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Remarks on the principle of least constraint

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1.

The collected mathematical works of **Riemann**, whose publication defines an enduring contribution by **R. Dedekind** and **H. Weber**, contains his response (written in Latin) to a problem that the Paris Academy posed in the year 1858 in regard to a problem in the distribution of heat, which was an article that was extended most remarkably to **Riemann**'s paper "Über die Hypothesen, welche der Geometrie zu Grunde liegen." In the second part of the aforementioned article, **Riemann** developed the conditions for a given quadratic form in n differentials whose coefficients are arbitrary functions of the n variables in question to be transformable into a form with constant coefficients, and in it, he summarized the conditions for all of the coefficients of a certain form [which is denoted by (II) on page 382 in the cited location] that is quadratic in two systems of differentials and covariant to the given form to vanish. That criterion can be combined with the one that I derived for that question in my treatise "Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen" (vol. 70 of this journal, page 71). The form Ψ, which was defined on page 84 of that reference by four systems of linear forms, which were also equal to the form that **Christoffel** denoted by G_4 in the same volume of the journal on page 58, will go to **Riemann**'s aforementioned form (II) as long as two plus two of the associated systems of differentials to the former system are set equal to each other, and from page 94 of the cited volume, the *necessary* and *sufficient condition* for the given quadratic form in n differentials to be convertible into a quadratic form with constant coefficients consists of the identical vanishing of the associated quadrilinear form Ψ. With the help of the aforementioned form (II), **Riemann** exhibited an analytical expression (III) for the concept of the curvature in a manifold of n^{th} order in that reference under which the given quadratic form in n differentials represented the square of the line element, but the expression (III) emerged from the analytical expression for that concept that was given in the treatise: "Fortgesetze Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen," vol. 72 of this journal, page 1, and especially page 24 (which was the quantity k_0 in that treatise), and it coincided completely with the expression that was cited in vol. 4 of **Darboux**'s Bulletin on page 150 and reiterated in vol. **81** of this journal on page 241.

Riemann communicated yet another way of representing the form that he denoted by (II), once the law of defining its coefficients is given, in which different types of variation signs were used, and three second-order equations in the variations were prescribed. That curious algorithm is explained by a fact that has been known to me for some years now. The form (II), which is covariant to the given quadratic form in n differentials, can in fact be regarded as the aggregate of two covariants, one of which equals **Riemann**'s second expression that is in question, while the other one will vanish, due to the equations that **Riemann** indicated. Now the essential components of the latter covariant make it the same covariant that the principle of least constraint required to be a minimum, and that will define the subject of the present article.

The result that was just quoted can be concluded from an equation that appeared as (37) in the second-cited treatise in vol. 72 of this journal on page 16. As in that location, let the given quadratic form in n differentials $dx_{\mathfrak{a}}$, whose coefficients are arbitrary functions of the n variables $x_{\mathfrak{a}}$, and in which the indices $\mathfrak{a}$, $\mathfrak{b}$, … go from 1 to n, be the following one:

$$(1) \qquad f(dx) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}} .$$

Let the bilinear form that is derived from it be:

$$(2) \qquad f(dx, \overset{1}{d}x) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, \overset{1}{d}x_{\mathfrak{b}} .$$

One further has the equation:

$$(3) \qquad -\delta f(dx, \overset{1}{d}x) + df(\delta x, \overset{1}{d}x) + \overset{1}{d}f(\delta x, dx) = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, d\,\overset{1}{d}x_{\mathfrak{b}}\, \delta x_{\mathfrak{a}} + \sum_{\mathfrak{a}} f_{\mathfrak{a}}(dx, \overset{1}{d}x)\, \delta x_{\mathfrak{a}}$$

for the unrestricted application of the three variation symbols. $f_{\mathfrak{a}}(dx, \overset{1}{d}x)$ is then a form that is bilinear in the differentials $dx_{\mathfrak{a}}$ and $\overset{1}{d}x_{\mathfrak{b}}$ that has this developed form:

$$(4) \qquad f_{\mathfrak{a}}(dx, \overset{1}{d}x) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{g},\mathfrak{b}} f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}}\, dx_{\mathfrak{g}}\, \overset{1}{d}x_{\mathfrak{b}} = \tfrac{1}{2}\sum_{\mathfrak{g},\mathfrak{b}} \left(\frac{\partial a_{\mathfrak{a},\mathfrak{g}}}{\partial x_{\mathfrak{b}}} + \frac{\partial a_{\mathfrak{a},\mathfrak{b}}}{\partial x_{\mathfrak{g}}} - \frac{\partial a_{\mathfrak{g},\mathfrak{b}}}{\partial x_{\mathfrak{a}}} \right) dx_{\mathfrak{g}}\, \overset{1}{d}x_{\mathfrak{b}} .$$

In addition, let:

$$(5) \qquad \sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, d\,\overset{1}{d}x_{\mathfrak{b}} + f_{\mathfrak{a}}(dx, \overset{1}{d}x) = \Psi_{\mathfrak{a}}(dx, \overset{1}{d}x),$$

so

$$(6) \qquad -\delta f(dx, \overset{1}{d}x) + df(\delta x, \overset{1}{d}x) + \overset{1}{d}f(\delta x, dx) = \sum_{\mathfrak{b}} \Psi_{\mathfrak{a}}(dx, \overset{1}{d}x)\, \delta x_{\mathfrak{a}} .$$

If one forms the same linear expressions in the coefficients $a_{\mathfrak{a},1}$, $a_{\mathfrak{a},2}$, …, $a_{\mathfrak{a},n}$ that one finds in $\Psi_{\mathfrak{a}}(dx, \overset{1}{d}x)$, i.e.:

$$(7) \qquad \sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} [d\,\overset{1}{d}x_{\mathfrak{b}} + \xi_{\mathfrak{a}}(dx, \overset{1}{d}x)] = \Psi_{\mathfrak{a}}(dx, \overset{1}{d}x),$$

then the combinations $\xi_{\mathfrak{a}}(dx, \overset{1}{dx})$ will be defined by:

$$\xi_{\mathfrak{a}}(dx, \overset{1}{dx}) = \sum_{\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} f_{\mathfrak{c}}(dx, \overset{1}{dx}), \tag{8}$$

with the help of the non-zero determinant $| a_{\mathfrak{a},\mathfrak{b}} | = \Delta$ and the adjoint element $\partial\Delta / \partial a_{\mathfrak{a},\mathfrak{b}} = A_{\mathfrak{a},\mathfrak{b}}$. The aforementioned form $\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x)$, which is linear in the four systems of differentials $\overset{1}{dx}_{\mathfrak{a}}$, $\overset{1}{\delta x}_{\mathfrak{a}}$, $dx_{\mathfrak{g}}$, $\delta x_{\mathfrak{b}}$ and covariant to the given form f (dx), will then be expressed in terms of the likewise-cited equation (37) as:

$$\begin{cases} \tfrac{1}{2}\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x) = \sum_{\mathfrak{b}} [d\Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\,\overset{1}{\delta x}_{\mathfrak{b}} - \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\,\xi_{\mathfrak{b}}(dx, \overset{1}{\delta x})] \\ \qquad\qquad - \sum_{\mathfrak{b}} [\delta\Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\,\overset{1}{\delta x}_{\mathfrak{b}} - \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\,\xi_{\mathfrak{b}}(\delta x, \overset{1}{\delta x})]. \end{cases} \tag{9}$$

One now obtains the desired conversion when one replaces the expression $\xi_{\mathfrak{a}}(dx, \overset{1}{\delta x})$ with the combination of the expressions $-d\,\overset{1}{\delta x}_{\mathfrak{b}}$ and $d\,\overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(dx, \overset{1}{\delta x})$, and replaces the expression $\xi_{\mathfrak{a}}(dx, \overset{1}{\delta x})$ with the combination of the corresponding expressions $-\delta\,\overset{1}{\delta x}_{\mathfrak{b}}$ and $\delta\,\overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(\delta x, \overset{1}{\delta x})$. In that way, the form $\tfrac{1}{2}\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x)$ will be equal to the aggregate of the combination:

$$\sum_{\mathfrak{b}} [d\Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\,\overset{1}{\delta x}_{\mathfrak{b}} + \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\,d\overset{1}{\delta x}_{\mathfrak{b}}] - \sum_{\mathfrak{b}} [\delta\Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\,\overset{1}{\delta x}_{\mathfrak{b}} + \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\,\delta\overset{1}{\delta x}_{\mathfrak{b}}] \tag{10}$$

and the combination:

$$-\sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})[d\,\overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(dx, \overset{1}{\delta x})] + \sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})[\delta\,\overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(\delta x, \overset{1}{\delta x})]. \tag{11}$$

