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The foundations of the theory of infinite continuous transformation groups – I.

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The continuous transformation groups decompose into two categories: finite and infinite. For the theory of the finite ones, much has happened already; apart from my own numerous investigations, in recent times, many other mathematicians have been concerned with them and have made important discoveries. Much less has been done for the infinite groups; apart from my own papers [1]), in which I have developed the fundamentals of their theory, there is only the treatise of Engel, who was concerned with the defining equations of their infinitesimal transformations.

It is now my wish to direct attention to the infinite continuous groups, since they define an extended and more rewarding realm than the finite ones. Admittedly, the theory is difficult. Whereas it already seems possible to bring the theory of finite groups to a conclusion, the wide variety of infinite groups has still not even been roughly surveyed, although many general theorems about such groups can be posed.

Hopefully, before long this realm will also have been approached from many sides. In particular, this is very desirable for the theory of differential equations.

In the following, I will give an outline of the infinite groups. While it is not also possible to develop this theory with the same completeness as that of finite groups, I still believe that the present summary yields the main facts for a rigorous foundation of the theory of infinite groups.

§ 1. Definition of the infinite continuous groups.

1. We define an infinite continuous group as follows:

Definition. *A family of transformations:*

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n) \tag{1}$$

[1]) Verh. d. Ges. d. Wiss. zu Christiana, 1883 and 1889, and part of the treatise on differential invariants in Bd. 24 of the Math. Ann., 1884 [here, Bd. V, Abh. XIII, XXIV; Bd. VI, Abh. II]. In the first paper, I established the concept of "infinite group" for the first time, and at the same time determined all infinite groups of the plane.

shall be called an infinite continuous group when $F_1, \ldots, F_n$ *are the most general solutions of a system of partial differential equations:*

$$W_k\left(x_1,\cdots,x_n,\mathfrak{r}_1,\cdots,\mathfrak{r}_n,\frac{\partial\mathfrak{r}_1}{\partial x_1},\cdots,\frac{\partial\mathfrak{r}_n}{\partial x_n},\frac{\partial^2\mathfrak{r}_1}{\partial x_1^2},\cdots\right)=0 \qquad (k = 1, 2, \ldots), \tag{2}$$

and when this system possesses the following properties:

1. *The most general solutions of the system* (2) *do not depend upon merely a finite number of arbitrary constants.*

2. *Whenever:*

$$\mathfrak{r}_i = F_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$$

and:

$$\mathfrak{r}_i = \Phi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$$

are any two systems of solutions of the differential equations (2):

$$\mathfrak{r}_i = \Phi_i(F_1(x), \ldots, F_n(x)), \qquad (i = 1, \ldots, n)$$

is likewise a system of solutions of these differential equation. In other words: When two arbitrary transformations of the family that is defined by (2) *are performed in sequence this always yields another transformation of the family.*

We call the differential equations (2) the defining equations of the **finite** transformations of the group in question.

In addition, one must remark that in the sequel we always think of system (2) as having been, from the outset, brought into such a form that nothing new is obtained by differentiation. More precisely: If m is the order of system (2) then all differential equations of order m or less that can be derived from (2) by differentiations and eliminations already follow from the system (2) without differentiations.

§ 2. General remarks.

2. When we define the infinite continuous groups in the manner that was just discussed, we exclude from the outset all groups that cannot be defined by differential equations. There are very good grounds for this.

In general, there are infinite continuous groups that cannot be defined by differential equations: One defines such a group, for example, by the totality of all transformations of the plane that leave a given point invariant. However, at first it seems to be difficult to pose general theorems on such groups. For example, it is impossible to distinguish from the outset whether a group that is not defined by differential equations does or does not possess differential invariants. Secondly, the continuous groups that can be defined by differential equations are indeed the only ones for which the general theory of differential equations is meaningful.

The totality of all transformations that leave a given system of differential equations invariant always defines a group that can always be defined by differential equations, when it is continuous.

It generally seems conceivable infinitely many differential equations might be required to define a group in question. If there are groups with the aforementioned properties then the general theory must also be extended from them. That would not be difficult, since in any case the number of differential equations of order *m* and less would be bounded, so the following developments would carry over from such groups almost without alteration. However, as long as I am uncertain of whether there actually are continuous groups that can only be defined by infinitely many differential equations, I shall regard it as natural to restrict myself to the ones whose definition demands a finite number of differential equations.

3. In order to simplify the theory, we introduce yet a second assumption, namely, we would like to consider only such infinite continuous groups whose transformations are pair-wise inverse to each other. This assumption likewise implies the fact that the groups in question include the identity transformation. If we then perform two mutually inverse transformations of a group then we again obtain a transformation of the group that is, in fact, the identity transformation.

The assumption that was just introduced appears to be a restriction, but it is not. Namely, one may prove that any infinite continuous group that can be defined by differential equations of the form (2) contains the identity transformation and consists of pair-wise inverse transformations. However, our assumption is still completely justified when this is not the case, since only groups with pair-wise inverse transformations enter into the applications.

4. The validity of the assertion that was just made can be explained as follows:

If:

$$\mathfrak{x}_i = f_i(x_1, \dots, x_n) \qquad (i = 1, \dots, n) \tag{A}$$

and:

$$\mathfrak{z}_i = \varphi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) \qquad (i = 1, \dots, n) \tag{B}$$

are two transformations of an infinite continuous group then the transformation:

$$\mathfrak{z}_i = \varphi_i(f_1(x), \dots, f_n(x)) \qquad (i = 1, \dots, n) \tag{C}$$

also belongs to the group. In particular, if (A) is a well-defined transformation of the group, while (B) is completely arbitrary, then (C) is also a completely arbitrary transformation of the group. One convinces oneself of this when one substitutes the independent variables $\mathfrak{x}_1, \dots, \mathfrak{x}_n$ for $x_1, \dots, x_n$ by means of the general transformation (B) of the group in the defining equations:

$$W_k\left(\mathfrak{x}_1, \cdots, \mathfrak{x}_n, \mathfrak{z}_1, \cdots, \mathfrak{z}_n, \frac{\partial \mathfrak{z}_1}{\partial \mathfrak{x}_1}, \cdots, \frac{\partial \mathfrak{z}_n}{\partial \mathfrak{x}_n}, \frac{\partial^2 \mathfrak{z}_1}{\partial \mathfrak{x}_1^{\,2}}, \cdots\right) = 0 \qquad (k = 1, 2, \dots).$$

We now select from the transformations of our group any family of ∞^1 of them:

(D) $$\bar{\mathfrak{x}}_i = F_i(x_1, \ldots, x_n, \bar{a}) \qquad (i = 1, \ldots, n)$$

that yield the transformation (A) for $\bar{a} = a_0$ and the transformation (B) for $\bar{a} = a$. If (A) and (D) then yield:

(E) $$\bar{\mathfrak{x}}_i = \Psi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \bar{a}) \qquad (i = 1, \ldots, n)$$

perhaps by removing the x, then, from the statements above, these equations represent nothing but transformations of our group when $\bar{a}$ remains in the neighborhood of a. Therefore, $\Psi_1, \ldots, \Psi_n$ are solutions of the defining equations of our group as long as $\bar{a}$ remains in the neighborhood of a. However, from this, it follows that $\Psi_1, \ldots, \Psi_n$ represent solutions of these defining equations for absolutely all values of $\bar{a}$ and the equations (E) represent transformations of our group for all values of $\bar{a}$. Now, since equations (E) yield the identity transformation for $\bar{a} = a_0$ then our group includes the identity transformation.

Finally, if:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

is any such transformation of our group that lies in the neighborhood of the identity then, from the remarks that were made above, there is always a second transformation of the group:

$$\mathfrak{z}_i = \Phi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad (i = 1, \ldots, n)$$

such that the equations:

$$\mathfrak{z}_i = \Phi_i(F_1(x), \ldots, F_1(x)) \qquad (i = 1, \ldots, n)$$

represent the identity transformation.

The transformations of our group are then actually pair-wise inverse to each other.

5. In the next paragraphs, we shall confirm that any infinite continuous group with the properties that were described contains certain infinitesimal transformations and that it likewise subsumes the one-parameter groups that are generated by these infinitesimal transformations.

Through the introduction and fundamental use of the infinitesimal transformations, the theory of infinite continuous groups now takes on a surprising simplicity. Here, as in the theory of finite groups, the infinitesimal transformations define the actual foundations of the theory.

6. One would not, moreover, wish to lose sight of one fact. The greater part of the following developments (§ 2, *et seq.*) is entirely independent of the fact that the group that is being examined in infinite; almost all of the considerations still remain valid when the group in question is finite. Thus, the following discussion is, at the same time, a new foundation for the theory of finite continuous groups.

§ 3. Infinitely small and infinitesimal transformations.

7. In order to make the following more understandable and to make it possible to express everything clearly, in the present paragraph we next introduce a concept that subsumes the concept of "infinitesimal transformation" as a special case.

