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On an algebraic type of condition for a system of moving masses

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1.

If a system of material of points is acted upon by a system of forces for which a force function exists and is subject to a system of condition equations that do not depend upon time then, as is known, there is always an integral into whose expression the individual nature of the constraints does not enter, namely, the *vis viva* integral. However, as far as I know, up to now it has not been noticed that one can derive a second result from the laws of motion that is independent of the nature of the constraints, as long as a certain restriction of a general sort is imposed upon them. If one calls the points of the system $P_1, P_2, \ldots, P_n$, respectively, and further lets $(B_1, B_2, \ldots, B_n)$ denote the position that the point P_α assumes at the location B_α, and denotes a certain distinguished position by $(A_1, A_2, \ldots, A_n)$, and one then denotes the coordinates of the locations B_α and A_α in a rectangular coordinate system by $x_\alpha, y_\alpha, z_\alpha$ and $a_\alpha, b_\alpha, c_\alpha$, resp., and finally define the reigning constraints by the equations:

$$(1) \qquad \Phi_1 = 0, \qquad \Phi_2 = 0, \ \ldots, \qquad \Phi_l = 0$$

then the restriction that was spoken of can be expressed by saying the $\Phi_1, \Phi_2, \ldots, \Phi_n$ should be homogeneous algebraic functions of the $3n$ coordinate differences:

$$x_\alpha - a_\alpha, \qquad y_\alpha - b_\alpha, \qquad z_\alpha - c_\alpha .$$

Now, $(B_1, B_2, \ldots, B_n)$ means the position that the mass-system will assume during its motion at time t. If one then forms the product of the mass m_α that belongs to each point P_α with the square of the distance $A_\alpha B_\alpha$ and takes the sum of those products over the n points, which might be equal to $2G$, then one will have the theorem that second differential quotient with respect to time t of the sum $2G$ will have a value that does not depend upon the nature of the individual constraints, but only on the position of the moving mass-system. I will prove that theorem, and with its help, I will present the conditions for the stability of the motion for certain problems of motion.

In regard to the general nature of a homogeneous algebraic function W of given elements, one can mention that since its degree is equal to a fraction $\mathfrak{p} / \mathfrak{q}$ whose numerator and denominator are whole numbers that are powers $W^{\mathfrak{q}}$ of a homogeneous function of degree $\mathfrak{p}$, and for that reason, they must always satisfy an algebraic equation:

$$E\,W^{\mathfrak{q}s} + E_1\,W^{\mathfrak{q}(s-1)} + \ldots + E_s = 0,$$

whose coefficients $E, E_1, \ldots, E_s$ are rational entire homogeneous functions of the elements with degrees $\mathfrak{e}$, $\mathfrak{e} + \mathfrak{p}$, …, $\mathfrak{e} + \mathfrak{p}s$, resp. Among the $\mathfrak{q}s$ roots of the equation in W, two roots will have equal values if and only if the discriminant Δ of the equation vanishes. For that reason, if one prescribes continuous changes for which the function Δ does not go through the value zero then the corresponding changes in either of the roots can ensue without becoming double-valued. Therefore, under certain assumptions those equations whose discriminant has the property that its sign does not change for any real system of elements and vanishes only when all elements likewise do will have the advantage that each of their real roots can be regarded as a real, continuous, and single-valued function of the elements for all combinations of real values of the elements. It is tacitly assumed that we shall speak of only algebraic functions with that property in what follows.

The geometric meaning of the given type of condition equations is easy to recognize: By assumption, each function Φ_β (where the index β runs from 1 to l) has the property that whenever one substitutes the expressions $a_\alpha + p\,(x_\alpha - a_\alpha)$, $b_\alpha + p\,(y_\alpha - b_\alpha)$, $c_\alpha + p\,(z_\alpha - c_\alpha)$ for the variables a_α, b_α, c_α, respectively, it will go to the expression $p^{\mathfrak{g}\beta}\,\Phi_\beta$ for each p, when the degree of Φ_β is equal to $\mathfrak{g}_\beta$. Therefore, as long as the l equations $\Phi_\beta = 0$ are fulfilled for some position (B_1, B_2, …, B_n), they will also be satisfied for each position $(B'_1, B'_2, \ldots B'_n)$ that relates to the former in such a way that the three locations A_α, B_α, B'_α of each point P_α lie along a straight line, and that the location B'_α divides the line segment $A_\alpha\,B_\alpha$ in a ratio that is the same for all n lines.