The combination (10) is obviously equal to the complete variation $d\sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\,\overset{1}{\delta x}_{\mathfrak{b}}$, minus the complete variation $\delta\sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\,\overset{1}{\delta x}_{\mathfrak{b}}$. However, the sums to be varied can be represented as aggregates of first variations by means of formula (6). Thus, the combination (10) will appear to be the aggregate of second variations:

$$d\,\overset{1}{df}\,(\delta x, \overset{1}{\delta x}) + \delta\,\overset{1}{\delta f}\,(dx, \overset{1}{dx}) - d\,\overset{1}{\delta f}\,(dx, \overset{1}{dx}) - \delta\,\overset{1}{df}\,(dx, \overset{1}{\delta x}). \tag{12}$$

The expressions $d\,\overset{1}{dx}_{\mathfrak{b}} + \xi_{\mathfrak{b}}\,(dx,\ \overset{1}{dx})$ can be represented as follows:

$$(7^*) \qquad d\,\overset{1}{dx}_{\mathfrak{b}} + \xi_{\mathfrak{b}}\,(dx,\ \overset{1}{dx}) = \sum_{\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} \Psi_{\mathfrak{c}}(dx, \overset{1}{dx}),$$

so the combination (11) will assume the form:

$$(13) \qquad \sum_{\mathfrak{b},\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} [\Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\Psi_{\mathfrak{c}}(\delta x, \overset{1}{\delta x}) - \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\Psi_{\mathfrak{c}}(dx, \overset{1}{\delta x})] \ .$$

The combination (12), as well as the combination (13), is a covariant of the form $f\,(dx)$, and the basis for that is given in the cited location (vol. 72 of this journal, pages 16 and 17). *One therefore has the theorem that one-half the value of the form* $\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x)$ *equals the aggregate of the covariant* (12) *and the covariant* (13). Both covariants contain first and second variations of the variables, but all of the second variations will cancel when one forms their aggregate, and all that will remain are the first variations.

In order to make the transition to **Riemann**'s formulas, the variation symbol $\overset{1}{d}$ must now be set equal to the variation symbol d, and the symbol $\overset{1}{\delta}$ must be set equal to the symbol δ. The covariant (12) will then be converted into the expression:

$$(14) \qquad \tfrac{1}{2} dd \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, \delta x_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} - d\delta \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} + \tfrac{1}{2} \delta\delta \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}}$$

by the complete representation of the quadratic and bilinear forms, and the covariant (13) will be converted into the expression:

$$(15) \qquad \sum_{\mathfrak{b},\mathfrak{c}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} [\Psi_{\mathfrak{b}}(dx, dx)\Psi_{\mathfrak{c}}(\delta x, \delta x) - \Psi_{\mathfrak{b}}(dx, \delta x)\Psi_{\mathfrak{c}}(dx, \delta x)] .$$

At the same time, as a result of the theorem that was proved, the aggregate of the two expressions (14) and (15) will be equal to one-half the value of the form $\Psi\,(dx,\ \delta x,\ dx,\ \delta x)$. In the investigations that are connected with the present ones, the square of the line element for the manifold of n variables $x_{\mathfrak{a}}$ was denoted by $2f\,(dx) = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}}$ in vol. 72 of this journal, page 24. Therefore, that quadratic form will have the same meaning as **Riemann**'s form $\sum b_{\mathrm{i},\mathrm{i}'}\, ds_{\mathrm{i}}\, ds_{\mathrm{i}'}$, and at the same time, the form $\Psi\,(dx,\ \delta x,\ dx,\ \delta x)$ will correspond to the first representation of **Riemann**'s form (II). One thus recognizes that one-half the value of **Riemann**'s form (II) is equal to the aggregate of the covariants (14) and (15). Furthermore, with the notation that was introduced, the expression (14) above will be equal to one-half the expression that one will find on page 381 of **Riemann**'s paper in the fifth line from the bottom, and will define the second representation of his form (II). Now, **Riemann**'s three equations that appear at the bottom of that page say that the combinations $\Psi_{\mathfrak{b}}\,(dx,\ \delta x)$, $\Psi_{\mathfrak{b}}\,(dx,\ dx)$, $\Psi_{\mathfrak{b}}\,(\delta x,\ \delta x)$ should be taken to be zero, since on the basis of equation (6) above, the left-hand sides of the three equations will coincide with the three expressions:

$$-2\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}(dx, \delta x)\, \overset{1}{\delta} x_{\mathfrak{a}}\ ,$$

$$-2\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}(dx, dx)\, \overset{1}{\delta} x_{\mathfrak{a}}\ ,$$

$$-2\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}(\delta x, \delta x)\, \overset{1}{\delta} x_{\mathfrak{a}}\ ,$$

respectively, so those sums must vanish independently of the n variations $\overset{1}{\delta} x_{\mathfrak{a}}$, and that can happen only if the combinations that were spoken of themselves vanish. That confirms that the covariant (15) will be equal to zero as a result of the three equations that **Riemann** exhibited. Now, since one-half the value of the form $\Psi\,(dx,\ \delta x,\ dx,\ \delta x)$ is equal to the aggregate of the covariants (14) and (15), under the stated assumption, the covariant (14) will yield a representation of one-half the value of the form $\Psi\,(dx,\ \delta x,\ dx,\ \delta x)$ in its own right. As was mentioned before, the form $\Psi\,(dx,\ \delta x,\ dx,\ \delta x)$ is equal to **Riemann**'s form (II), and the covariant (14) is equal to one-half the value of **Riemann**'s second representation of the form (II). We have then derived **Riemann**'s second way of representing his form (II) from the property of the form $\Psi(\overset{1}{d}x, \overset{1}{\delta}x, dx, \delta x)$ that was proved just now that it equals an aggregate of two covariants.

2.

We shall now address the development of the connection between the covariant (15) of the previous article and the expression that must be a minimum under the principle of least constraint. However, there is a certain complication that must be overcome. As is known, **Gauss** expressed his principle in words, but not analytical symbols, in vol. 4 of this journal (page 232), and then used synthetic considerations to reduce it to **d'Alembert**'s principle and the principle of virtual velocities. Thus, **Gauss** himself lacked an analytical formulation for his own principle, and that is all the more regrettable, since the words that **Gauss** used in his formulation admitted more than one interpretation at one point. Namely, he did not establish from the outset what sense he was imparting to the expression "the free motion of a point." In order to shed some light upon the question, we imagine that the mass-points of the system that is in motion are referred to a system of rectangular coordinates. For the first mass-point, they might be z_1, z_2, z_3, for the second one z_4, z_5, z_6, …, and for the last mass-point, they might be z_{n-2}, z_{n-1}, z_n. The mass of the first point will be denoted by $m_1 = m_2 = m_3$, the mass of the second point $m_4 = m_5 = m_6$, … Let the components of the applied forces, when decomposed along those three axes, be Z_1, Z_2, Z_3, respectively, for the first point, Z_4, Z_5, Z_6, for the second point, … Let the system of points be subject to a sequence of condition equations Φ_1 = const., Φ_2 = const., …, $\Phi_{\mathfrak{f}}$ = const., which depend upon only the coordinates and contain neither time t nor the derivatives of the coordinates with respect to time. Now, there can be no doubt that the given values of the coordinates of all mass-points at a moment in time t must satisfy the $\mathfrak{l}$ condition equations:

$$\Phi_1 = \text{const.}, \quad \Phi_2 = \text{const.}, \quad \ldots, \Phi_{\mathfrak{l}} = \text{const.}$$

By contrast, as far as the components of the velocities of the individual mass-points relative to the rectangular coordinate system are concerned, there exist two possibilities: Either the components of the velocity are chosen such that they are found to agree with those $\mathfrak{l}$ condition equations and satisfy the $\mathfrak{l}$ equations:

$$\frac{d\Phi_1}{dt} = 0, \qquad \frac{d\Phi_2}{dt} = 0, \qquad \ldots, \qquad \frac{d\Phi_{\mathfrak{l}}}{dt} = 0,$$

which follow from them, or they are chosen in such a way that they contradict them. The expression "free motion of a point" that **Gauss** used is consistent with both of those assumptions. Therefore, in order to ascertain the true content of the principle of least constraint, nothing seems to remain but to formulate it analytically under each of the two assumptions using **Gauss**'s words and examine whether the principle leads to a correct representation of the problem of motion in both cases. That exercise would show that the principle is valid only for the first assumption.

