"Über Integralinvarianten und ihre Verwertung für die Theorie der Differentialgleichungen," Leip. Ber. **4** (1897), 369-410; *Gesammelte Abhandlungen,* v. VI, art. 28, pp. 664-701.

On integral invariants and their use in the theory of differential equations

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1. In a treatise on integral invariants and differential invariants of continuous groups, I recently showed (cf., these Berichte 1897, pp. 342 [here, art. XXVII, pp. 649]) that all integral invariants:

$$\int \varphi\, d\omega$$

of a given continuous group can be found by integrating a complete system.

Now, the desired quantity φ itself appears in the complete system in question as an independent variable, while on the other hand, a solution Φ of that complete system will yield an integral invariant only if Φ actually includes the quantity φ. Moreover, since the complete system in question generally possesses solutions Φ that do not include φ, and in fact it can often happen that all solutions Φ are independent of φ, a discussion of the matrix of the complete system in question in each individual case will be required before a definitive answer can be given to the question that was posed of the existence of integral invariants of order m.

Now, from the nature of things, it is generally impossible to formulate *a general* theorem that resolves the question of the integral invariants of an *arbitrary* continuous group definitively, and at the same time, in an exhaustive way. In fact, several essentially different possibilities can occur. At least, one is in a position to present general theorems on the existence of integral invariants.

We shall prove such theorems in the first chapter of this treatise, and at the same time, we shall go into the connection between the two concepts of differential and integral invariants more closely.

2. In the second chapter, we imagine that we are given a linear, partial differential equation in *three* independent variables:

$$0 = \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z} \equiv Xf$$

to be integrated, and that we, perchance, know a first-order surface integral:

$$\int \varphi(x, y, z, p, q)\, dx\, dy$$

or a first-order curve integral:

$$\int \varphi(x, y, z, y', z')\, dx \qquad \left(y' = \frac{dy}{dx}, z' = \frac{dz}{dx}\right)$$

that remains invariant under the (infinitesimal) transformation *Xf*.

We show that the existence of such an integral invariant will always imply a simplification of the integration of the equation $Xf = 0$. However, at the same time, we recognize that various possibilities are conceivable. Whereas it can happen that under certain circumstances the integration of $Xf = 0$ can be converted into performable operations, in other case, one can find that the existence of a certain integral invariants will offer no advantage over the discovery of *Jacobi* multipliers.

No matter how remarkable and elegant that the theorems of the second chapter might seem, they therefore do not completely resolve the – in itself specialized – question of simplifying the integration of an equation:

$$0 = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} \equiv Xf$$

that follows from the existence of a known, invariant, first-order surface or curve integral. On the other hand, one can by no means foresee whether the theories in the second chapter can be extended to *n*-dimensional spaces and to integral invariants of second and higher orders.

3. Therefore, in the third chapter, we shall embark upon another path, and indeed we shall apply *the general methods of my theory of invariants.* In that way, in each individual case, we will find that *the given integration problem will lead back to a completely bounded problem that arises completely in my theories, namely, the integration of differential equations of the finite transformations of a continuous group.*

Finally, in the *fourth* chapter, we extend the concept of integral invariants to groups of *contact* transformations, and in that way we will find an opportunity to develop some general theories that merit a certain interest in their own right.

4. In my previous treatise, I already saw fit to strongly emphasize that the theory of integral invariants must be considered to be a chapter in my general theory of differential invariants. Each chapter of the present treatise will confirm the validity of that way of looking at things. In fact, the theories of this treatise are basically only consequences of my general theory of transformations. If I nonetheless give weight to the results of this paper then that is based primarily upon the fact that my current developments will yield instructive illustrations of my general theory.

I reserve the right to discuss various problems that are posed by the developments of this treatise, if only implicitly. At the same time, I will even generalize the general concept of an integral invariant in several directions.

Chapter I.

General theorems on the existence of integral invariants.

5. If one is given a continuous group then several sets of differential invariants will always belong to that group whose form depends, not only upon the form of the group by itself, but also upon the circumstances that are associated with the transformations of the group.

For example, if the group in question consists of point transformations of the space x, y, z then these transformations can be performed on *curves*, *surfaces*, *differential equations*, families of surfaces, and in fact, many different things. In particular, if the group is *finite* and continuous then there will always exist an infinite set of associated differential invariants in every case, that is, regardless of whether one considers curves, surface, differential equations, and so forth.

Things are quite different when the group is *infinite*. Then, one cannot assert from the outset that infinitely many differential invariants will belong to this group, which might relate to surfaces (or curves), for example, while one generally knows that one can always apply the transformations of the group in situations that correspond to infinitely many differential invariants.

6. If one considers, for example, the group of all point transformations $x_1 = X(x, y, z)$, $y_1 = Y$, $z_1 = Z$, under which all volumes remain unchanged – that is, all transformations whose functional determinant:

$$\sum \pm \frac{\partial x_1}{\partial x} \frac{\partial y_1}{\partial y} \frac{\partial z_1}{\partial z} = 1,$$

then one will easily recognize that no differential invariants of surfaces:

$$J\,(x, y, z, p, q, r, s, t, \ldots)$$

that include z as a function of x, y can belong to that group, indeed, not even an invariant partial differential equation:

$$\Omega\,(x, y, z, p, q, r, s, t, \ldots) = 0.$$

Namely, if $z = \psi\,(x, y)$ were an integral surface of such an invariant differential equation $\Omega = 0$ and $z = \chi\,(x, y)$ were any surface that did not fulfill $\Omega = 0$ then the transformation:

$$x_1 = x, \qquad y_1 = y, \qquad z_1 = z + \chi(x, y) - \psi(x, y),$$

which obviously belongs to the group, would take the integral surface $z = \psi(x, y)$ of the invariant differential equation $\Omega = 0$ to a surface $z = \chi(x, y)$ that does not satisfy $\Omega = 0$, and that will contradict the assumed invariance of $\Omega = 0$. There is then no invariant partial differential equation $\Omega(x, y, z, p, q, \ldots) = 0$.

Surfaces then have no differential invariants $J(x, y, z, p, q, r, s, t, \ldots)$ *under the group of all transformations* $x_1 = X, y_1 = Y, z_1 = Z$ *whose functional determinant is equal to* 1. A completely analogous reasoning shows that curves can also have no properties that remain invariant under that group, so no differential invariants of the form:

$$U(x, y, z, y', z', y'', z'', \ldots) \qquad \left(y' = \frac{dy}{dx}, \ldots\right)$$

can exist.

7. On the other hand, consider the group:

$$x_1 = X, \quad y_1 = Y, \quad z_1 = Z(z, x, y),$$

whose infinitesimal transformations possess the general form:

$$\zeta(x, y, z)\,\frac{\partial f}{\partial z}.$$

If one now seeks all associated differential invariants of surfaces then if one reasons precisely as one did in the previous case, one will, in turn, recognize that differential invariants of first order or higher $J(x, y, z, p, q, r, \ldots)$ do not exist. By contrast, x and y will be differential invariants of order zero of our group, as well as every function of x and y.

8. It is probably worth observing that *there are actually infinite groups in* x, y, z *for which there exist* ***two and only two*** *differential invariants of surfaces.* That is noteworthy due to the fact that as long as three independent differential invariants u, v, w of surfaces exist, one can always construct infinitely many differential invariants:

$$\frac{\partial w}{\partial u} = \begin{vmatrix} v & w \\ x & y \end{vmatrix} : \begin{vmatrix} v & u \\ x & y \end{vmatrix}, \qquad \frac{\partial w}{\partial v} = \begin{vmatrix} u & w \\ x & y \end{vmatrix} : \begin{vmatrix} u & v \\ x & y \end{vmatrix}, \ldots$$

On the other hand, if we imagine the transformations of the group:

$$\zeta(x, y, z)\,\frac{\partial f}{\partial z}$$

being applied to *curves* then x and y will, in turn, be differential invariants of order zero. However, the existence of these two invariants will now suffice for the construction of infinitely many further invariants:

$$y' = \frac{dy}{dx}, y'', y'''.$$

9. Thus, if one is given an (infinite) continuous group of point transformations of space x, y, z, and if its transformations were applied to surfaces then the following distinct cases could occur:

It is conceivable that absolutely no surface invariant $J\ (x, y, z, p, q, r, s, t, \ldots)$ exists, or that a single such invariant exists, or that there are two such invariants u, v, and none that are independent of u, v, or finally, that there are infinitely many independent surface invariants that can then be always derived from a *complete* system:

$$I_1, I_2, J_1, \ldots, J_m$$

by differentiation. In that last case, the general form of the associated surface invariant will be:

$$\Omega\left(I_1, I_2, J_1, \ldots, J_1, \frac{\partial J_1}{\partial I_1}, \ldots\right).$$

This example suffices to show that *many essentially different cases can come about in the search for all differential invariants of a q-fold manifold under groups of point transformations of an n-fold space.*

10. Under these circumstances, it should not be surprising that no simple and general law can be given for the appearance of integral invariants under (infinite) continuous groups that exhausts all possibilities. However, we will present a series of theorems that afford an actual insight into the state of affairs.

In order to ease the discussion, we will first remain in three-fold space x, y, z.

If we first consider the group of all translations and imagine that these transformations are applied to surfaces then we can recognize immediately that p, q, r, s, t, and absolutely all differential quotients of z with respect to x and y will remain invariant. If we set:

$$p = I_1, \qquad q = I_2, \qquad r = J_1, \qquad s = J_2, \qquad t = J_3$$

then we can bring each such differential invariant of a surface into the form:

$$\Omega\left(I_1, I_2, J_1, \ldots, J_1, \frac{\partial J_1}{\partial I_1}, \ldots\right).$$

If we are given a non-developable surface then we choose $I_1 = p$ and $I_2 = q$ as *Gaussian* coordinates of the point on the surface. Every (closed) curve on that surface will then be defined by a relation between I_1 and I_2 (that is, between p and q). Any surface integral can then be brought into the form:

$$\int W \, dI_1 \, dI_2 \, ,$$

regardless of whether it remains invariant under the group, and the associated domain of integration will then be defined by a certain equation between I_1 and I_2 .

However, it is immediately obvious that a surface integral $\int W \, dI_1 \, dI_2$ will remain invariant under all transformations of our group if and only if W is itself a differential invariant of the group, and therefore the general form of an invariant surface integral will be:

$$\int J\left(I_1, I_2, J_1, J_2, J_3, \frac{\partial J_1}{\partial I_1}, \frac{\partial J_1}{\partial I_2}, \ldots\right) dI_1 \, dI_2$$

in the present case.

