“Die Entwicklung der Lehre von den Berührungstransformationen,” from the Jahresbericht der deutschen Mathematiker-Vereinigung, v. 5, H. Liebmann and F. Engel, *Die Berührungstransformationen: Geschichte und Invariantentheorie*, B. G. Teubner, 1914.

The development of the theory of contact transformations.

By HEINRICH LIEBMANN in Munich [1]).

Translated by: D. H. Delphenich

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The original, life-giving well-spring for mathematical research is, and remains for all time, the free exercise of one’s imagination. Might the demands of the neighboring realms direct the unbound rushing mountain stream along certain courses, and should one like to know its descent solely for one’s own purposes, then the individual factions would always raise their voices once more, voices that always drown out the rushing of any mountain stream with their puritanical shouts of “Either-Or” and “Correct or untenable,” so the power of unbounded fantasy as the source of scientific progress would then remain entirely indispensible.

By comparison, in the prescribed context the will that is stubborn or compelled to pursue a fixed goal often fails to create. In order to bloom, if it is to also be capable of developing not merely tenable and tangible fruits of full blossoms, it must choose its problems themselves and might alter them in such a way – one must also occasionally admit to such things – that he has constructed his buildings on foundations whose load capacity has still not been shown to comply with all the rules of the building code.

Admittedly, many beautiful structures will then later collapse, and many others must be buttressed by aspirations that first evoke a somewhat unfamiliar impression.

Yet another disillusionment seldom fails to materialize: the knowledge that almost any concept will forfeit its originality before the unbiased and analytical view of the historian. Just as the prism resolves the bright rays of the sun into a spectrum, so do we find the basic thoughts whose union will first define the work under scrutiny.

From these somewhat timid-sounding considerations, history would compel the existence of perhaps less disciplines than precisely the domain of contact transformations did, which, as the total output that is inseparably linked with the name of Sophus Lie, has been developed in detail, and indeed evolved into something that the ambitious Norwegian researcher could scarcely have imagined. Mercifully, he has been long since buried, although perhaps resignation had prematurely paralyzed his drive, and did not allow him to attain his greatest discoveries at all.

These general remarks give the guidelines for orienting oneself in our domain. We must ask ourselves: How far had the theory of contact transformations advanced before

[1]) Talk submitted to the German Society of Mathematicians and excerpts presented at its meetings in Vienna, September 1913.

Lie? What did he – with or without the knowledge of the previous accomplishments – create from it? Which problems did he bequeath to the future?

The appointed authorities and colleagues of the work of Lie have addressed the first two questions many times, and several articles of the mathematical encyclopedias have already discussed them so thoroughly that the pages of this article will serve, on the one had, as a recapitulation of them, and on the other, as a modest survey of *Berührunstransformationen*. Nevertheless, at this time, much will be said for the sake of completeness that most of you will not appreciate the novelty of, although hopefully a brief summary of that book and sharper accentuation is not completely unwelcome.

With Klein, I would like to make two sources explicit: First, the *formal* problem of presenting canonical substitutions for canonical differential equations, as they appear in mechanics, and then, however, the *free mobility in the manipulation of the space elements*, both realms that were already thoroughly explored by Lie.

1. Canonical substitutions. Any partial differential equation of first order:

$$F(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n) = 0 \qquad \left(p_i = \frac{\partial z}{\partial x_i}\right)$$

is linked with a system of ordinary differential equations whose integration succeeds in describing *every* solution of the partial differential equation itself. It is the associated "canonical system," which defines the characteristics.