In order to prove the property of the function G that was expressed if the force function is called U then the differential equations for the motion of the mass-system might be written as follows:

$$(2)\qquad \left\{\begin{aligned} m_\alpha \frac{d^2 x_\alpha}{dt^2} &= \frac{\partial U}{\partial x_\alpha} + \lambda_1 \frac{\partial \Phi_1}{\partial x_\alpha} + \cdots + \lambda_l \frac{\partial \Phi_l}{\partial x_\alpha}, \\ m_\alpha \frac{d^2 y_\alpha}{dt^2} &= \frac{\partial U}{\partial y_\alpha} + \lambda_1 \frac{\partial \Phi_1}{\partial y_\alpha} + \cdots + \lambda_l \frac{\partial \Phi_l}{\partial z_\alpha}, \\ m_\alpha \frac{d^2 z_\alpha}{dt^2} &= \frac{\partial U}{\partial z_\alpha} + \lambda_1 \frac{\partial \Phi_1}{\partial y_\alpha} + \cdots + \lambda_l \frac{\partial \Phi_l}{\partial z_\alpha}, \end{aligned}\right.$$

when one uses the *Lagrange* method, which applies l undetermined multipliers λ_1, λ_2, …, λ_l . I multiply those three equations by the factors $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, in turn, add them, and take the sum of the aggregate over the n points of the system, which will produce the following equation:

(3)
$$\begin{cases} \sum_\alpha m_\alpha \left[(x_\alpha - a_\alpha)\dfrac{d^2 x_\alpha}{dt^2} + (y_\alpha - b_\alpha)\dfrac{d^2 y_\alpha}{dt^2} + (z_\alpha - c_\alpha)\dfrac{d^2 z_\alpha}{dt^2} \right] \\ = \sum_\alpha \left[(x_\alpha - a_\alpha)\dfrac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\dfrac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\dfrac{\partial U}{\partial z_\alpha} \right] \\ + \sum_\alpha \sum_\beta \lambda_\beta \left[(x_\alpha - a_\alpha)\dfrac{\partial \Phi_\beta}{\partial x_\alpha} + (y_\alpha - b_\alpha)\dfrac{\partial \Phi_\beta}{\partial y_\alpha} + (z_\alpha - c_\alpha)\dfrac{\partial \Phi_\beta}{\partial z_\alpha} \right]. \end{cases}$$

However, the basic property of homogeneous functions yields the equation:

(4)
$$\sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial \Phi_\beta}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial \Phi_\beta}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial \Phi_\beta}{\partial x_\alpha} \right] = \mathfrak{g}_\beta \, \Phi_\beta ,$$

whose right-hand side vanishes, due to equations (1). Thus, the double sum on the right-hand side of (3) will take on the value zero, and when one converts the left-hand side of that equation by means of relations that correspond to the three coordinates and take the form:

$$\frac{d^2 (x_\alpha - a_\alpha)^2}{dt^2} = 2\left(\frac{dx_\alpha}{dt}\right)^2 + 2(x_\alpha - a_\alpha)\frac{d^2 x_\alpha}{dt^2}$$

one will get:

(5)
$$\begin{cases} \tfrac{1}{2} \sum_\alpha m_\alpha \left[\dfrac{d^2 (x_\alpha - a_\alpha)^2}{dt^2} + \dfrac{d^2 (y_\alpha - b_\alpha)^2}{dt^2} + \dfrac{d^2 (z_\alpha - c_\alpha)^2}{dt^2} \right] - \sum_\alpha m_\alpha \left[\left(\dfrac{dx_\alpha}{dt}\right)^2 + \left(\dfrac{dy_\alpha}{dt}\right)^2 + \left(\dfrac{dz_\alpha}{dt}\right)^2 \right] \\ = \sum_\alpha \left[(x_\alpha - a_\alpha)\dfrac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\dfrac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\dfrac{\partial U}{\partial z_\alpha} \right]. \end{cases}$$