11. One sees, with no further analysis, that these considerations and results can be extended to n-dimensional spaces and q-dimensional manifolds. We can thus immediately state the following general theorem:

Theorem I. *If a one is given a (finite or) infinite continuous group of point transformations of space* $x_1, \ldots x_q, z_1, \ldots, z_m$ *that can be applied to the* q*-dimensional manifolds:*

$$z_1 = \varphi_1 (x_1, \ldots, x_q), \ldots, \qquad z_m = \varphi_m (x_1, \ldots, x_q)$$

in that space then it will always be possible to give a general form for all associated integral invariants in two cases:

If the group possesses more than q *independent differential invariants:*

$$U\left(x_1, \ldots x_q, z_1, \ldots z_m, \frac{\partial z_1}{\partial x_1}, \ldots\right)$$

then if:

$$I_1, \ldots, I_q, J_1, \ldots, J_s$$

denote suitably-chosen invariants then all further differential invariants can be brought into the form:

$$\Omega\left(I_1, \ldots, I_q, J_1, \ldots, J_s, \frac{\partial J_1}{\partial I_1}, \ldots\right);$$

one will then have:

$$\int W\left(I_1, \ldots, I_q, J_1, \ldots, J_s, \frac{\partial J_1}{\partial I_1}, \ldots\right) dI_1 \ldots dI_q$$

for the general form of the integral invariants of a q-dimensional manifold.

On the other hand, if the given group possesses q and only q mutually-independent differential invariants:

$$I_\kappa\left(x_1, \dots x_q, z_1, \dots z_m, \frac{\partial z_1}{\partial x_1}, \dots\right) \qquad (\kappa = 1, \dots, q)$$

then:

$$\int \Phi(I_1, \dots I_q)\, dI_1 \dots dI_q$$

will be the general form of an integral invariant of the q-fold manifolds.

12. If we apply this general theorem to the group:

$$\zeta(x, y, z)\,\frac{\partial f}{\partial z},$$

which admits no surface invariant $\Phi(x, y, z, p, q, r, \dots)$ besides x and y, for example, then we can deduce that the general formula:

$$\int \psi(x, y)\, dx\, dy$$

will yield all associated integral invariants of surfaces.

By contrast, if we consider the group of all transformations:

$$(A) \qquad z_1 = z, \qquad x_1 = \varphi(x, y, z), \qquad y_1 = \psi(x, y, z),$$

whose infinitesimal transformations are:

$$\xi(x, y, z)\,\frac{\partial f}{\partial x} + \eta(x, y, z)\,\frac{\partial f}{\partial y},$$

then we will find no other surface invariant $\Omega(x, y, z, p, q, \dots)$ than z. Therefore, the theorem above will give no information about the existence of invariant surface integrals.

We can always say one thing from the outset: Namely, if:

$$\int \varphi_1(x, y, z, p, q, r, \dots)\, dx\, dy$$

and

$$\int \varphi_2(x, y, z, p, q, r, \dots)\, dx\, dy$$

are two invariant surface integrals then $\varphi_1 : \varphi_2$ will be a differential invariant, and therefore a function of z:

$$\frac{\varphi_1}{\varphi_2} = W(z).$$

However, we can now resolve the current question in the following way. If:

$$\int \varphi\,(x, y, z, p, q, r, \ldots)\,dx\,dy$$

is an invariant surface integral then, no matter what functions of x, y, z the ξ and η might be (these Berichte, pp. 347 [here, art. XXVII, pp. 653, *et seq.*]), the relation:

$$X'\varphi + (\xi_x + \eta_y + \xi_s\,p + \eta_z\,q)\,\varphi = 0$$

will exist, and as a result of the equation $\varphi = 0$, it will remain invariant under every transformation of the form (A). Now, if φ were a function of only x, y, z then $\varphi = 0$ would be an invariant surface, and in fact a plane: z = const. However, our surface integral would then possess the form:

$$\int \omega(z)\,dx\,dy,$$

so it would not be an integral invariant, accordingly. On the other hand, if the quantity φ actually includes differential quotients of z then $\varphi = 0$ would be an invariant partial differential equation; however, our group has no invariant differential equations.

There thus exist no integral invariants of surfaces at all for the infinite group:

$$\xi(x, y, z)\,\frac{\partial f}{\partial x} + \eta(x, y, z)\,\frac{\partial f}{\partial y},$$

and only a single differential invariant of surfaces, namely, z.

13. We have previously considered the group of point transformation of space x, y, z whose functional determinant is equal to 1.

We saw that either surfaces or curves can have differential invariants under that group. However, our considerations at the time can lead us further. Namely, if there exists an invariant surface integral $\int \varphi\,dx\,dy$ or an invariant curve integral $\int \psi\,dx$ then the equation $\varphi = 0$ (the equation $\psi = 0$, respectively) would be invariant under the transformation of our group. However, there are no such invariant equations. Hence, one has:

Theorem 1. *Surfaces and curves in space have neither differential not integral invariants under the group of all transformations* $x_1 = X$, $y_1 = Y$, $z_1 = Z$ *whose functional determinant is equal to* 1, *and the only invariant space integral is* $\int dx\,dy\,dz$.

14. We are close to posing the problem of finding all transformation groups Xf in three variables x, y, z that possess either *no*, or only *one*, or only *two* independent surface differential invariants:

$$U(x, y, z, p, q, r, s, t, \ldots),$$

or even all groups Xf that leave invariant either no surface integral:

$$\int V(x, y, z, p, q, r, s, t, \ldots)\, dx\, dy$$

or a single such integral. The problem that is formulated here meets up with no significance difficulties, although I have still not addressed the corresponding problem in n dimensions.

If the formula:

$$\int \Psi(u, v)\;\; w\, dx\, dy,$$

with the arbitrary function Ψ of two arguments u, v, yields all invariant surface integrals of a group in space x, y, z then, from the previous developments, that integral can have the form $\int X(u, v)\, du\, dv$, and u and v will be the only differential invariants of surfaces for the group in question.

15. In this treatise, we shall restrict ourselves to the developments that were given here on the existence of integral invariants. However, we regard it as convenient to derive the most important of the results that we just presented from new considerations that are worthy of interest from several standpoints. In order to simply the language and formulas, we restrict ourselves to groups in x, y, z. However, it will not escape the attention of an intelligent reader that our developments can be extended immediately to n dimensions.

16. We imagine the coordinates x, y, z of a point on a surface have been given as functions of two parameters u, v that remain invariant under the transformations:

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}$$

of a given group. On the other hand, we introduce the following notations for the partial derivatives of x, y, z with respect to u and v:

$$\frac{\partial x}{\partial u} = x', \qquad \frac{\partial y}{\partial u} = y', \qquad \frac{\partial z}{\partial u} = z',$$

$$\frac{\partial x}{\partial v} = x_, , \qquad \frac{\partial y}{\partial v} = y_, , \qquad \frac{\partial z}{\partial v} = z_, ,$$

$$\frac{\partial^2 x}{\partial u^2} = x'', \qquad \frac{\partial^2 x}{\partial u\,\partial v} = x'_, , \qquad \dots$$

Every function of x, y, z, p, q, r, s, t, … can be expressed as a function of the quantities:

$$x, y, z, \; x_, , y', \dots, x'', \; x'_, , \; x_{,,} , \dots,$$

while the converse is in no way always the case.

17. If one wishes that the quantities:

$$\Omega\,(x, y, z, \; x_, , \dots, x'', \dots)$$

should be functions of x, y, z, p, q, … (so Ω should be independent of the choice of the parameters u, v) then one would only have to demand that Ω must be a differential invariant of the infinite group:

$$u_1 = U\,(u, v), \quad v_1 = V\,(u, v),$$

whose transformations define the transition from a system of parameters u, v to another system of parameters u_1, v_1 in the most general way. We therefore set:

$$\delta u = \alpha\,(u, v)\;\delta t, \qquad \delta v = \beta\,(u, v)\;\delta t$$

and understand α and β to mean arbitrary functions of u, v. We calculate the corresponding increments of x', $x_,$, y, $y_,$, … when we consider that x, y, z remain invariant under changes in the parameters, and we demand that the corresponding increase in Ω:

$$\delta\Omega = \frac{\partial\Omega}{\partial x'}\delta x' + \frac{\partial\Omega}{\partial x_,}\delta x_, + \dots + \frac{\partial\Omega}{\partial x''}\delta x'' + \dots$$

should be equal to zero.

One will get:

$$dx - x'\,du - x_,\,dv = 0, \qquad \delta(dx - x'\,du - x_,\,dv) = 0$$

and

$$\delta x' : \delta t = -\,x'\,\alpha' - x_,\,\beta', \qquad \delta x_, : \delta t = -\,x'\,\alpha_, - x_,\,\beta_, ,$$

and further:

$$\delta y' : \delta t = -y'\,\alpha' - y_{,}\,\beta', \qquad \delta y_{,} : \delta t = -y'\,\alpha_{,} - y_{,}\,\beta_{,}\,,$$
$$\delta z' : \delta t = -z'\,\alpha' - z_{,}\,\beta', \qquad \delta z_{,} : \delta t = -z'\,\alpha_{,} - y_{,}\,\beta_{,}\;.$$

We calculate the increments $\delta x''$, $\delta x'_{,}$, and so on, in a corresponding way. We will have:

$$\delta(dx' - x''\,du - x'_{,}\,dv) = 0$$

and

$$d\delta x' - x''\,d\alpha\,\delta t - x'_{,}\,d\beta\,\delta t - \delta x''\,du - \delta x'_{,}\,dv = 0,$$

and furthermore:

$$d\,(-x'\alpha' - x_{,}\,\beta') - x''(\alpha'\,du + \alpha_{,}\,dv) - x'_{,}\,(\beta'\,du + \beta_{,}\,dv) - (\delta x'' : \delta t)\,du - (\delta x'_{,} : \delta t)\,dv = 0,$$

from which:

$$\delta x'' : \delta t = -2x''\,\alpha' - 2\,x'_{,}\,\beta' - x'\,\alpha'' - x_{,}\,\beta'',$$
$$\delta x'_{,} : \delta t = -2x''\,\alpha' - 2\,x'_{,}\,\beta' - x'\,\alpha'' - x_{,}\,\beta'',$$
$$\delta x_{,,} : \delta t = -2\,x'_{,}\,\alpha_{,} - 2\,x_{,,}\beta_{,} - x'\,\alpha_{,,} - x_{,}\,\beta_{,,}\,,$$

and so on.

