-\eta_{\delta\nu}\partial_\gamma\partial_\rho)$$
$$+(\eta_{\alpha\nu}\partial_\beta\partial_\rho+\eta_{\beta\rho}\partial_\alpha\partial_\nu-\eta_{\alpha\rho}\partial_\beta\partial_\nu-\eta_{\beta\nu}\partial_\alpha\partial_\rho)\,(\eta_{\gamma\lambda}\partial_\delta\partial_\mu+\eta_{\delta\mu}\partial_\gamma\partial_\lambda-\eta_{\gamma\mu}\partial_\delta\partial_\lambda-\eta_{\delta\lambda}\partial_\gamma\partial_\mu)$$
$$-(\eta_{\alpha\gamma}\partial_\beta\partial_\delta+\eta_{\beta\delta}\partial_\alpha\partial_\gamma-\eta_{\alpha\delta}\partial_\beta\partial_\gamma-\eta_{\beta\gamma}\partial_\alpha\partial_\delta)\,(\eta_{\lambda\nu}\partial_\mu\partial_\rho+\eta_{\mu\rho}\partial_\lambda\partial_\nu-\eta_{\lambda\rho}\partial_\mu\partial_\nu-\eta_{\mu\nu}\partial_\lambda\partial_\rho)\,\}$$
$$\times D(\mathbf{x}-\mathbf{x}')\,.$$

We have thus obtained the commutation relation for the field H. One easily verifies that if H satisfies the identity (47-2) then the relation that one obtains is compatible with this identity; it is naturally compatible with the field equations (47-3). The study of the "little group," à la WIGNER, easily shows that the preceding quantization defined by (47-9) generally gives a mixture of particles that have null masses and spin 0 or 2. If one introduces the supplementary condition $H_{\alpha\beta} = 0$ then one necessarily $H_{\alpha\beta} = K^\rho{}_{\alpha,\rho\beta} = 0$, and one deduces from (47-7):

$$\sum_{i,j}\delta_{ij}a(i,j,\boldsymbol{l})=0,$$

namely, the trace condition:

$$a(1,1,\boldsymbol{l})+a(2,2,\boldsymbol{l})=0,$$

which leaves only the representation that is characterized by a null mass and a spin equal to two.

49. **Interpretation of the tensor H in terms of the metric.**

1) In Minkowski spacetime, which we refer to a frame that is orthonormal relative to $\eta_{\alpha\beta}$, we consider an ordinary tensor field $H^{\lambda}{}_{\mu,\beta\alpha}$ that satisfies the algebraic identities (47-1) and (47-2) and the field equations (47-3), as well as the supplementary condition that $H_{\alpha\beta} = 0$. From (47-3)a, there exists a system of quantities $\Gamma^{\lambda}_{\alpha\beta}$ such that:

(49-1) $$H^{\lambda}{}_{\mu,\beta\alpha} = \partial_\alpha \Gamma^{\lambda}_{\mu\beta} - \partial_\beta \Gamma^{\lambda}_{\mu\alpha},$$

a system of quantities that is found to be defined up to a transformation of the form:

(49-2) $$\Gamma^{\lambda}_{\alpha\beta} \to \Gamma^{\lambda}_{\alpha\beta} + \partial_\beta A^{\lambda}_{\alpha},$$

in which the A^{λ}_{α} are arbitrary. We set:

$$S^{\lambda}_{\alpha\beta} = \tfrac{1}{2}(\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\alpha}).$$

From (49-1), when written in the form:

$$H^{\lambda}{}_{\alpha,\beta\gamma} = \partial_\beta \Gamma^{\lambda}_{\alpha\gamma} - \partial_\gamma \Gamma^{\lambda}_{\alpha\beta},$$

one infers, after summation over cyclic permutations of α, β, γ and from the identity (47-2), that:

$$S \partial_\gamma S^{\lambda}_{\alpha\beta} = -\tfrac{1}{2} S H^{\lambda}{}_{\alpha,\beta\gamma} = 0.$$

From this, it results that there exists a system of quantities B^{λ}_{α} such that:

(49-3) $$2 S^{\lambda}_{\alpha\beta} = \partial_\alpha B^{\lambda}_{\beta} - \partial_\beta B^{\lambda}_{\alpha},$$

a system of quantities that is defined up to the transformation:

(49-4) $$B^{\lambda}_{\alpha} \to B^{\lambda}_{\alpha} + \partial_\alpha \varphi^{\lambda}.$$

From (49-3), it results that:

(49-5) $$\Gamma^{\lambda}_{\alpha\beta} + \partial_\beta B^{\lambda}_{\alpha} = \Gamma^{\lambda}_{\beta\alpha} + \partial_\alpha B^{\lambda}_{\beta}.$$

If we replace the original $\Gamma^{\lambda}_{\alpha\beta}$ with the left-hand side of (49-5), which is permissible, by the transformation (49-2), one sees that the $\Gamma^{\lambda}_{\alpha\beta}$ that satisfy (49-1) may be restricted to be symmetric in their lower two indices. They are then found to be defined up to a transformation:

(49-6) $$\Gamma^{\lambda}_{\alpha\beta} \to \Gamma^{\lambda}_{\alpha\beta} + \partial_\beta \partial_\alpha \varphi^{\lambda}.$$

We raise and lower the indices by means of the tensor $\eta_{\lambda\mu}$, and, in particular, we set:

$$\Gamma_{\alpha\mu\beta} = \eta_{\lambda\mu}\Gamma^{\lambda}_{\alpha\beta}\,.$$

The relation (49-1) may be put into the form:

$$H_{\lambda\mu,\beta\alpha} = \partial_\alpha \Gamma_{\mu\lambda\beta} - \partial_\beta \Gamma_{\mu\lambda\alpha}\,.$$

From the identity (47-1) that H verifies, it results that:

$$H_{\lambda\mu,\alpha\beta} + H_{\mu\lambda,\alpha\beta} = \partial_\alpha(\Gamma_{\mu\lambda\beta} + \Gamma_{\mu\lambda\alpha}) - \partial_\beta(\Gamma_{\lambda\mu\alpha} + \Gamma_{\mu\lambda\alpha}) = 0.$$

As a result, there exists a system of quantities $\psi_{\lambda\mu}$ that are symmetric in the indices λ and μ such that:

(49-7) $$\Gamma_{\lambda\mu\alpha} + \Gamma_{\mu\lambda\alpha} = \partial_\alpha \psi_{\lambda\mu}\,.$$

On account of (49-6), $\psi_{\lambda\mu}$ is defined up to the transformation:

(49-8) $$\psi_{\lambda\mu} \to \psi_{\lambda\mu} + \partial_\lambda \varphi_\mu + \partial_\mu \varphi_\lambda + \text{const.}$$

From the symmetry properties of Γ and (49-7), one deduces that:

(49-9) $$\Gamma_{\lambda\nu\mu} = [\lambda\mu,\, \nu],$$

in which the right-hand side denotes the CHRISTOFFEL algorithm applied to the system of $\psi_{\lambda\mu}$. Therefore, from the hypotheses that were made on H, there exists a system of quantities $\psi_{\lambda\mu}$ (with $\psi_{\lambda\mu} = \psi_{\mu\lambda}$), which is defined up to the transformation (49-8) and is such that the quantities (49-9) that are derived from them satisfy:

(49-10) $$H_{\lambda\mu,\alpha\beta} = \partial_\alpha \Gamma_{\mu\lambda\beta} - \partial_\beta \Gamma_{\mu\lambda\alpha}\,.$$

Consider the tensor that is defined, relative to the adopted coordinates, by:

$$g_{\lambda\mu} = \eta_{\lambda\mu} + \varepsilon\psi_{\lambda\mu}\,,$$

in which ε is an infinitesimal. If we effect the change of coordinates:

$$x^{\rho'} = x^\rho + \varepsilon x^\rho(x)$$

then one sees that $\psi_{\lambda\mu}$ is subjected to the transformation (called a *gauge transformation*):

(49-11) $$\psi_{\lambda\mu} \to \psi_{\lambda\mu} + \partial_\lambda \varphi_\mu + \partial_\mu \varphi_\lambda\,.$$

We may thus interpret the preceding results by giving spacetime the Riemannian metric that is defined by $g_{\lambda\mu}$. This metric is close to the Minkowskian metric that is defined by $\eta_{\lambda\mu}$ in local coordinates that are orthonormal for the Minkowski spacetime. This deviation persists under the change of coordinates (49-11), and $H_{\lambda\mu,\alpha\beta}$ is the principal part of the curvature tensor for $g_{\lambda\mu}$.

2) It remains for us to examine the equations:

$$H_{\lambda\mu} = 0.$$

By a convenient choice of φ^ρ, one may restrict the $\Gamma^\lambda_{\alpha\beta}$ to the coordinate conditions:

$$\eta^{\alpha\beta}\ \Gamma^\lambda_{\alpha\beta} = 0, \tag{49-12}$$

which is nothing but the principal part of the *isothermal conditions*. One then has:

$$\eta^{\alpha\beta}\ (\partial_\alpha \psi_{\beta\mu} + \partial_\beta \psi_{\alpha\mu} - \partial_\mu \psi_{\alpha\beta}) = 0, \tag{49-13}$$

and in order to respect (49-12) φ_λ must be restricted by the conditions:

$$\eta^{\alpha\beta}\ \partial_{\alpha\beta}\varphi_\lambda = 0. \tag{49-14}$$

From (49-10), one deduces (see (16-4)):

$$H_{\lambda\mu} = \tfrac{1}{2}\ \eta^{\alpha\beta}\ (\partial_{\alpha\lambda}\psi_{\beta\mu} + \partial_{\beta\mu}\psi_{\alpha\lambda} - \partial_{\alpha\beta}\psi_{\lambda\mu} - \partial_{\lambda\mu}\psi_{\alpha\beta}).$$