For example, in three-dimensional space the canonical system that is associated with:

$$(1) \qquad F(x, y, z, p, q) = 0 \qquad \left(p_i = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right)$$

reads like:

$$(2) \qquad \begin{gathered} \frac{dx}{dt} = \frac{\partial F}{\partial p}, \quad \frac{dy}{dt} = \frac{\partial F}{\partial q}, \quad \frac{dz}{dt} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q}, \\ \frac{dp}{dt} = -\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial z}, \quad \frac{dq}{dt} = -\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial z}. \end{gathered}$$

If one now completely ignores the fact that p and q should actually mean the partial differential quotients of a function of z, and considers the five quantities x, y, z, p, q as simply variables, moreover, then the following question is fully justified: *How must the functions:*

$$(3) \qquad \begin{aligned} x_1 &= X(x, y, z, p, q), \\ y_1 &= Y(x, y, z, p, q), \\ z_1 &= Z(x, y, z, p, q), \\ p_1 &= P(x, y, z, p, q), \\ q_1 &= Q(x, y, z, p, q), \end{aligned}$$

be arranged in order for this transformation of (2) *to again produce a system of the same type: viz., a canonical system?*

If one denotes the introduction of the new variables by including them in square brackets then this gives:

$$(4)\qquad \frac{d[f]}{dt} =$$

$$= \left(\frac{\partial[F]}{\partial x_1} + p_1\frac{\partial[F]}{\partial z_1}\right)[X, f] + \left(\frac{\partial[F]}{\partial y_1} + q_1\frac{\partial[F]}{\partial z_1}\right)[Y, f] + \frac{\partial[F]}{\partial p_1}[P, f] + \frac{\partial[F]}{\partial q_1}[Q, f].$$

In this, we have used the abbreviation:

$$[U, V] =$$

$$= \frac{\partial U}{\partial p}\left(\frac{\partial V}{\partial x} + p\frac{\partial V}{\partial z}\right) + \frac{\partial U}{\partial q}\left(\frac{\partial V}{\partial y} + q\frac{\partial V}{\partial z}\right) - \frac{\partial V}{\partial p}\left(\frac{\partial U}{\partial x} + p\frac{\partial U}{\partial z}\right) - \frac{\partial V}{\partial q}\left(\frac{\partial U}{\partial y} + q\frac{\partial U}{\partial z}\right).$$

One now needs only to replace $[f]$, in sequence, with x_1, y_1, z_1, p_1, q_1 in order to arrive at the desired conditions that the functions (3) must fulfill in order for *every* canonical system (i.e., one belonging to an arbitrary $F(x, y, z, p, q)$) to again go to another canonical system. The conditions read:

$$(5)\qquad \begin{aligned} &[X, Y] = [X, Z] = [Y, Z] = [X, Q] = [Y, P] = [P, Q] = 0, \\ &[P, X] = [Q, Y] = \rho, \\ &[P, Z] = \rho P, \quad [Q, Z] = \rho Q, \end{aligned}$$

and we remark in passing that according to Darboux they can be most simply obtained by the requirement that the system of equations:

$$(6)\qquad \begin{aligned} &dz - p\,dx - q\,dy = 0, \\ &\delta z - p\,\delta x - q\,\delta y = 0, \\ &dx\,\delta p + dy\,\delta q - \delta x\,dp - \delta y\,dq = 0 \end{aligned}$$

remains invariant. One can calculate most comfortably with this system, and when it is coupled with the equations:

$$\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q$$

$$= \left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)\delta x + \left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)\delta y + \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q = 0$$

that one derives from (1), since it leads to exactly the canonical system (2) the conservation of the system of equations (6) itself can be made the paramount demand.

Thus, we have arrived at a foundation:

Conservation of the canonical form of the system (2) *comes about when the conditions* (5) *are fulfilled, or when the system of equations* (6) *remains invariant.*

The third equation of the system (6) appeared for the very first time in Schering, and was not noticed. Instead of it, Lie demanded that:

$$dz_1 - p_1\,dx_1 - q_1\,dy_1 = \rho(dz - p\,dx - q\,dy), \tag{7}$$

and, in so doing, found a new setting for things.

Incidentally, the position is perhaps justified that with this demand the transformations that one arrives at were elevated to the status of an *autonomous class* that was no longer a *mere auxiliary construction* to the canonical equations.