We would now like to understand an infinitely small transformation to be a transformation that differs from the identity transformation only by infinitely little. If δt means an infinitely small quantity then the general form of an infinitely small transformation is this one:

$$x_i' = x_i + \delta t \cdot \varphi_i(x_1, \ldots, x_n) + (\delta t)^2 \cdot \psi_i(x_1, \ldots, x_n) + \ldots \qquad (i = 1, \ldots, n),$$

where the coefficients of δt, $(\delta t)^2$, … are arbitrary functions of $x_1, \ldots, x_n$.

8. Any infinitesimal transformation is a special kind of infinitely small transformation and can be defined simply as the infinitely small transformation of the one-parameter group that it generates. Namely, if:

$$Xf = \sum_{l=1}^{n} \xi_i(x_1, \cdots, x_n) \frac{\partial f}{\partial x_i}$$

is an infinitesimal transformation then the finite equations of the one-parameter group that is generated by Xf read as follows:

$$x_i' = x_i + \frac{t}{1}\xi_i + \frac{t^2}{1 \cdot 2} X\xi_i + \frac{t^3}{3!} XX\xi_i + \ldots \qquad (i = 1, \ldots, n).$$

The infinitely small transformation of this one-parameter group is now produced when one assigns an infinitely small value δt to the parameter t; it thus possesses the form:

$$x_i' = x_i + \delta t \cdot \xi_i + \frac{(\delta t)^2}{1 \cdot 2} X\xi_i + \frac{(\delta t)^3}{3!} XX\xi_i + \ldots \qquad (i = 1, \ldots, n).$$

However, when all infinitely small quantities of second and higher order are omitted, this has precisely the form:

$$x_i' = x_i + \delta t \cdot x_i \qquad (i = 1, \ldots, n),$$

in which we ordinarily prefer to write the equations of the infinitesimal transformation Xf.

We can therefore also say: An infinitesimal transformation is an infinitely small transformation in whose equations the infinitely small terms of second higher order are determined completely by the infinitely small terms of first order. In this, one also finds the grounds for the fact that the equations of an infinitesimal transformation can be satisfied by the given of the terms of first order, while dropping the higher order terms. This process is not allowed for an arbitrary infinitely small transformation with no further assumptions.

9. The introduction of the general concept of "infinitely small transformation" is indispensable for the following. Namely, before we can show that any infinite continuous group includes infinitesimal transformations, we must first prove that it possesses infinitely small transformations; this shall be demonstrated in the next paragraphs. It is first on the basis of the presence of infinitely small transformations that we can also prove the presence of infinitesimal transformations.

§ 4. The infinitely small transformations of an infinite continuous group.

10. Among the finite transformations of an infinite continuous group with the previously-defined properties, we choose any family of ∞^1 transformations:

$$\mathfrak{x}_i = f_i(x_1, \ldots, x_n; a) \qquad (i = 1, \ldots, n). \tag{3}$$

The ∞^1 associated inverse transformations, which are likewise contained in our group, might read:

$$\mathfrak{x}_i = \varphi_i(x_1, \ldots, x_n; a_0) \qquad (i = 1, \ldots, n). \tag{4}$$

Furthermore, the f_i, as well as the φ_i, might remain regular in the neighborhood of $a = a_0$.

If we first perform the transformation:

$$\mathfrak{x}_i = \varphi_i(x_1, \ldots, x_n; a_0) \qquad (i = 1, \ldots, n)$$

and then the transformation:

$$\mathfrak{x}_i' = f_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; a_0 + \varepsilon) \qquad (i = 1, \ldots, n)$$

with the arbitrary parameter ε then we obtain ∞^1 transformations:

$$\mathfrak{x}_i' = f_i(\varphi_1(x, a_0), \ldots, \varphi_n(x, a_0); a_0 + \varepsilon) \qquad (i = 1, \ldots, n), \tag{5}$$

which, in turn, belongs to our group. Here, the right-hand sides can be developed in powers of ε. Upon considering the identities:

$$f_i(\varphi_1(x, a_0), \ldots, \varphi_n(x, a_0); a_0 + \varepsilon) = x_i \qquad (i = 1, \ldots, n),$$

we then obtain the following representation for the transformation (5):

(5′) $$\mathfrak{x}_i' = x_i + \varepsilon\left[\frac{\partial f_i(\varphi_1(x,a_0),\cdots,\varphi_n(x,a_0);a_0+\varepsilon)}{\partial\varepsilon}\right]_{\varepsilon=0} + \dots \qquad (i = 1, \dots, n).$$

11. With this, we have found an associated family of transformations for our group whose equation for $\varepsilon = 0$ yields the identity transformation and remains regular everywhere in the neighborhood of $\varepsilon = 0$. We would like to say of such a family that it lies in the neighborhood of the identity transformation.

In particular, if we assign an infinitely small value to the parameter ε in equation (5′) then we find an infinitely small transformation that belongs to our group.

Obviously, the infinitely small transformation that we just derived is not an infinitesimal transformation, in general; it is one only when the ∞^1 transformations (3) belong to a one-parameter group. If this case comes up then the one-parameter group in question is naturally generated by the infinitesimal transformation in question.

12. From the family (3), one may derive still more infinitely small transformations that belong to our group; it is therefore more convenient to choose the family (5′) in place of the family (3) to be our starting point, because the former already include the identity transformation.

We would like to write the transformations (5′) briefly as:

(6) $$\mathfrak{x}_i = F_i(x_1, \dots, x_n; \varepsilon) \qquad (i = 1, \dots, n);$$

the solution of these equations for $x_1, \dots, x_n$ might read:

$$x_i = \Phi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n; \varepsilon) \qquad (i = 1, \dots, n).$$

If we first perform the transformation:

$$\mathfrak{x}_i = \Phi_i(x_1, \dots, x_n; \varepsilon) \qquad (i = 1, \dots, n)$$

and then the transformation:

$$\mathfrak{x}_i' = F_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n; \varepsilon + \omega) \qquad (i = 1, \dots, n)$$

then we again obtain a transformation of our group, namely, this one:

$$\mathfrak{x}_i' = F_i(\Phi_1(x, \varepsilon), \dots, \Phi_n(x, \varepsilon); \varepsilon + \omega) \qquad (i = 1, \dots, n).$$

Since the F_i remain regular in a certain neighborhood of $\varepsilon = 0$ we can develop them in powers of ω and find:

$$\mathfrak{x}_i' = F_i(\Phi_1(x, \varepsilon), \dots, \Phi_n(x, \varepsilon); \varepsilon + \omega) + \frac{\omega}{1}\left[\frac{\partial F_i(\Phi_1(x,\varepsilon),\cdots,\Phi_n(x,\varepsilon);\alpha)}{\partial\alpha}\right]_{\alpha=\varepsilon} + \dots$$

If we finally consider that the term on the right-hand side that is free of ω equals x_i, and we set, moreover:

$$\left[\frac{\partial F_i(\Phi_1(x,\varepsilon),\cdots,\Phi_n(x,\varepsilon);\alpha)}{\partial\alpha}\right]_{\alpha=\varepsilon} = \xi_i(x_1, \dots, x_n;\, \varepsilon) \qquad (i = 1, \dots, n) \tag{7}$$

then we obtain the following representation for our transformation:

$$\mathfrak{x}_i' = x_i + \omega \cdot \xi_i(x_1, \dots, x_n;\, \varepsilon) + \dots \qquad (i = 1, \dots, n), \tag{8}$$

where the coefficients of the omitted higher powers of ω are likewise functions of $x_1, \dots, x_n$ and ε.

From now on, we assign an infinitely small value to the parameter ω and immediately obtain an infinitely small transformation that is associated with our group whose analytical expression includes an arbitrary parameter, namely, ε. This infinitely small transformation is independent of ε only when the family of ∞^1 transformations (6) defines a one-parameter group. In this case, it is naturally nothing but the infinitesimal transformation that will generate this one-parameter group; by contrast, in any other case, we have ∞^1 different infinitely small transformations of our infinite group corresponding to the ∞^1 values of ε.

We express the results obtained as follows:

Theorem 1. *From any family of ∞^1 transformations:*

$$\mathfrak{x}_i = f_i(x_1, \dots, x_n;\, a) \qquad (i = 1, \dots, n) \tag{3}$$

that belongs to an infinite continuous group with pair-wise inverse transformations, one may derive a family of transformations that is associated with the group:

$$\mathfrak{x}_i' = x_i + \omega \cdot \xi_i(x_1, \dots, x_n;\, \varepsilon) + \omega^2 \cdot \vartheta_i(x_1, \dots, x_n;\, \varepsilon) + \dots \qquad (i = 1, \dots, n),$$

which includes the identity and formally two parameters, in addition. If one chooses the ω in this family to be infinitely small then one obtains either one or ∞^1 different infinitely small transformations of the infinite group, and indeed the first case comes up when and only when the ∞^1 transformations (3) *belong to a one-parameter group.*

13. One can also proceed with the family:

$$\mathfrak{x}_i = F_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n;\, \varepsilon + \omega) \qquad (i = 1, \dots, n) \tag{6}$$

in another way.