Under the first assumption that was pointed out, the coordinates of the individual mass-points, as well as their first derivatives with respect to time t, must be considered to be given at the moment in time t. By contrast, the second derivatives of the coordinates with respect to time t are considered to be unknown and must be determined by precisely that principle of least constraint. If τ means a small increment in time t then the rectangular coordinates of a mass-point that belongs to a moving system (for example, the first mass-point) at the time $t + \tau$ will assume the values:

$$z_1 + \frac{dz_1}{dt}\tau + \frac{1}{2}\frac{d^2 z_1}{dt^2}\tau^2,$$

$$z_2 + \frac{dz_2}{dt}\tau + \frac{1}{2}\frac{d^2 z_2}{dt^2}\tau^2,$$

$$z_3 + \frac{dz_3}{dt}\tau + \frac{1}{2}\frac{d^2 z_3}{dt^2}\tau^2,$$

respectively, for a well-defined actual motion that is described with a precision that goes up to order τ^2. By contrast, the coordinates of that point at the same time when the motion that results from the influence of the given applied force at that point proves to be free would be:

$$z_1 + \frac{dz_1}{dt}\tau + \frac{1}{2}\frac{Z_1}{m_1}\tau^2,$$

$$z_2 + \frac{dz_2}{dt}\tau + \frac{1}{2}\frac{Z_2}{m_2}\tau^2,$$

$$z_3 + \frac{dz_3}{dt}\tau + \frac{1}{2}\frac{Z_3}{m_3}\tau^2,$$

respectively. Therefore, the square of the deviation of the first point from its free motion will be measured by the sum of the squares of the corresponding coordinate differences, and will have the expression:

$$\left[\left(\frac{d^2 z_1}{dt^2} - \frac{Z_1}{m_1}\right)^2 + \left(\frac{d^2 z_2}{dt^2} - \frac{Z_2}{m_2}\right)^2 + \left(\frac{d^2 z_3}{dt^2} - \frac{Z_3}{m_3}\right)^2\right]\frac{\tau^4}{4}.$$

From the rule that **Gauss** gave, that will be multiplied by the mass of the point in question, which we have called $m_1 = m_2 = m_3$, and the sum of the products that are defined in the same way for all of points of the system will then represent the expression that must be a minimum. That sum will be equal to the product of the factor $\tau^4 / 4$, which is considered to be unvarying, with the combination:

(1) $$\sum_{\mathfrak{a}} m_{\mathfrak{a}} \left(\frac{d^2 z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)^2,$$

in which the symbol $\mathfrak{a}$ runs through the sequence of numbers from 1 to n, as in art. **1**. The principle of least constraint can then be expressed by saying that for the given values of $z_{\mathfrak{a}}$ and $\frac{dz_{\mathfrak{a}}}{dt}$, the quantities $\frac{d^2 z_{\mathfrak{a}}}{dt^2}$ can be determined in such a way that the combination (1) will become a minimum.

In order to address that problem, above all, one must ponder the equations that the desired quantities $\frac{d^2 z_{\mathfrak{a}}}{dt^2}$ must satisfy. If any of the functions $\Phi_1, \Phi_2, \ldots, \Phi_{\mathfrak{l}}$ is denoted by $\Phi_{\mathfrak{a}}$ then what will follow first from each condition equation $\Phi_{\mathfrak{a}}$ = const. is the aforementioned equation:

(2) $$\frac{d\Phi_\alpha}{dt} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial z_{\mathfrak{a}}} \frac{dz_{\mathfrak{a}}}{dt} = 0,$$

which is fulfilled by the first derivatives of the coordinates, and then secondly, the equation:

(3) $$\frac{d^2\Phi_\alpha}{dt^2} = \sum_{\mathfrak{a}} \frac{\partial\Phi_\alpha}{\partial z_{\mathfrak{a}}}\frac{d^2 z_{\mathfrak{a}}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2\Phi_\alpha}{\partial z_{\mathfrak{a}}\,\partial z_{\mathfrak{b}}}\frac{dz_{\mathfrak{a}}}{dt}\frac{dz_{\mathfrak{b}}}{dt} = 0,$$

which the second derivatives of the coordinates must fulfill. However, since the values of $z_{\mathfrak{a}}$ and $\frac{dz_{\mathfrak{a}}}{dt}$ are fixed by the prevailing relations, the $\mathfrak{l}$ equations (3) express only the idea that the aggregate $\sum_{\mathfrak{a}} \frac{\partial\Phi_\alpha}{\partial z_{\mathfrak{a}}}\frac{d^2 z_{\mathfrak{a}}}{dt^2}$ that appears in them must have unvarying values. When one applies the undetermined multipliers $\lambda_1, \lambda_2, \ldots, \lambda_{\mathfrak{l}}$, the minimum problem to be solved will lead to the n equations:

(4) $$m_{\mathfrak{a}}\left(\frac{d^2 z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right) = \lambda_1\frac{\partial\Phi_1}{\partial z_{\mathfrak{a}}} + \lambda_2\frac{\partial\Phi_2}{\partial z_{\mathfrak{a}}} + \cdots + \lambda_{\mathfrak{l}}\frac{\partial\Phi_{\mathfrak{l}}}{\partial z_{\mathfrak{a}}},$$

from the well-known rules. However, these are nothing but the differential equations of the problem of motion that was posed. The principle of least constraint is therefore justified for the first assumption that was made.

The second assumption can be characterized by saying that the given velocity components *do not* correspond to the equations (2). The values of the velocity components might be called $\zeta_1, \zeta_2, \zeta_3$ for the first point, $\zeta_4, \zeta_5, \zeta_6$ for the second point, etc. Therefore, values of the first derivatives of the coordinates for the motion of the individual points cannot, in fact, prove to be equal to the given values, and on those grounds, the first, as well as the second, derivatives of the coordinates with respect to time must now be regarded as unknowns. For that reason, the rectangular coordinates of the first mass-point at time $t + \tau$ will have the previously-exhibited expressions for the actual motion that is to be determined, up to a precision that goes to order τ^2. By contrast, the coordinates in question of that point will have the following expressions:

$$z_1 + \zeta_1\,\tau + \frac{1}{2}\frac{Z_1}{m_1}\tau^2,$$

$$z_2 + \zeta_2\,\tau + \frac{1}{2}\frac{Z_2}{m_2}\tau^2,$$

$$z_3 + \zeta_3\,\tau + \frac{1}{2}\frac{Z_3}{m_3}\tau^2,$$

for the free motion that is now supposed to result with the given velocity components and under the influence of the associated applied force. The square of the deviation of the first point from its motion will then equal the sum of the squares of the coordinate differences:

$$\left[\left(\frac{dz_1}{dt}-\zeta_1\right)\tau+\frac{1}{2}\left(\frac{d^2z_1}{dt^2}-\frac{Z_1}{m_1}\right)\tau^2\right]^2+\left[\left(\frac{dz_2}{dt}-\zeta_2\right)\tau+\frac{1}{2}\left(\frac{d^2z_2}{dt^2}-\frac{Z_2}{m_1}\right)\tau^2\right]^2$$
$$+\left[\left(\frac{dz_3}{dt}-\zeta_3\right)\tau+\frac{1}{2}\left(\frac{d^2z_3}{dt^2}-\frac{Z_3}{m_3}\right)\tau^2\right]^2.$$

When one multiplies that by the mass $m_1 = m_2 = m_3$ of the point in question, deals with all points of the system similarly, and takes the sum of the resulting expressions, what one will get is the combination to be minimized:

$$\sum_{\mathfrak{a}} m_{\mathfrak{a}}\left[\left(\frac{dz_{\mathfrak{a}}}{dt}-\zeta_{\mathfrak{a}}\right)+\frac{1}{2}\left(\frac{d^2z_{\mathfrak{a}}}{dt^2}-\frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)\tau\right]^2, \tag{5}$$

multiplied by the factor τ^2.

The conception of the principle of least constraint that is assumed in that is the basis for the generalization of that principle that **Schering** presented in the essay "**Hamilton-Jacobische** Theorie für Kräfte, deren Mass von der Bewegung der Körper abhängt," in volume XVIII of the Abh. de K. G. d. Wiss. zu Göttingen. In order to obtain the expression (5) above from **Schering**'s formulas, the more general assumptions that he made in them must be replaced with the simpler assumptions for a problem that actually arises in mechanics. With that simplification, the specifics of **Schering**'s conception of the principle of least constraint will emerge quite clearly, and one can make a more confident decision about the justification for that concept.

