"Die Theorie der Integralinvarianten ist ein Korollar der Theorie der Differentialinvarianten," Leipziger. Berichte (1897), Heft III, submitted on 14-7-1897, pp. 342-357. Presented at the session on 3-5-1897; *Gesammelte Abhandlungen*, v. IV, art. XXVII, pp. 649-663.

The theory of integral invariants is a corollary to the theory of differential invariants

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1. In this Note, I shall first show that my general theory of *differential invariants* implies the entire theory of *integral invariants* with no further assumptions. That will allow me to clarify their dependency relationship, in which not only the relevant investigations of **Zorawsky**, **Cartan**, and **Hurwitz** ([1]), but also some papers of **Poincaré**, and especially a Note of **Königs**, relate to my older work.

As far as I can see, **Cartan** is the only one whose has presented the matter correctly. At the same time, however, I would like to express my thanks to **Zorawsky** and **Hurwitz** for the fact that they have, in any event, referred to the relationship with my investigations in their own work. As far as the work of **Königs** is concerned, I must assume that my paper that goes back to the beginning of the year 1877 in the Norwegian Archiv, Bd. II: "Die Störungstheorie und die Berührungstransformationen" [this collection, v. III, art. XX] has escaped his attention.

I.

General definition of the concept of integral invariant.

2. If we have been given a *finite* or *infinite* continuous transformation group in the variables:

$$x_1', \ldots, x_{n'}', \; x_1'', \ldots, x_{n''}'', \; \ldots, \; x_1^{(q)}, \ldots, x_{n^{(q)}}^{(q)},$$
$$z_1', \ldots, z_m', \; z_1'', \ldots, z_{m''}'', \; \ldots, \; z_1^{(q)}, \ldots, z_{n^{(q)}}^{(q)}$$

then we will say that an integral:

([1]) **Zorawsky**, Akademie zu Krakau, April 1895. – **Königs**, Comptes rendus, Paris 9 Dec. 1895. – **Cartan**, Bulletin de la Société math., Paris 1896. – **A. Hurwitz**, Gesellsch. d. Wiss. zu Göttingen, March 1897. – **Poincaré**, Acta math., t. 13, 1890. – **Sophus Lie**, Norweg. Archiv, Bd. II, 1877, and *Theorie der Trans.*, Leipzig 1888-1893, Bd. I and III; Ges. d. Wiss. zu Christiana 1883 [this collection, v. III, art. XX, and art. XLI]

$$\int \Omega\left(x_1', \ldots, x_{n'}', \ldots, x_{n^{(q)}}^{(q)}, z_1', \ldots, z_{m^{(q)}}^{(q)}, \frac{\partial z_1'}{\partial x_1'}, \ldots, \frac{\partial^2 z_1'}{\partial x_1'^2}, \ldots \right) d\omega$$

$$\left(d\omega = dx_1', \ldots, dx_{n'}', \ldots, dx_{n^{(q)}}^{(q)} \right)$$

represents an *integral invariant* of the group in question when the variation of the integral:

$$\delta \int \Omega(\cdots)\, dx_1' \cdots dx_{n^{(q)}}^{(q)}$$

vanishes for all infinitesimal transformations of the group – in other words, when the form of the integral is preserved for all finite transformations of the group.

In this, we have assumed that each $z_k^{(i)}$ can be considered to be a function of $x_1^{(i)}$, $x_2^{(i)}$, …, $x_{n^{(i)}}^{(i)}$, and furthermore that Ω includes not just the x and z, but also higher-order derivatives of the z with respect to x, and finally that the quantities $x^{(i)}$, $z^{(i)}$ transform amongst themselves.

The fact that *both* of the definitions of the concept of an integral invariant that will be formulated here are *equivalent* is a special case of my general theorem that every finite transformation of an arbitrary (finite or infinite), continuous group of transformations can be replaced by the composition of a sequence of infinitely-many infinitesimal transformations of the group.