2. The changing of spatial elements. For every initial *formal* train of thought (*conservation of the canonical form!*), there is another one that follows an entirely separate path. We are accustomed to considering the point as a spatial element, so we see any line or arbitrary curve as a structure that is composed of ∞^1 points and the plane or any arbitrary surface as a structure that is composed of ∞^2 points. However, duality, especially the transition from a pole to a polar for a conic section, from pole to polar plane for a surface of second degree leads one to look upon the lines or the planes as the constituent spatial elements. As a further example of a consideration that leads to a *change in spatial element*, we cite the *base point transformation.* If one associates a point P with the structure that consists of the totality of base points for the perpendicular that goes from a fixed point O to the lines through P then this necessitates a *change of spatial element*. The point x, y, z corresponds to the sphere:

$$x_1^2 + y_1^2 + z_1^2 - xx_1 - yy_1 - zz_1 = 0.$$

How is this "change of spatial element" connected with the previously-found transformations?

The change of spatial element, which implies only *one* equation:

$$\Omega(x, y, z, x_1, y_1, z_1) = 0,$$

then takes a point to a surface and a surface:

$$z = f(x, y)$$

to ∞^2 surfaces, which themselves possess an *enveloping surface* in $R_1(x_1, y_1, z_1)$.

Here, one finds the simple, but prior to Lie, not sufficiently emphasized or exploited, theorem: If two surfaces in the space $R(x, y, z)$ have the tangential plane:

$$(\zeta - z) - \frac{\partial f}{\partial x}(\xi - x) - \frac{\partial f}{\partial y}(\eta - y) = 0$$

in common at a point – or also a system of values:

$$x, y, z, p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y},$$

then this property remains preserved under the map from the surfaces to the associated enveloping surfaces.

In fact, if:

$$z_1 = f_1(x_1, y_1)$$

is the associated enveloping surface then the function f_1 must be determined by the elimination of x and y from:

$$\Omega(x, y, z, x_1, y_1, z_1) = 0,$$

$$\text{(8)} \qquad \frac{\partial \Omega}{\partial x} + p\frac{\partial \Omega}{\partial z} = 0,$$

$$\frac{\partial \Omega}{\partial y} + q\frac{\partial \Omega}{\partial z} = 0.$$

One then finds the partial differential quotients:

$$p_1 = \frac{\partial z_1}{\partial x_1}, \qquad q_1 = \frac{\partial z_1}{\partial y_1}$$

from:

$$d\Omega = \left(\frac{\partial \Omega}{\partial x} + p\frac{\partial \Omega}{\partial z}\right)dx + \left(\frac{\partial \Omega}{\partial y} + q\frac{\partial \Omega}{\partial z}\right)dy + \left(\frac{\partial \Omega}{\partial x_1} + p_1\frac{\partial \Omega}{\partial z}\right)dx_1 + \left(\frac{\partial \Omega}{\partial y_1} + q_1\frac{\partial \Omega}{\partial z_1}\right)dy_1 = 0;$$

i.e., p_1 and q_1 are given by:

$$\text{(9)} \qquad \frac{\partial \Omega}{\partial x_1} + p_1\frac{\partial \Omega}{\partial z_1} = 0,$$

$$\frac{\partial \Omega}{\partial y_1} + q_1\frac{\partial \Omega}{\partial z_1} = 0.$$

Thus, x_1, y_1, z_1, p_1, q_1 are completely determined by x, y, z, p, q, and we have therefore proved: *Two surfaces that a definite tangential plane in common at a definite point go over to just such enveloping surfaces.*

The same thing may be confirmed for the point-curve transformation that is given by two equations:

$$(10)\qquad \begin{aligned}\Omega_1(x, y, z, x_1, y_1, z_1) = 0,\\ \Omega_2(x, y, z, x_1, y_1, z_1) = 0.\end{aligned}$$

A surface in a space thus corresponds to a focal surface in another space that envelops a two-fold infinitude of curves, and the theorem is true for the relationship between surfaces and associated focal surfaces.