The problem that was posed of minimizing the expression (5) above differs from the minimum problem that one solves for the expression (1) by the fact that the values of $\frac{d^2z_{\mathfrak{a}}}{dt^2}$, as well as the values of $\frac{dz_{\mathfrak{a}}}{dt}$, must be determined for the former, while only the values of $\frac{d^2z_{\mathfrak{a}}}{dt^2}$ must be determined for the latter. For that reason, the $\mathfrak{l}$ condition equations (2) and the $\mathfrak{l}$ condition equations (3) must be considered in such a way that the $\frac{d^2z_{\mathfrak{a}}}{dt^2}$ and the $\frac{dz_{\mathfrak{a}}}{dt}$ prove to be variable. Therefore, if one introduces $\mathfrak{l}$ multipliers ρ_α, as well as $\mathfrak{l}$ multipliers that might be denoted by $\tau\,\sigma_\alpha$, then the requirement in question might be expressed by saying that the expression:

$$\left\{\begin{aligned}&\sum_{\mathfrak{a}} m_{\mathfrak{a}}\left[\left(\frac{dz_{\mathfrak{a}}}{dt}-\zeta_{\mathfrak{a}}\right)+\frac{1}{2}\left(\frac{d^2z_{\mathfrak{a}}}{dt^2}-\frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)\tau\right]^2\\ &\qquad -2\sum_\alpha\rho_\alpha\sum_{\mathfrak{a}}\frac{\partial\Phi_\alpha}{\partial z_{\mathfrak{a}}}\frac{dz_{\mathfrak{a}}}{dt}-\tau\sum_\alpha\sigma_\alpha\left(\sum_{\mathfrak{a}}\frac{\partial\Phi_\alpha}{\partial z_{\mathfrak{a}}}\frac{dz_{\mathfrak{a}}}{dt}+\sum_{\mathfrak{a},\mathfrak{b}}\frac{\partial^2\Phi_\alpha}{\partial z_{\mathfrak{a}}\,\partial z_{\mathfrak{b}}}\frac{dz_{\mathfrak{a}}}{dt}\frac{dz_{\mathfrak{b}}}{dt}\right)\end{aligned}\right. \tag{6}$$

must be a minimum.

From the well-known rules, that will yield the equations:

$$(7)\qquad \begin{cases} m_{\mathfrak{a}}\left(\dfrac{dz_{\mathfrak{a}}}{dt}-\zeta_{\mathfrak{a}}\right)+\dfrac{m_{\mathfrak{a}}}{2}\left(\dfrac{d^2z_{\mathfrak{a}}}{dt^2}-\dfrac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)\tau=\sum_{\alpha}\rho_{\alpha}\dfrac{\partial\Phi_{\alpha}}{\partial z_{\mathfrak{a}}}+\tau\sum_{\alpha}\sigma_{\alpha}\sum_{\mathfrak{b}}\dfrac{\partial^2\Phi_{\alpha}}{\partial z_{\mathfrak{a}}\,\partial z_{\mathfrak{b}}}\dfrac{dz_{\mathfrak{b}}}{dt}, \\ m_{\mathfrak{a}}\left(\dfrac{dz_{\mathfrak{a}}}{dt}-\zeta_{\mathfrak{a}}\right)+\dfrac{m_{\mathfrak{a}}}{2}\left(\dfrac{d^2z_{\mathfrak{a}}}{dt^2}-\dfrac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)\tau=\sum_{\alpha}\sigma_{\alpha}\dfrac{\partial\Phi_{\alpha}}{\partial z_{\mathfrak{a}}}. \end{cases}$$

The fact that these equations do not embody the nature of the corresponding mechanical problem can be seen with no further discussion. When one subtracts the two equations, it will follow that:

$$\sum_{\alpha}(\rho_{\alpha}-\sigma_{\alpha})\frac{\partial\Phi_{\alpha}}{\partial z_{\mathfrak{a}}}+\tau\sum_{\alpha}\sigma_{\alpha}\sum_{\mathfrak{b}}\frac{\partial^2\Phi_{\alpha}}{\partial z_{\mathfrak{a}}\,\partial z_{\mathfrak{b}}}\frac{dz_{\mathfrak{b}}}{dt}=0,$$

and in the case where $\mathfrak{l} = 1$, in which *only one* condition equation Φ_1 = const. is given, that will demand that the quotient:

$$\frac{1}{\dfrac{\partial\Phi_1}{\partial z_{\mathfrak{a}}}}\sum_{\mathfrak{b}}\frac{\partial^2\Phi_1}{\partial z_{\mathfrak{a}}\,\partial z_{\mathfrak{b}}}\frac{dz_{\mathfrak{b}}}{dt}$$

must have the same value for each index $\mathfrak{a}$. However, such a prescription is entirely foreign to theoretical mechanics. On those grounds, the principle of least constraint cannot be applied when **Gauss**'s words are formulated under the assumption that we called the second one.

Earlier I said that in the expression that **Gauss** gave to his principle, the meaning of the words "free motion of a point" *could not be established from the outset.* However, that opinion was based upon the fact that **Gauss** could only recognize the meaning that those words should have from the *proof* that he carried out. **Gauss** based his proof on the principle of virtual velocities, and that depended upon whether the virtual displacements of the points could all be regarded as small quantities of the same order under the principle of virtual velocities. If that were allowed then those quantities, which Gauss called $c\gamma$, $c'\gamma'$, $c''\gamma''$, … in his proof, would all have to be small quantities of the same order. However, that assumption will hold true only under our first assumption, where the curve that a point of the system of masses that is considered to be in motion describes and the curve that point of the fictitious free system would describe would have the same tangent, while the convention that relates to the second assumption would not hold true unconditionally. For myself, I do not doubt that the principle of virtual velocities must include the convention that was referred to intrinsically. However, due to the nature of the principle of virtual velocities, since a rigorous proof of the necessity of that condition probably cannot be achieved, I have preferred an analytical discussion that is completely convincing, as far as I can see.

At the same time, the consideration that was just discussed shows the way by which the principle of least constraint must be modified in order to deduce results that are acknowledged in theoretical mechanics for those values of the velocity components that are compatible with the condition equations of the problem. It suffices that for a first application of the principle, the coordinates of the mass-points that relate to the time-point $t + \tau$ should be taken with a precision that goes up to only order τ. The second derivatives of the coordinates will not enter into consideration of the actual motion that results, nor will the given applied forces come into play for the consideration of the fictitious free motion, and the first derivatives of the motion that actually results will be determined by the requirement that the expression:

$$\sum_{\mathfrak{a}} m_{\mathfrak{a}} \left(\frac{dz_{\mathfrak{a}}}{dt} - \zeta_{\mathfrak{a}} \right)^2 \tag{8}$$

must be a minimum. The second requirement can then be satisfied when one includes the values $\frac{dz_{\mathfrak{a}}}{dt}$ that are obtained in that way, namely, that the expression (1) can be minimized by a choice of the second derivatives $\frac{d^2 z_{\mathfrak{a}}}{dt^2}$. Meanwhile, I cannot suppress a remark in regard to that subject that does not refer to the treatment of the mechanical problem in question as much as it does to the essence of it. If one assumes that the velocity components of a system of moving mass-points contradict the governing condition equations at some time-point and that after a vanishingly-small time τ has elapsed the individual points of the system will assume velocities that satisfy the condition equations, and under which the motion will proceed according to the given applied forces, then the conversion of the given velocity components into the velocity components that are actually maintained can take place only in such a way that the sum of the *vis vivas* that are impressed upon the system inside of the vanishingly-small time τ will experience a loss. However, while the formulas of theoretical mechanics can represent such a process, the approximation to the true state of affairs must be much closer than the one that is attained in those mechanical problems whose representation does not assume a momentary violation of continuity and momentary loss of *vis viva*.

3.

Once it has been emphasized that the principle of least constraint should refer to those quantities that were referred to in (1) of the previous article, their expressions should be ascertained under the assumption that an arbitrary system of n independent variables must be introduced in place of the rectangular coordinates $z_{\mathfrak{a}}$ for the mass-points of the system in motion. It is known that the transformation of the system of differential equations (4) in article **2** depends upon only the fact that the quadratic form $\frac{1}{2}\sum_{\mathfrak{a}} m_{\mathfrak{a}}\, dz_{\mathfrak{a}}^2$ in the n differentials $dz_{\mathfrak{a}}$ and the expression $\sum_{\mathfrak{a}} Z_{\mathfrak{a}}\, dz_{\mathfrak{a}}$ that is defined by the components of the applied forces must be represented in the new variables. Consistent with the notation in article **1**, one will have:

(1) $$\tfrac{1}{2}\sum_{\mathfrak{a}} m_{\mathfrak{a}}\, dz_{\mathfrak{a}}^2 = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}} = f\,(dx),$$

and furthermore:

(2) $$\sum_{\mathfrak{a}} Z_{\mathfrak{a}}\, dz_{\mathfrak{a}} = \sum_{\mathfrak{a}} X_{\mathfrak{a}}\, dx_{\mathfrak{a}}\,.$$

The $\mathfrak{l}$ functions Φ_α of the coordinates $z_{\mathfrak{a}}$ are converted into functions of the variables $x_{\mathfrak{a}}$, and equations (2) and (3) of the previous article will be replaced by the equations:

(3) $$\frac{d\Phi_\alpha}{dt} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{dx_{\mathfrak{a}}}{dt} = 0,$$

(4) $$\frac{d^2\Phi_\alpha}{dt^2} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{d^2 x_{\mathfrak{a}}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 \Phi_\alpha}{\partial x_{\mathfrak{a}}\, \partial x_{\mathfrak{b}}} \frac{dx_{\mathfrak{a}}}{dt} \frac{dx_{\mathfrak{b}}}{dt}\,.$$

