"Bemerkungen über das Prinzip der virtuellen Verrückungen in der Hydrodynamik inkompressibler Flüssigkeiten," Ann. soc. polon. de math **3** (1924), 20-28.

Remarks on the principle of virtual displacements in the hydrodynamics of incompressible fluids

By

Leon Lichtenstein in Leipzig

Translated by D. H. Delphenich

---

1. – Let T be any region that is filled with a homogeneous or heterogeneous incompressible fluid whose boundary S consists of, say, a finite number of pieces of analytic and regular surfaces. Let S' be one part of the surface that is defined by rigid walls, while the remaining part S'' is free. ρ might denote the density, which is assumed to be continuous on the pieces, while $d\tau$ denotes the volume element, and $d\sigma$ denotes the surface element. The fluid considered might be found to be in equilibrium under the action of volume forces $\rho\, X\, d\tau$, $\rho\, Y\, d\tau$, $\rho\, Z\, d\tau$, as well as the surface forces $X_\sigma\, d\sigma$, $Y_\sigma\, d\sigma$, $Z_\sigma\, d\sigma$. The unit forces X, Y, Z will be assumed to be continuous functions of position in the interior of T and on its boundary, or more briefly, in $T + S$, while X_σ, Y_σ, Z_σ are the same sort of functions of position on S''.

Now, let ξ, η, ζ be any functions that are declared to be continuous in $T + S$ and have first-order derivatives that are continuous or just piece-wise continuous and satisfy the equation:

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0 , \tag{1}$$

along with the relation:

$$\xi \cos(n, x) + \eta \cos(n, y) + \zeta \cos(n, z) = 0 . \tag{2}$$

In (2), (n) denotes the direction of the interior normal. That equation states that the vector ξ, η, ζ falls in the tangent plane to the surface S'. On some edges, it is possibly tangent to that direction, while it is equal to zero on the corners of the body.

Let ε be a real parameter. We set $\delta x = \varepsilon\, \xi$, $\delta y = \varepsilon\, \eta$, $\delta z = \varepsilon\, \zeta$. The transformation:

$$x^* = x + \delta x , \quad y^* = y + \delta y , \quad z^* = z + \delta z , \tag{3}$$

viz., a "virtual displacement," associates T with a region T^* in a continuous, single-valued, and invertible way for all sufficiently-small $|\varepsilon|$, as one easily shows. (1) and (2) imply the relations:

$$\frac{\partial\,\delta x}{\partial x}+\frac{\partial\,\delta y}{\partial y}+\frac{\partial\,\delta z}{\partial z}=0 \tag{4}$$

in T, and

$$\delta x \cos(n, x) + \delta y \cos(n, y) + \delta z \cos(n, z) = 0 \tag{5}$$

on S'.

The principle of virtual displacements says that as long as the volume and surface forces remain in equilibrium, as was assumed, the work that they do under all virtual displacements will vanish:

$$\int_T \rho(X\,\delta x+Y\,\delta y+Z\,\delta z)\,d\tau+\int_{S''}(X_\sigma\,\delta x+Y_\sigma\,\delta y+Z_\sigma\,\delta z)\,d\sigma=0\,. \tag{6}$$

One can, with **Lagrange**, derive the equilibrium conditions from that relation when one appeals to the use of **Lagrange** multipliers that are customary in the mechanics of systems of mass-points and understands λ to mean a function that is continuous in $T + S$ and has piece-wise continuous first-order partial derivatives there, and one exhibits the conditions for that by saying that:

$$\int_T\left\{\rho(X\,\delta x+Y\,\delta y+Z\,\delta z)+\lambda\left(\frac{\partial\,\delta x}{\partial x}+\frac{\partial\,\delta y}{\partial y}+\frac{\partial\,\delta z}{\partial z}\right)\right\}d\tau+\int_{S''}(X_\sigma\,\delta x+Y_\sigma\,\delta y+Z_\sigma\,\delta z)\,d\sigma=0\,. \tag{7}$$

Partial integration will yield, in the known way:

$$\begin{aligned}&\int_T\left\{\left(\rho X-\frac{\partial\lambda}{\partial x}\right)\delta x+\left(\rho Y-\frac{\partial\lambda}{\partial y}\right)\delta y+\left(\rho Z-\frac{\partial\lambda}{\partial z}\right)\delta z\right\}d\tau\\&+\int_{S''}\{(X_\sigma-\lambda\cos(n,x))\,\delta x+(Y_\sigma-\lambda\cos(n,y))\,\delta y+(Z_\sigma-\lambda\cos(n,z))\,\delta z\}\,d\sigma=0\,,\end{aligned} \tag{8}$$

from which the equilibrium conditions will follow in T:

$$\rho X=\frac{\partial\lambda}{\partial x},\qquad \rho Y=\frac{\partial\lambda}{\partial y},\qquad \rho Z=\frac{\partial\lambda}{\partial z}, \tag{9}$$

and

$$\lambda\cos(n,x)=X_\sigma,\qquad \lambda\cos(n,y)=Y_\sigma,\qquad \lambda\cos(n,z)=Z_\sigma \tag{10}$$

on S''. The multiplier λ has the meaning of the fluid pressure.