If one considers the $\mathfrak{x}_i$ as functions of ε then (6) yields upon differentiation with respect to ε:

$$\frac{d\mathfrak{x}_i}{d\varepsilon} = \frac{\partial F_i(x_1, \cdots, x_n; \varepsilon)}{\partial \varepsilon},$$

or, when one makes the substitution:

$$x_\nu = \Phi_\nu(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon) \qquad (\nu = 1, \ldots, n)$$

in the right-hand side:

$$\frac{d\mathfrak{x}_i}{d\varepsilon} = \left[\frac{\partial F_i(x_1, \cdots, x_n; \varepsilon)}{\partial \varepsilon}\right]_{x_\nu = \Phi_\nu(\mathfrak{x}, \varepsilon)} = \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon) \qquad (i = 1, \ldots, n),$$

where the ξ_i are obviously the same functions of their arguments as in equations (7). Conversely, if one now integrates the simultaneous system:

(9) $$\frac{d\mathfrak{x}_1}{\xi_1(\mathfrak{x}_1, \cdots, \mathfrak{x}_n; \varepsilon)} = \ldots = \frac{d\mathfrak{x}_n}{\xi_n(\mathfrak{x}_1, \cdots, \mathfrak{x}_n; \varepsilon)} = d\varepsilon$$

while adding the initial conditions:

(10) $$[\mathfrak{x}_i]_{\varepsilon = 0} = x_i \qquad (i = 1, \ldots, n)$$

then the $\mathfrak{x}_i$ become completely determined functions of $x_1, \ldots, x_n; \varepsilon$. However, we know, on the other hand, that the equations (6), from which the simultaneous system (9) is derived by differentiation, assume the form:

$$\mathfrak{x}_i = x_i, \qquad (i = 1, \ldots, n)$$

for $\varepsilon = 0$. As a result, we can conclude that we must obtain precisely equations (6) by integrating the simultaneous system (9) with the initial conditions (10).

From this, we next infer that the functions ξ_i cannot all vanish; otherwise, we would not, in fact, obtain the family of ∞^1 transformations (6) by the aforementioned integration of the simultaneous system (9), but merely the identity transformation:

$$\mathfrak{x}_i = x_i, \qquad (i = 1, \ldots, n).$$

Furthermore, one deduces that the family (6) always defined a one-parameter group when, but also only when, the functions ξ_i can be represented in the form:

$$\xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon) = \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \cdot \chi(\varepsilon) \qquad (i = 1, \ldots, n),$$

where the ξ_i are completely free of ε. Whether the infinitely small transformation (8) is or is not independent of ε and whether it is or is not an infinitesimal transformation may be decided already by considering the terms of first order in ω.

14. The well-known relation between the family of transformations (6) and the simultaneous system (9) also admits an intuitive explanation.

Namely, if a one-parameter group is generated by the infinitesimal transformation:

$$Yf = \sum_{i=1}^{n} \eta_i(x_1, \cdots, x_n) \frac{\partial f}{\partial x_i}$$

then its finite transformations would be obtained by integrating the simultaneous system:

$$\frac{dx_1'}{\eta_1(x_1', \cdots, x_n')} = \ldots = \frac{dx_n'}{\eta_n(x_1', \cdots, x_n')} = dt,$$

with the assumption of the initial condition $x_i' = x_i$ for $t = 0$. One can thus think of the finite transformations of this one-parameter group as arising by performing the infinitesimal transformation Xf in sequence infinitely often.

In a corresponding way, one can think of the finite transformations of the family (6) as arising by performing infinitely many different infinitesimal transformations in sequence. To this end, one merely needs to interpolate a continuous sequence of values between $\varepsilon = 0$ and $\varepsilon = \varepsilon$; if one now assigns the sequence of all these values for the ε in the infinitesimal transformation:

$$\sum_{i=1}^{n} \xi_i(x_1, \cdots, x_n; \varepsilon) \frac{\partial f}{\partial x_i},$$

and one thinks of the resulting infinitude of infinitesimal transformations as being performed in succession then one obtains precisely the general finite transformation of the family (6).

The foregoing is naturally only an intuitive rationalization of the integration process by which equations (6) arise from the simultaneous system (9).

15. In order to clarify how a one-parameter group of an infinitesimal transformation:

$$\delta x_i = \xi_i(x_1, \ldots, x_n)\, \delta t \qquad (i = 1, \ldots, n)$$

is generated, I have resorted to the following manner of presentation on several occasions: I think of a compressible fluid that is chosen to be in a state of stationary motion. The velocity components of the particle that is found at the location $x_1, \ldots, x_n$ will then always be determined by the equations:

$$\frac{dx_i}{dt} = \xi_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n).$$

On the other hand, if:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon) \qquad (i = 1, \ldots, n)$$

are the equations of ∞^1 transformations of an infinite group then these equations, as we found above, are the integral for of the equations of a certain simultaneous system:

$$\text{(9)} \qquad \frac{d\mathfrak{x}_i}{d\varepsilon} = \xi_i(x_1, \ldots, x_n; \varepsilon) \qquad (i = 1, \ldots, n).$$

If we thus imagine a compressible fluid whose motion (which is, in general, non-stationary) is defined by saying that at the time t the fluid particles possess the velocity components:

$$\frac{d\mathfrak{x}_i}{dt} = \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; t) \qquad (i = 1, \ldots, n)$$

at the location $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$, while the same fluid particles assume the initial position $x_1, \ldots, x_n$ at the time $t = 0$ then the aforementioned fluid particles will be found at the position:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon) \qquad (i = 1, \ldots, n)$$

at the time $t = \varepsilon$.

Therefore, if the equations: $\mathfrak{x}_i = F_i(x; \varepsilon)$ represent ∞^1 transformations of an infinite group, in which one finds the identity transformation, in particular, then one can always make a simple kinematic picture of these ∞^1 transformations such that one describes them as ∞^1 successive positions of a compressible fluid that is found in a certain – in general, non-stationary – motion.

16. Whether or not the infinitesimal transformations that appeared above belong to our group is a question that will first find its answer in the paragraphs after the next one, and in fact, in the affirmative. First, we would like to develop another important relation that exists between these infinitesimal transformations and the family (6).

Let $\Omega(x_1, \ldots, x_n)$ be a function that remains invariant under all transformations of the family:

$$\text{(6)} \qquad \mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon) \qquad (i = 1, \ldots, n).$$

Among them, by assumption, there exists an identity of the form:

$$\Omega(F_1(x, \varepsilon), \ldots, F_n(x, \varepsilon)) = \Omega(x_1, \ldots, x_n).$$

Differentiating this with respect to ε yields the identity:

$$\sum_{i=1}^{n} \frac{\partial \Omega(F_1(x,\varepsilon),\cdots,F_n(x,\varepsilon))}{\partial F_i(x,\varepsilon)} \cdot \frac{\partial F_i(x,\varepsilon)}{\partial \varepsilon} \equiv 0,$$

or, when we make the substitution:

$$x_\nu = \Phi_\nu(\mathfrak{x}_1, \ldots, \mathfrak{x}_\nu; \varepsilon) \qquad (n = 1, \ldots, n)$$

and take into account the identities:

$$F(\Phi_1(x, \varepsilon), \ldots, \Phi_n(x, \varepsilon)) \equiv \mathfrak{x}_i,$$
$$\left[\frac{\partial F_i(x,\varepsilon)}{\partial \varepsilon}\right]_{x_\nu = \Phi_\nu(\mathfrak{x},\varepsilon)} \equiv \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon),$$

this yields the following identities:

$$\sum_{i=1}^{n} \frac{\partial \Omega(\mathfrak{x}_1,\cdots,\mathfrak{x}_n)}{\partial \mathfrak{x}_i}\, \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon) \equiv 0.$$

However, this says that the function $\Omega(\mathfrak{x}_1, \ldots, \mathfrak{x}_n)$ admits ∞^1 infinitesimal transformations:

$$\sum_{i=1}^{n} \xi_i(\mathfrak{x}_1,\cdots,\mathfrak{x}_n;\varepsilon) \frac{\partial f}{\partial \mathfrak{x}_i}.$$

We can thus state the theorem:

Theorem 2. *Let:*

(6) $$\mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon) \qquad (i = 1, \ldots, n)$$

be a family of ∞^1 *transformations that include the identity transformation and indeed, for* $\varepsilon = 0$; *furthermore, let:*

$$\frac{d\mathfrak{x}_1}{\xi_1(\mathfrak{x}_1,\cdots,\mathfrak{x}_n;\varepsilon)} = \ldots = \frac{d\mathfrak{x}_n}{\xi_n(\mathfrak{x}_1,\cdots,\mathfrak{x}_n;\varepsilon)} = d\varepsilon,$$

be the simultaneous system from which one obtains equations (6) *by integration with the initial conditions:*

$$[\mathfrak{x}_1]_{\varepsilon=0} = x_1, \ldots, [\mathfrak{x}_n]_{\varepsilon=0} = x_n .$$

If the function $\Omega(x_1, \ldots, x_n)$ *now admits the* ∞^1 *transformations* (6) *then at the same time it also admits the* ∞^1 *transformations:*

$$\sum_{i=1}^{n} \xi_i(\mathfrak{x}_1, \cdots, \mathfrak{x}_n; \varepsilon) \frac{\partial f}{\partial \mathfrak{x}_i}.$$

17. Up to now, we have made absolutely no use of the fact that our group is infinite. The foregoing developments will thus work just as well for an arbitrary finite continuous group whose transformations are pair-wise inverse to each other. We will now also take into account that we are dealing with an infinite continuous group, and will show that our group includes infinitely many different infinitely small transformations.

