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On a problem in analysis that is attached to the equations of dynamics

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In a note on 11 April of last year, Painlevé proposed to completely study a question that is the immediate generalization of the problem that was solved by Dini, namely, the problem of the geodesic representation of surfaces.

The results that Painlevé announced for a system of points that is subject to forces that admit a potential were presented by me in some other research, but since I mentioned only one of them in particular towards the end of my note on 8 April 1891, I would not have returned to that subject since the method that I used seems very different from the one that Painlevé employed, but at the same, it has led me to some results that do not seem to have been pointed out up to now. I request the permission of the Academy to say a few words about them.

When a material system is subject to the action of forces that derive from a potential, if one lets $x_1, x_2, \ldots, x_m$ denote the variables upon which the position depends at an arbitrary instant then the equations that determine the trajectories at its various points can be represented by:

$$(1) \qquad dx_i\, d^2x_k - dx_k\, d^2x_i = \sum_{(h,h')} (p_{h,h'}^{(k)}\, dx_i - p_{h,h'}^{(i)}\, dx_k)\, dx_h\, dx_{h'} \,.$$

The coefficients $p_{h,h'}^{(k)}$, $p_{h,h'}^{(i)}$ are functions of $x_1, x_2, \ldots, x_m$, and not all of them enter in a distinct fashion in equations (1). The latter include only certain combinations, which are given. I shall suppose that one has completed the definition of those coefficients by the identities:

$$\sum_{(i)} p_{i,k}^{(i)} = 0 \,,$$

which are m in number, and whose significance is easy to understand. A set of equations of type (1) does not always belong to the trajectories of the points of a material system: In order for that to be the case, it is necessary that certain conditions must be satisfied, which one might propose to obtain. They result from the following theorem:

In order for equations (1) *to define the trajectories of a points of a material system that is subject to the action of forces that derive from a potential, it is necessary and sufficient that following equations, which are:*

$$\tfrac{1}{2}m(m-1)(m+2)$$

in number, must be satisfied by the $\tfrac{1}{2}m(m+1)$ *unknowns* f_{i^2}, $f_{i,k}$, *which they include* ***linearly****:*

$$(2)\qquad \begin{cases} \dfrac{\partial f_{i^2}}{\partial x_{i'}} - 2\sum\limits_{(h)} p_{h,i'}^{(i)}\, f_{i,h} = 0 = \dfrac{\partial f_{i,k}}{\partial x_{i'}} - \sum\limits_{(h)} (p_{h,i'}^{(k)}\, f_{i,h} + p_{h,i'}^{(i)}\, f_{i,k}), \\ \dfrac{\partial f_{i^2}}{\partial x_{i'}} - 2\dfrac{\partial f_{i,k}}{\partial x_k} - 2\sum\limits_{(h)} (p_{i,h}^{(i)}\, f_{i,h} - p_{k,h}^{(k)}\, f_{i,k} - p_{h,k}^{(i)}\, f_{h,k}) = 0. \end{cases}$$

That being the case, let Δ be the symmetric determinant of the quantities $f_{i,k}$. The following quadratic form:

$$(3)\qquad 2T = \sum_{(i,k)} \frac{\partial \Delta}{\Delta^2\, \partial f_{i,k}}\, dx_i\, dx_k$$

represents the *vis viva* of the motion, so equations (1) will then give the definition of the corresponding second-degree integral that is expressed by the formula:

$$2T = \text{constant}.$$

It can happen that equations (2) possess several distinct solutions. The study of the case in which that is true is an immediate extension of the Dini problem. If one then lets $f_{i,k}^{(0)}$ denote the functions that constitute a second set that satisfies equations (2), and lets $\Delta^{(0)}$ denote the determinant that is analogous to Δ then the ratio:

$$(4)\qquad \frac{\sum \dfrac{\partial \Delta}{\partial f_{i,k}}\, dx_i\, dx_k}{\sum \dfrac{\partial \Delta^{(0)}}{\partial f_{i,k}^{(0)}}\, dx_i\, dx_k},$$

which is equal to a constant, will be an integral of the differential equations of the trajectories.

That is Painlevé's proposition.

Moreover, since the relations (2) are linear, they are also satisfied by the functions $f_{i,k} + c\, f_{i,k}^{(0)}$, in which the constant c is up to one's discretion. Upon introducing the latter in place of the quantities $f_{i,k}$ in the expression (4), its numerator will become an entire function of c that contains the power $m - 1$. The integral (4) then decomposes into several others and will give, in general, a complete system of first integrals of the problem under study. Those are the circumstances that I alluded to in the last indented line of a note that I presented to the Academy on 14 December 1891, and to which Painlevé kindly responded.

In that same note, I pointed out that the existence of a second-degree integral that differs from that of *vis viva* is not sufficient for the corresponding quadratic form (3) to verify the conditions of the Dini problem: The existence of several second-degree integrals is not characteristic either.

Indeed, there is one case that has been known for a long time in which the equations of motion of a free material system admit a complete system of second-degree first integrals. That case will not satisfy all of the conditions for Dini's problem when the number of variables is greater than two.

Moreover, the preceding considerations establish only that solving the problem is possible. However, that results from a later work (*Comptes rendus*, 6 April 1891).

Even when the relations (2) are not true, equations (1) will not lose their essential form under a change of variables $x_1, x_2, \ldots, x_m$. Their invariants under those transformations are obtained by a very simple generalization of the method that was indicated in a paper that is already old to the case in which the number of variables reduces to two.