Trying to establish the method of **Lagrange** multipliers directly raises certain difficulties. When one rises from equations (9) and (10) to formulas (7) and (6), one can use the present argument to prove that the relations (9) and (10) are sufficient for the vanishing of the virtual work (6) when the condition equations (4) and (5) are fulfilled, but not that they are also necessary. That this is actually the case, so the system of equations (9) and (10) is completely equivalent to the statement of principle of virtual displacements, can be shown with no difficulty as long as one assumes that ρX, ρY, ρZ have continuous first-order derivatives. The proof when one drops that assumption is not as obvious. In that case, a lemma of **Haar** will provide the required tool for that, and a very simple proof of it will be given at the conclusion of this article.

2. – Let (x_0, y_0, z_0) be any point in T at which ρX, ρY, ρZ behave continuously, and let K be a cube whose edges have length $2h$ and are parallel to the coordinate axes and lies completely in the interior of T with (x_0, y_0, z_0) as its midpoint. We now take the functions δx, δy, δz, which are continuous in $T + S$, as before, have piece-wise continuous first-order derivatives and satisfy equations (4) and (5), and in particular, they equal zero in $T - K$. We must then have:

(11) $$\int_K \rho(X\,\delta x + Y\,\delta y + Z\,\delta z)\,d\tau = 0\,.$$

Now, let δx and δy be any pair of infinitely-small functions that are continuous in the square:

(12) $$x_0 - h \leq x \leq x_0 + h\,, \qquad y_0 - h \leq y \leq y_0 + h\,,$$

along with their first-order partial derivatives, satisfy the conditions:

(13) $$\frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} = 0\,,$$

and vanish on its periphery. We assert that for all z in:

(14) $$z_0 - h \leq z \leq z_0 + h\,,$$

we have:

(15) $$\int_Q \rho(X\,\delta x + Y\,\delta y)\,dx\,dy = 0\,.$$

In contrast to that assertion, for some value z_1 in (14) and a certain pair of functions δx, δy that satisfy the present conditions, one might have, say:

(16) $$\int_Q \rho(X\,\delta x + Y\,\delta y)\,dx\,dy > 0\,.$$

In the interval:

(17) $$z_0 - \varepsilon \leq z \leq z_0 + \varepsilon \qquad (\varepsilon < h),$$

we then choose:

(18) $$\delta x = \delta x\left(1 - \frac{|z - z_1|}{\varepsilon}\right), \qquad \delta y = \delta y\left(1 - \frac{|z - z_1|}{\varepsilon}\right),$$

while $\delta x = \delta y = 0$ for all other z in (14). Due to (13), the present functions δx, δy, as well as the function $\delta z = 0$, define a system of virtual displacements. For sufficiently-small ε, we will have:

(19) $$\int_K \rho(X\,\delta x + Y\,\delta y + Z\,\delta z)\,d\tau > 0,$$

due to (16), which is not possible. Thus, the relation (15) is, in fact, true.

The condition (13) is obviously the condition for there to be a function Θ that is continuous in the interior and boundary of Q, along with its first-order partial derivatives, such that one will have:

(20) $$\delta x = \frac{\partial \Phi}{\partial y}, \qquad \delta y = -\frac{\partial \Phi}{\partial x}.$$

Let (x_0, y_0) be any point on the boundary of Q. One can set:

(21) $$\Theta = -\int_{(x^0, y^0)}^{(x, y)} (\delta y\,dx - \delta x\,dy).$$

Due to (20), one has:

(22) $$\Theta = \frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial y} = 0.$$

When that is substituted in (15), that will give:

(23) $$\int_Q \left(\rho X \frac{\partial \Theta}{\partial y} - \rho Y \frac{\partial \Theta}{\partial x}\right) dx\,dy = 0,$$

or, in the event that ρX and ρY have continuous, or at least piece-wise continuous, first-order partial derivatives, after a partial integration, one will have:

(24) $$\int_Q \left\{\frac{\partial}{\partial y}(\rho X) - \frac{\partial}{\partial x}(\rho Y)\right\} \Theta\,dx\,dy = 0,$$

due to (22). However, it will follow from this that:

$$\frac{\partial}{\partial y}(\rho X) - \frac{\partial}{\partial x}(\rho Y) = 0 . \tag{25}$$