The coordinate conditions and principal parts of the isothermal conditions permit us to give a simple expression for $H_{\lambda\mu}$, the principal part of the RICCI tensor. By differentiating (49-13) one deduces:

$$\eta^{\alpha\beta}\ \partial_{\lambda\mu}\psi_{\alpha\beta} = \eta^{\alpha\beta}\,(\partial_{\alpha\lambda}\psi_{\beta\mu} + \partial_{\beta\lambda}\psi_{\alpha\mu}) \qquad \eta^{\alpha\beta}\ \partial_{\alpha\beta}\psi_{\lambda\mu} = \eta^{\alpha\beta}\,(\partial_{\alpha\mu}\psi_{\beta\lambda} + \partial_{\beta\mu}\psi_{\alpha\lambda}).$$

From this, one deduces that, with the conditions on the coordinates, one has:

$$H_{\lambda\mu} = -\tfrac{1}{2}\ \eta^{\alpha\beta}\,\partial_{\lambda\mu}\psi_{\alpha\beta}.$$

Therefore, with the coordinate conditions that were introduced, the equations $H_{\lambda\mu} = 0$ translate into:

$$\eta^{\alpha\beta}\,\partial_{\lambda\mu}\psi_{\alpha\beta} = 0. \tag{49-15}$$

50. Relationship with the theory of the graviton. One may finally show that our process is equivalent to the classical process that leads to the theory of the graviton; however, it reveals itself to be more coherent and, as a result, more satisfying, both mathematically and physically.

Under the simple hypotheses that permit us to introduce the FOURIER transformation, one has, by virtue of (49-14) and (49-15):

$$\psi_{\lambda\mu}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_C \chi_{\lambda\mu}(\boldsymbol{l})\, e^{i\boldsymbol{l}\cdot\mathbf{x}}\, d\Omega(\boldsymbol{l})$$

and:

$$\varphi_{\mu}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_C f_{\mu}(\boldsymbol{l})\, e^{i\boldsymbol{l}\cdot\mathbf{x}}\, d\Omega(\boldsymbol{l}),$$

in which C is the isotropic cone that is defined by $\eta_{\lambda\mu}$: $\psi_{\lambda\mu}$ is thus defined up to a gauge transformation, and $\chi_{\lambda\mu}$ is defined up to the transformation:

(50-1) $$\chi_{\lambda\mu}(\boldsymbol{l}) - \chi_{\mu\lambda}(\boldsymbol{l}) + l_\lambda f_\mu(\boldsymbol{l}) + l_\mu f_\lambda(\boldsymbol{l}),$$

in which $f_\mu(\boldsymbol{l})$ is then restricted by the condition:

$$l^\mu f_\mu(\boldsymbol{l}) = 0.$$

If $\mathbf{n}^{(1)}(\boldsymbol{l})$, $\mathbf{n}^{(2)}(\boldsymbol{l})$ are two vectors that are normal and orthogonal and tangent to C along $\boldsymbol{l}$ then $f_\mu(\boldsymbol{l})$ is an arbitrary vector of the form:

(50-3) $$f_\mu(\boldsymbol{l}) = \tfrac{1}{2}\, a(\boldsymbol{l}) l_\mu + \sum_i b(i, \boldsymbol{l})\mathbf{n}_\mu^{(i)} \qquad (i, j = 1, 2).$$

In order for the coordinate conditions to be satisfied it is necessary and sufficient that:

$$l^\lambda c_{\lambda\mu}(\boldsymbol{l}) = 0.$$

From this, it results that there exist scalars $a(\boldsymbol{l})$, $b(i, \boldsymbol{l})$, $c(i, j, \boldsymbol{l})$ (with $c(i, j, \boldsymbol{l}) = c(j, i, \boldsymbol{l})$) such that:

$$\chi_{\lambda\mu}(\boldsymbol{l}) = a(\boldsymbol{l}) l_\lambda l_\mu + \sum_i b(i, \boldsymbol{l})(l_\lambda n_\mu^{(i)} + l_\mu n_\lambda^{(i)}) + \sum_{i,j} c(i, j, \boldsymbol{l}) n_\lambda^{(i)} n_\mu^{(j)}.$$

By using the transformation (50-1) with (50-3), one sees that $\chi_{\lambda\mu}$ may be restricted to be of the form:

(50-4) $$\chi_{\lambda\mu}(\boldsymbol{l}) = \sum_{i,j} c(i, j, \boldsymbol{l}) n_\lambda^{(i)} n_\mu^{(j)},$$

and is then entirely determined. In order to have $\chi(\boldsymbol{l}) = 0$ it is necessary and sufficient that one have:

(50-5) $$c(1, 1, \boldsymbol{l}) + c(2, 2, \boldsymbol{l}) = 0.$$

According to our process, the quantization of the field $\chi_{\lambda\mu}$ is effected by imposing conditions that are analogous to (47-8) on the operators that are substituted for the scalars $c(i, j, \boldsymbol{l})$ in (50-4). A somewhat long calculation then permits us to deduce the commutation relations (48-2).

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