If we denote a summation over x, y, z by $\sum$ then we will obtain the following expression for $\delta\Omega$:

$$\begin{aligned}
-\delta\Omega : \delta t \;&= \sum \frac{\partial\Omega}{\partial x'}(x'\alpha' + x_{,}\beta') + \sum \frac{\partial\Omega}{\partial x_{,}}(x'\alpha_{,} + x_{,}\beta_{,}) \\
&+ \sum \frac{\partial\Omega}{\partial x''}(2x''\alpha' + 2x'_{,}\beta' + x'\alpha'' + x_{,}\beta'') \\
&+ \sum \frac{\partial\Omega}{\partial x'_{,}}(x''\alpha_{,} + x'_{,}(\alpha' + \beta_{,}) + x_{,,}\,\beta' + x'\alpha'_{,} + x_{,}\beta'_{,}) \\
&+ \sum \frac{\partial\Omega}{\partial x_{,,}}(2x'_{,}\,\alpha_{,} + 2x_{,,}\beta_{,} + x'\alpha_{,,} + x_{,}\beta_{,,}) \\
&+ \ldots,
\end{aligned}$$

in which we have now written out the increments in the first and second order derivatives explicitly.

18. Now, should $\delta\Omega$ equal zero then α', $\alpha_{,}$, β', $\beta_{,}$, α'', …, might also have that value. Therefore (from my general theory of differential invariants), Ω must fulfill all of the linear, partial differential equations that we obtain when we successively set the coefficients of α', $\alpha_{,}$, β', $\beta_{,}$, α'', … equal to zero. By a slight conversion, we succeed in bringing these equations into the following form:

$$U_1 f = \sum x' \frac{\partial \Omega}{\partial x_{,}} + \sum x'' \frac{\partial \Omega}{\partial x'_{,}} + 2\sum x'_{,} \frac{\partial \Omega}{\partial x_{,,}} + \ldots = 0,$$

$$U_2 f = \sum \left(x' \frac{\partial \Omega}{\partial x'} - x_{,} \frac{\partial \Omega}{\partial x_{,}} \right) + 2\sum \left(x'' \frac{\partial \Omega}{\partial x''} - x_{,,} \frac{\partial \Omega}{\partial x_{,,}} \right) + \ldots = 0,$$

$$U_3 f = \sum x_{,} \frac{\partial \Omega}{\partial x'} + 2\sum x'_{,} \frac{\partial \Omega}{\partial x''} + \sum x_{,,} \frac{\partial \Omega}{\partial x'_{,}} + \ldots = 0,$$

$$U_4 f = \sum \left(x' \frac{\partial \Omega}{\partial x'} + x_{,} \frac{\partial \Omega}{\partial x_{,}} \right) + 2\sum \left(x'' \frac{\partial \Omega}{\partial x''} + x'_{,} \frac{\partial \Omega}{\partial x'_{,}} + x_{,,} \frac{\partial \Omega}{\partial x_{,,}} \right) + \ldots = 0,$$

$$\sum x' \frac{\partial \Omega}{\partial x''} + \ldots = 0, \qquad \sum x' \frac{\partial \Omega}{\partial x'_{,}} = 0, \qquad \sum x' \frac{\partial \Omega}{\partial x_{,,}} + \ldots = 0,$$

$$\sum x_{,} \frac{\partial \Omega}{\partial x''} + \ldots = 0, \qquad \sum x_{,} \frac{\partial \Omega}{\partial x'_{,}} = 0, \qquad \sum x_{,} \frac{\partial \Omega}{\partial x_{,,}} + \ldots = 0,$$

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We expressly emphasize that the three expressions $U_1 f$, $U_2 f$, $U_3 f$ fulfill relations of the form:

$$(U_1\, U_2) = -\,2U_1 f, \qquad (U_1\, U_3) = U_2 f, \qquad (U_2\, U_3) = -\,2U_3 f,$$

and that they therefore generate a three-parameter group that is composed the same as the general projective group of a simple manifold. We further remark that we have:

$$(U_1\, U_4) = (U_2\, U_4) = (U_3\, U_4) = 0,$$

and that $U_1 f$, $U_2 f$, $U_3 f$, $U_4 f$ therefore generates a four-parameter group that has the same composition as the general linear homogeneous group of a two-fold manifold.

19. We would now like to assume that our group Xf possesses differential invariants that depend upon only x, y, z, p, q, r, s, t, and that Φ and Ψ are two such invariants that are given as functions of $x, y, z, x', x_{,}, \ldots, x'', \ldots$ Φ and Ψ are then solutions of the aforementioned linear, partial differential equations. We can also say that Φ and Ψ are differential invariants of the infinite group that subsumes all Xf of the originally-given group, as well as all possible parameter transformations:

$$\alpha(u, v)\,\frac{\partial f}{\partial u} + \beta(u, v)\,\frac{\partial f}{\partial v}.$$

In any case, it is clear that Φ, as well as Ψ, remain invariant under every infinitesimal transformation of the form:

$$\delta u = \alpha(u, v)\,\delta t, \qquad \delta v = \beta(u, v)\,\delta t, \qquad \delta x = 0, \quad \delta y = 0, \quad \delta z = 0,$$

and that one therefore has:

$$\delta\Phi = 0, \qquad \delta\Psi = 0.$$

If we now set:

$$d\Phi - \Phi'\,du - \Phi_{,}\,dv = 0, \qquad d\Psi - \Psi'\,du - \Psi_{,}\,dv = 0$$

then we will recognize, by means of calculations that are identical in form to the ones on pp. 11, that the derivatives Φ', $\Phi_{,}$, Ψ', $\Psi_{,}$ will take on the increments:

$$\delta\Phi' = -(\Phi'\alpha' + \Phi_{,}\beta')\,\delta t, \qquad \delta\Phi_{,} = -(\Phi'\alpha_{,} + \Phi_{,}\beta_{,})\,\delta t,$$
$$\delta\Psi' = -(\Psi'\alpha' + \Psi_{,}\beta')\,\delta t, \qquad \delta\Psi_{,} = -(\Psi'\alpha_{,} + \Psi_{,}\beta_{,})\,\delta t.$$

If we introduce the notation:

$$\begin{vmatrix} \Phi' & \Psi' \\ \Phi_{,} & \Psi_{,} \end{vmatrix} = \Delta$$

for the functional determinant of Φ and Ψ and then calculate the increment $\delta\Delta$ of Δ then that will imply that $\delta\Delta$ possesses the value:

$$\delta\Delta = -(\alpha' + \beta_{,})\,\Delta\,\delta t.$$

We state, and will prove, that it emerges from this that the integral:

$$\int \Delta\;du\,dv$$

remains invariant under all transformations Xf of the original group, and therefore represents an integral invariant of that group.

20. In order to prove that, we would like to look for analytical criteria that a function:

$$\varphi(x, y, z, x', x_{,}, \ldots, x'', \ldots)$$

must fulfill when the integral:

$$\int \varphi\ (x, y, z, x', x_{,}, \ldots, x'', \ldots)\, du\, dv$$

admits all transformations Xf of the originally-given group.

From our previous developments, it is first requisite that φ should represent a differential invariant of that group. However, some further conditions get added to this that characterize the behavior of the quantity φ under changes of the parameters u, v.

As we know (these Berichte, pp. 353 [here, art. XXVII, pp. 658, *et seq.*, no. 12]), there exists an equation of the form:

$$\varphi = F\,(x, y, z, p, q, r, s, t, \ldots) \begin{vmatrix} x' & y' \\ x_{,} & y_{,} \end{vmatrix} \equiv F \cdot D,$$

and we know the behavior of the two quantities F and D under the transition from the parameters u, v to the new parameters:

$$u_1 = U\,(u, v), \qquad v_1 = V\,(u, v).$$

F indeed keeps its form under such a change, while one has:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} \\ \dfrac{\partial x}{\partial v_1} & \dfrac{\partial y}{\partial v_1} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} : \begin{vmatrix} \dfrac{\partial u_1}{\partial u} & \dfrac{\partial v_1}{\partial u} \\ \dfrac{\partial u_1}{\partial v} & \dfrac{\partial v_1}{\partial v} \end{vmatrix}.$$

Under the infinitesimal transformation:

$$\delta u = \alpha\,(u, v)\ \delta t, \qquad \delta v = \beta\,(u, v)\ \delta t, \qquad \delta x = \delta y = \delta z = 0,$$

the functional determinant:

$$D = \begin{vmatrix} x' & y' \\ x_{,} & y_{,} \end{vmatrix}$$

will then take on the increment:

$$\delta D = -\,(\alpha' + \beta_{,})\ D\ \delta t,$$

so the quantity $\varphi = F \cdot D$ will take on the increment:

$$\delta\varphi = \delta F \cdot D + F \cdot \delta D = -\,(\alpha' + \beta_{,})\ F \cdot D \cdot \delta t,$$

or, what amounts to the same thing:

$$\delta\varphi = -(\alpha' + \beta_,) \cdot \varphi \cdot \delta t.$$

In this way, we next get the:

Theorem 2. *The requirement that the quantity:*

$$\int \varphi(x, y, z, x', x_,, y', \ldots, x'', \ldots)\ du\, dv$$

should be an integral invariant of a continuous group whose infinitesimal transformations are represented by the general symbol:

$$Xf = \xi(x, y, z)\,\frac{\partial f}{\partial x} + \eta(x, y, z)\,\frac{\partial f}{\partial y} + \zeta(x, y, z)\,\frac{\partial f}{\partial z}$$

will be formulated analytically when we, on the one hand, demand that φ should be a differential invariant of the group Xf, and on the other hand, that under any infinitesimal change of the parameters:

$$\delta u = \alpha(u, v)\ \delta t, \qquad \delta v = \beta(u, v)\ \delta t,$$

φ shall take on the increment:

$$\delta\varphi = -\varphi(\alpha' + \beta_,)\ \delta t.$$

21. It is now easy to see that the previously-given functional determinant:

$$\Delta = \begin{vmatrix} \Phi' & \Psi' \\ \Phi_, & \Psi_, \end{vmatrix}$$

actually fulfills all requirements that are placed upon the quantity φ, in which it is obviously assumed that Φ and Ψ represent the differential invariants of the group Xf that remain invariant under changes of the parameters.