The following equations will enter in place of equations (4) of the previous article, which are likewise constructed with the undetermined multipliers $\lambda_1, \lambda_2, \ldots, \lambda_{\mathfrak{l}}$, and in which the notations for differentials that were defined in article $\mathfrak{l}$ are adapted to differential quotients:

(5) $$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \left[\frac{d^2 x_{\mathfrak{b}}}{dt^2} + \xi_{\mathfrak{b}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right)\right] - X_{\mathfrak{a}} = \lambda_1 \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} + \cdots + \lambda_{\mathfrak{l}} \frac{\partial \Phi_{\mathfrak{l}}}{\partial x_{\mathfrak{a}}}\,.$$

By means of equation (7) of article **1**, they can also assume the form:

(6) $$\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{a}} = \lambda_1 \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} + \cdots + \lambda_{\mathfrak{l}} \frac{\partial \Phi_{\mathfrak{l}}}{\partial x_{\mathfrak{a}}}\,.$$

Now, it follows from equation (6) of article **1** that the sum $\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) \delta x_{\mathfrak{a}}$, and from the nature of the quantities $X_{\mathfrak{a}}$, the sum $\sum_{\mathfrak{a}} X_{\mathfrak{a}}\, \delta x_{\mathfrak{a}}$, will go over to an analogously-constructed expression when one introduces another arbitrary system of variables; that is, the expression is covariant to the form $f\,(dx)$ and the current problem. Therefore the sum:

(7) $$\sum_{\mathfrak{a}} \left[\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{a}} \right] \delta x_{\mathfrak{a}}$$

will also have that property. The form $f\,(dx)$ might now go over to the form $g\,(dy) = \tfrac{1}{2}\sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, dy_{\mathfrak{k}}\, dy_{\mathfrak{l}}$ by the introduction of another arbitrary system of variables $y_{\mathfrak{l}}$, and one further lets the determinant be $| e_{\mathfrak{k},\mathfrak{l}} | = E$ and the adjoint element by $\partial E / \partial e_{\mathfrak{k},\mathfrak{l}} = E_{\mathfrak{k},\mathfrak{l}}$. When the linear expressions:

$$a_{\mathfrak{a},1}\, dx_1 + a_{\mathfrak{a},2}\, dx_2 + \ldots + a_{\mathfrak{a},n}\, dx_n = p_{\mathfrak{a}}$$

and

$$e_{\mathfrak{a},1}\, dy_1 + e_{\mathfrak{a},2}\, dy_2 + \ldots + e_{\mathfrak{a},n}\, dy_n = q_{\mathfrak{l}}$$

are introduced into the forms in question, each of the forms will be converted into its adjoint form as follows:

$$2\, f\,(dx) = \sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\, p_{\mathfrak{a}}\, p_{\mathfrak{b}}\,, \qquad 2\, g\,(dy) = \sum_{\mathfrak{k},\mathfrak{l}} \frac{E_{\mathfrak{k},\mathfrak{l}}}{E}\, q_{\mathfrak{k}}\, q_{\mathfrak{l}}\,,$$

and likewise the equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\, p_{\mathfrak{a}}\, p_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{E_{\mathfrak{k},\mathfrak{l}}}{E}\, q_{\mathfrak{k}}\, q_{\mathfrak{l}} \tag{8}$$

must be true, as a result of the equation $f\,(dx) = g\,(dy)$. As soon as one applies another system of independent variations $dx_{\mathfrak{a}}$ and the corresponding system of variations $dy_{\mathfrak{l}}$, one will also get the equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, dy_{\mathfrak{k}}\, \delta y_{\mathfrak{l}}$$

from the transformation that was performed, which can be replaced with the equation:

$$\sum_{\mathfrak{a}} p_{\mathfrak{a}}\, \delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} q_{\mathfrak{k}}\, \delta y_{\mathfrak{k}} \tag{9}$$

by means of the quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$. That equation exhibits the linear dependency that exists between the quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$, and it illuminates the fact that as long as equation (9) between two systems of quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$ is fulfilled, equation (8) must follow.

The fact that the sum (7) above is covariant to the form $f\,(dx)$ in our mechanical problem means that the expressions $\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{a}}$ have the same relationship to the corresponding expressions that are formed with a new system of variables that is prescribed between the quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$ in (9). Therefore, the expression that is found on the left-hand side of (8):

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} \left[\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{a}} \right] \left[\Psi_{\mathfrak{b}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{b}} \right] \tag{10}$$

must be equal to the analogously-constructed expression when one introduces a system of new variables. *The expression* (10) *is then covariant to the form* $f\,(dx)$ *and the mechanical problem in question.*

One will likewise get the form that the combination (10) assumes from that property as soon as one again introduces the rectilinear coordinates $z_{\mathfrak{a}}$ in place of the variables $x_{\mathfrak{a}}$. Due to equation (1), for the case when $\mathfrak{a}$ and $\mathfrak{b}$ are different from each other, the adjoint elements $A_{\mathfrak{ab}} / \Delta$ must be replaced with $1 / m_{\mathfrak{a}}$, which are, by contrast, zero for the case of $\mathfrak{a} = \mathfrak{b}$. From (6), $\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right)$ will go to the expression $m_{\mathfrak{a}} \frac{d^2 z_{\mathfrak{a}}}{dt^2}$, and from (2), the quantities $X_{\mathfrak{a}}$ will go to the quantities $Z_{\mathfrak{a}}$. The expression:

$$(11) \qquad \sum_{\mathfrak{a}} \frac{1}{m_{\mathfrak{a}}}\left(m_{\mathfrak{a}} \frac{d^2 z_{\mathfrak{a}}}{dt^2} - Z_{\mathfrak{a}}\right)^2$$

will then arise, which is equal to the expression (1) in article **2** identically. *The quantity that is minimized by the principle of least constraint will then be represented by the covariant* (10) *of the given mechanical problem.* A comparison of the covariant (10) with the covariant (13) in article **1** will show that their structures agree completely. Both covariants are based upon the form that is adjoint to the form $2f(dx)$. The covariant (13) in article **1** is equal to a difference of two values of the adjoint form, one of which is constructed from the variables $\Psi_{\mathfrak{b}}(dx, \overset{1}{dx})$ and $\Psi_{\mathfrak{k}}(\delta x, \overset{1}{\delta x})$, while the other is constructed from the system of variables $\Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})$ and $\Psi_{\mathfrak{k}}(dx, \overset{1}{\delta x})$. The covariant (10) above is equal to a value of the adjoint form in which *only one* system of variables appears that is represented by the difference $\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{a}}$.

One can further convert the covariant (10) by consulting the differential equations (6) when one replaces $\Psi_{\mathfrak{a}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{a}}$ with the sum $\sum_{\alpha} \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}}$ and replaces $\Psi_{\mathfrak{b}}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_{\mathfrak{b}}$ with the sum $\sum_{\beta} \lambda_\beta \frac{\partial \Phi_\beta}{\partial x_{\mathfrak{b}}}$, which amounts to the same thing. One will then get the expression:

$$(12) \qquad \sum_{\mathfrak{a},\mathfrak{b}} \sum_{\alpha,\beta} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \lambda_\beta \frac{\partial \Phi_\beta}{\partial x_{\mathfrak{b}}},$$

If one now introduces the schema that was given on page 277 of vol. 71 of this journal:

$$(13) \qquad \sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{\partial \Phi_\beta}{\partial x_{\mathfrak{b}}} = (\alpha, \beta)$$

then the covariant (10) will be equal to the following double sum, in which the indices α and β go from 1 to $\mathfrak{l}$:

$$\sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta (\alpha, \beta) . \tag{14}$$

I cited this expression for the quantity that must be minimized by the principle of least constraint under the assumption that the system of mass-points in motion is subject to condition equations, but no accelerating forces, in vol. 81 of this journal on page 231.

4.

When one extends the problem in mechanics in such a way that for each mass-point, the square of the line element in space is equal to an arbitrary essentially-positive quadratic form in the coordinate differentials, such that consistent with that, the sum of the *vis vivas* of all mass-points of the system in motion will be equal to an essentially-positive quadratic form $2f\left(\frac{dx}{dt}\right)$ in the differential quotients of the coordinates x_α with respect to time t, and such that the expression $\sum_\alpha X_\alpha \, \delta x_\alpha$ and the ι condition equations Φ_α = const. take on a corresponding meaning, one will obtain a system of differential equations by means of the fundamental theorems that were developed in the treatise "Untersuchung eines Problem der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist" (vol. 74 of this journal, page 116, *et seq.*) that has exactly the same form as the system of differential equations (6) of the previous article. An expression that is constructed under the cited assumptions from the given quadratic form $2f(dx)$ and the sum $\sum_\alpha X_\alpha \, \delta x_\alpha$ by the prescription that was given for constructing the combination (10) must, on the same grounds, be a covariant relative to the extended mechanical problem, and that covariant must be considered to be the extension of that concept that was represented by the expression (10) for the original mechanical problem. That also easily shows that when one considers the values x_α and dx_α / dt to be given and imposes the demand that the values of $d^2 x_\alpha / dt^2$ must be determined in such a way that the covariant that was spoken of becomes a minimum, a system of equations will arise that coincides with the system of differential equations of the extended mechanical problem in question. However, that embodies the associated extension of the principle of least constraint.