3. I will now show that *my general theory of differential invariants achieves the determination of all integral invariants that are present in every individual case* ***immediately*** *because it will convert the problem into the integration of a complete system.*

In order to make the validity of this noteworthy general theorem clear to the reader, I find it convenient to first consider an example that I have long since treated thoroughly. Namely, I will first recall how I approached the question at the time of whether a given group of point transformations of space $x_1, \ldots, x_n$ leaves a differential expression:

$$H(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$$

invariant that is *homogeneous of* ***first*** *order* in the differentials $dx_1, \ldots, dx_n$, and *thus represents an element of arc length*, in the ***Riemann*** sense of the word. If any such expression H is given then:

$$\int H\left(x_1, \ldots, x_n, 1, \frac{dx_2}{dx_1}, \ldots, \frac{dx_n}{dx_1} \right) dx_1$$

will obviously be an integral invariant of the group in question.

II.

Transformation groups that leave an arc length invariant.

4. Already in the year 1886 ([1]), I concerned myself in detail with all groups of transformations:

$$x'_k = f_k (x_1, \ldots, x_n, a_1, \ldots, a_r) \qquad (k = 1, \ldots, n)$$

for which a function:

$$H (x_1, \ldots, x_n, dx_1, \ldots, dx_n)$$

that is homogeneous of first degree in the differentials $dx_1, \ldots, dx_n$ remains invariant. In chap. 25 of the first volume of my *Theorie der Transformationsgruppen* and chaps. 22, 23 of the third volume (1888 and 1893, resp.), I gave the most elementary presentation of this special theory that is possible, which is contained implicitly as a special case in my general theory of differential invariants. I will now permit myself to summarize the development that I gave in the aforementioned place.

5. If $X_1 f, \ldots, X_r f$, where:

$$X_k f = \sum_i \xi_{ki}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i},$$

are r independent infinitesimal transformations of an r-parameter group, and if:

$$X'_k f = \sum_i \xi_{ki}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} + \sum_{i,\nu} \frac{\partial \xi_{ki}}{\partial x_\nu} x'_\nu \frac{\partial f}{\partial x'_i} \qquad (k = 1, \ldots, r)$$

are the infinitesimal transformations of the once-extended group then the $r + 1$ infinitesimal transformations $X'_1 f, \ldots, X'_r f$, and:

$$X'_0 f = \sum_{i=1}^{n} x'_i \frac{\partial f}{\partial x'_i}$$

will also generate a group in their own right, since the r expressions:

$$X'_0 X'_i f - X'_i X'_0 f$$

will all vanish identically.

However, two cases can come about here:

It is, at first, conceivable that $X'_0 f$ can be represented as a sum of the $X'_i f$, when they are multiplied by suitable quantities:

([1]) Cf., Leipziger Berichte 1886, pps. 341, 342; 1890, pps. 292, 293. *Grundlagen der Geometrie* [this collection, v. II, art. V, at the end; art. VI, § 2].

(1) $$X_0' f = \varphi_1 X_1' f + \cdots + \varphi_r X_r' f .$$

In that case, all of the invariant differential expressions:

$$\Omega\,(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$$

that remain invariant under the r-parameter group $X_1 f, \ldots, X_r f$ will be of order *zero* in the differentials $dx_1, \ldots, dx_n$. There will then exist no invariant differential expressions of first order in the dx_i, and therefore no invariant arc length, either. The curves in space $x_1, \ldots, x_n$ will then have no arc length that is invariant under the group.

By contrast, if no relation of the form (1) exists then the transformations $X_1' f$, …, $X_r' f$, and $X_0' f$ will generate an $(r + 1)$-parameter group in the variables $x_1, \ldots, x_n, x_1'$, …, x_n', and it possible, moreover, to find solutions of the equations:

(2) $$\begin{cases} X_1' f = 0, \ldots, X_r' f = 0 \\ 0 = X_0' f + \varphi \dfrac{\partial f}{\partial \varphi} \equiv \sum x_i' \dfrac{\partial f}{\partial x_i'} + \varphi \dfrac{\partial f}{\partial \varphi} \equiv Yf ; \end{cases}$$

the $X_1' f$, …, $X_r' f$, and Yf will obviously generate an $(r + 1)$-parameter group in $x_1, \ldots, x_n, x_1', \ldots, x_n'$, and φ.