Since our group is infinite, its most general transformation depends upon not just a finite number of arbitrary parameters; there are therefore transformations in the group that contain infinitely many parameters. For example, we consider a transformation:

(11) $$\mathfrak{x}_i = f_i(x_1, \ldots, x_n; a_1, \ldots, a_l) \qquad (i = 1, \ldots, n)$$

of our group in which precisely l essential parameters appear. We can thus choose the whole number l to be arbitrarily large:

(12) $$\mathfrak{x}_i = \varphi_i(x_1, \ldots, x_n; a_1, \ldots, a_l) \qquad (i = 1, \ldots, n),$$

and let $a_1^0, \ldots, a_l^0$ be a system of values in whose neighborhood the f_i, as well as the φ_i, remain regular. If we now first perform the transformation:

$$\mathfrak{x}_i = \varphi_i(x_1, \ldots, x_n; a_1^0, \ldots, a_l^0) \qquad (i = 1, \ldots, n)$$

and then the transformation:

$$\mathfrak{x}'_i = f_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; a_1^0 + \varepsilon_1, \ldots, a_l^0 + \varepsilon_l) \qquad (i = 1, \ldots, n)$$

then we obtain a transformation:

$$\mathfrak{x}'_i = f_i(\varphi_1(x, a^0), \ldots, \varphi_n(x, a^0); a_1^0 + \varepsilon_1, \ldots, a_l^0 + \varepsilon_l) \qquad (i = 1, \ldots, n)$$

that again belongs to our group, and which, like (11), contains l essential parameters, namely: $\varepsilon_1, \ldots, \varepsilon_l$.

We thus write this transformation briefly as:

(13) $$\mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_l) \qquad (i = 1, \ldots, n);$$

$F_1, \ldots, F_n$ are then ordinary power series in $\varepsilon_1, \ldots, \varepsilon_l$ and reduce to $x_1, \ldots, x_n$, respectively, for $\varepsilon_1 = 0, \ldots, \varepsilon_l = 0$. The solutions of the equations (13) for $x_1, \ldots, x_n$ might read:

(14) $$x_i = \Phi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1, \ldots, \varepsilon_l) \qquad (i = 1, \ldots, n).$$

If we now first perform the transformation:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_l) \qquad (i = 1, \ldots, n),$$

and then the transformation:

$$\mathfrak{x}_i' = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1 + \omega_1, \ldots, \varepsilon_l + \omega_l) \qquad (i = 1, \ldots, n)$$

then we obtain a transformation of our group that can be written, with the help of the abbreviation:

(15) $$\left[\frac{\partial F_i(\Phi_1(x,\varepsilon),\ldots,\Phi_n(x,\varepsilon);\varepsilon_1+t_1,\ldots,\varepsilon_l+t_l)}{\partial t_k}\right]_{t_1=\cdots=t_l=0} = \xi_{ki}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_l),$$

as:

(16) $$\mathfrak{x}_i' = x_i + \sum_{k=1}^{l} \omega_k\, \xi_{ki}(x_1, \ldots, x_n; \varepsilon_1, \ldots, \varepsilon_l), \qquad (i = 1, \ldots, n),$$

where the unwritten terms are of higher order in $\omega_1, \ldots, \omega_l$.

If we assign ω_k infinitely many values here then we obtain infinitely many transformations of our group that depend upon the l parameters $\varepsilon_1, \ldots, \varepsilon_l$, as well as the $l - 1$ arbitrary ratios of the ω_k. Since l is arbitrary, our infinite group includes infinitely many small transformations.

18. One can also come to the functions ξ_{ki} that enter into (16) in yet another way.

Namely, if one differentiates equations (13) with respect to ε_k:

$$\frac{\partial \mathfrak{x}_i}{\partial \varepsilon_k} = \frac{\partial F_i(x_1,\ldots,x_n;\varepsilon_1,\ldots,\varepsilon_l)}{\partial \varepsilon_k}$$

and makes the substitution (14) on the right-hand side then one obtains:

$$\frac{\partial \mathfrak{x}_i}{\partial \varepsilon_k} = \left[\frac{\partial F_i(x_1,\ldots,x_n;\varepsilon_1,\ldots,\varepsilon_l)}{\partial \varepsilon_k}\right]_{x_\nu=\Phi_\nu(\mathfrak{x},\varepsilon)} = \xi_{ki}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1, \ldots, \varepsilon_l),$$

where the ξ_{ki} are obviously the same functions of their arguments as in equations (15). We can thus conclude that equations (13) are obtained when one integrates the system of partial differential equations:

(17) $$\frac{\partial \mathfrak{x}_i}{\partial \varepsilon_k} = \xi_{ki}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1, \ldots, \varepsilon_l), \qquad (i = 1, \ldots, n;\ k = 1, \ldots, l),$$

while adding the initial conditions:

$$[\mathfrak{x}_i]_{\varepsilon_k=\cdots=\varepsilon_l=0} = x_i .$$

From this, one ultimately infers an important property of the functions ξ_{ki} .

Namely, since the l parameters $\varepsilon_1, \ldots, \varepsilon_l$ are essential in equations (13), the n functions $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ of the l variables $\varepsilon_1, \ldots, \varepsilon_l$ can never satisfy one and the same differential equation of the form:

$$\sum_{k=1}^{l} \alpha_k(\varepsilon_1, \ldots, \varepsilon_l) \frac{\partial \Omega}{\partial \varepsilon_k} = 0,$$

in which the α_k are functions of only the ε. It follows that the ln functions $\xi_{ki}(\mathfrak{x}, \varepsilon)$ can also never satisfy n relations of the form:

$$\sum_{k=1}^{l} \alpha_k(\varepsilon_1, \ldots, \varepsilon_l)\ \xi_{ki}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1, \ldots, \varepsilon_l), \qquad (i = 1, \ldots, n).$$

In other words: The l infinitesimal transformations:

$$\sum_{i=1}^{n} \xi_{ki}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1, \ldots, \varepsilon_l) \frac{\partial f}{\partial \mathfrak{x}_i} \qquad (k = 1, \ldots, n)$$

in the variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ are independent of each other.

19. We would not like to overlook the formulation of this important result as a theorem. In order to do this in the most convenient manner, we remark that in general the l infinitesimal transformations:

$$\sum_{i=1}^{n} \xi_{ki}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n; \varepsilon_1, \ldots, \varepsilon_l) \frac{\partial f}{\partial \mathfrak{x}_i} \qquad (k = 1, \ldots, n)$$

also remain independent of each other when we assign any well-defined numerical values to the ε. We further remark that l different transformations with one parameter can be defined from the transformation (16), when we, in fact, set all ω up to ω_1 equal to zero in succession, then all of them up to ω_2, and so forth. We can then say:

Theorem 3. *No matter how large one sets the positive whole number l, an infinite continuous group with pair-wise inverse transformations always contains l families:*

$$\mathfrak{x}_i = x_i + \omega\, \xi_{ki}(x_1, \ldots, x_n) + \omega^2\ \vartheta_{ki}(x_1, \ldots, x_n) + \ldots \qquad (i = 1, \ldots, n)$$

of ∞^1 *transformations such that the l expressions:*

$$\sum_{i=1}^{n} \xi_{ki}(x_1, \ldots, x_n) \frac{\partial f}{\partial \mathfrak{x}_i} \qquad (k = 1, \ldots, n)$$

determine just as many independent infinitesimal expressions.

§ 5. The defining equations of the finite transformations of an infinite continuous group.

20. We imagine that the finite transformations:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

of an infinite continuous group with pair-wise inverse transformations is defined by a system of partial differential equations:

$$W_k\left(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1}{\partial x_1^2}, \ldots\right) = 0 \qquad (k = 1, 2, \ldots). \tag{18}$$

We will now show that this system of differential equations, in a certain sense, admits the infinite group that it defines.