The sum whose general term is the product of the mass m_α with the square of the distance $A_\alpha B_\alpha$ was previously set to:

$$\sum_\alpha m_\alpha \left[(x_\alpha - a_\alpha)^2 + (y_\alpha - b_\alpha)^2 + (z_\alpha - c_\alpha)^2 \right] = 2G,$$

but now the sum of the *vis viva* is assumed to be:

$$\sum_\alpha m_\alpha \left[\left(\frac{dx_\alpha}{dt}\right)^2 + \left(\frac{dy_\alpha}{dt}\right)^2 + \left(\frac{dz_\alpha}{dt}\right)^2 \right] = 2T,$$

and equation (5) will be converted into the following one:

(6)
$$\frac{d^2 G}{dt^2} - 2T = \sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial U}{\partial z_\alpha} \right].$$

One gets the *vis viva* integral from this in the following form:

$$T - U = T(0) - U(0), \tag{7}$$

in which $T(0)$ and $U(0)$ mean the values of the functions T and U that arise when one substitutes the values of the quantities in the expressions x_α, y_α, z_α, dx_α / dt, dy_α / dt, dz_α / dt that are given at a certain moment in time t. The elimination of the function T from equations (6) and (7) then yields the result:

$$\frac{d^2G}{dt^2} = 2U + \sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial U}{\partial z_\alpha} \right] + 2T(0) - 2U(0), \tag{8}$$

and according to the assertion above, that will represent the second differential quotient with respect to time of the quantity G as a pure function of the position of the moving mass-system.

Equations (6) and (8) will assume an even simpler form when the force function U, like the functions Φ_β, is a homogeneous function of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$. If one then denotes the order of the function U by $\mathfrak{k}$ and writes U instead of Φ_β and $\mathfrak{k}$ instead of $\mathfrak{g}_\beta$ in equations (4) then the foregoing relation will lead to the new equations:

$$\frac{d^2G}{dt^2} - 2T = \mathfrak{k}\, U, \tag{6*}$$

$$\frac{d^2G}{dt^2} = (2 + \mathfrak{k})\, U + 2T(0) - 2U(0)\,. \tag{8*}$$

Under the assumption that the motion of the mass-system is free of conditions, the considerations that were just presented coincide essentially with the ones that *Jacobi* developed in Bd. XVII, page 120 of this journal and on page 21 of his *Vorlesungen über Dynamik*. However, there, the points that were called $A_1, A_2, \ldots, A_n$ here were combined into one, and that point was chosen to be the origin of the rectangular coordinates. For the application that *Jacobi* gave to a system of masses that were under the influence of only a mutual attraction, one has the special circumstance that the choice of that coordinate origin is irrelevant for the problem, and for that reason the second differential quotient with respect to time t of the function G will always take on the same value for it, at which one might also fix the point $A_1, A_2, \ldots, A_n$.

2.

When one assumes that all of the Φ_β are entire rational functions of degree one of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, but the force function U is an entire rational function of degree two of those elements, one will have the general assumption that will pertain as soon as the mass-system describes small oscillations about the position $(A_1, A_2, \ldots, A_n)$ (*). From the theory of that problem,

(*) *Lagrange*, *Mécanique analytique*, Part Two, section VI.

one knows the necessary and sufficient conditions for completely arbitrary initial positions and velocities of the mass-point to never exceed certain fixed values for the distance from the mass-point to the position ($A_1, A_2, \dots, A_n$) or the velocity of the mass-point at any time, respectively, so that the mass-system would not be at rest in any other position besides ($A_1, A_2, \dots, A_n$). However, it is a far-reaching property of the algebraic type of condition equations that were characterized above that the question of the stability of the motion can be resolved just as simply as it can in the case of small oscillations when they apply, and at the same the force function U is a homogeneous algebraic function of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ that is equal to zero whenever all elements likewise vanish and always increases to infinity whenever an element exceeds its corresponding limit. Namely, the associated answer can be summarized as follows: When all of the functions Φ_β, as well as the force function U, are homogeneous algebraic functions of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, and the function U will be equal to zero whenever all of its elements go to the value zero and will grow to infinity whenever any element increases beyond measure such that the degree of the function is equal to a positive number, the necessary and sufficient condition for neither the distance from a mass-point to the position ($A_1, A_2, \dots, A_n$) nor the velocity of a mass-point at any time to exceed certain fixed limits when one is given completely arbitrary initial positions and velocities and for it to be impossible that the mass-system is at rest anywhere but the position ($A_1, A_2, \dots, A_n$) will then consist of saying that the force function U takes on a finite negative value for all finite, real systems of those elements that that are compatible with the conditions $\Phi_\beta = 0$, except for the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$. The justification for the criterion for the stability of motion shall follow directly.