On the one hand, we have, in fact, already seen that under the infinitesimal transformation:

$$\delta u = \alpha(u, v)\ \delta t, \qquad \delta v = \beta(u, v)\ \delta t, \qquad \delta x = \delta y = \delta z = 0,$$

Δ will take on the increment:

$$\delta\Delta = -(\alpha' + \beta_,)\ \Delta \cdot dt.$$

On the other hand, since Φ and Ψ, like u and v, remain invariant under the infinitesimal transformation of the given group Xf:

$$\delta u = 0, \qquad \delta v = 0, \qquad \delta x = \xi\,\delta t, \qquad \delta y = \eta\,\delta t, \qquad \delta z = \zeta\,\delta t,$$

we can, with no further analysis, possibly by following through on the equations:

$$\delta(d\Phi - \Phi'\,du - \Phi_{,}\,dv) = 0, \quad \delta(d\Psi - \Psi'\,du - \Psi_{,}\,dv) = 0,$$

conclude that the derivatives of Φ and Ψ with respect to u and v also represent differential invariants of our group Xf, and it follows immediately from this that Δ also represents a differential invariant of the group Xf.

22. The functional determinant Δ thus actually fulfills all requirements that must be placed upon the desired quantity φ, and we can therefore state the following noteworthy theorem:

Theorem II. *If Φ and Ψ are differential invariants of a surface $z = f(x, y)$ under a certain continuous group of point transformations of space x, y, z then the formula:*

$$\int \begin{vmatrix} \dfrac{\partial\Phi}{\partial u} & \dfrac{\partial\Psi}{\partial u} \\ \dfrac{\partial\Phi}{\partial v} & \dfrac{\partial\Psi}{\partial v} \end{vmatrix} du\,dv$$

will always yield an integral invariant of the group. It is therefore our assumption that u and v will refer to parameters that remain invariant under the transformations of the group in question.

Moreover, if $W(x, y, z, p, q, r, s, t, \ldots)$ is any differential invariant of the group then:

$$\int W \begin{vmatrix} \dfrac{\partial\Phi}{\partial u} & \dfrac{\partial\Psi}{\partial u} \\ \dfrac{\partial\Phi}{\partial v} & \dfrac{\partial\Psi}{\partial v} \end{vmatrix} du\,dv$$

will always be an integral invariant, and all integral invariants of the group in question that relate to surfaces will be found in that way.

The integral invariant of a surface that is presented here:

$$\int W \begin{vmatrix} \frac{\partial \Phi}{\partial u} & \frac{\partial \Psi}{\partial u} \\ \frac{\partial \Phi}{\partial v} & \frac{\partial \Psi}{\partial v} \end{vmatrix} du\, dv$$

can obviously be brought into the form:

$$\int W \; d\Phi \, d\Psi,$$

and the theorem that was just stated, which can be extended to n dimensions with no further analysis, will therefore coincide with the general theorem I that was formulated on pp. 6.

Chapter II

The use of invariant first-order surface integrals.

23. We would like to assume that we have been given a linear, partial differential equation in x, y, z:

$$0 = \xi(x, y, z) \frac{\partial \xi}{\partial x} + \eta(x, y, z) \frac{\partial \xi}{\partial y} + \zeta(x, y, z) \frac{\partial \xi}{\partial z} \equiv Xf$$

to be integrated, and that we happen to know an associated first-order integral invariant:

$$\text{(1)} \qquad \int \varphi(x, y, z, p, q) \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv$$

that refers to a general surface: $z = Z(x, y)$. We would like to show that this situation will always yield essential simplifications of the integration of the equation $Xf = 0$.

However, we expressly point out that the developments of this chapter, as interesting as they might seem from the standpoint of function theory, are to be in no way regarded as definitive from a group-theoretic viewpoint. As we will show in the next chapter, the *general theory of invariants,* which I have already based upon my theory of continuous groups for about twenty years now, then allows us to not just prove that one can employ the existence of an integral invariant to simplify the difficulties in integration, but it also *allows us to decide which simplifications can be achieved in which individual cases.*

It is precisely in the latter situation that one finds the most essential part of the *profound significance of my general theory of invariants,* whose essence – indeed, whose existence – continues to remain unknown to mathematicians.

24. As we know, our assumption that the infinitesimal transformation Xf leaves the surface integral $\int \varphi\, dx\, dy$ invariant finds its analytical expression in the existence of the equation:

(2) $$X'\varphi + (\xi_x + \eta_y + \xi_z\, p + \eta_z\, q)\, \varphi = 0,$$

in which $X'f$ arises from Xf by a single extension, and possesses the form:

(2′) $$\begin{cases} X' f = \xi \dfrac{\partial f}{\partial x} + \eta \dfrac{\partial f}{\partial y} + \zeta \dfrac{\partial f}{\partial z} \\ \quad + \{\zeta_x + p\zeta_z - p(\xi_x + p\xi_y) - q(\eta_x + p\eta_y)\} \dfrac{\partial f}{\partial p} \\ \quad + \{\zeta_y + q\zeta_z - p(\xi_y + p\xi_z) - q(\eta_y + p\eta_z)\} \dfrac{\partial f}{\partial q}. \end{cases}$$

The fundamental equation (2) next asserts that the first-order partial differential equation:

$$\varphi(x, y, z, p, q) = 0$$

admits the infinitesimal transformation, and it then follows from my general theory that the two first-order partial differential equations:

$$\varphi(x, y, z, p, q) = 0, \qquad \xi p + \eta q - \zeta = 0$$

(if they do not coincidentally reduce to one equation) will possess ∞^1 common integral surfaces:

$$u(x, y, z) = \text{const.},$$

which can, at any rate, be found by integrating a total differential equation:

$$dz - P(x, y, z)\, dx - Q(x, y, z)\, dy = 0,$$

and therefore by integrating a first-order ordinary differential equation. In this, $u(x, y, z)$ is *eo ipso* a solution of the linear, partial differential equation $Xf = 0$, and we can thus find the missing solution v, in any event, by integrating a new first-order differential equation.

It is, moreover, always possible, to avoid this last integration in such a way that the integration of the equation:

$$0 = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z} \equiv Xf$$

can be reduced to the solution of a single first-order ordinary differential in any event, as long as an integral invariant:

$$\int \varphi\ (x, y, z, p, q)\, dx\, dy$$

of Xf is given, and φ does not vanish at the same time as $\zeta p + \eta q - \zeta$.

We would like to confirm the validity of that assertion.

To that end, we must generally go back quite a ways, and indeed, we will appeal to my theory of infinitesimal contact transformations, on the one hand, and to the connection between *Jacobi's* theory of multipliers and my general theory of transformations that I discovered, on the other.

25. In my theory of contact transformations, I have shown that every infinitesimal contact transformation of the space x, y, z is determined completely by a single function W of x, y, z, p, q – viz., my so-called *characteristic function* – and that the symbol Bf of the infinitesimal transformation question possesses the form:

$$Bf = \frac{\partial W}{\partial p}\frac{\partial f}{\partial x} + \frac{\partial W}{\partial q}\frac{\partial f}{\partial y} + \left(p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} - W\right)\frac{\partial f}{\partial z} - \left(\frac{\partial W}{\partial x} + p\frac{\partial W}{\partial z}\right)\frac{\partial f}{\partial p} - \left(\frac{\partial W}{\partial y} + q\frac{\partial W}{\partial z}\right)\frac{\partial f}{\partial q},$$

moreover, or, with the use of the *Poisson* bracket symbol [], the form:

$$Bf = [W\, f] - W\,\frac{\partial f}{\partial z}.$$

I further show that from two infinitesimal contact transformations:

$$B_1 f = [W_1\, f] - W_1\frac{\partial f}{\partial z}, \qquad B_2 f = [W_2\, f] - W_2\frac{\partial f}{\partial z},$$

one can always derive a well-defined third contact transformation:

$$Bf = [\mathfrak{W}\, f] - \mathfrak{W}\frac{\partial f}{\partial z},$$

whose characteristic function $\mathfrak{W}$ possesses the form:

$$\mathfrak{W} = [W_1\, W_2] - W_1\frac{\partial W_2}{\partial z} + W_2\frac{\partial W_1}{\partial z},$$

and indeed the connection between $B_1 f$, $B_2 f$, and Bf is given by the formula:

$$Bf = B_1\,(B_2\,(f)) - B_2\,(B_1\,(f))\,.$$

In particular, if we set:

$$W_1 = \varphi, \qquad W_2 = \xi p + \eta q - \zeta$$

then we will have:

$$\mathfrak{W} = [\varphi,\ \xi p + \eta q - \zeta] - \varphi\,(\xi_z\, p + \eta_z\, q - \zeta_z) + \varphi_z\,(\xi p + \eta q - \zeta)\,.$$

However, from our assumption, equation (2) will be valid:

$$\begin{aligned}\xi\,\varphi_x + \eta\,\varphi_y + \zeta\,\varphi_z &+ \{\zeta_x + \zeta_z\,p - p\,(\xi_x + \xi_z\,p) - q\,(\eta_x + \eta_z\,p)\}\,\varphi_p \\ &+ \{\zeta_y + \zeta_z\,q - p\,(\xi_y + \xi_z\,q) - q\,(\eta_y + \eta_z\,p)\}\,\varphi_q \\ &+ (\xi_x + p\,\xi_z + \eta_y + q\,\eta_z)\,\varphi = 0,\end{aligned}$$

or, what amounts to the same thing, the equation:

(3) $$0 = -[\varphi, \xi\,p + \eta\,q - \zeta] - \varphi_z\,(p\xi + q\eta - \zeta) + (\xi_x + p\xi_x + \eta_y + q\,\eta_z)\,\varphi,$$

and thus the expression above for $\mathfrak{W}$ will take on the remarkable form:

(4) $$\mathfrak{W} = \varphi\,(\xi_x + \eta_y + \zeta_z)\,.$$

We note the result that we just obtained as a theorem:

Theorem 3. *If the first-order surface integral:*

$$\int \varphi\;(x, y, z, p, q)\;dx\;dy$$

admits the infinitesimal point transformation:

$$Xf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}$$

then the two infinitesimal contact transformations:

$$Af = [\varphi, f] - \varphi\,\frac{\partial f}{\partial z},$$

$$Bf = [\xi\,p + \eta\,q - \zeta, f] - (\xi\,p + \eta\,q - \zeta])\frac{\partial f}{\partial z}$$

will fulfill the relation:

(5) $$A\,(B(f)) - B\,(A(f)) = [\varphi\,(\xi_x + \eta_y + \zeta_z), f] - \varphi\,(\xi_x + \eta_y + \zeta_z)\frac{\partial f}{\partial z},$$

as well as the equivalent relation:

(5′) $$A\,(B(f)) - B\,(A(f)) = (\xi_x + \eta_y + \zeta_z)\,Af + \varphi\,[\xi_x + \eta_y + \zeta_z, f]\,.$$

26. In order to be able to define a formula from the important formula that was just found (which is, for us, even more important, if also specialized), we would next like to prove that the system of equations:

$$(6) \qquad \varphi(x, y, z, p, q) = 0, \qquad \xi p + \eta q - \zeta = 0$$

admit the transformation *Bf*, as well as the infinitesimal transformation *Af*.