21. If:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{19}$$

and

$$\mathfrak{x}_i' = \Phi_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{20}$$

are any two transformations of our infinite group then:

$$\mathfrak{x}_i' = \Phi_i(F_1(x), \ldots, F_n(x)) \qquad (i = 1, \ldots, n) \tag{21}$$

is always a transformation of the group, as well. $\mathfrak{x}_1', \ldots, \mathfrak{x}_n'$, when regarded as functions of $x_1, \ldots, x_n$, thus satisfy the differential equations:

$$W_k\left(x_1, \ldots, x_n, \mathfrak{x}_1', \ldots, \mathfrak{x}_n', \frac{\partial \mathfrak{x}_1'}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n'}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1'}{\partial x_1^2}, \ldots\right) = 0 \qquad (k = 1, 2, \ldots) \tag{18′}$$

identically.

On the other hand, $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$, when regarded as functions of $x_1, \ldots, x_n$, fulfill equations (18) identically. Thus, if we imagine that the variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ are removed from (18), by means of (20), then we must obtain a system of differential equations:

$$V_k\left(x_1, \ldots, x_n, \mathfrak{x}_1', \ldots, \mathfrak{x}_n', \frac{\partial \mathfrak{x}_1'}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n'}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1'}{\partial x_1^2}, \ldots\right) = 0 \qquad (k = 1, 2, \ldots) \tag{22}$$

that are satisfied identically, just as the system (18′) is with the substitution (21).

Now, if (19) and (20) are completely arbitrary transformations of our infinite group, and since we can let (19), in particular, coincide with the identity transformation, then (21) is also an entirely arbitrary transformation of the group; equations (21) thus represent a completely arbitrary system of solutions of the differential equations (18′). We then see that any system of solutions of the differential equations (18′) likewise satisfies the differential equations (22); from this, it follows immediately that the system (22) is equivalent to the system (18′).

In fact, the system (22) arises from (18) when $\mathfrak{x}_1', \ldots, \mathfrak{x}_n'$ are introduced in place of $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ by means of the transformation (20). It is therefore of the same order as the system (18′) and also contains exactly as many independent equations as it. Finally, since the system (18′) yields no new equations of the same or lower order by differentiation (cf., pp. 318 [here, pp. 301]), the totality of solutions of (18′) can also be solutions of (22) only when these two systems of differential equations are equivalent, and thus when all of equations (22) follow from equations (18′), and conversely.

With that, we have proved that the system (18) always preserves its form when one introduces the new variables $\mathfrak{x}_1', \ldots, \mathfrak{x}_n'$ in place of $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ by means of any transformation (20) of our group. We express this result in the following way:

Theorem 4. *If an infinite continuous group with pair-wise inverse transformations is defined by a system of partial differential equations:*

(18) $$W_k\left(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1}{\partial x_1^2}, \ldots\right) = 0 \qquad (k = 1, 2, \ldots),$$

and if:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

is any transformation that belongs to the group then the system of differential equations (18) *always admits the transformation:*

$$\mathfrak{x}_i' = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad x_i' = x_i \qquad (i = 1, \ldots, n).$$

22. This important theorem may be inverted.

Namely, if the system (18) admits the transformation:

(23) $$\mathfrak{x}_i' = \Pi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad x_i' = x_i \qquad (i = 1, \ldots, n),$$

and if:

(19) $$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

is any transformation of the infinite group that is defined by (18) then, along with the functions (10), the n functions:

(24) $$\mathfrak{x}_i' = \Pi_i(F_1(x), \ldots, F_n(x)) \qquad (i = 1, \ldots, n)$$

also define a system of solutions of the differential equations (18). From this, it then follows that equations (24) represent a transformation of our infinite group.

In order to express this conveniently, we would now like to introduce the symbols S and T for the transformations (19) and (23), respectively. The aforementioned result may then be expressed briefly as: If the system (18) admits the transformation T and if S is a transformation of the group that is defined by (18) then ST is always a transformation of the group, as well. However, the group contains the transformation S^{-1} at the same time as S, and therefore also the transformation $S^{-1}ST$; that is, it contains T.

With that, we have proved the important theorem:

Theorem I. *If an infinite continuous group with pair-wise inverse transformations is defined by a system of partial differential equations:*

$$(18) \qquad W_k\left(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1}{\partial x_1^2}, \ldots\right) = 0 \qquad (k = 1, 2, \ldots)$$

then this system of differential equations always admits a transformation of the form:

$$\mathfrak{x}'_i = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad x'_i = x_i \qquad (i = 1, \ldots, n)$$

when, but also only when, the equations:

$$\mathfrak{x}'_i = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad (i = 1, \ldots, n)$$

represent a transformation of the infinite group in question.

23. It is perhaps useful to clarify this theorem by an example.

The differential equation:

$$(25) \qquad \sum \pm \frac{\partial \mathfrak{x}_1}{\partial x_1} \frac{\partial \mathfrak{x}_2}{\partial x_2} \cdots \frac{\partial \mathfrak{x}_n}{\partial x_n} = 1$$

defines an infinite continuous group. Namely, if $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ are functions of $x_1, \ldots, x_n$ that satisfy equation (25) then $\mathfrak{x}'_1, \ldots, \mathfrak{x}'_n$, as functions of $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$, satisfy the equation:

$$\sum \pm \frac{\partial \mathfrak{x}'_1}{\partial x_1} \frac{\partial \mathfrak{x}'_2}{\partial x_2} \cdots \frac{\partial \mathfrak{x}'_n}{\partial x_n} = 1$$

that arises from (25) and (26) by multiplication.

Now, if:

$$(27) \qquad \mathfrak{x}'_i = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n), \qquad x'_i = x_i \qquad (i = 1, \ldots, n)$$

is any transformation for which the differential equation (25) remains invariant then equation (25) retains the form:

$$\sum \pm \frac{\partial \mathfrak{x}_1'}{\partial x_1} \frac{\partial \mathfrak{x}_2'}{\partial x_2} \cdots \frac{\partial \mathfrak{x}_n'}{\partial x_n} = 1 \tag{28}$$

under the transformation (27), where $x_1, \ldots, x_n$ are written for $x_1', \ldots, x_n'$; however, equation (25) comes into play, and because of that, as is known, one has:

$$\sum \pm \frac{\partial x_1}{\partial \mathfrak{x}_1} \frac{\partial x_2}{\partial \mathfrak{x}_2} \cdots \frac{\partial x_n}{\partial \mathfrak{x}_n} = 1. \tag{29}$$

Thus, if we multiply (28) and (19) together then we get:

$$\sum \pm \frac{\partial \mathfrak{x}_1'}{\partial \mathfrak{x}_1} \frac{\partial \mathfrak{x}_2'}{\partial \mathfrak{x}_2} \cdots \frac{\partial \mathfrak{x}_n'}{\partial \mathfrak{x}_n} = 1.$$

This immediately illuminates the fact that the equations:

$$\mathfrak{x}_i' = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n), \qquad (i = 1, \ldots, n)$$

actually represent a transformation of the infinite group that is defined by (25).

24. Theorem I on pp. 336 (here, pp. 317) says that the defining equation (18) of an invariant group remains invariant when one performs an arbitrary transformation of the group on the variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$, but, by contrast, only the identity transformation on the variables $x_1, \ldots, x_n$. One may now easily show – although we omit the very simple proof – that Theorem I is distinct from the following one:

Theorem II. *If an infinite continuous group with pair-wise transformations is defined by a system of partial differential equations:*

$$W_k\left(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1}{\partial x_1^2}, \ldots\right) = 0 \qquad (k = 1, 2, \ldots) \tag{18}$$

then this system of differential equations always admits a transformation of the form:

$$\mathfrak{x}_i' = \mathfrak{x}_i, \qquad x_i' = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

when, but also only when, the equations:

$$x_i' = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

represent a transformation of the infinite group in question.

Here, $\mathfrak{x}_1, \dots, \mathfrak{x}_n$ are then transformed by the identity transformation, while the $x_1, \dots, x_n$ are transformed by an arbitrary transformation.

§ 6. The infinitesimal transformations of a finite continuous group.

With theorem I, we are now in a position to prove that any infinite continuous group with pair-wise inverse transformations contains certain infinitesimal transformations and certain one-parameter groups.

25. Let:

$$\mathfrak{x}_i' = \mathfrak{x}_i + \varepsilon\, \xi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \varepsilon^2\, \vartheta_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \dots \qquad (i = 1, \dots, n) \tag{30}$$

be any family of ∞^1 transformations that belongs to our infinite group, and then we assume, as the form of equations (30) shows, that this family includes the identity transformation.

If we now add the following equations to equations (30):

$$x_1' = x_1, \dots, x_n' = x_n \tag{31}$$

then as a result of Theorem I we must obtain a family of ∞^1 transformations that remains invariant under our group:

$$W_k\left(x_1, \dots, x_n, \mathfrak{x}_1, \dots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \dots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \frac{\partial^2 \mathfrak{x}_1}{\partial x_1^2}, \dots\right) = 0 \qquad (k = 1, 2, \dots). \tag{18}$$

All of the following conclusions rest upon this fact.