It will first be shown that when the motion of the mass-system preserves the prescribed stability character, the function U in question must have the given character. To that end, I eliminate the function U from equation (6^*), which is true by assumption, and from the *vis viva* integral (7) and obtain the equation:

(9) $$(2+\mathfrak{k})T = \frac{d^2G}{dt^2} + \mathfrak{k}[T(0) - U(0)] \,.$$

The two sides of it, which represent the motion at every moment as completely-determined functions of time t, can be multiplied by the elements dt and integrated from the value $t = \sigma$ to the value $t = \tau$. That gives the result:

(10) $$(2+\mathfrak{k})\int_\sigma^\tau T\,dt = \left(\frac{dG}{dt}\right)_\sigma^\tau + \mathfrak{k}[T(0) - U(0)](\tau - \sigma),$$

where the function dG / dt has the meaning of:

(11) $$\frac{dG}{dt} = \sum_\alpha m_\alpha \left[(x_\alpha - a_\alpha)\frac{dx_\alpha}{dt} + (y_\alpha - b_\alpha)\frac{dy_\alpha}{dt} + (z_\alpha - c_\alpha)\frac{dz_\alpha}{dt} \right],$$

and the expression $\left(\frac{dG}{dt}\right)_\sigma^\tau$ means the difference of the two values of dG / dt that correspond to the substitutions $t = \sigma$ and $t = \tau$. Now, when none of the quantities $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, dx_α

$/ dt$, dy_α / dt, dz_α / dt exceeds a fixed limit at any time during the motion, by assumption, the same thing will also be true for the expression dG / dt, and it is obvious that the ratio $\dfrac{\left(\dfrac{dG}{dt}\right)_\sigma^\tau}{\tau - \sigma}$ must have the value zero as a limit when the quantity τ increases beyond measure, but the quantity σ remains fixed. Equation (10) will now be divided by the expression $\mathfrak{k}\,(\tau - \sigma)$, and one can increase the value of τ beyond any number, while the value σ is fixed, and the following equation will arise in the limiting case:

$$(12) \qquad \frac{2+\mathfrak{k}}{\mathfrak{k}} \lim \frac{1}{\tau - \sigma} \int_\sigma^\tau T\, dt = T(0) - U(0) .$$

By its nature, the sum of the *vis vivas* T can never be equal to a negative value, but its value at any time is contained between fixed limits under the existing assumptions and can vanish continually only in the case where one continually has $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$. For that reason, as long as the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$ is not continually true, the expression $\lim \frac{1}{\tau - \sigma} \int_\sigma^\tau T\, dt$ will always have a finite positive value, and the ratio $\frac{2+\mathfrak{k}}{\mathfrak{k}}$ will always be a positive quantity in any event as a consequence of the hypothesis. If one would now like to assume that the function U assumes either a positive or vanishing or infinitely large value for a system of finite real values x_α, y_α, z_α that satisfy the equations $\Phi_\beta = 0$ and is different from the system $x_\alpha = a_\alpha$, $y_\alpha = b_\alpha$, $z_\alpha = c_\alpha$ then one would need only to choose the relevant values of x_α, y_α, z_α to be the coordinates of the corresponding point P_α for the moment in time $t = t_0$ (which is allowed by hypothesis) in order to imply a contradiction. On the given grounds, the left-hand side of equation (12) will then have a finite positive value, but when the value of $U(0)$ is negative, vanishing, or infinitely large for the system of values x_α, y_α, z_α in question, from the hypothesis, one can, in each case, assign the velocities of the mass-points that correspond to the time-point $t = t_0$, and as a result the value of $T(0)$, such that the right-hand side of equation (12) becomes negative or vanishing or infinitely large, respectively. The contradiction that emerges then necessarily proves that the function U will take on a finite negative value for all finite real systems of elements that are compatible with the equations $\Phi_\beta = 0$, except for the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$.