If $\Phi(x_1, \ldots, x_n, x_1', \ldots, x_n', \varphi)$ is an arbitrary solution of the complete system (2), and if the equation:

$$\Phi(x_1, \ldots, x_n, x_1', \ldots, x_n', \varphi) = \text{const.}$$

yields:

$$\varphi = \varphi(x_1, \ldots, x_n, x_1', \ldots, x_n')$$

upon solving it then φ will be a differential invariant of the group $X_1 f, \ldots, X_r f$, that is homogeneous of first degree in the x_k'. Moreover:

$$ds = \varphi\,(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$$

is then an arc length that is invariant under the group $X_1 f, \ldots, X_r f$.

Therefore:

$$\int \varphi\left(x_1, \ldots, x_n, x_1', \ldots, x_n'\right) \frac{dx_1}{x_1'}$$

is an integral invariant of our group. Obviously, all integral invariants of our group that relate to a curve in space $x_1, \ldots, x_n$ will be found to have *degree one*.

I presented the matter in an essentially more general way in the cited place in volume three of my *Transformationsgruppen*. On the one hand, I presented the theory in a form that was general enough that the *infinite* groups were dealt with, but on the other hand, it was *suggested* that I had carried out the determination of *all* integral invariants.

III.

A general theorem on integral invariants.

6. As we did in § I, we would like to assume that we are given a finite or infinite continuous transformation group in the variables:

$$x_1^{(i)}, \ldots, x_{n^{(i)}}^{(i)}, z_1^{(j)}, \ldots, z_{m^{(j)}}^{(j)} \qquad (i, j = 1, 2, \ldots, q),$$

and that we are seeking all of the associated integral invariants:

$$J = \int \Omega\left(x_1', \ldots, x_{n^{(q)}}^{(q)}, z_1', \ldots, z_{m^{(q)}}^{(q)}, \frac{\partial z_1'}{\partial x'}, \ldots \right) dx_1' \ldots dx_{n^{(q)}}^{(q)};$$

it is therefore our assumption in this that Ω includes not just the independent variables x and their functions z, but also the associated derivatives of order one, two, …, up to v.

If we now set:

$$dx_1' \, dx_2' \ldots dx_n' \ldots dx_{n^{(q)}}^{(q)} \equiv d\omega,$$

for brevity, and correspondingly:

$$J = \int \Omega \, d\omega,$$

then the variation:

$$\delta \int \Omega \, d\omega = \int (\delta\Omega \cdot d\omega + \Omega \cdot d\delta\omega)$$

must vanish for all infinitesimal transformations of our group. From the rules of the calculus of variations, in order for this to be true, it is necessary and sufficient that the equation:

$$\delta(\Omega \, d\omega) = \delta\Omega \cdot d\omega + \Omega \cdot \delta d\omega = 0,$$

or the equivalent one:

$$X (\Omega \, d\omega) = X \, \Omega \cdot d\omega + \Omega \cdot X (d\omega) = 0,$$

must always be true, no matter which infinitesimal transformation Xf of the group we also choose to be the variation operator.

Now, if:

$$Xf = \sum \xi(x, z) \frac{\partial f}{\partial x} + \sum \zeta(x, z) \frac{\partial f}{\partial z}$$

is the general symbol of an infinitesimal transformation of our group then, from a known theorem of hydrodynamics (which we shall derive later on in a group-theoretic way), moreover, one will have:

$$X (d\omega) = d\omega \cdot \sum \left(\frac{\partial \xi}{\partial x} \right),$$

in which one must observe that every derivative $(\partial\xi : \partial x)$ in the calculation, like the z and their derivatives with respect to the x, should be regarded as functions of x.