26. We extend the ∞^1 transformations (30), (31) by taking into account the fact that the differential quotients of the $\mathfrak{x}$ with respect to the x are transformed. In this way, we obtain a number of equations of the form:

$$\begin{cases} \dfrac{\partial \mathfrak{x}_i'}{\partial x_\nu'} = \dfrac{\partial \mathfrak{x}_i}{\partial x_\nu} + \varepsilon \displaystyle\sum_{\tau=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\tau} \frac{\partial \mathfrak{x}_\tau}{\partial x_\nu} + \cdots, \\ \dfrac{\partial^2 \mathfrak{x}_i'}{\partial x_\mu' \,\partial x_\nu'} = \dfrac{\partial^2 \mathfrak{x}_i}{\partial x_\mu \,\partial x_\nu} + \varepsilon \left\{ \displaystyle\sum_{\tau=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\tau} \frac{\partial^2 \mathfrak{x}_\tau}{\partial x_\mu \,\partial x_\nu} + \sum_{\tau=1}^{n} \frac{\partial^2 \xi_i}{\partial \mathfrak{x}_\tau \,\partial \mathfrak{x}_\pi} \frac{\partial \mathfrak{x}_\tau}{\partial x_\nu} \frac{\partial \mathfrak{x}_\pi}{\partial x_\mu} \right\} + \cdots \end{cases} \tag{32}$$

and we then take all differential quotients of first, second, …, up to m^{th} order when the system of differential equations (18) is of m^{th} order.

If we now add the equations (32) to (30) and (31) then we obtain a family of ∞^1 transformations in the variables:

$$x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial^m \mathfrak{x}_n}{\partial x_1^m}, \tag{33}$$

under which the system of equations (18) in these variables remains invariant. That is only another way of expressing the fact that the system of differential equations (18) remains invariant under the ∞^1 transformations (30), (31).

In order to be able to analytically express the invariance of the system of equations (18) in a convenient way, we now introduce the infinitesimal transformation:

$$Xf = \sum_{i=1}^{n} \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \frac{\partial f}{\partial \mathfrak{x}_i},$$

and also extend it by taking all differential quotients of the $\mathfrak{x}$ with respect to the x up to m^{th} order. The infinitesimal transformation that is thus extended:

$$X^{(m)}f$$

in the variables (33) is very easy to compute; namely, it must leave invariant the system of Pfaffian equations in the variables (33):

$$d\mathfrak{x}_i - \sum_{\nu=1}^{n} \frac{\partial \mathfrak{x}_i}{\partial x_\nu} dx_\nu = 0,$$

$$d\frac{\partial \mathfrak{x}_i}{\partial x_\nu} - \sum_{\mu=1}^{n} \frac{\partial^2 \mathfrak{x}_i}{\partial x_\nu\, \partial x_\mu} dx_\mu = 0,$$

and since it does not transform $x_1, \ldots, x_n$ at all, one finds very easily that:

$$X^{(m)}\left(\frac{\partial \mathfrak{x}_i}{\partial x_\nu}\right) = \sum_{\tau=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\tau} \frac{\partial \mathfrak{x}_\tau}{\partial x_\nu},$$

$$X^{(m)}\left(\frac{\partial^2 \mathfrak{x}_i}{\partial x_\mu\, \partial x_\nu}\right) = \sum_{\tau=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\tau} \frac{\partial^2 \mathfrak{x}_\tau}{\partial x_\mu \partial x_\nu} + \sum_{\pi,\tau}^{1\cdots n} \frac{\partial^2 \xi_i}{\partial \mathfrak{x}_\tau \partial \mathfrak{x}_\pi} \frac{\partial \mathfrak{x}_\tau}{\partial x_\nu} \frac{\partial \mathfrak{x}_\pi}{\partial x_\mu}.$$

In other words: One can briefly write the equations of the ∞^1 transformations (30), (31), (32) as:

$$
(34) \qquad \begin{cases}
x'_i = x_i, \\
\mathfrak{x}'_i = \mathfrak{x}_i + \varepsilon X^{(m)}(\mathfrak{x}_i) + \cdots, \\
\dfrac{\partial \mathfrak{x}'_i}{\partial x'_\nu} = \dfrac{\partial \mathfrak{x}_i}{\partial x_\nu} + \varepsilon X^{(m)}\left(\dfrac{\partial \mathfrak{x}_i}{\partial x_\nu}\right) + \cdots, \\
\dfrac{\partial^2 \mathfrak{x}'_i}{\partial x'_\mu \, \partial x'_\nu} = \dfrac{\partial^2 \mathfrak{x}_i}{\partial x_\mu \partial x_\nu} + \varepsilon X^{(m)}\left(\dfrac{\partial^2 \mathfrak{x}_i}{\partial x_\mu \partial x_\nu}\right) + \cdots, \\
\qquad \ldots
\end{cases}
$$

We do not generally know how the coefficients of the higher powers of ε behave; however, that is also entirely irrelevant.

27. We therefore now go on to draw new conclusions from the aforementioned invariance of the system of equations (18).

The system of equations (18) admits the ∞^1 transformations (34). Thus, if we think of all the variables in (18) as now being written with primes then if we express the primed quantities in the equations thus obtained:

$$
(35) \qquad W'_1 = 0, \qquad W'_2 = 0, \ldots
$$

in terms of unprimed ones everywhere by means of (34) then the system of equations that emerges must exist for every ε by means of equations (18). However, we now obtain the following system of equations:

$$
W_k + \varepsilon X^{(m)}(W_k) + \ldots = 0 \qquad (k = 1, 2, \ldots)
$$

from (35) by the given process, where once more only the first powers of ε are taken into account. Should this system of equations exist for any value of ε due to (18) then it is, in any case, necessary that all of the expressions:

$$
X^{(m)}(W_k)
$$

vanish due to (18). However, this is nothing more than the fact that the system of equations (18) in the variables (33) admits the infinitesimal transformation $X^{(m)}f$ in these variables, in addition to the ∞^1 transformations (34).

28. Since the system of equations (18) admits the infinitesimal transformation $X^{(m)}f$, it also admits any transformation of the one-parameter group that is generated by the $X^{(m)}f$. If we then revert to the original standpoint, from which we regarded (18) as a system of differential equations, then we can say: The system of the differential equations (18) admits the infinitesimal transformation Xf and likewise any transformation of the one-parameter group that is generated by Xf.

However, from Theorem I, one thus proves that our infinite group subsumes the infinitesimal transformation Xf, as well as the one-parameter group that is generated by Xf.

Theorem III. *If an infinite continuous group with pair-wise inverse transformations contains* ∞^1 *transformations of the form:*

$$\mathfrak{x}_i = x_i + \varepsilon\,\xi_i(x_1, \ldots, x_n) + \varepsilon^2\,\vartheta_i(x_1, \ldots, x_n) + \ldots \qquad (i = 1, \ldots, n), \tag{30}$$

where ε *denotes an arbitrary parameter, then it likewise contains the infinitesimal transformation:*

$$\sum_{i=1}^{n} \xi_i(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i},$$

and above all, the one-parameter group that it generates.

29. This important theorem is capable of yet another extension. It can indeed happen that the coefficients of $\varepsilon, \varepsilon^2, \ldots, \varepsilon^{l-1}$ in the equations (30) can all vanish, while it is first the coefficients of ε^l that are not all zero. In this case, one obtains by exactly the same reasoning as above:

Theorem 5. *If an infinite continuous group with pair-wise transformations contains* ∞^1 *transformations of the form:*

$$\mathfrak{x}_i = x_i + \varepsilon^l\,\xi_i(x_1, \ldots, x_n) + \varepsilon^{l+1}\,\psi_i(x_1, \ldots, x_n) + \ldots \qquad (i = 1, \ldots, n),$$

where l *denotes a positive whole number that is* ≥ 1, *then it also contains the infinitesimal transformation:*

$$\sum_{i=1}^{n} \xi_i(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i},$$

and the one-parameter group that it generates.

With that, we have then proved that any infinite continuous group with pair-wise inverse transformations contains not just infinitely small, but also infinitesimal transformations.

30. Finally, we can prove very easily with the help of Theorem 3 on pp. 333 [here, pp. 315] that our group contains infinitely many independent [infinitesimal] transformations.

In fact, as a result of this theorem our group always contains l transformations of the form:

$$\mathfrak{x}_i = x_i + \omega\,\xi_{ki}(x_1, \ldots, x_n) + \ldots \qquad (i = 1, \ldots, n),$$

no matter how large l might be, and in which ω denotes an arbitrary parameter, and where the l expressions:

$$\sum_{i=1}^{n} \xi_{ki}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} \qquad (k = 1, \ldots, l) \tag{36}$$

represent just as many independent infinitesimal transformations. However, from Theorem III, the l infinitesimal transformations (36) and the one-parameter groups that they generate belong to our infinite group, so we have:

Theorem 6. *Any infinite continuous group with pair-wise inverse transformations contains infinitely many independent infinitesimal transformations, and likewise infinitely many different one-parameter groups that are generated by these infinitesimal transformations.*

§ 7. The finite transformations of an infinite group are generated by infinitesimal transformations of the group.