Therefore, the "change of spatial element" is converted into to a *contact transformation*.

We now also recognize the close connection between our two considerations: The enlargement of the transformations (8) or (9) that are linked with the change of spatial element to a contact transformation is, in fact, nothing but the construction of a transformation that fulfills the condition (7).

This shall be shown for the point-surface transformation.

From:

$$\Omega(x, y, z, x_1, y_1, z_1) = 0$$

$$\frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz + \frac{\partial\Omega}{\partial x_1}dx_1 + \frac{\partial\Omega}{\partial y_1}dy_1 + \frac{\partial\Omega}{\partial z_1}dz_1 = 0$$

$$dz_1 - p_1\, dx_1 - q_1\, dy_1 = \rho(dz - p\, dx - q\, dy),$$

it indeed follows that:

$$\frac{\partial\Omega}{\partial x} : \frac{\partial\Omega}{\partial y} : \frac{\partial\Omega}{\partial z} : \frac{\partial\Omega}{\partial x_1} : \frac{\partial\Omega}{\partial y_1} : \frac{\partial\Omega}{\partial z_1} = \rho p : \rho q : -p_1 : -q_1 : 1,$$

and from this follow exactly the previous equations (8) and (9).

The intrinsic basis for this is easy to see: When we extend contact transformations, we actually require that a system of values:

$$z = f(x, y), \qquad p = u(x, y), \qquad q = v(x, y)$$

for which:

$$u = \frac{\partial f}{\partial x}, \qquad v = \frac{\partial f}{\partial y},$$

or:

$$dz - p\, dx - q\, dy = 0$$

is true go to a system of values for which the demand:

$$dz_1 - p_1\, dx_1 - q_1\, dy_1 = 0$$

is fulfilled.

That is: *Transformations* (3) *that preserve the form of the canonical equations* (2) *are identical with the contact transformations that arise from the change of spatial element.*

If Jacobi had also been previously led from the construction of a system of equations:

$$\Omega_i(z, x_1, \ldots, x_n, z', x_1', \ldots, x_1') = 0 \qquad \begin{pmatrix} i = 1, 2, \cdots, m \\ m < n+1 \end{pmatrix}$$

to the canonical substitutions then *the main idea of geometry breaks down*, which was first realized by Lie.

3. The infinitesimal element and the union of elements. We now encounter an objection that we have expressly suppressed up to now. *Does the equation:*

$$dz - p\,dx - q\,dy = 0 \tag{11}$$

actually deliver only the totality of points of a surface with their associated tangential planes? Furthermore: *Do surfaces actually go to other surfaces?*

The one of these situations is so much less often the case than the other!

In order to explain this, one avails oneself of the concept of an (infinitesimal) *surface element.* By this, it shall be understood that we mean just the system of values x, y, z, p, q, where p and q serve to represent the plane:

$$(\zeta - z) - p(\xi - x) - q(\eta - y) = 0$$

that includes x, y, z. The surface element is the *combination of a point and a plane that goes through it*, where it simplifies the presentation when one considers only the points in the immediate vicinity of that point, so the plane becomes a limiting case of a planar surface piece.

As Lie said, (11) defines a *union of surface elements*, and there are three different classes of two-dimensional unions of surface elements:

1. *The elements of an arbitrary surface:*

$$z = f(x, y), \qquad p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y}.$$

They alone were considered up to now.

2. *The elements of a curve:*

$$y = f(x), \qquad z = g(x), \qquad g' - p - qf' = 0.$$

3. *The elements of a point:*

x, y, z are given constants, while p and q are arbitrary.

With this, the second question also finds its complete resolution: For a contact transformation, the surface elements of a surface do not need to again go to those of a surface, since the union that they define can go to a union that has a *curve* or a *point* as its *carrier*.

The only characteristic of contact transformations is that all elements again go to other elements.