Now, if:

$$\int \Omega_1\, d\omega \qquad \text{and} \qquad \int \Omega_2\, d\omega$$

are two such integral invariants, and correspondingly, the two relations:

$$\left(X\Omega_1 + \Omega_1 \sum \left(\frac{\partial\xi}{\partial x}\right)\right) d\omega = 0, \qquad \left(X\Omega_2 + \Omega_2 \sum \left(\frac{\partial\xi}{\partial x}\right)\right) d\omega = 0$$

are verified for each infinitesimal transformation Xf of the group in question (or the equivalent ones:

$$X\Omega_1 + \Omega_1 \sum \left(\frac{\partial\xi}{\partial x}\right) = 0, \qquad X\Omega_2 + \Omega_2 \sum \left(\frac{\partial\xi}{\partial x}\right) = 0,$$

for that matter), then one will have:

$$X\left(\frac{\Omega_2}{\Omega_1}\right) = 0.$$

That is, the ratio of the two quantities Ω_1 and Ω_2 is a *differential invariant of the group that is subject to no other restriction on its form whatsoever beside that it should depend upon only the x, the z, and the derivatives of the z with respect to the x.*

7. One then has the following theorem:

Theorem. *If one is given a finite or infinite continuous transformation group in the variables:*

$$x_1^{(i)}, \ldots, x_{n^{(i)}}^{(i)}, \; z_1^{(j)}, \ldots, z_{m^{(j)}}^{(j)} \qquad (i, j = 1, 2, \ldots, q),$$

and if:

$$\int \Omega \cdot dx_1' \cdots dx_{n'}' \cdots dx_{n^{(q)}}^{(q)}$$

is a known integral invariant, while:

$$U\left(x_1', \ldots, x_{n'}', \ldots, x_{n^{(q)}}^{(q)}, z_1', \ldots, z_{m^{(q)}}^{(q)}, \frac{\partial z_1'}{\partial x_1'}, \cdots, \frac{\partial^2 z_1'}{\partial x_1'^2}, \cdots\right)$$

denotes any differential invariant of the group in question that depends upon only the x, the z, and the derivatives of the z with respect to the x, then the general formula:

$$\int U \cdot \Omega \cdot dx_1' \cdots dx_{n'}' \cdots dx_{n^{(q)}}^{(q)}$$

will yield all of the associated integral invariants of the group.

8. Several sets of integral invariants belong to any continuous group, each of which will encompass infinitely many members.

If:

$$\int \Omega \cdot dx'_1 \cdots dx^{(q)}_{n^{(q)}}$$

is an integral invariant then Ω itself will not be a differential invariant, in general; from the discussion above, the expression $X\,\Omega$ will then have the value:

$$-\Omega \cdot \sum\left(\frac{\partial \xi}{\partial x}\right),$$

and therefore, it is only when all of the expressions:

$$\sum\left(\frac{\partial \xi}{\partial x}\right)$$

vanish identically that Ω will represent a differential invariant of our group.

By contrast, the expression:

$$\Omega\ dx'_1 \dots dx'_n \dots dx^{(q)}_{n^{(q)}} \equiv \Omega\ d\omega$$

is, in turn, a differential invariant of the group; that differential invariant is specified by the fact that it contains the quantity $d\omega$ *as a factor.*

Later, we shall find a more precise expression for this state of affairs.

IV.

Any continuous group has infinitely many integral invariants.

9. Before we go further, we now set:

$$n' + n'' + \dots + n^{(q)} = n, \qquad m' + m'' + \dots + m^{(q)} = m,$$

and correspondingly denote the independent variables $x_k^{(i)}$ by:

$$x_1, x_2, \dots, x_n,$$

and, on the other hand, denote the functions $z_v^{(j)}$ by:

$$z_1, z_2, \ldots, z_m .$$

We denote the associated first-order derivatives by $p_1, p_2, \ldots$, the second-order derivatives by $r_1, r_2, \ldots$, and so on.

Now, if:

$$Uf = \sum \xi_k(x_1, \ldots, x_n, z_1, \ldots, z_m)\frac{\partial f}{\partial x_k} + \sum \zeta_i(x, z)\frac{\partial f}{\partial z_i}$$

is the general symbol of the infinitesimal transformations of a continuous (finite or infinite) group, and on the other hand if:

$$\int \Omega\,(x_1, \ldots, x_n, z_1, \ldots, z_m, p_1, \ldots, r_1, \ldots)\, dx_1 \ldots dx_n \equiv \int \Omega\; d\omega$$

denotes any associated integral invariant then, as we saw, the equation:

$$U(\Omega) + \Omega \sum \left(\frac{\partial \xi}{\partial x}\right) = 0$$

will be verified, and conversely, any solution Ω of this equation will give an integral invariant.