31. If one has a finite continuous – perhaps r-parameter – group:

$$x_i' = f_i(x_1, \ldots, x_n; a_1, \ldots, a_r) \qquad (i = 1, \ldots, n)$$

with pair-wise inverse transformations then one knows that the transformations of this group can be arranged into one-parameter groups: Any transformation whose parameter lies in a certain neighborhood of the identity transformation belongs to a certain one-parameter subgroup of the r-parameter group and is therefore generated by an infinitesimal transformation of this group.

In the present paragraph, we have now generally shown that any infinite continuous group with pair-wise inverse transformations contains infinitely many one-parameter groups, but it was in no way proved that the group consists of nothing but one-parameter subgroups. Whether that is the case, i.e., whether any transformation of the infinite group belongs to a one-parameter subgroup of the group, and is thus generated by an infinitesimal transformation of the group, is a function-theoretic question whose resolution does not seem to be simple. Here, we leave this question completely aside, and are all the more justified in doing this because we can show that the transformations of an infinite continuous group are still generated by the infinitesimal transformations of the group, in a certain sense.

32. We recall the developments of pp. 327, *et seq.* [here, pp. 309, *et seq.*]. There, we considered a family:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n; \varepsilon) \qquad (i = 1, \ldots, n) \tag{6}$$

of ∞^1 transformations that belongs to an infinite group and contains the identity transformation. The identity transformation belongs to the value $\varepsilon = 0$ and indeed the F_i are regular in a certain neighborhood of $\varepsilon = 0$.

We have thus shown that any of the ∞^1 transformations (6) can be obtained when one performs the ∞^1 different infinitesimal transformations:

$$(6') \qquad \sum_{i=1}^{n} \xi_i(x_1, \dots, x_n; \varepsilon) \frac{\partial f}{\partial x_i}$$

in succession. On the other hand, we found that our infinite group contains a family of transformations that are represented by equations of the form:

$$(8) \qquad \mathfrak{x}'_i = x_i + \omega\, \xi_i(x_1, \dots, x_n; \varepsilon) + \dots \qquad (i = 1, \dots, n).$$

We can therefore conclude from theorem III, pp. 342 [here, pp. 322] that the ∞^1 infinitesimal transformations (6′) all belong to our infinite group in such a way that each of the ∞^1 transformations (6) can be obtained by performing the ∞^1 different infinitesimal transformations of our group in succession.

We must still remember that any transformation that belongs to the family (6) and exhibits the aforementioned behavior can be briefly referred to as a transformation that lies in the neighborhood of the identity transformation (cf., pp. 323 [here, pp. 306]), so we can state the following:

Theorem 7. *Any transformation of an infinite continuous group with pair-wise inverse transformations that lies in a neighborhood of the identity can be obtained by performing* ∞^1 *different infinitesimal transformations of this group in succession.*

33. A very simple example will suffice to explain things.

The infinite continuous group of all point transformations in a plane will be represented by two equations of the form:

$$(37) \qquad \mathfrak{x} = F_1(x, y), \qquad \mathfrak{y} = F_2(x, y),$$

in which we understand F_1, F_2 to mean completely arbitrary functions of x, y; the differential equations that define the finite equations of this group thus consist of nothing but the identity $0 = 0$.

Among the transformations of the group (37), we choose any well-defined one of them – say:

$$(37') \qquad \mathfrak{x} = f_1(x, y), \qquad \mathfrak{y} = f_2(x, y),$$

and we would like to prove that it is generated by infinitesimal transformations of the group.

We next construct a family of ∞^1 transformations of our group in which the identity transformation is contained, along with the transformation (37′). To that end, we connect any point x, y with the point $\mathfrak{x}$, $\mathfrak{y}$ that it goes to under the transformation (37′) through a line and determines a point x', y' on this line whose distance from x, y relates to the distance between the points x, y and $\mathfrak{x}$, $\mathfrak{y}$ like λ: 1. The point x', y' will then be represented by the two equations:

$$\begin{cases} x' = x + \lambda(f_1(x, y) - x), \\ y' = y + \lambda(f_2(x, y) - y), \end{cases} \tag{38}$$

and it is clear that these equations with the arbitrary parameter λ represent a family of ∞^1 transformations that possess the desired properties.

The point x, y goes to x', y' under the transformation (38); if we replace λ with $\lambda + d\lambda$ then x, y goes to a point that is infinitely close to x', y': $x' + dx'$, $y' + dy'$. However, we can obviously arrive at the point $x + dx$, $y + dy$ by an infinitely small transformation of our group. In order to find this infinitely small transformation, we need only to express dx', dy' in terms of x', y', λ, and $d\lambda$. We thus differentiate equations (38) with respect to λ:

$$\frac{dx'}{d\lambda} = f_1(x, y) - x, \qquad \frac{dy'}{d\lambda} = f_2(x, y) - y,$$

and then eliminate x and y by means of (38), which yields equations of the form:

$$\frac{dx'}{d\lambda} = \xi(x', y', \lambda), \qquad \frac{dy'}{d\lambda} = \eta(x', y', \lambda).$$

With that, we have found the desired infinite small transformation.

Obviously, we can, however, say: Starting from the point x', y', we arrive at the infinitely neighboring point $x' + dx'$, $y' + dy'$ when we perform the infinitesimal transformation with the symbol:

$$\xi(x', y', \lambda)\frac{\partial f}{\partial x'} + \eta(x', y', \lambda)\frac{\partial f}{\partial y'} \tag{39}$$

on x', y'. If we now imagine that the λ in (39) has been replaced with all real numbers from 0 to 1, in succession, then we obtain ∞^1 infinitesimal transformations that yield precisely the transformation (37′) when performed in succession.

With that, we have proved that the transformation (37′) is generated by ∞^1 infinitesimal transformations of the group (37). This group then contains absolutely all point transformations of the plane, so it also contains the ∞^1 infinitesimal transformations (39), in particular.

§ 8. Relations between the infinitesimal transformations of an infinite continuous group.

34. As before, an infinite continuous group with pair-wise inverse transformations can be defined by a system of partial differential equations:

$$(40)\qquad W_k\left(x_1,\dots,x_n,\mathfrak{x}_1,\dots,\mathfrak{x}_n,\frac{\partial\mathfrak{x}_1}{\partial x_1},\dots,\frac{\partial\mathfrak{x}_n}{\partial x_n},\frac{\partial^2\mathfrak{x}_1}{\partial x_1{}^2},\dots\right)=0\qquad (k=1,2,\dots).$$

If:

$$Xf=\sum_{i=1}^{n}\xi_i(\mathfrak{x}_1,\dots,\mathfrak{x}_n)\frac{\partial f}{\partial\mathfrak{x}_i},\qquad Yf=\sum_{i=1}^{n}\eta_i(\mathfrak{x}_1,\dots,\mathfrak{x}_n)\frac{\partial f}{\partial\mathfrak{x}_i}$$

are two infinitesimal transformations of our group then the system of equations (40) admits the two extended infinitesimal transformations:

$$X^{(m)}f,\qquad Y^{(m)}f$$

in the variables:

$$x_1,\ \dots,\ x_n,\ \mathfrak{x}_1,\ \dots,\ \mathfrak{x}_n,\ \frac{\partial\mathfrak{x}_1}{\partial x_1},\ \dots,\ \frac{\partial\mathfrak{x}_n}{\partial x_n},\ \frac{\partial^2\mathfrak{x}_1}{\partial x_1{}^2},\ \dots,\ \frac{\partial^m\mathfrak{x}_n}{\partial x_n{}^m}$$

(cf., pp. 339, *et seq.* [here, pp. 319, *et seq.*]). However, the system (40) then also admits the infinitesimal transformation:

$$a\,X^{(m)}f+b\,Y^{(m)}f=(a\,Xf+b\,Yf)^{(m)}$$

that arises from $a\,Xf + b\,Yf$ by extension. We therefore have the theorem:

Theorem 8. *If X_1f, X_2f, X_3f, … are infinitesimal transformations of an infinite continuous group with pair-wise inverse transformations then this group likewise contains any infinitesimal transformation of the form:*

$$c_1\,X_1f+c_2\,X_2f+c_3\,X_3f+\dots$$

for whatever values one assigns to the constants c_1, c_2, c_3, ...

On the other hand, along with $X^{(m)}f$ and $Y^{(m)}f$, the system of equations (40) also admits the infinitesimal transformation:

$$X^{(m)}f\,Y^{(m)}f-Y^{(m)}f\,X^{(m)}f,$$

which arises from:

$$XY\,f-YX\,f=(X\,Y)$$

by extension. We have thus arrived at the fundamental:

Theorem IV. *If X_1f, X_2f, X_3f, … are infinitesimal transformations of an infinite continuous group with pair-wise inverse transformations then every infinitesimal transformation:*

$$X_i\,X_k\,f-X_k\,X_i\,f=(X_i\,X_k),\qquad (i,k=1,2,\dots)$$

also belongs to the group.