The treatise that is found in volume 74 of this journal refers to a further extension of the mechanical problem that replaces the line element for each mass-point in space with the p^{th} root of an essentially-positive form of degree p in the coordinate differentials, and the *vis viva* of each mass-point is measured by multiplying the mass of the point by the p^{th} power of the line element and dividing by the p^{th} power of the time-element. The p^{th} part of the sum of the *vis vivas* of all mass-points of a moving system is then equal to an essentially-positive form $f(dx / dt)$ of degree p in the differential quotients with respect to time t of the coordinates x_α of all points, a function U of the variables x_α that represents the force function, and ι condition equations Φ_α = const., and the analogy with

Hamilton's variational problem will lead to the requirement that the first variation of the integral:

$$\int_{t_0}^{t_1}\left[f\left(\frac{dx}{dt}\right)+U\right]dt \tag{1}$$

must be made to vanish. In the cited place, it was assumed that the variables in the problem were chosen in such a way that they fulfilled the given condition equations. By contrast, the problem is formulated precisely as above in my treatise "Sätze aus dem Grenzgebiet der Mechanik und Geometrie" in vol. VI of **Clebsch** and **Neumann**'s Mathematischen Annalen on page 416. There, the function U that appears under the integral sign in the integral (1) was added to the expression $\lambda_1\,\Phi_1 + \lambda_2\,\Phi_2 + \ldots + \lambda_{\mathfrak{l}}\,\Phi_{\mathfrak{l}}$ that is formed with the undetermined multipliers. The system of differential equations that is associated with the variational problem will then read as follows:

$$\frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial\frac{dx_{\mathfrak{a}}}{dt}}\right]-\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}}=\frac{\partial U}{\partial x_{\mathfrak{a}}}+\lambda_1\frac{\partial\Phi_1}{\partial x_{\mathfrak{a}}}+\cdots+\lambda_{\mathfrak{l}}\frac{\partial\Phi_{\mathfrak{l}}}{\partial x_{\mathfrak{a}}}. \tag{2}$$

Now, that shows that the principle of least constraint is sufficiently robust that it will also be valid in this domain. When one separates the terms in the left-hand side of (2) that contain the second differential quotients $d^2 x_{\mathfrak{a}} / dt^2$ from the terms in which only the first differential quotients occur, the following expression will arise:

$$\sum_{\mathfrak{b}}\frac{\partial^2 f\left(\frac{dx}{dt}\right)}{\partial\frac{dx_{\mathfrak{a}}}{dt}\,\partial\frac{dx_{\mathfrak{b}}}{dt}}\frac{d^2x_{\mathfrak{b}}}{dt^2}+f_{\mathfrak{a}}\left(\frac{dx}{dt}\right), \tag{3}$$

in which $f_{\mathfrak{a}}\left(\frac{dx}{dt}\right)$ means a homogeneous function of degree p in the $dx_{\mathfrak{a}} / dt$. That representation is taken from vol. 70 of this journal on page 76, and, at the same time, formula (10) there says that the sum:

$$\sum_{\mathfrak{a}}\left[\sum_{\mathfrak{b}}\frac{\partial^2 f\left(\frac{dx}{dt}\right)}{\partial\frac{dx_{\mathfrak{a}}}{dt}\,\partial\frac{dx_{\mathfrak{b}}}{dt}}\frac{d^2x_{\mathfrak{b}}}{dt^2}+f_{\mathfrak{a}}\left(\frac{dx}{dt}\right)\right]\delta x_{\mathfrak{a}} \tag{4}$$

is covariant with the form $f(dx)$. Similarly, the sum:

$$(5) \qquad \sum_{\mathfrak{a}} \frac{\partial U}{\partial x_{\mathfrak{a}}} \delta x_{\mathfrak{a}}$$

has a value that is independent of the chosen system of variables $x_{\mathfrak{a}}$. As long as the system of differential equations (2) must be convertible into the system of differential equations (5) of the previous article in such a way that the arbitrary functions $X_{\mathfrak{a}}$ will enter in place of the $\partial U / \partial x_{\mathfrak{a}}$, one must maintain the condition that the sum $\sum_{\mathfrak{a}} X_{\mathfrak{a}}\, \delta x_{\mathfrak{a}}$ is likewise independent of the chosen system of variables $x_{\mathfrak{a}}$ in order for the system of differential equations to be meaningful independently of it, respectively. For the sake of brevity, I will introduce the notation:

$$(6) \qquad \frac{\partial^2 f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}\, \partial \frac{dx_{\mathfrak{b}}}{dt}} = a_{\mathfrak{a},\mathfrak{b}}$$

for the second derivatives of the form $f\left(\frac{dx}{dt}\right)$. As long as $p = 2$, the expressions $a_{\mathfrak{a},\mathfrak{b}}$ will coincide with the coefficients of the $2f\left(\frac{dx}{dt}\right)$ form and take on their previous meaning accordingly. However, for a form of degree p when $p > 2$, they will be equal to forms of degree $p - 2$ in the elements $dx_{\mathfrak{a}} / dt$. Furthermore, let the determinant be $| a_{\mathfrak{a},\mathfrak{b}} | = \Delta$ and let the adjoint element be $\partial\Delta / \partial a_{\mathfrak{a},\mathfrak{b}} = A_{\mathfrak{a},\mathfrak{b}}$.

If the form $f(dx)$ goes to the form $g(dy)$ when one substitutes a new system of arbitrary variables $y_{\mathfrak{l}}$, and the variations $\delta x_{\mathfrak{a}}$ again correspond to the variations $\delta y_{\mathfrak{l}}$, then a basic algebraic property of homogeneous functions will imply the equation:

$$(7) \qquad \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 f(dx)}{\partial\, dx_{\mathfrak{a}}\, \partial\, dx_{\mathfrak{b}}} \delta x_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{\partial^2 g(dy)}{\partial\, dy_{\mathfrak{k}}\, \partial\, dy_{\mathfrak{l}}} \delta y_{\mathfrak{k}}\, \delta y_{\mathfrak{l}} ,$$

and by means of the notation (6) and the corresponding notation:

$$\frac{\partial^2 g\left(\frac{dy}{dt}\right)}{\partial \frac{dy_{\mathfrak{k}}}{dt}\, \partial \frac{dy_{\mathfrak{l}}}{dt}} = e_{\mathfrak{k},\mathfrak{l}}$$

that will lead to the following equation:

$$(7) \qquad \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, \delta x_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, \delta x_{\mathfrak{k}}\, \delta x_{\mathfrak{l}} .$$

One can now conclude from this, in the way that was developed in the previous article, that as long as the determinant $| e_{\mathfrak{k},\mathfrak{l}} |$ equals E and the adjoint element $\partial E / \partial e_{\mathfrak{k},\mathfrak{l}}$ is set to $E_{\mathfrak{k},\mathfrak{l}}$, and when the equation:

$$\sum_{\mathfrak{a}} p_{\mathfrak{a}}\, \delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} q_{\mathfrak{k}}\, \delta y_{\mathfrak{k}}$$

is true for the arbitrary variations $\delta x_{\mathfrak{a}}$ and $\delta y_{\mathfrak{k}}$ that correspond to the two systems of quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$, the equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\, p_{\mathfrak{a}}\, p_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{E_{\mathfrak{k},\mathfrak{l}}}{E}\, q_{\mathfrak{k}}\, q_{\mathfrak{l}}$$

will exist. Therefore, since the sums (4) and (5) in our problem are covariant, from the remarks that were made, and the same thing will also be true for the sum $\sum_{\mathfrak{a}} X_{\mathfrak{a}}\, \delta x_{\mathfrak{a}}$ that replaces (5), the expression:

(8) $$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\left[\sum_{\mathfrak{c}} a_{\mathfrak{a},\mathfrak{c}}\frac{d^2 x_{\mathfrak{c}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) - \frac{\partial U}{\partial x_{\mathfrak{a}}}\right]\left[\sum_{\mathfrak{b}} a_{\mathfrak{b},\mathfrak{b}}\frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{b}}\left(\frac{dx}{dt}\right) - \frac{\partial U}{\partial x_{\mathfrak{b}}}\right]$$

will be *a covariant for the variational problem that was posed*, and the expression:

(9) $$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\left[\sum_{\mathfrak{c}} a_{\mathfrak{a},\mathfrak{c}}\frac{d^2 x_{\mathfrak{c}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) - X_{\mathfrak{a}}\right]\left[\sum_{\mathfrak{b}} a_{\mathfrak{b},\mathfrak{b}}\frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{b}}\left(\frac{dx}{dt}\right) - X_{\mathfrak{b}}\right]$$

will be *a covariant for the system of differential equations that arises from the system* (2) *when one substitutes $X_{\mathfrak{a}}$ for $\partial U / \partial x_{\mathfrak{a}}$* , namely:

(10) $$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) = X_{\mathfrak{a}} + \lambda_1\frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} + \cdots + \lambda_{\mathfrak{l}}\frac{\partial \Phi_{\mathfrak{l}}}{\partial x_{\mathfrak{a}}}\ .$$

As long as the form f (dx) is a quadratic form and the assumptions that are actually true for mechanics are accepted, the covariant (9) will be converted into the covariant (10) of article **3**, and as the general rules of minimization problems will show, the former will have the common property with the latter that the associated system of differential equations (10) will emerge from the requirement that for given values of $x_{\mathfrak{a}}$ and $dx_{\mathfrak{a}} / dt$, the value of (9) will be minimized when one determines the values of the $d^2 x_{\mathfrak{a}} / dt^2$. In fact, as a result of the $\mathfrak{l}$ condition equations $\Phi_{\mathfrak{a}}$ = const., the equations that correspond to equations (3) and (4) of article **3** will also be definitive here:

$$\frac{d\Phi_{\mathfrak{a}}}{dt} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{dx_{\mathfrak{a}}}{dt} = 0,$$

$$\frac{d^2\Phi_{\mathfrak{a}}}{dt^2} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{d^2 x_{\mathfrak{a}}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 \Phi_\alpha}{\partial x_{\mathfrak{a}}\, \partial x_{\mathfrak{b}}} \frac{dx_{\mathfrak{a}}}{dt} \frac{dx_{\mathfrak{b}}}{dt} = 0.$$

Moreover, since the quantities $x_{\mathfrak{a}}$ and $dx_{\mathfrak{a}} / dt$ can be considered to be unvarying in the minimum problem, the combinations $a_{\mathfrak{a},\mathfrak{b}}$ will also figure in a form of degree p, where $p > 2$, only as unvarying quantities, and for the same reason, the sums $\sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{d^2 x_{\mathfrak{a}}}{dt^2}$ will have unvarying values, as before. That is the basis for the assertion that was made that the principle of least constraint can be adapted to that extension of the problem in mechanics that is included in the system of differential equations (10) by the use of the covariant (9).

5.

The study by **Schering** that was cited in article **2** refers to the assumption that the square of the line element in space is equal to an essentially-positive quadratic form in the coordinate differentials and pursues the objective of arriving at the system of differential equations in that domain that would be obtained from the corresponding generalization of **Hamilton**'s variational problem by means of an extension of the concept of force and an extension of the principle of least constraint. Under the intended application of the principle of least constraint, **Schering**'s deduction is connected precisely with the expressions that **Gauss** chose, but it also raises the aforementioned objection yet again. That deduction must satisfy the requirement that it represents no other concept than the one that was included in the original train of thought for the case in which the square of the line element in space can be represented as an aggregate of squares of three differentials, so the space in question must be **Euclidian** space itself. However, for the case of **Euclidian** space, the coordinates that were used in **Schering**'s deduction, which were not assigned any special properties, would differ by a small amount from the general coordinates by which a point in **Euclidian** space is determined. In **Schering**'s way of looking at things, the square of the deviation of a point from its free motion would be equal to the square of the distance between two points whose coordinates differ from each other by quantities that are not all of only first order. Now, that sheds some light upon the fact that when a point in **Euclidian** space is referred to rectangular coordinates z_1, z_2, z_3, the square of the line element in space will have the expression:

(1) $$dz_1^2 + dz_2^2 + dz_3^2,$$

and at the same time, that the square of the distance between two arbitrary points $z_1^{(1)}$, $z_2^{(1)}$, $z_3^{(1)}$ and $z_1^{(2)}$, $z_2^{(2)}$, $z_3^{(2)}$ will have the expression:

$$(z_1^{(2)} - z_1^{(1)})^2 + (z_2^{(2)} - z_2^{(1)})^2 + (z_3^{(2)} - z_3^{(1)})^2 . \tag{2}$$

By contrast, as soon as the same point z_1, z_2, z_3 in space is referred to arbitrary coordinates x_1, x_2, x_3, and the square of the line element (1) goes over to the form:

$$a_{11}\,dx_1^2 + a_{22}\,dx_2^2 + a_{33}\,dx_3^2 + 2a_{23}\,dx_2\,dx_3 + 2a_{31}\,dx_3\,dx_1 + 2a_{12}\,dx_1\,dx_2 , \tag{3}$$

it cannot be asserted that the square of the distance between the points $z_1^{(1)}$, $z_2^{(1)}$, $z_3^{(1)}$ and $z_1^{(2)}$, $z_2^{(2)}$, $z_3^{(2)}$ will generally be expressed correctly when one forms the expression:

$$\begin{cases} a_{11}(x_1^{(2)} - x_1^{(1)})^2 + a_{22}(x_2^{(2)} - x_2^{(1)})^2 + a_{33}(x_3^{(2)} - x_3^{(1)})^2 + 2a_{23}(x_2^{(2)} - x_2^{(1)})(x_3^{(2)} - x_3^{(1)}) \\ \qquad + 2a_{31}(x_3^{(2)} - x_3^{(1)})(x_1^{(2)} - x_1^{(1)}) + 2a_{12}(x_1^{(2)} - x_1^{(1)})(x_2^{(2)} - x_2^{(1)}) \end{cases} \tag{4}$$

from the associated coordinates $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$ and $x_1^{(2)}$, $x_2^{(2)}$, $x_3^{(2)}$ of those points. The transformation from rectangular coordinates to polar coordinates already suffices to show that this process is inadmissible. However, it was precisely that process that **Schering** appealed to on page 11 of his paper, where he wished to express the square of the deviation of a possible motion of a point from the free motion. There, **Schering** called the coordinate differences differentials, in general. However, the expressions that he gave for the coordinate differences as aggregates of terms that were of first and second order in the element of time and the points that would follow from those terms in the formulas alluded to terms of even higher order. In order for the process to also be valid for the terms of first and second order, one would generally have to be able to draw the conclusion from the equality of the expressions (1) and (3) that when the differences $z_\mathfrak{a}^{(2)} - z_\mathfrak{a}^{(1)}$ are replaced with the aggregate $dz_\mathfrak{a} + \frac{1}{2} d^2 z_\mathfrak{a}$ for the values $\mathfrak{a}$ = 1, 2, 3, and at the same time, the differences $x_\mathfrak{a}^{(2)} - x_\mathfrak{a}^{(1)}$ are replaced with the corresponding aggregate $dx_\mathfrak{a} + \frac{1}{2} d^2 x_\mathfrak{a}$, the expression (2) will be equal to the expression (4). Thus, as a result of setting terms of the same higher order equal to each other, the equation:

$$\sum_\mathfrak{a} dz_\mathfrak{a}\, d^2 z_\mathfrak{a} = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_\mathfrak{a}\, d^2 x_\mathfrak{b}$$

must exist, which would also be incorrect in the cited example of the transformation from rectangular coordinates to polar coordinates.