In this, one must observe that Ω must first take the form of a function of the x, the z, the first-order derivatives p, the second-order derivatives r, and so on; however, for our purposes, the z will be functions of the x, and therefore we shall also regard Ω as a function of the x. If we would therefore like to calculate the expression $U(\Omega)$ then we would first have to exhibit the v-fold extended infinitesimal transformation:

$$U^{(v)} f \equiv \sum \xi \frac{\partial f}{\partial x} + \sum \zeta \frac{\partial f}{\partial z} + \sum \pi \frac{\partial f}{\partial p} + \sum \rho \frac{\partial f}{\partial r} + \ldots,$$

and then set:

$$U(\Omega) = U^{(v)}\,\Omega.$$

On the other hand, we must understand the $\sum \left(\frac{\partial \xi}{\partial x}\right)$ to mean the quantities that we denote more specifically by:

$$\sum_i \left\{\frac{\partial \xi_i}{\partial x_i} + \sum_j \frac{\partial \xi_i}{\partial z_j} p_i^{(j)}\right\}.$$

If Uf represents the general symbol of an infinitesimal transformation of our group then the desired function Ω of the $x, z, p, r, \ldots$ will be defined by the fact that every equation of the form:

$$0 = U^{(v)}\,\Omega + \Omega \sum_i \left\{\frac{\partial \xi_i}{\partial x_i} + \sum_j \frac{\partial \xi_i}{\partial z_j} p_i^{(j)}\right\}$$

should be fulfilled. We have to examine whether these equations possess a common solution.

That is the question that shall be answered.

10. From principles that have been known for some time, this question coincides with the question of whether the homogeneous, linear, partial differential equations:

$$U^{(v)}\,\Omega - \Omega \sum_i \left\{ \frac{\partial \xi_i}{\partial x_i} + \sum_j \frac{\partial \xi_i}{\partial z_j} p_i^{(j)} \right\} \frac{\partial f}{\partial \Omega} = 0$$

possess common solutions, and the latter question will be dealt with in a general way by my theory.

Namely, since Uf implies the general symbol of the infinitesimal transformations of a group, and $U^{(v)}f$ represents the associated v-fold extended transformation, the infinitesimal transformations:

$$U^{(v)}\,\Omega - \Omega \sum_i \left\{ \frac{\partial \xi_i}{\partial x_i} + \sum_j \frac{\partial \xi_i}{\partial z_j} p_i^{(j)} \right\} \frac{\partial f}{\partial \Omega}$$

will also generate an infinite group ([1]) in the variables:

$$x, z, p, r, \ldots, \Omega \, .$$

One then has only to decide whether this infinite group possesses invariants by using my usual rules. Any such invariant that includes Ω will yield an integral invariant:

$$\int \Omega \; dx_1 \ldots dx_n \, ,$$

and one will find all integral invariants in that way.

In particular, if the given group is finite then it can be proved that v can be chosen to be large enough that one has integral invariants of order v.

11. By contrast, if the group of Uf is infinite then a special analysis will be required. For example, if we take the group of all point transformations:

$$\xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y} ,$$

and if we apply these transformation to the pair of curves:

([1]) Cf., Leipziger Berichte, 11 April 1895, pp. 307-308 [here, art. XX, pp. 579, *et seq.*]

$$y_1 = \varphi_1(x_1), \qquad y_2 = \varphi_2(x_2)$$

then we will have to examine whether our group possesses integral invariants of the form:

$$\int \Omega(x_1, x_2, y_1, y_1', y_1'', \ldots, y_2, y_2', y_2'')\ dx_1\, dx_2\ .$$