Lie himself first built the concepts only gradually, although today many mathematicians also apply the well-known dictum here of the privilege of genius, whose train of thought seems no longer *complicated* in later times, but *trivial*.

The first point-curve transformation of Lie is his celebrated line-sphere transformation, with the two defining *aequationes directrices:*

$$x_1 + iy_1 + xz_1 + z = 0,$$
$$x(x_1 - iy_1) - z_1 - y = 0,$$

through which, the lines of the space x, y, z go to the spheres in the space x_1, y_1, z_1 . In amicable competition with his friend Klein, he simultaneously solved the problem of determining the principal tangent curves of the Kummer singularity surfaces by means of these transformations, but he had still not developed the general notion of contact transformation, which is so simple and self-explanatory for us.

Hopefully it is the very fact that the concept of a contact transformation, which is becoming classical moreover, is also only gradually becoming clear to the researchers and discoverers that were cited in regard to it that will also justify the elaborate discussion of its development.

4. The finished philosophical system of contact transformations. As Lie himself said, he could go into the elaboration of the details by which the foundations were obtained "alive with music." In place of this, in light of the presentation in the works published by Engel and Scheffers, and with hindsight of the encyclopedia article of E. von Weber and G. Fano, we are allowed a briefer discussion, in which the emphasis is on the concise characterization of the main points.

a) *The complete founding of the system of formulas* on the basic requirement:

$$dz' - p_1' \, dz_1' - \ldots - p_n' \, dz_n' = \rho(dz - p_1 \, dx_1 - \ldots - p_n \, dx_n). \tag{12}$$

To this, belongs the proof that the $2n + 1$ functions:

$$Z, X, \ldots, X_n , P, \ldots, P_n$$

of:

$$z, x, \ldots, x_n , p, \ldots, p_n$$

define a contact transformation when the bracket relations:

$$[Z, X_i] = [X_i, X_k] = [P_i, X_k] = 0 \qquad (i = k),$$
$$[P_i, X_i] = \rho, \qquad [P_i, Z] = \rho\, P_i \qquad (i, k = 1, 2, \ldots, n)$$

are fulfilled, in which, one has:

$$[u, v] = \sum_{\nu=1}^{n} \left\{ \frac{\partial u}{\partial p_\nu} \left(\frac{\partial v}{\partial p_\nu} + p_\nu \frac{\partial v}{\partial z} \right) - \frac{\partial v}{\partial p_\nu} \left(\frac{\partial u}{\partial x_\nu} + p_\nu \frac{\partial u}{\partial z} \right) \right\}$$

and furthermore, the generation from the *aequationes directrices:*

$$\Omega_\mu(z, x_1, \ldots, x_n, z_1', x_1', \ldots, x_n') = 0 \qquad \begin{pmatrix} \mu = 1, 2, \cdots, m \\ m < n+1 \end{pmatrix},$$

and the given, which was missing from Jacobi, of the independence conditions that the functions Ω_μ are subject to.

b) *Invariant theory of contact transformations*; i.e., establishing the criteria that are necessary and sufficient for a system of functions:

$$F_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n) \qquad (i = 1, 2, \ldots, m)$$

to be convertible into the system:

$$\mathfrak{F}_i(x_1', \cdots . x_n', z', p_1', \cdots, p_n') \qquad (i = 1, 2, \ldots, m)$$

by a contact transformation.

This can be decided by differentiations and eliminations, and indeed by the detour to a *homogeneous* system of functions that is obtained by the substitutions:

$$z = y_{n+1}, \qquad x_i = y_i, \qquad p_i = \frac{-q_i}{q_{n+1}},$$

$$z' = y_{n+1}', \qquad x_i' = y_i', \qquad p_i' = \frac{-q_i'}{q_{n+1}'} \qquad (i = 1, 2, \ldots, n).$$

The homogeneous functions must be extended to a *function group* by bracket operations – i.e., to a system that produces no new independent functions by bracket operations – and the same *composition* must be verified for the system that is derived in this way from the homogenized F_i and the associated extension as the one that takes place in the corresponding system that is derived from the F_i – i.e., the bracket operation on any two functions of the first (extended) one – so it must produce the same function of these functions as the second one did.

c) *Infiitesimal contact transformations and group theory*. The theory of groups of contact transformations is a chapter in the theory of groups of finite continuous point transformations in $2n + 1$ variables:

$$z, x_1, \ldots, x_n, p_1, \ldots, p_n .$$

One only adds the auxiliary condition (12).