Once that point is reached, there is no difficulty to proving that when the force function U possesses the prescribed property, equilibrium will be impossible for the system except for the position $(A_1, A_2, \ldots, A_n)$, and that the distances from the mass-points to that position, as well as their velocities, is contained within fixed limits. First of all, if it is to be possible for equilibrium to exist at a position of the system that is different from $(A_1, A_2, \ldots, A_n)$ then one would have $T = 0$, and for that reason, due to the equation (12), one would have $U(0) = 0$, which is contrary to assumption. The validity of the second part of the assertion is a consequence of the *vis viva* integral:

$$T - U = T(0) - U(0) .$$

From the assumption that was made, the function U can assume only finite negative values for all finite systems of values for $x_\alpha, y_\alpha, z_\alpha$ that are compatible with the equations $\Phi_\beta = 0$, excluding the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$, but for the present system $U = 0$. Therefore, the expression $T - U$ is an aggregate of the two functions T and $- U$ that remains finite and never negative for all allowable systems of finite values $x_\alpha, y_\alpha, z_\alpha, dx_\alpha / dt, dy_\alpha / dt, dz_\alpha / dt$, and the same thing will then be true for the special value of the aggregate $T(0) - U(0)$. Now, by its very nature, the function T has the property of increasing when one of the quantities $dx_\alpha / dt, dy_\alpha / dt, dz_\alpha / dt$ increases beyond measure, and according to the general assumption that was made above, the function U has the property that it likewise increases without end when one of the quantities $x_\alpha - a_\alpha, y_\alpha - b_\alpha, z_\alpha - c_\alpha$ increases beyond measure. Therefore, under the assumption that prevails above, the values $x_\alpha - a_\alpha, y_\alpha - b_\alpha, z_\alpha - c_\alpha, dx_\alpha / dt, dy_\alpha / dt, dz_\alpha / dt$ will be coupled with certain finite limits that are not exceeded at any time by the equation $T - U = T(0) - U(0)$, and that is what was asserted. The criterion that was proposed for the stability of the motion of the problem of motion in question is thus established completely.

3.

From the viewpoint that has been reached, one can also be consider problems of motion for which the functions Φ_β, as well as the function U, are homogeneous algebraic functions of the elements $x_\alpha - a_\alpha, y_\alpha - b_\alpha, z_\alpha - c_\alpha$, and in addition the function U has the property that it is positive for all finite systems of elements that are compatible with the conditions $\Phi_\beta = 0$, it will vanish only when an element increases to infinity, and will be infinitely large if and only if all elements vanish simultaneously, but behave in such a way that the product of the function U and the function G always converges to zero when all elements decrease. The number $\mathfrak{k}$, which denotes the degree of the function U will then lie between the limits 0 and – 2, exclusive of them. Under those assumptions, it can be proved that for a finite value of the function G, the mass-system cannot remain in equilibrium, that the constant $T(0) - U(0)$ must have a finite negative value when the functions G and T remains smaller than certain finite limits at any time in the motion (*), and that the function G cannot exceed a finite limit when that constant has a finite negative value.

The validity of the first and second part of the assertion follows from the fact that as long as the functions G and T cannot increase, equation (12) will be in force. If T is continually equal to zero then, as was also remarked above, $U(0)$ would likewise have to be equal to zero, and that would be incompatible with the assumption that the function G has a finite value. However, in the case of an actual motion, the constant $T(0) - U(0)$ will be equal to the product of a finite negative value $\frac{2+\mathfrak{k}}{\mathfrak{k}}$ with the finite positive value $\lim \frac{1}{\tau - \sigma} \int_\sigma^\tau T\, dt$. The third part of the assertion will be resolved by the *vis viva* equation of $T - U = T(0) - U(0)$, because if the function G is to exceed all limits for a negative value of the constant $T(0) - U(0)$ then, by assumption, the function U must approach zero, so the never-negative function T would be equal to the negative constant $T(0) - U(0)$, which would be impossible.

(*) Cf., the places cited above in *Jacobi*.

Due to the equation $T - U = T(0) - U(0)$, the function T can become infinite only when the function U likewise increases to infinity, and by assumption, that can once more happen only when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ approach zero. If one then multiplies the *vis viva* equation times the function G and observes that, by assumption, the product GU will always have zero for a limit when all elements vanish then one will see that under the relationships that exist, the function T can only be infinitely large in the case when the mass-system goes to the position $(A_1, A_2, \ldots, A_n)$ at the same time and the product GT converges to zero. For that reason, the function T can never grow to infinity whenever some situation obstructs the process of the product GT converging to zero when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ decrease.