35. We now give yet another elementary proof of this important theorem.

If Xf and Yf are two infinitesimal transformations of our group then this group also contains the two associated one-parameter groups whose finite equations read as follows when one considers only the terms of first and second order:

$$\mathfrak{x}'_i = \mathfrak{x}_i + \varepsilon\,\xi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \frac{\varepsilon^2}{1\cdot 2}\,X\,\xi_i + \dots \qquad (i = 1, \dots, n) \tag{41}$$

and:

$$\bar{\mathfrak{x}}'_i = \mathfrak{x}_i + \varepsilon'\,\eta_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \frac{\varepsilon'^2}{1\cdot 2}\,Y\,\eta_i + \dots \qquad (i = 1, \dots, n). \tag{42}$$

We now first bring the point $\mathfrak{x}_i$ into the new position $\mathfrak{x}'_i$ by means of a transformation of the one-parameter group (41), and we then take the point $\mathfrak{x}'_i$ to the new position $\mathfrak{x}''_i$ by a transformation of the one-parameter group (42):

$$\mathfrak{x}''_i = \mathfrak{x}'_i + \varepsilon'\eta_i(\mathfrak{x}'_1, \dots, \mathfrak{x}'_n) + \frac{\varepsilon'^2}{1\cdot 2}Y'\eta'_i + \dots \qquad (i = 1, \dots, n).$$

If we then substitute the values of $\mathfrak{x}'_i$ from (41) then we obtain the $\mathfrak{x}''_i$, as expressed in terms of the $\mathfrak{x}_i$:

$$\mathfrak{x}''_i = \mathfrak{x}_i + \varepsilon\xi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \frac{\varepsilon^2}{1\cdot 2}X\xi_i + \varepsilon'\{\eta_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \varepsilon X\eta_i\} + \frac{\varepsilon'^2}{1\cdot 2}Y\eta_i + \dots \qquad (i = 1, \dots, n). \tag{43}$$

The equations (43) then naturally represent a transformation of our group.

On the other hand, if we first bring the point $\mathfrak{x}_i$ to the new position $\bar{\mathfrak{x}}'_i$ by means of the transformation (42) and the $\bar{\mathfrak{x}}'_i$ to the new position $\bar{\mathfrak{x}}''_i$ with the help of (41) then the $\bar{\mathfrak{x}}''_i$ are expressed in terms of the $\mathfrak{x}_i$ as follows:

$$\bar{\mathfrak{x}}''_i = \mathfrak{x}_i + \varepsilon'\eta_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \frac{\varepsilon'^2}{1\cdot 2}Y\eta_i + \varepsilon\{\xi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) + \varepsilon' Y\xi_i\} + \frac{\varepsilon^2}{1\cdot 2}X\xi_i + \dots \qquad (i = 1, \dots, n). \tag{44}$$

This is also a transformation of our group.

We now consider the transformation that takes the point $\mathfrak{x}''_i$ to $\bar{\mathfrak{x}}''_i$. This transformation belongs to our group and will be obtained when one removes $\mathfrak{x}_1, \dots, \mathfrak{x}_n$ from (43) and (44). This next gives:

$$\bar{\mathfrak{x}}''_i = \mathfrak{x}''_i + \varepsilon\varepsilon'(Y\,\xi_i - X\,\eta_i) + \dots,$$

where the omitted terms are of third order and higher.

If one expresses the $\mathfrak{x}$ in terms of the $\mathfrak{x}''$ by means of (43) here then one comes to:

(45) $$\overline{\mathfrak{x}}_i'' = \mathfrak{x}_i'' + \varepsilon\varepsilon'(Y''\xi_i'' - X''\eta_i'') + \ldots \qquad (i = 1, \ldots, n).$$

These are, except for terms of third and higher order, the equations for the transformation that takes the point $\mathfrak{x}_i''$ to $\overline{\mathfrak{x}}_i''$.

If one sets $\varepsilon = \varepsilon'$ then one obtains a family of transformations of our group, to which one can apply Theorem 5, pp. 343 [here, pp. 322] with no further assumptions. This immediately yields the fact that our group contains the infinitesimal transformations:

$$\sum_{i=1}^{n} (X\eta_i - Y\xi_i)\frac{\partial f}{\partial \mathfrak{x}_i} = XYf - YXf = (X\,Y).$$

With that, the promised second proof of Theorem IV is delivered.

36. If the finite transformations (41) and (42) of the two one-parameter groups Xf and Yf are denoted by S and T, respectively, then equations (45) obviously represent the transformation:

(46) $$T^{-1}\,S^{-1}\,T\,S.$$

If one then chooses S and T to both be infinitely small, in particular – perhaps when one sets $\varepsilon = \varepsilon' = \delta t$ – then the transformation (46) takes on precisely the form:

$$x_i' = x_i + (\delta t)^2\,(Y\xi_i - X\eta_i) + \ldots \qquad (i = 1, \ldots, n),$$

up to terms of second order. This remark, as also might be emphasized here, explains the important role that the Poisson bracket expression:

$$(X\,Y) = \sum_{i=1}^{n} (X\eta_i - Y\xi_i)\frac{\partial f}{\partial x_i}$$

plays in group theory.

We would like to further mention that the developments of the last pages can also find application to such groups with pair-wise inverse transformations that are not definable through differential equations.

If one knows, on whatever basis, that such a group contains the two infinitesimal transformations Xf and Yf, and therefore also the associated one-parameter groups, then one can conclude from the above that it simultaneously contains the infinitely small transformation:

$$x_i' = x_i + (\delta t)^2\,(Y\xi_i - X\eta_i) + \ldots \qquad (i = 1, \ldots, n);$$

whether it indeed subsumes the infinitesimal transformation $(X\,Y)$ is another matter.

37. A third proof of Theorem IV is given by the following reasoning:

If we understand S and T to mean the finite transformations (41) and (42) of the one-parameter groups Xf and Yf, respectively, and we imagine that ε is chosen to be fixed, while ε' is arbitrary, then the ∞^1 transformations $S^{-1}\ T\ S$ define a one-parameter group that belongs to our infinite group and is generated by the infinitesimal transformation:

$$\mathfrak{Y}f = Yf + \varepsilon(X\ Y) + \frac{\varepsilon^2}{1 \cdot 2}((Y\ X)\ X) + \ldots$$

The system of differential equations that defines our infinite group thus admits the infinitesimal transformation $\mathfrak{Y}f$, and since ε can take on any value it likewise admits the infinitesimal transformation $(Y\ X) = -(X\ Y)$, and thus $(X\ Y)$, as well as the one-parameter group that is generated by $(X\ Y)$, also belongs to our infinite group. However, that is just the content of Theorem IV.

If one applies these considerations to arbitrary infinite continuous groups that are not actually definable by means of differential equations, but whose transformations are associated with each other pair-wise as inverses, then this yields the following:

If such a group contains the two infinitesimal transformations Xf and Yf then it likewise contains the infinitesimal transformation:

$$\mathfrak{Y}f = Yf + \varepsilon(X\ Y) + \frac{\varepsilon^2}{1 \cdot 2}((Y\ X)\ X) + \ldots,$$

no matter what the value of ε might be. Now, if the group contains, above all, the two infinitely small transformations:

$$x_i' = x_i + \varphi_i(x_1, \ldots, x_n)\ \delta t + \ldots \qquad (i = 1, \ldots, n),$$
$$x_i' = x_i + \psi_i(x_1, \ldots, x_n)\ \delta t + \ldots \qquad (i = 1, \ldots, n)$$

then, as one easily confirms, it also contains the infinitely small transformation:

$$x_i' = x_i + (a\varphi_i + b\psi_i)\ \delta t + \ldots \qquad (i = 1, \ldots, n)$$

for arbitrary a and b. Thus, under the assumptions that were made, we can conclude that our group contains the infinitely small transformations whose first-order terms coincide with the first-order terms of the infinitesimal transformation:

$$\varepsilon(X\ Y) + \frac{\varepsilon^2}{1 \cdot 2}((Y\ X)\ X) + \ldots$$

Since ε is arbitrary, we can see from this that our group contains absolutely any infinitely small transformation whose first-order terms coincide with the first-order terms of the infinitesimal transformation $(X\ Y)$. By contrast, we cannot actually prove that our group contains the infinitesimal transformation $(X\ Y)$ itself when the group is not defined by differential equations; still, one cannot indeed doubt that this is the case.

38. The foregoing treatise is, like the paper of Herrn Professor Engel on linear differential equations (these Berichte, pp. 253, *et seq.* [here, Bd. IV, Abh. V]), elaborated upon in a manuscript.