The fact that our system of equations (6) admits the infinitesimal contact transformation:

$$Bf = [\xi p + \eta q - \zeta, f] - (\xi p + \eta q - \zeta) \frac{\partial f}{\partial z}$$

is based upon the fact that this transformation, in turn, represents the extension of the infinitesimal point transformation:

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}.$$

One has:

$$B(\xi p + \eta q - \zeta) = [\xi p + \eta q - \zeta, \xi p + \eta q - \zeta] - (\xi p + \eta q - \zeta) \frac{\partial(\xi p + \eta q - \zeta)}{\partial z}$$

$$= -(\xi p + \eta q - \zeta) \frac{\partial(\xi p + \eta q - \zeta)}{\partial z}$$

and

$$B(\varphi) = [\xi p + \eta q - \zeta, \varphi] - (\xi p + \eta q - \zeta) \frac{\partial \varphi}{\partial z},$$

or, if one considers equation (3):

$$B(\varphi) = (\xi_x + p\xi_x + \eta_y + q\eta_z)\, \varphi,$$

and, as above:

$$B(\xi p + \eta q - \zeta) = -\frac{\partial(\xi p + \eta q - \zeta)}{\partial z} (\xi p + \eta q - \zeta).$$

The last two equations show that the expressions $B(\varphi)$ and $B(\xi p + \eta q - \zeta)$ will vanish, due to the system of equations $\varphi = 0$, $\xi p + \eta q - \zeta = 0$, and that this system of equations will actually admit the infinitesimal contact transformation *Bf*.

Moreover, one has:

$$A(\xi p + \eta q - \zeta) = [\varphi, \xi p + \eta q - \zeta] - \varphi \frac{\partial(\xi p + \eta q - \zeta)}{\partial z},$$

or, upon considering the formula (3):

$$A(\xi p + \eta q - \zeta) = -\varphi_z (p\xi + q\eta - \zeta) + \varphi(\xi_x + \eta_y + \zeta_z),$$

and on the other hand:

$$A(\varphi) = [\varphi\, \varphi] - \varphi \frac{\partial \varphi}{\partial z}.$$

The expressions $A(\xi p + \eta q - \zeta)$ and $A(\varphi)$ also vanish then, due to the system of equations $\varphi = 0$, $\xi p + \eta q - \zeta = 0$, which therefore also admits the infinitesimal contact transformation Af. We then have the:

Theorem 4. *If the first-order surface integral:*

$$\int \varphi\ (x, y, z, p, q)\, dx\, dy$$

admits the infinitesimal point transformation:

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}$$

then the system of two first-order partial differential equations $\varphi = 0$, $\xi p + \eta q - \zeta = 0$ *will admit one of the two infinitesimal contact transformations:*

$$Af = [\varphi f] - \varphi \frac{\partial f}{\partial z} \qquad \textit{and} \qquad Bf = [\xi p + \eta q - \zeta, f] - (\xi p + \eta q - \zeta) \frac{\partial f}{\partial z}.$$

The first-order differential equations $\varphi = 0$, $\xi p + \eta q - \zeta = 0$ *(when they are not coincidentally identical) defines a system in involution, moreover, since the left-hand side of the equation that we found before, namely:*

$$(3) \qquad [\varphi, \xi p + \eta q - \zeta] = -\varphi_z (\xi p + \eta q - \zeta) + (\xi_x + p\xi_x + \eta_y + q\, \eta_z)\, \varphi,$$

will vanish due to the fact that $\varphi = 0$, $\xi p + \eta q - \zeta = 0$. *This system in involution has, eo ipso,* ∞^1 *integral surfaces* $u(x, y, z) = c$.

27. A three-dimensional manifold then exists in the five-dimensional space x, y, z, p, q, whose equations:

$$\varphi = 0, \quad \xi p + \eta q - \zeta = 0$$

we solve for p and q:

$$p = P\ (x, y, z), \qquad\qquad q = Q\ (x, y, z).$$

We can then consider the quantities x, y, z to be coordinates of the individual points of that three-dimensional manifold.

As we know, that manifold admits the two infinitesimal transformations:

$$Af = \varphi_p \frac{\partial f}{\partial x} + \varphi_q \frac{\partial f}{\partial y} + (p\varphi_p + q\varphi_q - \varphi)\frac{\partial f}{\partial z} - (\varphi_x + p\varphi_z)\frac{\partial f}{\partial p} - (\varphi_y + q\varphi_z)\frac{\partial f}{\partial q}$$

and

$$Bf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}$$

$$+ \{\zeta_x + p\zeta_z - p(\xi_x + p\,\xi_z) - q(\eta_x + p\eta_z)\}\frac{\partial f}{\partial p}$$
$$+ \{\zeta_y + q\zeta_z - p(\xi_y + q\xi_z) - q(\eta_y + q\eta_z)\}\frac{\partial f}{\partial q},$$

which, as we saw, are related by:

$$A\,(B(f)) - B\,(A(f)) = (\xi_x + \eta_y + \zeta_z)\,Af + \varphi\,[\xi_x + \eta_y + \zeta_z\,,f]. \tag{5′}$$

If we now consider x, y, z to be the coordinates of the individual points of our three-dimensional manifold then we can say that the truncated infinitesimal transformations:

$$\bar{A}f = (\varphi_p)_{p=P,\,q=Q}\frac{\partial f}{\partial x} + (\varphi_q)_{p=P,\,q=Q}\frac{\partial f}{\partial y} + (p\varphi_p + q\varphi_q - \varphi)_{p=P,\,q=Q}\frac{\partial f}{\partial z}$$

and

$$\bar{B}f = \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z}$$

will show us how that three-dimensional manifold transforms.

We would like to show that $\bar{A}f$ and $\bar{B}f$ fulfill the relation:

$$\bar{A}(\bar{B}(f)) - \bar{B}(\bar{A}(f)) = (\xi_x + \eta_y + \zeta_z)\,\bar{A}f\,.$$

In order to verify that assertion, we remark that the coefficients of $\partial f : \partial x$, $\partial f : \partial y$, $\partial f : \partial z$ on both sides of the identity equation (5′) must agree, and therefore that one must have the three relations:

$$A(\xi) - B(\varphi_p) = (\xi_x + \eta_y + \zeta_z)\,\varphi_p\,,$$
$$A(\eta) - B(\varphi_q) = (\xi_x + \eta_y + \zeta_z)\,\varphi_p\,,$$
$$A(\zeta) - B(p\varphi_p + q\,\varphi_q - \varphi) = (\xi_x + \eta_y + \zeta_z)\,(p\varphi_p + q\,\varphi_q - \varphi),$$

respectively. We make the substitution $p = P(x, y, z)$, $q = Q(x, y, z)$ on both sides of these three relations and thus obtain (cf., my *Theorie der Transformationsgruppen*, Bd. I, pp. 110, formula (3)) the equations:

$$\bar{A}\xi - \bar{B}((\varphi_p)_{p=P,q=Q}) = (\xi_x + \eta_y + \zeta_z)\cdot(\varphi_p)_{p=P,\,q=Q}\,,$$
$$\bar{A}\eta - \bar{B}((\varphi_q)_{p=P,q=Q}) = (\xi_x + \eta_y + \zeta_z)\cdot(\varphi_q)_{p=P,\,q=Q}\,,$$
$$\bar{A}\zeta - \bar{B}((p\varphi + q\varphi_p - \varphi)_{p=P,q=Q}) = (\xi_x + \eta_y + \zeta_z)\cdot(p\varphi_p + q\varphi_q - \varphi)_{p=P,\,q=Q}\,,$$

which show us that the relation that we announced, namely:

$$\bar{A}(\bar{B}(f)) - \bar{B}(\bar{A}(f)) = (\xi_x + \eta_y + \zeta_z)\,\bar{A}f\,, \tag{7}$$

actually exists identically.

With that, we have the:

Theorem 5. *If the first-order surface integral:*

$$\int \varphi\ (x, y, z, p, q)\ dx\ dy$$

admits the infinitesimal point transformation:

$$Xf = \xi(x, y, z)\ \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}$$

of the space x, y, z, and thus the two equations $\varphi = 0$, $\xi p + \eta q - \zeta = 0$ *will give the values* $p = P\ (x, y, z)$, $q = Q\ (x, y, z)$ *when one solves them for p and q, then between Xf and the infinitesimal point transformation:*

$$\bar{A}f = (\varphi_p)_{p = P,\, q = Q}\ \frac{\partial f}{\partial x} + (\varphi_q)_{p = P,\, q = Q}\ \frac{\partial f}{\partial y} + (p\varphi_p + q\varphi_q - \varphi)\ \frac{\partial f}{\partial z}$$

there will exist the relation:

$$\bar{A}(\bar{B}(f)) - \bar{B}(\bar{A}(f)) = (\xi_x + \eta_y + \zeta_z)\, \bar{A}f\ . \tag{7}$$

28. Before we go any further, we would like denote the increments in the quantities *x*, *y*, *z* under the transformation by α, β, γ, resp., in order to simplify the formulas and correspondingly set:

$$\bar{A}f = \alpha(x, y, z)\ \frac{\partial f}{\partial x} + \beta(x, y, z) \frac{\partial f}{\partial y} + \gamma(x, y, z) \frac{\partial f}{\partial z}\ .$$

Formula (7) next shows that the two linear, partial differential equations:

$$Xf = 0, \qquad \bar{A}f = 0$$

always possess one (and, in general, only one) common solution $u\ (x, y, z)$, from which, it will be immediately clear that equation:

$$u = \text{an arbitrary constant}$$

will yield the aforementioned integral surfaces of the system in involution:

$$\varphi(x, y, z, p, q) = 0, \qquad \xi p + \eta q - \zeta = 0.$$

However, we can infer even more from formula (7), namely, that when $v\ (x, y, z)$ denotes any solution of $Xf = 0$ that does not simultaneously fulfill $\bar{A}f = 0$, the quantity $\bar{A}v = \Phi$, which obviously fulfills the relation:

$$X\Phi + (\xi_x + \eta_y + \zeta_z)\,\Phi = 0,$$

will represent a multiplier of the linear partial differential equation $Xf = 0$.