One can now give a simple condition for such a situation to arise, as long as the functions Φ_β and U are subject to a further restriction. Any function V that includes the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ only in the $n(n+1)/2$ homogeneous combinations:

$$(x_\alpha - a_\alpha)(x_{\alpha'} - a_{\alpha'}) + (y_\alpha - b_\alpha)(y_{\alpha'} - b_{\alpha'}) + (z_\alpha - c_\alpha)(z_{\alpha'} - c_{\alpha'}) ,$$

where α and α' run through all values from 1 to n, obviously satisfies the three partial differential equations:

$$(13) \qquad \left\{ \begin{aligned} &\sum_\alpha \left[\frac{\partial V}{\partial y_\alpha}(z_\alpha - c_\alpha) - \frac{\partial V}{\partial z_\alpha}(y_\alpha - b_\alpha) \right] = 0, \\ &\sum_\alpha \left[\frac{\partial V}{\partial z_\alpha}(x_\alpha - a_\alpha) - \frac{\partial V}{\partial x_\alpha}(z_\alpha - c_\alpha) \right] = 0, \\ &\sum_\alpha \left[\frac{\partial V}{\partial x_\alpha}(y_\alpha - b_\alpha) - \frac{\partial V}{\partial y_\alpha}(x_\alpha - a_\alpha) \right] = 0. \end{aligned} \right.$$

Therefore, if the functions Φ_β and U include the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ only in the $n(n+1)/2$ homogeneous combinations:

$$(x_\alpha - a_\alpha)(x_{\alpha'} - a_{\alpha'}) + (y_\alpha - b_\alpha)(y_{\alpha'} - b_{\alpha'}) + (z_\alpha - c_\alpha)(z_{\alpha'} - c_{\alpha'})$$

then they will satisfy the three partial differential equations for V, and one can derive the following equation from the differential equations of motion (2):

$$(14) \qquad \left\{ \begin{aligned} &\sum_\alpha m_\alpha \left[\frac{d^2 y_\alpha}{dt^2}(z_\alpha - c_\alpha) - \frac{d^2 z_\alpha}{dt^2}(y_\alpha - b_\alpha) \right] = 0, \\ &\sum_\alpha m_\alpha \left[\frac{d^2 z_\alpha}{dt^2}(x_\alpha - a_\alpha) - \frac{d^2 x_\alpha}{dt^2}(z_\alpha - c_\alpha) \right] = 0, \\ &\sum_\alpha m_\alpha \left[\frac{d^2 x_\alpha}{dt^2}(y_\alpha - b_\alpha) - \frac{d^2 y_\alpha}{dt^2}(x_\alpha - a_\alpha) \right] = 0, \end{aligned} \right.$$

which immediately implies the three integrals:

$$(15)\qquad \begin{cases} m_\alpha \left[\dfrac{dy_\alpha}{dt}(z_\alpha - c_\alpha) - \dfrac{dz_\alpha}{dt}(y_\alpha - b_\alpha) \right] = \mathfrak{A}, \\ m_\alpha \left[\dfrac{dz_\alpha}{dt}(x_\alpha - a_\alpha) - \dfrac{dx_\alpha}{dt}(z_\alpha - c_\alpha) \right] = \mathfrak{B}, \\ m_\alpha \left[\dfrac{dx_\alpha}{dt}(y_\alpha - b_\alpha) - \dfrac{dy_\alpha}{dt}(x_\alpha - a_\alpha) \right] = \mathfrak{C}. \end{cases}$$

They will go to the integrals that are known by the name of the area theorems when the n points $A_1, A_2, \ldots, A_n$ are united into one point. I will now prove the fact that when the sum of the squares of the three new integration constants $\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2$ is not equal to zero, the product GT cannot vanish when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ decrease, and that as a result the function T will never grow to infinity at any time under the given relationships.

For the sake of brevity, one might use the following notations:

$$(16)\qquad \begin{cases} \dfrac{dy_\alpha}{dt}(z_\alpha - c_\alpha) - \dfrac{dz_\alpha}{dt}(y_\alpha - b_\alpha) = p_\alpha, \\ \dfrac{dz_\alpha}{dt}(x_\alpha - a_\alpha) - \dfrac{dx_\alpha}{dt}(z_\alpha - c_\alpha) = q_\alpha, \\ \dfrac{dx_\alpha}{dt}(y_\alpha - b_\alpha) - \dfrac{dy_\alpha}{dt}(x_\alpha - a_\alpha) = r_\alpha, \\ (x_\alpha - a_\alpha)^2 + (y_\alpha - b_\alpha)^2 + (z_\alpha - c_\alpha)^2 = s_\alpha^2, \\ \left(\dfrac{dx_\alpha}{dt}\right)^2 + \left(\dfrac{dy_\alpha}{dt}\right)^2 + \left(\dfrac{dz_\alpha}{dt}\right)^2 = v_\alpha^2, \\ \left(\sum_\alpha m_\alpha p_\alpha\right)^2 + \left(\sum_\alpha m_\alpha q_\alpha\right)^2 + \left(\sum_\alpha m_\alpha r_\alpha\right)^2 = D^2, \end{cases}$$

