"Die infinitesimalen Berührungstransformationen der Optik," Leipziger Berichte (1896), Heft I, submitted 17-4-1896, pp. 131-133. Presented at the session on 13-1-1896; *Gesammelte Abhandlungen*, v. VI, art. XXIV.

The infinitesimal contact transformations of optics

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1. In the years 1869 and 1870, I recognized (cf., the Verh. d. Ges. d. Wiss. zu Christiania) that it is preferable to introduce the concept of infinitesimal point transformation, as well as the general concept of contact transformation, and to systematically exploit them in geometry and the theory of differential equations. By combining and extending these concepts, the general concept of *infinitesimal contact transformation, as well as that of one-parameter group of contact transformation,* emerged in the years 1871 and 1872 (cf., Gött. Nachrichten and Verh. d. Ges. d. Wiss. zu Christiana); in that way, the foundations for the invariant theory of contact transformations (1872), as well as for the theory of transformation groups, were laid.

2. In my lectures over many years ([1]) on the theory that I founded at the universities of Christiania (1872 to 1885) and Leipzig, I regularly directed the listener's attention to the fact that various domains in mechanics and physics (in particular, optics) *illustrate* the concept of a one-parameter group of point (contact, resp.) transformations in a beautiful way, and at the same, demand the explicit introduction of these concepts.

I also sought to induce other researchers to utilize my theories in their explanation of nature by talks (among others, ones that were presented to this society in the year 1893), as well as by oral communications to several mathematicians, astronomers, physicists, and chemists.

These endeavors of mine have not been completely lacking in effect. However, since the representatives of astronomy, physics, and chemistry can generally acquaint themselves with the newer and more advanced branches of mathematics only with some difficulty, I consider it appropriate to refer to a *particular*, concise note on my investigations that will come under consideration here.

3. The so-called wave motions give the simplest picture of a one-parameter group of contact transformations. For isotropic media (whose wave surfaces are *eo ipso* spheres),

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([1]) *Theorie der Transfor.*, Bd. I, pp. 58, Leipzig, 1888; Bd. II, pp. 250, 1890; Bd. III, Foreword, pp. vi, 1893 [Leipziger Berichte (1889), pp. 145 [this collection, art. VI, pp. 237]]; cf., also my accompanying remarks to the German translation of **Goursat**'s *Theorie der partiellen Differentialgleichungen erster Ordnung*, Leipzig (1893) [this coll., v. IV, art. VIII].

these groups will be generated by infinitesimal contact transformations, for whose symbol, I shall employ the expression:

$$\sqrt{p_1^2 + p_2^2 + p_3^2} \, .$$

The group property of all dilatations has the closest connection with **Huygens**'s principle ([1]).

Reflections are contact transformations that leave that infinitesimal contact transformation *invariant*, and *commute* with it.

Refractions during the transition to another isotropic medium are contact transformations that likewise leave that infinitesimal contact transformation invariant.

This follows immediately from the constructions and developments in the textbooks (e.g., *Jamin*) on reflections and refractions, combined with my old remark that every surface complex f (or curve complex) defines a contact transformation.

All of the ∞^2 surfaces f that contact an arbitrary surface φ will, in fact, generally envelop yet another surface Φ, and therefore the transformation from φ to Φ will be a contact transformation.

Furthermore, let me emphasize that all infinitesimal contact transformations that commute with a given contact transformation will define a homogeneous function group. Likewise, all contact transformations that leave such an infinitesimal transformation invariant will generate an infinite group.

As I have expressly emphasized recently in these Berichten (1895, pp. 499 [here, art. XXIII, pp. 607]), moreover, these general theories of mine can obviously be extended to *all* wave motions, regardless of whether the medium in question is isotropic or not.

4. In this way, I arrived at a generalization of the celebrated theorem of **Malus**, among other things. In it, I employed an extension of the usual theory of curvature that I developed in the year 1872 in volume five of the Mathematischen Annalen, pp. 196 [this coll., v. II, art. I, § 15, no. 46, penultimate paragraph]. With the use of this terminology, the generalized **Malus** theorem can be formulated in the following way:

Light rays that define a system of pseudo-normals will go to a system of pseudo-normals under any reflection or refraction. If the two pseudo-spheres (i.e., wave surfaces) that come under consideration for such a refraction differ essentially then each system of pseudo-normals will refer to the pseudo-sphere of the space in question.

([1]) In connection with that, permit me to point out that the concept of *dilatation* can also be related to *Huygens's* principle.