We formulate the result that is then obtained (which merits some attention, even though it will play no role in what follows) as follows:

Theorem 6. *If:*

$$\int \varphi(x, y, z, p, q)\, dx\, dy$$

is an integral invariant of the infinitesimal transformation:

$$Xf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z},$$

and if the two equations $\varphi = 0$, $\xi p + \eta q - \zeta = 0$ *yield* $p = P(x, y, z)$, $q = Q(x, y, z)$ *upon solving them then if* $v\ (x, y, z)$ *represents a solution of the linear, partial differential equation* $Xf = 0$ *that does not, at the same time, fulfill the equation:*

$$0 = (\varphi_p)_{p=P,q=Q}\frac{\partial f}{\partial x} + (\varphi_q)_{p=P,q=Q}\frac{\partial f}{\partial y} + (p\varphi_p + q\varphi_q - \varphi)_{p=P,q=Q}\frac{\partial f}{\partial z} \equiv \bar{A}f,$$

then the quantity $\bar{A}f$ *will always be a Jacobi multiplier of the equation* $Xf = 0$.

The theorem that was just presented will become invalid when a relation of the form:

$$\bar{A}f = \rho \cdot Xf$$

exists. However, the relation (7) will then have the form:

$$- X\rho \cdot Xf = (\xi_x + \eta_y + \zeta_z)\,\rho \cdot Xf,$$

and correspondingly one has the equation:

$$X(\rho) + (\xi_x + \eta_y + \zeta_z) \cdot \rho = 0,$$

which states directly that r represents a multiplier of $Xf = 0$. If we preserve the notations of Theorem 6 then we will have:

Theorem 7: *If the expressions* $\bar{A}f$ *and* Xf *of Theorem 6 are coupled by the relation:*

$$\bar{A}f = \rho \cdot Xf$$

then ρ *will be a multiplier of* $Xf = 0$.

29. We will now show that it is *always possible* to give a multiplier of our equation $Xf = 0$. To that end, we would like to employ the following theorem, which we presented some time ago (cf., e.g., Math. Ann., Bd. XI [pp. 508, this coll., v. IV, art. III, § 10, no. 23]):

Theorem 8: *If the linear, partial differential equation:*

$$0 = \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} + \gamma \frac{\partial f}{\partial z} \equiv Uf$$

admits the infinitesimal transformation:

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z},$$

and if, as a result, one has the relation:

$$X\,Uf - UX\,f = \lambda \cdot Uf,$$

and if M is a multiplier of the equation $Uf = 0$ *then the quantity:*

$$X(\log M) + \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) + \lambda$$

will be a solution of the equation $Uf = 0$.

We apply this theorem to the aforementioned equation $\bar{A}f = 0$ and set:

$$\bar{A}f = \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} + \gamma \frac{\partial f}{\partial z},$$

for brevity. We will now have the equation:

$$X(\bar{A}(f)) - \bar{A}(X(f)) = -(\xi_x + \eta_y + \zeta_z) \cdot \bar{A}f, \tag{7}$$

and in the present case we will have:

$$\xi_x + \eta_y + \zeta_z + \lambda = 0,$$

which once more says that the quantity $X(\log M)$ will represent a solution of $\bar{A}f = 0$, as long as M refers to a multiplier of $\bar{A}f = 0$:

$$\bar{A}\,(X \log M) = 0.$$

If we combine this result with the previous results (cf., Theorem 6) then we can say:

Theorem 9: *If the expressions:*

$$\bar{A}f = \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} + \gamma\frac{\partial f}{\partial z}, \qquad Xf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}$$

(as in the foregoing developments) are related by:

$$\bar{A}(X(f)) - X(\bar{A}(f)) = (\xi_x + \eta_y + \zeta_z)\cdot\bar{A}f$$

then every solution v of $Xf = 0$ *will yield the multiplier* $\bar{A}v$ *of* $Xf = 0$*, and every multiplier M of* $\bar{A}f = 0$ *will yield a solution of* $\bar{A}f = 0$*, namely, X* (log *M*).

If we set f equal to log M in the identity (7):

$$\bar{A}(X(f)) - X(\bar{A}(f)) = (\xi_x + \eta_y + \zeta_z)\cdot\bar{A}f$$

and remark that $\bar{A}(X(\log M))$ then vanishes then we will get the equation:

$$X(\bar{A}(\log M)) + (\xi_x + \eta_y + \zeta_z)\cdot\bar{A}(\log M) = 0,$$

which states that $\bar{A}(\log M)$ represents a multiplier of $Xf = 0$, if M denotes a multiplier of $\bar{A}f = 0$. However, under that assumption, one will have the equation:

$$\bar{A}(\log M) + \alpha_x + \beta_y + \gamma_z = 0,$$

and therefore the quantity $\alpha_x + \beta_y + \gamma_z$, which we can always exhibit, will be a multiplier of $Xf = 0$.

30. In order to stress the importance of the result that was just found, we formulate it as a theorem:

Theorem III. *If the surface integral:*

$$\int \varphi\,(x, y, z, p, q)\,dx\,dy$$

remains invariant under the infinitesimal transformation:

$$Xf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}$$

then it will be possible to find a multiplier of the equation $Xf = 0$. *Namely, if the equations:*

$$\varphi(x, y, z, p, q) = 0, \qquad \xi p + \eta q - \zeta = 0$$

yield $p = P(x, y, z)$, $q = Q(x, y, z)$ *when one solves them, and if we set:*

$$\left(\frac{\partial\varphi}{\partial p}\right)_{p=P,q=Q} = \alpha(x, y, z), \qquad \left(\frac{\partial\varphi}{\partial q}\right)_{p=P,q=Q} = \beta(x, y, z),$$

$$\left(p\frac{\partial\varphi}{\partial p} + q\frac{\partial\varphi}{\partial q} - \varphi\right)_{p=P,q=Q} = \gamma(x, y, z)$$

then the quantity $\alpha_x + \beta_y + \gamma_z$ *will be a multiplier of* $Xf = 0$.

If the equation $\varphi = 0$ *is linear in p and q:*

$$\varphi \equiv Ap + Bq - C,$$

and if the determinant:

$$\begin{vmatrix} A & B \\ \xi & \eta \end{vmatrix} \equiv 0$$

then our analysis will become invalid. By contrast, if φ *is linear in p and q, while the determinant above is not equal to zero then we will get an actual multiplier of* $Xf = 0$.

If the quantity φ *is not linear in p and q then values* $p = P(x, y, z)$, $q = Q(x, y, z)$ *that emerge by solving the equations* $\varphi = 0$, $\xi p + \eta q - \zeta = 0$ *will be multi-valued functions of x, y, z, in general. The multiplier* $\alpha_x + \beta_y + \gamma_z$ *will also represent a multi-valued function of x, y, z then inside of a domain in which* $\xi, \eta, \zeta, \ldots$ *are regular and single-valued. In general, we will then find several multipliers of* $Xf = 0$, *and correspondingly, by dividing two multipliers, a solution of* $Xf = 0$ *whose integration will amount to a quadrature in the adverse case.*

31. As we expressly pointed out, the theorem that was just stated will become invalid when φ is linear in p and q and possesses the special form:

$$\varphi \equiv \sigma(\xi p + \eta q) + \omega,$$

moreover. We then directly pose the question of what sort of advantage we can infer in this special case from the known integral invariants.

We easily recognize that this case actually occurs. Namely, if $u(x, y, z)$ and $v(x, y, z)$ are two solutions of $Xf = 0$ then, as we saw before, the integral:

$$\int \begin{vmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{vmatrix} dx\, dy,$$

or, when written out in detail, the integral:

$$\int \left\{ \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} + p \begin{vmatrix} u_z & v_z \\ u_y & v_y \end{vmatrix} + q \begin{vmatrix} u_x & v_x \\ u_z & v_z \end{vmatrix} \right\} dx\, dy,$$

will be an invariant of Xf, and there will actually exist a relation of the form:

$$\varphi = \sigma(\xi p + \eta q - \zeta)$$

here.

A special discussion of the aforementioned exceptional case might be in order.

If:

$$\int (Ap + Bq - C)\, dx\, dy$$

is an integral invariant of the transformation Xf, and therefore not all of the two-rowed determinants of the matrix:

$$(M) \qquad \begin{vmatrix} A & B & C \\ \xi & \eta & \zeta \end{vmatrix}$$

vanish then we can always arrange that:

$$\begin{vmatrix} A & B \\ \xi & \eta \end{vmatrix} \not\equiv 0$$

by a suitable permutation of the coordinates x, y, z. We therefore need to concern ourselves with only the assumption that all two-rowed determinants of the matrix (M) vanish, and that correspondingly:

$$\varphi \equiv Ap + Bq - C \equiv \omega(\xi p + \eta q - \zeta).$$

If we substitute this value for φ into the fundamental equation:

$$X'\varphi + (\xi_x + p\xi_z + \eta_y + q\eta_z)\,\varphi = 0$$

then we will obtain the relation:

$$\begin{aligned} X\omega \cdot (\xi p + \eta q - \zeta) &+ \omega(pX\xi + qX\eta - X\zeta) \\ &+ \omega\xi\,\{\zeta_x + p\zeta_z - p(\xi_x + p\xi_z) - q(\eta_x + p\eta_z)\} \\ &+ \omega\eta\,\{\zeta_y + q\zeta_z - p(\xi_y + q\xi_z) - q(\eta_y + q\eta_z)\} \\ &+ (\xi_x + p\xi_z + \eta_y + q\eta_z)\,\omega(\xi p + \eta q - \zeta) = 0 \end{aligned}$$

and ultimately the equation:

$$(\xi p + \eta q - \zeta)\,\{X(\omega) + (\xi_x + \eta_y + \zeta_z)\,\omega\} = 0,$$

which says that ω represents a multiplier of $Xf = 0$.

We have then succeeded in completing the previous results, and we can correspondingly state the following theorem:

Theorem IV: *If we know a first-order surface integral:*

$$\int \varphi\,(x, y, z, p, q)\,dx\,dy$$

that remains invariant under the [infinitesimal] transformation:

$$Xf = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}$$

then it will always be possible to find a multiplier of the equation $Xf = 0$. *If the rules of Theorem III give no multiplier of* $Xf = 0$, *even after permuting the symbols x, y, z, then* φ *will be linear in p and q and possess the form:*

$$\varphi = \rho\,(\xi p + \eta q - \zeta),$$

and then ρ *itself will represent a multiplier of* $Xf = 0$.