We now have to set:

$$Uf = \xi(x_1, y_1)\frac{\partial f}{\partial x_1} + \eta(x_1, y_1)\frac{\partial f}{\partial y_1} + \xi(x_2, y_2)\frac{\partial f}{\partial x_2} + \eta(x_2, y_2)\frac{\partial f}{\partial y_2}\,.$$

Now, if $U^{(v)}f$ is the v-fold extended infinitesimal transformation then the infinitesimal transformations:

$$U^{(v)}f - \Omega\ \sum_{k=1}^{2}\left(\frac{\partial \xi(x_k, y_k)}{\partial x_k} + \frac{\partial \xi(x_k, y_k)}{\partial y_k}\,y_k'\right)\frac{\partial f}{\partial \Omega}$$

will also generate an infinite continuous group that cannot be defined by differential equations, in general. If one would like to decide whether that group possesses invariants for a certain v then one would have to observe that there is no connection between the various first, second, …, up to v^{th}-order derivatives of $\xi(x_1, y_1)$ and the corresponding derivatives of $\xi(x_2, y_2)$. The desired invariants must therefore be, at the same time, invariants of an infinite group that one finds when one sets:

$$V f = \xi_1(x_1, y_1)\frac{\partial f}{\partial x_1} + \eta_1(x_1, y_1)\frac{\partial f}{\partial y_1} + \xi_2(x_2, y_2)\frac{\partial f}{\partial x_2} + \eta_2(x_2, y_2)\frac{\partial f}{\partial y_2}\,,$$

and then constructs the v-fold extended transformation $V^{(v)} f$, and finally, the equation:

$$V^{(v)} f - \Omega\sum_{k=1}^{2}\left(\frac{\partial \xi_k(x_k, y_k)}{\partial x_k} + \frac{\partial \xi_k(x_k, y_k)}{\partial y_k}\,y_k'\right)\frac{\partial f}{\partial \Omega} = 0. \tag{3}$$

Here, the left-hand side represents the infinitesimal transformation of an infinite group that is defined by differential equations.

Since the quantities ξ_1, η_1, ξ_2, η_2, and the associated derivatives of order one up to v are not linked by any (linear) relations, equation (3) will decompose into:

$$4\ (1 + 2 + 3 + \ldots + (v - 1)) = 2\ (v + 1)\ (v + 2)$$

different linear partial differential equations. The number of independent variables x_1, y_1, y_1', …, $y_1^{(v)}$, x_2, y_2, …, $y_2^{(v)}$ is, however, equal to:

$$2\ (v + 2),$$

and since v must be greater than zero, the number of independent variables will be small than the number of equations for any v. One can therefore expect that the group:

$$V^{(v)} - \Omega \sum (\cdots) \frac{\partial f}{\partial \Omega}$$

possesses no invariants, and the correspondingly, the infinite group Uf will possess no integral invariants that relate to a pair of curves in the x, y-plane in the desired way.

12. In order to give the simplest-possible form to the fact that the theory of differential invariants originates in the theory of integral invariants (as a special case, in fact) it will be better to express that idea in the following way:

If we keep the previous notations, should the integral:

$$\int \Omega\left(x_1, x_2, \ldots, z_1, z_2, \ldots, \frac{\partial z_1}{\partial x_1}, \ldots. \frac{\partial^2 z_1}{\partial x_1^2}, \ldots\right) dx_1\, dx_2 \ldots dx_n\,,$$

which can be brought into the form:

$$\int \Omega\left(\sum \pm \frac{\partial x_1}{\partial \tau_1}, \ldots. \frac{\partial x_n}{\partial \tau_n}\right) d\tau_1\, d\tau_2 \ldots d\tau_n$$

by the introduction of the parameters $\tau_1, \ldots, \tau_n$, keep its form for all transformations of a certain finite or infinite, continuous, discontinuous, or mixed group in the x and z, and thus represent a so-called integral invariant, then it would be necessary and sufficient that the product:

$$\Omega\left(x_1, x_2, \ldots, z_1, z_2, \ldots, \frac{\partial z_1}{\partial x_1}, \ldots\right) \cdot \left(\sum \pm \frac{\partial x_1}{\partial \tau_1} \frac{\partial x_2}{\partial \tau_2} \cdots \frac{\partial x_n}{\partial \tau_n}\right)$$

should represent a differential invariant of the group in question.

