"Zur Geometrie einer Monge'schen Gleichung," Ber. Verh. Kon Ges. Wiss. Leipzig **50** (1898), 1-3.

On the geometry of a Monge equation

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A **Monge** equation:

$$f(x, y, z\ ;\ dx : dy : dz) = 0$$

always has ∞^∞ integral curves, and many relations exist between all integral curves of a certain Monge equation, as I have often emphasized, whose totality might be regarded as the *geometry of the Monge equation.* That geometry includes the geometry of curves on a given surface: $w(x, y, z) = 0$ as a special case. The curves on that surface then all fulfill the linear Monge equation:

$$w_x\, dx + w_y\, dy + w_z\, dz = 0.$$

In my lectures, I have pointed out on many occasions that Meusnier's theorem in geometry has its analogue for a Monge equation. In the following note, I would like to follow through on that important remark in detail. In so doing, I will ignore the minimal curves of the equations:

$$dx^2 + dy^2 + dz^2 = 0,$$

for obvious reasons.

We choose the arc-length s of a curve to be the independent variable and set:

$$\frac{dx}{ds} = x', \qquad \frac{dy}{ds} = y', \qquad \frac{dz}{ds} = z', \qquad \frac{dx'}{ds} = x'', \qquad \dots$$

Upon differentiating the following equation:

$$f(x, y, z\ ;\ x' : y' : z') = 0,$$

we will then obtain the differential equation:

$$f_{x'}\, x'' + f_{y'}\, y'' + f_{z'}\, z'' + f_x\, x' + f_y\, y' + f_z\, z' = 0,$$

which we divide by the quantity:

$$\sqrt{x''^2 + y''^2 + z''^2} \cdot \sqrt{f_{x'}^2 + f_{y'}^2 + f_{z'}^2}\ .$$

If we then set:

$$\frac{1}{\sqrt{x''^2 + y''^2 + z''^2}} = R,$$

$$\frac{f_{x'}\, x'' + f_{y'}\, y'' + f_{z'}\, z''}{\sqrt{x''^2 + y''^2 + z''^2} \cdot \sqrt{f_{x'}^2 + f_{y'}^2 + f_{z'}^2}} = \sin\Theta$$

then we will immediately get the formula that was promised:

$$R = \sin\Theta \cdot \frac{\sqrt{f_{x'}^2 + f_{y'}^2 + f_{z'}^2}}{f_x\, x' + f_y\, y' + f_z\, z'},$$

which possesses the same form as a well-known formula of surface theory.

In this formula, R is the radius of curvature of an integral curve, Θ is the angle between the osculating plane of the curve and the plane that contacts the element of the Monge equation along the tangent curve.

That formula will become illusory when one has:

$$f_x\, x' + f_y\, y' + f_z\, z' = 0\ ;$$

i.e., when the Monge equation has the form:

$$0 = f \equiv F\,(x', y', z', y\,z' - y'z, z\,x' - z'x, x\,y' - x'y)$$

and thus defines a line complex. However, in that case, one also has $\Theta = 0$, and R will then have the indeterminate form:

$$R = \frac{0}{0}.$$

Clearly, we shall overlook the obvious analytically-exceptional cases. With that self-explanatory restriction, we might say that *the centers of curvature of all integral curves with a common line-element lie on a circle.*

Thus, a circle is associated with each line-element of a Monge equation $f = 0$.