On the other hand, if M is an arbitrary multiplier of $Xf = 0$ *then:*

$$\int M\,(\xi p + \eta q - \zeta)\,dx\,dy$$

will always be an integral invariant of Xf.

The *Jacobi* multiplier theory of an equation:

$$\xi f_x + \eta f_y + \zeta f_z = 0$$

thus takes the form of a special case, in a certain sense, of the theory of invariant surface integrals.

32. We summarize the most important results of the investigations in this chapter into the following theorem:

Theorem V: *If one knows a surface integral:*

$$\int \varphi(x, y, z, p, q)\,dx\,dy$$

that remains invariant under the infinitesimal transformation Xf then that fact can always be utilized for the integration of the equation $Xf = 0$.

The unfavorable case is then that φ *possesses the form:*

$$\varphi \equiv \rho\,(\xi\,p + \eta\,q - \zeta),$$

so ρ *will then be a Jacobi multiplier of* $Xf = 0$, *and since, on the other hand, the integral:*

$$\int M\,(\xi\,p + \eta\,q - \zeta)\,dx\,dy$$

always remains invariant when M denotes a multiplier of $Xf = 0$, *the known surface integral in the present case will accomplish only the determination of the latter solution by quadrature.*

If φ *does not possess the form* $\rho\,(\xi\,p + \eta\,q - \zeta)$ *then the equations:*

$$\varphi\,(x, y, z, p, q) = 0, \qquad \xi\,p + \eta\,q - \zeta = 0$$

can be solved for p and q in any event after a suitable permutation of the quantities x, y, z:

$$p = P\,(x, y, z), \qquad\qquad q = Q\,(x, y, z).$$

If one then sets:

$$(\varphi_p)_{p=P,q=Q} = \alpha, \qquad (\varphi_q)_{p=P,q=Q} = \beta, \qquad \alpha P + \beta Q = \gamma$$

then the quantity $\alpha_x + \beta_y + \gamma_z$ *will always be a Jacobi multiplier of the equation* $Xf = 0$. *Furthermore, the total differential equation:*

$$dz - P\,(x, y, z)\,dx - Q\,(x, y, z)\,dy = 0$$

will be integrable then, and its integral u (x, y, z) *will always be a solution of* $Xf = 0$, *whose missing solution will then be given by quadrature.*

If φ *is not linear in p and q then P and Q, as well as* α, β, γ, *will be multi-valued functions of x, y, z, in general, and the formula* $\alpha_x + \beta_y + \gamma_z$ *will then give several multipliers of* $Xf = 0$ *whose integration will require only performable operations in this case.*

The example in which a relation of the form:

$$\alpha p + \beta q - \gamma = \sigma\,(\xi\,p + \eta\,q - \zeta)$$

exists deserves special attention; namely, σ *would then be a multiplier, and* $\alpha_x + \beta_y + \gamma_z : \sigma$ *would be a solution of* $Xf = 0$.

One can now address the question of how the integration of an equation $Xf = 0$ simplifies when one knows a *curve integral* $\int \psi\,(x, y, z, y', z')$ that admits Xf from the outset in a completely analogous way.

Chapter III

Making the greatest possible use of known integral invariants.

33. In the previous chapter, we addressed the question of how the integration of the linear, partial differential equation:

$$0 = \xi(x, y, z)\,\frac{\partial f}{\partial x} + \eta(x, y, z)\,\frac{\partial f}{\partial y} + \zeta(x, y, z)\,\frac{\partial f}{\partial z} \equiv Xf$$

simplifies when a *first*-order integral invariant of the infinitesimal transformation *Xf*:

$$\int \varphi\,(x, y, z, p, q)\,dx\,dy \tag{1}$$

happens to be known from the outset. We succeeded in deriving several beautiful results, from which it emerged that one can always utilize the existence of such an invariant for the integration of the equation $Xf = 0$. In certain special cases, it was, moreover, possible to prove that our theorems allowed us to derive the greatest possible benefit from the situation in question. However, we have still not established precisely what simplification that the presence of a known first-order integral invariant (1) will imply in complete generality.

Now, my general theory of invariants allows one to resolve definitively not only the problem that was just described, but in fact any such problem.

34. In order to ease the discussion, we temporarily restrict ourselves to the following problem, which still possesses a very general character:

How does the integration of the linear, partial differential equation:

$$0 = \xi_1(x_1, \ldots, x_n, z_1, \ldots, z_m)\frac{\partial f}{\partial x_1} + \ldots + \xi_n(x, z)\frac{\partial f}{\partial x_n} + \zeta_1\frac{\partial f}{\partial z_1} + \ldots + \zeta_m\frac{\partial f}{\partial z_m} \equiv Xf$$

simplify when an integral invariant:

$$\int \varphi\left(x_1, \ldots, x_n, z_1, \ldots, z_m, \frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial^2 z_1}{\partial x_1^2}, \ldots\right) dx_1\,dx_2 \ldots dx_n$$

of the infinitesimal transformation Xf is known from the outset?

If we imagine, for the moment, that we already know the finite equations:

$$x'_\kappa = \chi_\kappa(x_1, \ldots, x_n, z_1, \ldots, z_m, c) \qquad (\kappa = 1, \ldots, n),$$
$$z'_\kappa = \psi_i(x_1, \ldots, x_n, z_1, \ldots, z_m, c) \qquad (i = 1, \ldots, m),$$

of the one-parameter group Xf then we will known how φ and Xf will behave by the introduction of new variables x', z'. Indeed, Xf remains invariant while φ will be reproduced with a factor, and one correspondingly has the equation:

$$(2) \qquad \varphi\left(x_1',\ldots,z_1',\ldots,\frac{\partial z_1'}{\partial x_1'},\ldots\right)=\varphi\left(x_1,\ldots,z_1,\ldots,\frac{\partial z_1}{\partial x_1'},\ldots\right):\sum\pm\left(\frac{\partial x_1'}{\partial x_1}\right)\cdots\left(\frac{\partial x_n'}{\partial x_n}\right),$$

where:

$$\left(\frac{\partial x_i'}{\partial x_\kappa}\right)=\frac{\partial x_i'}{\partial x_\kappa}+\sum_{j=1}^{m}\frac{\partial x_i'}{\partial z_j}\frac{\partial z_j}{\partial x_n},$$

and furthermore:

$$(3) \qquad \sum_i \xi_i(x',z')\frac{\partial f}{\partial x_i'}+\sum_j \zeta_j(x',z')\frac{\partial f}{\partial z_i'}=\sum_i \xi_i(x,z)\frac{\partial f}{\partial x_i}+\sum_j \zeta_j(x,z)\frac{\partial f}{\partial z_i}.$$

We will show that everything depends upon whether or not the transformations of the one-parameter group Xf are the only ones that fulfill our equations (2) and (3). In the former case, we can show that the integration of the equation $Xf = 0$ will not require quadratures, but it can be performed by *executable* operations.

35. If there are even more transformations:

$$x_i' = \chi_i(x, z), \quad z_j' = \psi_i(x, z)$$

that fulfill both of our condition equations (2) and (3) then the form of these condition equations will show that *all of these transformations define a group*, which can be mixed under some circumstances, but it will then include an invariant *continuous* subgroup G.

In calculations, everything will take the following form:

If one considers the quantities x_i' and z_j' in the two equations:

$$(2) \qquad \varphi\left(x_1',\ldots,z_1',\ldots,\frac{\partial z_1'}{\partial x_1'},\ldots\right)=\varphi\left(x_1,\ldots,z_1,\ldots,\frac{\partial z_1}{\partial x_1},\ldots\right):\sum\pm\left(\frac{\partial x_1'}{\partial x_1}\right)\ldots,$$

$$(3) \qquad \sum \xi_i(x',z')\frac{\partial f}{\partial x_i'}+\sum \zeta_\kappa(x',z')\frac{\partial f}{\partial z_\kappa'}=\sum \xi_\kappa(x,z)\frac{\partial f}{\partial x_\kappa}+\sum \zeta_j(x,z)\frac{\partial f}{\partial z_j}$$

to be unknown functions of the x and z, and if one gradually assigns the values x_1', …, x_n', z_1', …, z_m' to the quantities f in the latter equation then one will obtain a series of *partial differential equations that determine all of the x' and z' as functions of x and z. The differential equations, in turn, yield what I have preferred to call the defining equations of the finite transformations of the desired group G.*

Here, the most general system of solutions $\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \mathfrak{z}_1, \ldots, \mathfrak{z}_m$ will emerge from a special system of solutions $x_1', \ldots, x_n', z_1', \ldots, z_m'$ by way of equations:

$$\mathfrak{x}_i = F_i(x_1', \ldots, x_n', z_1', \ldots, z_m'), \quad \mathfrak{z}_i = \Phi_\kappa(x_1', \ldots, x_n', z_1', \ldots, z_m')$$

that define a group, and in fact the group G.

Our differential equations can then be brought into the form:

$$L_\kappa\left(x_1', \ldots, x_n', z_1', \ldots, z_m', \frac{\partial x_1'}{\partial x_1}, \ldots, \frac{\partial z_1'}{\partial z_1}, \ldots\right) = B_\kappa(x, z),$$

in which *the L_κ denote differential invariants of the group G, and they define a complete system of differential invariants, moreover.*

36. The integration of the equations $L_\kappa = B_\kappa$ will, however, be governed by the group G in a known way.

If the finite transformations of the group Xf are the only transformations that fulfill our demands – in other words, if the group G includes no other transformations besides the finite transformations of the one-parameter group Xf – then among the differential invariants L_κ, one will find $m + n - 1$ of them that are of order zero:

$$L_\kappa(x_1', \ldots, x_n', z_1', \ldots, z_m') \qquad (\kappa = 1, \ldots, m + n - 1),$$

namely, the solutions of the equation:

$$0 = \sum \xi_i(x_1', \ldots, x_n', z_1', \ldots, z_m') \frac{\partial f}{\partial x_i'} + \sum \zeta_\kappa(x', z') \frac{\partial f}{\partial z_\kappa'}.$$

One therefore derives $m + n - 1$ equations in the $x_1', \ldots, x_n', z_1', \ldots, z_m'$ and the $x_1', \ldots, x_n', z_1', \ldots, z_m'$:

$$\Omega_\kappa(x, z, x', z') = 0$$

from the equations $L_\kappa = B_\kappa$ (by eliminating all derivatives of the x' and z' from the x and z). *These equations determine all paths of the infinitesimal transformation Xf directly*, because they yield all ∞^1 positions x', z' that belong to an arbitrary initial position.