Schering's work aroused the desire in me to see how the concept of a force that acts upon a point in that domain would carry over to the one that pertained to the variational problem of the integral that was denoted by (1) in the previous article. It will now be assumed in that problem that only a single point of unit mass moves freely. The line element for the point that is referred to by the coordinates x_α will then have the expression $\sqrt[p]{p\,f(dx)}$, and the requirement that the first variation of the integral:

$$\int \sqrt[p]{p\,f(dx)} \tag{5}$$

must vanish will determine the first-order manifold that corresponds to the shortest line in the relevant space for the coordinates x_α. When one thinks of the variables x_α as independent of a variable t, the integral (5) can take on the form:

$$\int_{t_0}^{t_1} \sqrt[p]{p\,f\left(\frac{dx}{dt}\right)}\,dx, \tag{5*}$$

from which the differential drops out. Therefore, it still remains completely undetermined how the variables x_α should depend upon the variable t in the variational problem for the integral (5*). The associated system of differential equations that is given on page 124 of vol. 74 of this journal reads:

$$\frac{d}{dt}\left[\frac{\partial \sqrt[p]{p\,f\left(\frac{dx}{dt}\right)}}{\partial \frac{dx_\alpha}{dt}}\right] - \frac{\partial \sqrt[p]{p\,f\left(\frac{dx}{dt}\right)}}{\partial x_\alpha} = 0. \tag{6}$$

The quantities x_α will be determined completely by it when initial values $x_\alpha(0)$ and the $n - 1$ ratios of the initial differentials $dx_\alpha(0)$ are given for the value $t = t_0$, and we assume that the integration was carried out for that data. The value of the integral (5*), which might be called r, accordingly represents the length of the shortest line that goes from the point $x_\alpha(0)$ to the point x_α, or the distance between the point x_α and the point $x_\alpha(0)$, and when it is represented as a pure function of the system of values $x_\alpha(0)$ and x_α, it must satisfy the equation that is included in (7^b) in the cited location:

$$\delta r = \frac{1}{\left[p\,f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}} \sum_\alpha \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_\alpha}{dt}}\,\delta x_\alpha - \frac{1}{\left[p\,f_0\left(\frac{dx(0)}{dt}\right)\right]^{\frac{p-1}{p}}} \sum_\alpha \frac{\partial f_0\left(\frac{dx(0)}{dt}\right)}{\partial \frac{dx_\alpha(0)}{dt}}\,\delta x_\alpha(0). \tag{7}$$

The addition of the symbol 0 to the form $f(dx)$ means the substitution of the quantities $x_\alpha(0)$ for the corresponding x_α in the coefficients of the form. The differential dt can also appear in the foregoing equation (7) only formally, and will cancel by means of the homogeneity of the form $f(dx)$.

We shall now consider the variational problem of the integral:

$$\int_{t_0}^{t_1} \left[f\left(\frac{dx}{dt}\right) + U \right] dt \,, \tag{8}$$

under the assumption that the force function U is a pure function of the function r that is referred to a fixed system $x_\mathfrak{a}$ (0) and the moving system $x_\mathfrak{a}$; it will be called P (r). That problem defines a generalization of the problem for the free motion of a point in which the given force function is a pure function of the distance between the moving point and a fixed point. That problem was solved under the assumption that the form f (dx) is a quadratic form that belongs to a certain genus of forms in the treatise "Extension of the planet problem to a space of n dimensions and constant integral curvature" (Quarterly Journal of Mathematics, no. 48, pp. 349), and indeed under the condition that the values of $x_\mathfrak{a}$ and $dx_\mathfrak{a} / dt$ are given arbitrarily for a time-point $t = t_1$. **Schering** solved the same problem for a space of n dimensions and constant curvature by a somewhat more extended assumption in regard to the force function on page 35 of the treatise that was cited above. In the present discussion, the degree p of the form f (dx) can be arbitrary, although it was established that the variables $x_\mathfrak{a}$ should assume the values $x_\mathfrak{a} = x_\mathfrak{a}$ (0) for the time-point $t = t_0$, which was chosen for the definition of the quantity r. We will then focus our attention on the free motion of a point that moves under the influence of a force function P (r), and whose motion begins from that fixed point, and whose distance to that point is measured by r. The differential equations of the variational problem that was just posed can be obtained from the differential equations (2) of the previous article when one drops the condition functions and replaces U with P (r). They will then be these:

$$\frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_\mathfrak{a}}{dt}}\right] - \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_\mathfrak{a}} = \frac{\partial r}{\partial x_\mathfrak{a}} \frac{dP(r)}{dr} \,. \tag{9}$$

It can now be verified that under the assumption in question, by which the equations $x_\mathfrak{a} = x_\mathfrak{a}$ (0) must be true for $t = t_0$, those differential equation for the variables $x_\mathfrak{a}$ will prescribe the same first-order manifold that is determined by the system (6). That is, a point that is under the influence of the force function P (r) must move along a shortest line from the point $x_\mathfrak{a}$ (0). One can give the system (6) above the following form:

$$\frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_\mathfrak{a}}{dt}}\right] - \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_\mathfrak{a}} = \frac{1}{\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}} \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_\mathfrak{a}}{dt}} \frac{d}{dt}\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}} \tag{10}$$

by means of a conversion that was performed in (3^{b*}) on page 124 of vol. 74 of this journal. Due to equation (7), the partial differential quotient of the function r that is expressed in terms of the quantities $x_{\mathfrak{b}}$ and $x_{\mathfrak{b}}$ (0) with respect to the individual values $x_{\mathfrak{a}}$ will have the expression:

$$\frac{\partial r}{\partial x_{\mathfrak{a}}} = \frac{1}{\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}} \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}. \tag{11}$$

Therefore, the system (10) will be converted into this one:

$$\frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}\right] - \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}} = \frac{\partial r}{\partial x_{\mathfrak{a}}} \frac{d}{dt}\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}. \tag{12}$$

Since r is the value of the integral (5^*), one will have the equation:

$$\frac{dr}{dt} = \left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{1}{p}}. \tag{13}$$

For that reason, one will also have the following expression for the system (12):

$$\frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}\right] - \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}} = \frac{\partial r}{\partial x_{\mathfrak{a}}} \frac{d\left(\frac{dr}{dt}\right)^{p-1}}{dt}. \tag{14}$$

The left-hand side of equation (14) is identical to the left-hand side of equation (9) for every value of the index $\mathfrak{a}$, and similarly the factor that is found on the right-hand side will coincide with $\partial r / \partial x_{\mathfrak{a}}$. Now since the system (14) determined only the first-order manifold for the variables $x_{\mathfrak{a}}$, the dependency of the individual variables on the variable t will, however, remain undetermined, so it is possible to arrange that dependency in such a way that the system (6) coincides with the system (9), and that will come about when one assumes that the equation:

$$\frac{d}{dt}\left(\frac{dr}{dt}\right)^{p-1} = \frac{dP(r)}{dr} \tag{15}$$

is valid. The desired integration of the system (9), in which the equations $x_\alpha = x_\alpha$ (0) must be true for $t = t_0$ and the initial values $\frac{dx_\alpha}{dt} = \frac{dx_\alpha(0)}{dt}$ should be proportional to the corresponding differentials that were chosen in the integration of the system (6) or (14), will then yield the first-order manifold for the variables x_α that is predicted by the system (14), as was stated, while equation (15) determined the dependency of the path length *r*, which follows the shortest line, on time *t*. Here, the differential quotient of the function *P* (*r*) with respect to the quantity *r* will take over the role of the force that acts upon the point of unit mass. Equation (15) says that under the motion that is spoken of, which results in the shortest line that starts from the point x_α (0), the differential quotient with respect to time of the $(p-1)^{\text{th}}$ power of the first differential quotient of the path length *r* with respect to time will be equal to the given quantity $dP\,(r)\,/\,dr$. However, for the value $p = 2$, that rule will go over to the rule that under the motion in question, the second differential quotient with respect to time of the path length *r* must be equal to the given quantity $dP\,(r)\,/\,dr$. For the sake of simplicity of expression, I have assumed that the initial values $x_\alpha = x_\alpha$ (0) were prescribed in the integration of the system (9), and that the initial values $\frac{dx_\alpha}{dt} = \frac{dx_\alpha(0)}{dt}$ were given to be proportional to the values of the initial differentials that were chosen in the integration of the system (6). However, the reduction of the system (9) to the system (6) can be accomplished in the same way when one demands that for the system (9) at an arbitrary time-point $t = t_1$, only those values of $x_\alpha = x_\alpha$ (1) and $\frac{dx_\alpha}{dt} = \frac{dx_\alpha(1)}{dt}$ should be valid that are taken from the first-order manifold that is determined by the integration of the system (6) that was performed. That is, in a different language, the motion of point that is under the influence of the force function *P* (*r*) when it begins from an arbitrary point of a shortest line that goes through the fixed point x_α (0), and indeed in the direction of that shortest line, will always remain on that shortest line and obey equation (15).

When equation (15) is multiplied by $dr/\,dt$, it will take on the form:

$$(p-1)\left(\frac{dr}{dt}\right)^{p-1}\frac{d^2r}{dt^2} = \frac{dP(r)}{dr}.$$

That equation admits the undetermined integration:

$$\frac{p-1}{p}\left(\frac{dr}{dt}\right)^{p} = P\,(r) + H, \tag{16}$$

in which *H* means an arbitrary constant whose value is determined by the given initial values. Equation (16) will go to the equation:

$$(p-1)\, f\left(\frac{dr}{dt}\right) = P\,(r) + H, \tag{17}$$

by means of (13). For the present problem that is nothing but the equation that was denoted by (5^a) on page 123 of vol. 74 of this journal, and the integral will then represent the *vis viva*. Equation (16) will imply the equation:

$$dt = \frac{dr}{\sqrt[p]{\frac{p}{p-1}[P(r)+H]}}, \tag{18}$$

which will yield the dependency of the path length r on time t by performing a quadrature and inverting the resulting equation.

Bonn, 13 November 1876.