Namely, if the bracketed expression (24) is, say, > 0 at a point (x_1, y_1) in Q then one can choose a certain Θ (even one that does not satisfy any conditions that are introduced) to be positive in a neighborhood of (x_1, y_1), but otherwise equal to zero, such that the integral (24) will prove to be positive. One will then have:

$$\frac{\partial}{\partial y}(\rho X) = \frac{\partial}{\partial x}(\rho Y) \tag{26}$$

in K and analogously:

$$\frac{\partial}{\partial z}(\rho X) = \frac{\partial}{\partial x}(\rho Z), \qquad \frac{\partial}{\partial y}(\rho Z) = \frac{\partial}{\partial z}(\rho Y) . \tag{27}$$

The formulas (26) and (27) are valid in the neighborhood of any point in T where one has continuity. On the grounds of continuity, they will be valid in the interior and on the boundary of any region in which ρX, ρY, ρZ are continuous, and in particular, on S, as well then. Hence, there is one continuous function $\bar{p}$ in $T + S$ that is determined up to an additive constant and has continuous, or at least piece-wise continuous, first and second order partial derivatives, such that:

$$\rho X = \frac{\partial \bar{p}}{\partial x}, \qquad \rho Y = \frac{\partial \bar{p}}{\partial y}, \qquad \rho Z = \frac{\partial \bar{p}}{\partial z} . \tag{28}$$

Due to the fact that:

$$\begin{aligned}\int_T \rho(X\,\delta x + Y\,\delta y + Z\,\delta z)\,d\tau &= \int_T \rho\left(\frac{\partial \bar{p}}{\partial x}\delta x + \frac{\partial \bar{p}}{\partial y}\delta y + \frac{\partial \bar{p}}{\partial z}\delta z\right)d\tau \\ &= -\int_S \bar{p}\,(\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z))\,d\sigma \\ &= -\int_{S''} \bar{p}\,(\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z))\,d\sigma ,\end{aligned}$$

equation (6) will go to:

$$\int_{S''}\{(X_\sigma - \bar{p}\cos(n,x))\,\delta x + (Y_\sigma - \bar{p}\cos(n,y))\,\delta y + (Y_\sigma - \bar{p}\cos(n,z))\,\delta z\}\,d\sigma = 0 . \tag{29}$$

That formula is true for all δx, δy, δz on S'' that are continuous there and are arranged such that:

$$\int_{S''}[\delta x\cos(n,x) + \delta y\cos(n,y) + \delta z\cos(n,z)]\,d\sigma = 0 , \tag{30}$$

moreover. One gets the relation (30) from (4) by integrating over T :

$$\int_T \left[\frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z}\right] d\tau$$

(31) $$= - \int_S [\delta x \cos(n,x) + \delta y \cos(n,y) + \delta z \cos(n,z)]\, d\sigma$$
$$= - \int_{S''} [\delta x \cos(n,x) + \delta y \cos(n,y) + \delta z \cos(n,z)]\, d\sigma = 0 \,.$$

From known theorems, it follows from (29) and (31) that:

(32) $$X_\sigma = (\bar{p} + \alpha)\cos(n,x)\,, \qquad Y_\sigma = (\bar{p} + \alpha)\cos(n,y)\,, \qquad Z_\sigma = (\bar{p} + \alpha)\cos(n,z)\,,$$
(α = constant).

The formula brings us back to the statement that the pressure has been defined only up to an additive constant up to now. As is known, one cares to establish the value of the pressure in such a way that one lets it vanish at those points on the surface at which $X_\sigma^2 + Y_\sigma^2 + Z_\sigma^2 = 0$ (1). If one sets $\bar{p} + \alpha = p$ then one will find that:

(33) $$\rho X = \frac{\partial p}{\partial x}\,, \qquad \rho Y = \frac{\partial p}{\partial y}\,, \qquad \rho Z = \frac{\partial p}{\partial z}\,,$$

and

(34) $$X_\sigma = p \cos(n,x)\,, \quad Y_\sigma = p \cos(n,y)\,, \quad Z_\sigma = p \cos(n,z)$$

on S''. Formulas (33) and (34) are the basic equations for hydrostatics.