Infinitesimal contact transformations, and thus, the *one-parameter* group that arises from:

$$dx_i : dp_i : dz =$$

$$= \xi_i(x_1, \ldots, x_n, z, p_1, \ldots, x_n) : \pi_i(x_1, \ldots, x_n, z, p_1, \ldots, x_n) : \zeta(x_1, \ldots, x_n, z, p_1, \ldots, x_n),$$

by integration, will be generated with the help of a characteristic function:

$$W = p_1\, \xi_1 + p_2\, \xi_2 + \ldots + p_n\, \zeta - \zeta,$$

so one has:

$$\xi_i = \frac{\partial W}{\partial p_i}, \qquad \pi_i = -\frac{\partial W}{\partial x_i} - p_i \frac{\partial W}{\partial z}, \qquad z = \sum p_i \frac{\partial W}{\partial p_i} - W.$$

One is then dealing with an r-parameter group of contact transformations when and only when the r generating characteristic functions:

$$W_1, \ldots, W_r ,$$

which fulfill no *linear relation with constant coefficients*, fulfill the relations:

$$\{W_i\ W_k\} = \sum_{s=1}^{r} c_{iks} W_s .$$

Thus:

$$\{W_i\ W_k\} = [W_i\ W_k] - W_i \frac{\partial W_k}{\partial z} + W_k \frac{\partial W_i}{\partial z} .$$

5. Elaborating on the details. The load-bearing superstructure of the abstract theory is given by these three elements, so we again turn to the concrete questions and go from the theory to the applications, take up the finer distinctions, and – last, but not least – treat definite geometric problems.

a) *Group theory as the main idea.* From more detailed classifications, one must, above all, cite the distinction between *reducible* and *irreducible* groups; the former can be converted into point transformations by a contact transformation, but not the latter.

Lie has determined all irreducible groups of contact transformations in the plane. The largest one has ten parameters and can be converted into the group that takes the differential equation of certain parabolas, or also the differential equation of the circle, into itself. The other two are a seven-parameter and a six-parameter subgroup of it. The ten-parameter group can be converted into the projective group of a linear complex by mapping the line elements of the plane to the points of R_3 .

Lie had determined the three groups of irreducible contact transformations in R_n that are transitive as point transformations of the elements $x_1, \ldots, x_n, z, p_1, \ldots, p_n$ in R_{2n+1}, and second, amongst those infinitesimal transformations that leave the point in general position $x_i = p_i = z = 0$ invariant, obtain the greatest possible number of them – i.e., $n(2n + 1)$ such mutually independent ones – from which no infinitesimal can be linearly derived that adheres to each individual direction of the bundle $z' = 0$. One of these groups is primitive, and the other two are imprimitive.

In R_3 there is, except for this one primitive group, only the 14-parameter group of F. Engel.

Of the imprimitive groups, we cite the one that G. Scheffers determined, which leaves invariant a sheaf of partial differential equations of first order:

$$f(x, y, z, p, q) = c,$$

and the ones that Oseen determined with the parameter counts 8, 9, 11, and 12.

b) *Geometric starting point.* I would like to compare the rules and conceptual structures of group theory with the form that natural law takes for *crystals*. If it is permissible to remain in the picture, then we might add that the remaining mother liquor is a rich agar in which a lush *organic life* unfolds.

The issues of geometry are the germs and seeds of this life, and we might only mention the determination of the arc lengths of all surfaces, for which the curves of constant geodetic curvature admit an infinitesimal contact transformation, and furthermore, the determination of all contact transformations for which the points of one space correspond to the lines of a complex in another, and conversely, and finally, all contact transformations that again admit the rotation around a fixed point, which is meaningful in mechanics.