37. We then assume that the group G includes only *two* infinitesimal transformations, namely, Xf and Yf. The equations:

$$X'f = 0, \qquad Y'f = 0$$

then define a complete system with $m + n - 2$, or even $m + n - 1$, solutions:

$$L_\kappa(x', z'),$$

which can be regarded as differential invariants of order zero of the group G. One then certainly finds all of the equations of the form:

$$L_\kappa(x', z') = L_\kappa(x, z)$$

among the equations $L_\kappa = B_\kappa$. We will find these equations, or a system of equations that is equivalent to them, namely:

$$W_i(x, z, x', z') = 0,$$

when we drop all of the derivatives of the x' and z' with respect to the x and z from the equations $L_\kappa = B_\kappa$ by elimination. If one assigns fixed values to the x and z in the systems of equations $W_i = 0$ that are found in that way then one will obtain a two-dimensional region in the space of x', z', in general, that contains the ∞^1 paths of Xf. One then finds the paths by quadrature.

38. One finds the finite equations of the group G in all situations by integrating auxiliary equations whose number and properties are determined from the group G in a way that I gave some time ago. Once the finite equations of the group G have been found, one will determine the finite equations of the subgroup Xf, and thus, at the same time, the paths of Xf. In this, *if the group contains only a bounded number of parameters then it will only be necessary to perform certain quadratures.* That follows immediately from my general theorem that, as long as the finite equations of any continuous, finite group have been found, one can always find the finite equations of any subgroup – and in particular, the path of every infinitesimal transformation of the group, as well – by performing certain quadratures.

39. We now assume that the group G is infinite, and that we have already found its finite equations by integrating the required auxiliary equations. We will show that the determination of the paths of Xf then demands only certain quadratures.

If the group G is intransitive then one will find the zero-order invariants of that group by elimination.

We can find all finite transformations of the space x, z that commute with all transformations of the group G without integrating, moreover. These new transformations define a group Γ in their own right that includes Xf. If the group G is infinite and transitive then the group Γ must be intransitive. *Its zero-order invariants, which represent solutions of* $Xf = 0$, *eo ipso, will be found without integration.*

We now know all zero-order invariants of the group G, and likewise all zero-invariants of the group Γ. *One thus finds all invariants of the group g that consists of the common transformations of the groups G and Γ without integration.* At the same time, one finds the finite equations of the group g whose transformations commute with each other pair-wise, and thus with Xf, as well. *One then finds the still-missing solutions of the linear, partial differential equation* $Xf = 0$ *by certain quadratures that are independent of each other.*

Everything then comes down to the determination of the finite equations of the group G. If the transformations of that group have been found then the integration of $Xf = 0$ will require only certain quadratures, in any case.

We reserve the right to present the resolution of the problem that we posed here, which we sketched out in a brief form, in a more rigorous form, and at the same time, to illustrate it by some examples.

40. It is easy to see that the general theories that were developed here can be generalized in several directions with no further discussion.

For example, one can assume that we have certain infinitesimal transformations X_1f, X_2f, …, X_qf that determine a complete system:

$$X_1f = 0,\ X_2f = 0,\ \ldots,\ X_qf = 0,$$

and that one knows certain common integral invariants:

$$\int \varphi_1 d\omega_1 ,\int \varphi_2 d\omega_2 ,\ \ldots,\ \int \varphi_\nu d\omega_\nu$$

of all Xf from the outset. One can ask what sort of simplifications in the integration of the complete system could be gleaned from this situation.

Then again, one can assume that a complete system $X_1f = 0$, …, $X_q f = 0$ to be integrated has been given and that certain common integral invariants are known from the outset, and likewise, certain common differential invariants of all $X_\kappa f$. In every individual case, my general theory of invariants will allow one to decide how the circumstances that present themselves can be utilized for the integration of the complete system.

Finally, one can assume that one is dealing with a certain complete system $X_1f = 0$, …, $X_q f = 0$ with certain known infinitesimal transformations Y_1f, Y_2f, …, and certain known integral and differential invariants, as well as certain invariant systems of differential equations to be integrated. My theory of invariants will resolve every problem of that kind in a definitive way.

I have already dealt with various special problems of this nature thoroughly for some time now. For example, my integration of an equation $Af = 0$ with a known multiplier and known infinitesimal (or finite) transformations belongs to them.

41. We shall not refrain from expressly proving that the foregoing developments implicitly resolve an interesting problem, namely, that of the determination of all transformations under which a given integral or several such integrals (differential expressions or equations, resp.) remain invariant.

For example, if one wishes to find all transformations of space x, y, z under which a first-order surface integral:

$$\int \varphi\ (x, y, z, p, q)\, dx\, dy$$

remain invariant then one will define the equation:

$$\varphi(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{p}, \mathfrak{q}) = \frac{\varphi(x, y, z, p, q)}{\sum \pm \left(\frac{\partial \mathfrak{x}}{\partial x}\right)\left(\frac{\partial \mathfrak{h}}{\partial y}\right)}.$$

If one replaces the quantities $\mathfrak{p}$ and $\mathfrak{q}$ with their values as functions of:

$$p,\ q,\ \frac{\partial \mathfrak{x}}{\partial x},\ \frac{\partial \mathfrak{y}}{\partial y},\ \ldots,\ \frac{\partial \mathfrak{z}}{\partial z}$$

then one will obtain the defining equations of the most general group G whose transformations leave the given integral invariant.

42. In general, it is advantageous to look for the *infinitesimal* transformations of the group G first. To that end, one defines the equation:

$$X'\varphi + (\xi_x + \eta_y + p\ \xi_z + q\ \eta_z)\ \varphi = 0$$

and demands that it should be valid for the p and q identically ([1]).

Chapter IV

Integral invariants of groups of contact transformations.

43. In this chapter, we will extend the concept of integral invariant to groups of contact transformations. In order to simply the discussion and the formulas, we shall restrict ourselves to transformations in x, y, z, p, q. The extension to n dimensions, like the restriction to the truncated contact transformations (i.e., to transformations in $x_1, .., x_n$, $p_1, \ldots, p_n$), involves no complications.

We denote the infinitesimal transforms of a given group of contact transformations of the space x, y, z with the symbol:

([1]) *Carda*, who studied my theories under me at Leipzig, recently determined all point transformations of the space x, y, z under which the integral:

$$\int \sqrt{1+p^2+q^2}\ dx\, dy$$

of the surface space remains invariant. The aforementioned theory will give a simple resolution of that problem.

$$[W\,f] - W\,\frac{\partial f}{\partial z} = Xf,$$

and let:

$$\int \bar{\Omega}(x, y, z, p, q, r, s, t) \cdot \begin{vmatrix} x' & y' \\ x_{,} & y_{,} \end{vmatrix}\, du\, dv,$$

in which the parameters u, v are not transformed, be an invariant *second-order* surface integral of that group, while x', $x_{,}$, y', $y_{,}$ denote the partial derivatives of x and y with respect to u and v, as before.

If we now set:

$$\begin{vmatrix} x' & y' \\ x_{,} & y_{,} \end{vmatrix} = \Delta$$

then we will get:

$$X(\Delta) = \begin{vmatrix} (W_p)' & (W_q)' \\ x_{,} & y_{,} \end{vmatrix} + \begin{vmatrix} x' & y' \\ (W_p)_{,} & (W_q)_{,} \end{vmatrix},$$

and with the use of the symbols:

$$\Phi_x + \Phi_z\,p + \Phi_p\,r + \Phi_q\,s = A\,\Phi,$$
$$\Phi_y + \Phi_z\,p + \Phi_p\,s + \Phi_q\,t = B\,\Phi$$

that will yield:

$$X(\Delta) = \{A\,(W_p) + B\,(W_q)\}\Delta,$$

and correspondingly $\bar{\Omega}$ will be determined by the equations:

$$X(\bar{\Omega}) + (AW_p + BW_q)\,\bar{\Omega} = 0.$$

r, *s*, *t* will take on increments of:

$$\delta r = \rho\,\delta\omega, \qquad \delta s = \sigma\,\delta\omega, \qquad \delta t = \tau\,\delta\omega$$

under our infinitesimal transformations Xf, which can be found in a known way.

44. We define the infinitesimal transformation:

$$[Wf] - Wf_z + \rho f_r + \sigma f_s + \tau f_t - \{AW_p + BW_q\}\,\Omega\,\frac{\partial f}{\partial \Omega} \equiv Uf$$

and remark that this transformation in the variables x, y, z, p, q, r, s, t, and Ω leaves not only the two systems of equations:

$$dz - p\,dx - q\,dy = 0$$

and

$$dz - p\,dx - q\,dy = 0, \quad dp - r\,dx - s\,dy = 0, \quad dq - s\,dx - t\,dy = 0$$

invariant, but, at the same time, the equation:

$$\Omega - \bar{\Omega}(x, y, z, p, q, r, s, t) = 0.$$

Therefore, if $U_1\,f$ and $U_2\,f$ are two such transformations with the characteristic functions W_1 and W_2 then the transformation:

$$U_1(U_2(f)) - U_2(U_1(f))$$

will also leave each of the three systems of equations above invariant, and this transformation $U_1(U_2(f)) - U_2(U_1(f))$ will then possess a form that is entirely similar to that of $U_1(f)$ and $U_2(f)$. From my rules, its characteristic function will be the quantity:

$$[W_1\ W_2] - W_1\,\frac{\partial W_2}{\partial z} + W_2\,\frac{\partial W_1}{\partial z}.$$

45. In that way, we recognize that all Uf that belong to the given group of contact transformations define an extended group in the variables $x, y, z, p, \ldots, \Omega$.

If that extended group has invariants Φ that are not all free of Ω, and one finds, by solving an equation:

$$\Phi(x, \ldots, \Omega) = a = \text{const.}$$

for Ω, the value:

$$\Omega = \bar{\Omega}(x, y, z, p, q, r, s, t)$$

then:

$$\int \bar{\Omega}\ dx\,dy$$

will be an invariant second-order surface integral of our group of contact transformations.

One determines all invariant surface integrals of order three and higher in an entirely analogous way. One sees, with no further analysis, how the general theorems on groups of contact transformations that were presented in chapters I and III can be extended.

The application of these theories to first-order partial differential equations, canonical systems, and so on, deserves special attention.