In general, the previously-chosen form is often preferable for practical calculations.

13. As one sees, the connection between the theory of integral invariants and the theory of differential invariants is immediately obvious for any mathematician that has availed himself of the first elements of my theory of transformation groups.

At no point have I ever doubted that so distinguished a mathematician as **Poincaré** has clearly recognized that state of affairs on the first glance. If he did not necessarily discover what was proved here then that would probably be due to the fact that he addressed only one-parameter groups. However, no matter how self-explanatory my conception of the expression:

$$Xf = \sum \xi_k(x_1, \ldots, x_n) \frac{\partial f}{\partial x_k}$$

as the symbol of an infinitesimal transformation and the generator of a one-parameter group might seem, it still remains for one to consider that it is precisely that conception of mine that defines the starting point for investigations that have already exerted a powerful influence on the development of pure, as well as applied, mathematics.

As far as the theory of n bodies that **Poincaré** has treated is concerned, allow me to show that my older investigations into function groups have led to that theory in the same way that **Lagrange**'s investigations into algebraic equations of degree three and four led to the general theory of algebraic equations.

V.

On canonical systems and differential (integral, resp.) invariants.

14. In my older studies (cf., e.g., "Die Störungstheorie und die Berührungstransformationen," Norweg. Archiv, Bd. II, January 1877 [this coll., v. III, art. XX, pp. 309-313]), I showed (among other things) that the reduction of a simultaneous system:

$$\frac{dx_k}{dt} = \alpha_k(x_1, \ldots, x_{2n}) \qquad (k = 1, \ldots, 2n)$$

to the so-called canonical form can be accomplished if and only if a certain **Pfaffian** expression:

$$X_1(x)\, dx_1 + \ldots + X_{2n}\, dx_{2n} = \sum X\, dx$$

exists that fulfills an equation of the form:

$$\frac{d}{dt}\left(\sum X_k dx_k\right) = d\Omega\,(x_1, \ldots, x_{2n}),$$

and furthermore, that $2n$ quantities:

$$y_1, \ldots, y_n, q_1, \ldots, q_n$$

must be given that fulfill a relation of the form:

$$\sum q_k dy_k = \sum X_k dx_k .$$

In particular, that would imply that the characteristic function of the canonical system in question will be *homogeneous of first degree* in the q when the two differentials $d\Omega$ and dV are equal to zero.

In the latter case, it is obvious that the integral:

$$\int (X_1 dx_1 + \cdots + X_{2n} dx_{2n}),$$

when extended along an arbitrary curve in the space of x, preserves its value under the infinitesimal transformation:

$$\delta x_k = \alpha_k (x_1, \ldots, x_{2n})\ \delta t,$$

and, at the same time, for all finite transformation of the associated one-parameter group.

I did not find it necessary to make this last remark explicitly; if my work, especially my older work, is not, in fact, known to the reader then it would not be clear immediately that any transformation that leaves a differential $d\omega$ invariant will simultaneously reproduce the integral $\int d\omega$.

In the treatise in question, I also gave (among other things) the general determination of all transformation in $x_1, \ldots, x_n, p_1, \ldots, p_n$ that take a *given* canonical system into a canonical system. I thus found transformations that generally did not represent any contact transformation in the x, p.

15. Recently (Comptes rendus, Paris, Dec. 1895), **Königs** has addressed precisely some of these problems that I treated in general in his note: "Application des invariants intégraux à la reduction au type canonique…" His note might have some formal value. As far as reality is concerned, he has not added anything to my own developments, in my opinion.

Now, I realize that **Königs** was probably still unaware of my aforementioned work in the editing of his note. In my opinion, however, he must, in any event, establish the relationships that exist between his note and still-older work that likewise goes back to myself and is hardly known to him.

In order to illustrate how much further that my studies at the time went beyond the note of **Königs**, permit me to reproduce the following final theorem of my paper [this coll., v. III, art. XX, pp. 317, lines 3-12].