The relations (20) are included in the known general formulas:

(35) $$\delta x = \frac{\partial V}{\partial z} - \frac{\partial W}{\partial y}\,, \qquad \delta y = \frac{\partial V}{\partial z} - \frac{\partial W}{\partial y}\,, \qquad \delta z = \frac{\partial V}{\partial z} - \frac{\partial W}{\partial y}$$

for the components of a vector δx, δy, δz whose divergence $\frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z}$ vanishes as the case where $U = 0$, $V = 0$. Naturally, one can also arrive at formulas (33) and (34) in such a way that one substitutes the expressions (35) for δx, δy, δz in (11) and partially integrates, as before. **Herglotz** arrived at the equations of motion for an electron from **Hamilton**'s principle along a similar path in a paper that already goes back a long way in time (2). The method breaks down as

---

(1) One recalls **Torricelli**'s classical experiment.

(2) Cf., **G. Herglotz**, "Zur Elektronentheorie," Gött. Nachr. (1903), 357-382.

soon as ρX, ρY, ρZ do not have piece-wise continuous first-order derivatives. One can do without that assumption when one appeals to a lemma due to **Haar** ([1]), which reads as follows:

Let F be any bounded, simply-connected planar region whose boundary C has piece-wise continuous tangent. Let U and V two functions that are continuous $F + C$, and let:

$$\int_F \left(U \frac{\partial \Psi}{\partial x} + V \frac{\partial \Psi}{\partial y} \right) dx\, dy = 0 \tag{36}$$

for all Ψ that are continuous in $F + C$, vanish on C, and have continuous first-order partial derivatives in F. Thus, the integral that is extended along an arbitrary, closed, continuous curve Γ in F :

$$\int_\Gamma (U\, dy - V\, dx) = 0 . \tag{37}$$

There is then a function $\omega(x, y)$ that is continuous in $F + C$, along with its first-order partial derivatives, such that:

$$U = \frac{\partial \omega}{\partial y}, \qquad V = -\frac{\partial \omega}{\partial x}. \tag{38}$$

As will be shown below, that theorem is also true when one assumes, for the time being, that the functions U and V are piece-wise continuous in $F + C$.

Obviously, in order to arrive at the relations (33), is it sufficient to replace U, V, Ψ, and ω with $-\rho Y$, ρX, Θ, $-p$, respectively, and in so doing, to demonstrate the equivalence of the equilibrium conditions (33) and the statement of the principle of virtual displacements when it is based upon piece-wise continuous ρX, ρY, ρZ.

Now, there is a simple proof of the **Haar** lemma. Let Γ be a closed, continuously-curved curve in F that has no double points, let Γ_1 be a curve that is parallel to it at a distance of ε, and let D be the finite region that is bounded by Γ, while D_1 is the one that is bounded by Γ_1. Moreover, Γ might lie in D_1. Furthermore, let $\chi(h)$ be any function that is continuous in $< 0, \varepsilon >$, along with its derivative, and satisfies the following conditions:

$$\chi(0) = 1, \qquad \chi(\varepsilon) = 0, \qquad \chi'(0) = 0, \qquad \chi'(\varepsilon) = 0, \qquad \chi'(0) < 0, \qquad \text{for } 0 < h < \varepsilon . \tag{39}$$

We take $\Psi = 1$ in D, equal to 0 in $F - D_1$, and equal to $\chi(h)$ along any curve Γ_h that is parallel to Γ in $D_1 - D$ and whose distance from Γ has the value $h \leq \varepsilon$. Let (n) be an arbitrary normal to Γ that points outward and let α be the angle that (n) makes with the x-axis. Finally, let P_h be the point of intersection of (n) with Γ_h. We convince ourselves almost immediately that we have:

([1]) **A. Haar**, "Über die Variation der Doppelintegrale," J. reine angew. Math. **149** (1919), 1-18.

$$(40) \qquad \frac{\partial \Psi}{\partial x} = \cos\alpha\, \chi'(h), \quad \frac{\partial \Psi}{\partial y} = \sin\alpha\, \chi'(h)$$

in P_h . If we now introduce the arc-length s along Γ and the distance h as new independent variables then, due to (40), we will get:

$$(41) \qquad \int_\Gamma \int_0^\varepsilon ds\,(U\cos\alpha + V\sin\alpha)\,\chi'(h)\left(1+\frac{h}{\mathfrak{r}}\right)dh = 0$$

for all sufficiently-small ε, in which $\mathfrak{r}$ is understood to mean the radius of curvature of the curve Γ at the point s. Since:

$$(42) \qquad \sin\alpha = -\frac{dx}{ds}, \qquad \cos\alpha = \frac{dy}{ds}, \quad \left|\int_0^\varepsilon \chi'(h)\,h\,dh\right| < \varepsilon\left|\int_0^\varepsilon \chi'(h)\,dh\right| = \varepsilon,$$

(41) will give the desired formula as $\varepsilon \to 0$:

$$\int_\Gamma (U\,dy - V\,dx) = 0 .$$

____