With that, we have reached the end of our overview of the accomplishments of Lie in the realm of contact transformations, which can only represent an incomplete sketch.

Since then, research – if we ignore individual investigations, which have more the character of an extension – moved in two directions, which – if we must cite two names – were picked up on by Engel and Study and excised from the wealth of questions that Lie left behind.

6. Renewed interest in line geometry. For Lie, line geometry defined the gateway from his line-sphere transformation to the realm of contact transformations. In connection with this, it would itself be linked with a developable property that conscientious criticism must treat with the greatest initiative.

Criticism demands that one recognize as full-fledged only such transformations that are single-valued, and the *uncertainty* with which the response to *general* questions is often afflicted, must be cast aside by new creative works. Above all, *coordinates* that fail nowhere and whose carrying capacity omits all "exceptional cases" belong to these coordinates.

Vast new classes of contact transformations are suitable for this field of endeavor – for example, the *equilongs*, which take "spheres" to "spheres" and therefore leave the distance between two oriented line elements on a sphere invariant. G. Fano has gone into that topic thoroughly, while referring to the work of Laguerre and Scheffers. Study has presented a program for investigations of that sort in a rather obscure place. His own work and the work of Coolidge, Blaschke, et al., that he inspired shows the fruitfulness of the issues that were addressed there.

7. The invariant theory of differential equations. In the second place, one must enter into the invariant theory of differential equations. The effect of the *groups* that have emerged from the general theory of transformations has been so informative that the restriction to them can be a fetter, and the general *theory of equivalence* must move more freely than that. *The simplification of the integration* of differential equations in the sense that Cauchy already defined, whereby the integration of a partial differential equation of first order would come down to the treatment of a system of ordinary differential equations that is constructible by means of just differentiations and eliminations – this goal must not be lost from sight. The theory of equivalence – that is, the *invariant theory* of the problem for *infinite groups* of point or contact transformations – affords the leverage.

An example might clarify this, which we have extracted from what has long since become the realm of the greater common good: The integration of a partial differential equation of first order, when that equation is linear, and thus takes the form:

$$A_1\,p_1 + A_2\,p_2 + \ldots + A_n\,p_n = 0,$$

demands only the integration of:

$$\frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \ldots = \frac{dx_n}{A_n} = \frac{dz}{A},$$

and not that of a system of $2n$ equations. The intrinsic basis for this is that the ∞^{2n-1} *characteristic strips* that are generated are arranged into sheaves of ∞^{n-1}, each of which possesses one of the ∞^n characteristic curves as common carrier. Between the two extremes – namely, ∞^n and ∞^{2n-1} – *carrier curves* for the characteristic curves given one all of the possible intermediate cases, and each of them must, by already being the simplest, lead to a simplification of the integration problem. One will then arrive at *criteria* for the appearance of such a case, and that is a question for *invariant theory*, which must be decidable in any case by differentiations.

Furthermore, it can happen, and also with no reduction of the manifold of carrier curves, that integral manifolds appear that have a lower dimension as point manifolds. Their determination will be simpler than the integration of the partial differential equations itself, and one will arrive at criteria for these special cases.

Necessary and sufficient criteria for these and kindred cases, *equivalence theory* within the classes thus found, *simplifying the integration* of a problem in a class – these are three essential problems to whose required solution, which was in part completely carried out and in part tentative, but far-reaching, the third section of article III.D.7:

Berührungstransformationen in the Math. Enzyklopädie was directed in detail, using a general procedure that was contributed by Engel.

However, this report might be concluded with the well-founded hope that the refined methods in the lever of definite contact transformations and the new lemmas that were worked out for general equivalence theory will still bring to light important developments for a long time. A look backward at the work that had already been accomplished since the premature death of Lie justifies that perspective on the future.