"Conversely, let a complete system:

$$A_1 f = 0, \ldots, A_q f = 0$$

be given. I assume that I know an expression:

$$X_1\, dx_1 + \ldots + X_m\, dx_m$$

that fulfills q relations of the form:

$$A_i \left(\sum X_k dx_k\right) = d\Omega_i \qquad \text{(or = 0)}.$$

I pose the problem of evaluating this situation to the greatest extent possible. If $q = 1$, $m = 2n$, and therefore the normal form for $\sum X\, dx$ contains $2n$ independent functions, then the integration of $A_1 f = 0$ will require only:

$$2n - 2,\ 2n - 4,\ \dots,\ 6,\ 4,\ 2$$

operations."

VI.

Historical remarks.

16. The concept of "integral invariant" had already occurred to the older mathematicians, who, in fact, completely understood the concepts of arc length, surface area, volume, and so on, in seemingly greater generality, and obviously their relationships with the family of all motions, as well.

By and by, the concept of integral invariant appeared in its general form. Thus, for example, **Lobatscheffsky**, **Bolyai**, **Gauss**, and **Riemann** also introduced the concepts of arc length, surface, area, and so on into non-Euclidian geometry.

From the nature of things, extending the concept of integral invariant to arbitrary transformation groups was first possible only after I introduced the general concept of a continuous group.

17. In all of my studies of contact transformations and **Pfaffian** expressions, which began in the year 1871 (and especially in the aforementioned note in the year 1877), I have exclusively addressed linear, *homogeneous first-order* differential equations:

$$\sum X_k\,(x_1, \dots, x_n)\, dx_k$$

that remain invariant under one-parameter groups. In particular, I have developed theories in the year 1877 that **Königs** recently reproduced, and which were, in fact, derived from **Poincaré's** entirely specialized considerations on integral invariants that were published in the year 1890.

18. Moreover, in the year 1886, in these Berichten [this coll, v. II, art. V], *I hinted at investigations that relate to transformation groups for which the arc length integral remains invariant.* I gave a rigorous presentation of these investigations in various places, and the most thorough of them was in volume three of my *Theorie der Transformationsgruppen* (in particular, see pp. 499, *et seq.*)

I rigorously discussed the connection between that theory and my general studies into the invariants of several finitely-separated points under all transformations of a continuous group. In particular, I direct your attention to the following theorem [*Th. d. Trans.*, Bd. III, pp. 506, line 17-23], which clearly shows that I was in possession of the general theory of integral invariants at the time:

"The foregoing considerations regarding the relations between the invariants of finitely-separated points and those of infinitely-close points can be completed substantially; one can also generalize them when one introduces three and more points, which bring with them the differentials of the x_v of order two and higher, and so on. We reserve the thorough treatment of these questions to another occasion."

16. I regularly touched upon the concept of integral invariant in my older lectures; but then, I also only touched upon it.

In the year 1895 (11 April), I present a treatise to the local society (whose content I had already submitted to the society in the year 1894, moreover) that exhibited a formula on pp. 307-308 that yielded the foundation for the actual calculation of the integral invariants [here, art. XX, pp. 579, *et seq.*]. At precisely the same time (April 1895), **Zorawsky**, who had studied my theories with me in the years 1889 and 1891, published an undoubtedly worthwhile paper on integral invariants. Unfortunately, I was able to read only the formulas, but not the text, of his Polish paper.

Later, **Cartan** published a very interesting paper in that theory. **Cartan**, whose excellent papers on the composition of groups assume an especially pre-eminent place among the recent group-theoretic investigations, said, and with complete justification, in my opinion:

"The question thus-posed reduces to purely a problem in the calculus, thanks to the theories of Lie that permit one to form all of these quantities (integral invariants)."

On another occasion, I would like to be permitted to show that **Cartan**'s beautiful remarks on the optics of non-Euclidian space can be connected with my conception of **Huygens**'s wave theory as a chapter in the theory of contact transformations, and on the other hand, with by studies of surfaces whose principal tangent curves belong to line complexes.

In a following note, I will (among other things) definitively address the question of the existence of integral invariant for infinite continuous groups, as well.