which will make:

$$\sum_\alpha m_\alpha\, s_\alpha^2 = 2G, \qquad \sum_\alpha m_\alpha\, v_\alpha^2 = 2T\,.$$

Now, when the symbol α runs through the values from 1 to n, as before, while the symbols α and α' in the double summation represent only pairs of unequal numbers from the sequence from 1 to n, one will have the equation:

$$(17)\qquad \begin{cases} 4GT - D^2 = \sum_\alpha m_\alpha^2 (s_\alpha^2 v_\alpha^2 - p_\alpha^2 - q_\alpha^2 - r_\alpha^2) \\ \qquad + \sum_\alpha \sum_{\alpha''} m_\alpha m_{\alpha'} (s_\alpha^2 v_{\alpha'}^2 + s_{\alpha'}^2 v_\alpha^2 - 2 p_\alpha p_{\alpha'} - 2 q_\alpha q_{\alpha'} - 2 r_\alpha r_{\alpha'}). \end{cases}$$

Since one generally has the inequality:

$$(p_\alpha\, p_{\alpha'} + q_\alpha\, q_{\alpha'} + r_\alpha\, r_{\alpha'})^2 \leq (p_\alpha^2 + q_\alpha^2 + r_\alpha^2)(p_{\alpha'}^2 + q_{\alpha'}^2 + r_{\alpha'}^2)\,,$$

and the known relation between the quantities p_α, q_α, r_α that results from the way that they were defined:

$$p_\alpha^2 + q_\alpha^2 + r_\alpha^2 = s_\alpha^2\, v_\alpha^2 - s_\alpha^2 \left(\frac{ds_\alpha}{dt}\right)^2, \tag{18}$$

one will have the consequences:

$$s_\alpha^2\, v_\alpha^2 - p_\alpha^2 - q_\alpha^2 - r_\alpha^2 = s_\alpha^2 \left(\frac{ds_\alpha}{dt}\right)^2 \geq 0,$$

$$(p_\alpha\, p_{\alpha'} + q_\alpha\, q_{\alpha'} + r_\alpha\, r_{\alpha'})^2 < s_\alpha^2\, v_\alpha^2\, s_{\alpha'}^2\, v_{\alpha'}^2\,,$$

$$s_\alpha^2\, v_{\alpha'}^2 + s_{\alpha'}^2\, v_\alpha^2 - 2\, p_\alpha\, p_{\alpha'} - 2 q_\alpha q_{\alpha'} - 2 r_\alpha r_{\alpha'} \geq 0\,.$$

Hence, the combination $4GT - D^2$ is equal to an aggregate of nothing but non-negative quantities. As a result of the three integrals (15), D^2 will be equal to the constant, and that will explain the fact that when it has a non-zero value, the product GT cannot vanish when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ decrease, either; however, that was to be proved.

When the time $t = t_0$ corresponds to a system of finite values x_α, y_α, z_α that satisfies the conditions $\Phi_\beta = 0$, and is different from the system $x_\alpha = a_\alpha$, $y_\alpha = b_\alpha$, $z_\alpha = c_\alpha$, but entirely arbitrary, by assumption, the associated value of $U(0)$ will be finite and positive. For that reason, the choice of velocity components dx_α / dt, dy_α / dt, dz_α / dt for the time $t = t_0$ can just as well be arranged to make the constant $T(0) - U(0)$ take on a negative value and to make the constant $\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2$ take on a non-zero one, as to make the constant $T(0) - U(0)$ take on a positive value. One reaches the conclusion from this that when the position of the mass-system under the prevailing relationships is given arbitrarily, but such that the function G has a non-zero value, whether the motion does or does not have the character of stability will depend upon simply the associated values of the velocity components.

Bonn, 16 October 1866.

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