"Sätze aus dem Grenzgebiet der Mechanik und Geometrie," Math. Ann. **6** (1874), 416-435.

Theorems at the interface between mechanics and geometry

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Since mechanics considers bodies in a state of motion, it must take advantage of geometry in order to establish its foundations and must always exploit it in order to achieve its goals.

However, it was first by the ongoing construction of mechanics and geometry that attention was directed to questions whose exploration both sciences have an equal interest in, and which depend upon precisely the same algorithms when they are expressed analytically. The development of a concept that touches upon the interface of mechanics and geometry can be found in the treatise "Über einen algebraischen Typus der Bedingungsgleichungen eines bewegten Massensystems" (Borchardt's Journal for Mathematics, vol. 66, pp. 363). One imagines a system of material points whose masses might be called m_1, m_2, …, m_q, respectively. The position of each individual mass m_e will be referred to the rectangular coordinates x_e, y_e, z_e. For a certain arrangement of the system, the coordinates of the individual points will take on the well-defined values $x_e = a_e$, $y_e = b_e$, $z_e = c_e$. The concept in question can then be defined to be the *sum over all q points of the products of those masses m_e with the squares of the distance from the position* (x_e, y_e, z_e) *to the position* (a_e, b_e, c_e) :

$$2G = \sum_e m_e[(x_e - a_e)^2 + (y_e - b_e)^2 + (z_e - c_e)^2]. \tag{1}$$

The assumption that the system of material points m_e moves in space free from the influence of accelerating forces and with no restricting conditions leads to a new definition of that concept. The uniform advance of each point along a straight line will necessarily follow from the requirement that the first variation of the associated *integral of least action:*

$$R = \int \sqrt{\sum_e m_e(dx_e^2 + dy_e^2 + dz_e^2)}, \tag{2}$$

must vanish at the initial and final positions of the individual points. Now when the coordinates (a_e, b_e, c_e) determine the initial position of the mass m_e and the coordinates (x_e, y_e, z_e) determine the final position that will easily imply the equation:

(3) $$R^2 = 2G.$$

The function $2G$ *is then equal to the square of the associated integral of least action for a motion of the system from the point* m_e *from the position* (a_e, b_e, c_e) *to the position* (x_e, y_e, z_e), *respectively*. On the other hand, the representation of the integral of least action by the equation $R = \sqrt{2G}$ coincides with the representation that **Hamilton** introduced as the *characteristic function of the mechanical problem in question.*

In order to derive the relationship of the function $2G$ to a more general motion of the system of points m_e, it will be assumed that the system is under the influence of accelerating forces for which there exists a force function U and that it is subject to a series of condition equations:

(4) $$\Phi_1 = \text{const.}, \quad \Phi_2 = \text{const.}, \quad \ldots, \quad \Phi_l = \text{const.},$$

in which the l functions contain only the coordinates of the q individual points and not time t. In that case, the principle that goes back to **Hamilton** prescribes that the first variation of the integral:

(5) $$\int \left\{ \tfrac{1}{2} \sum_e m_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2 \right] + U + \lambda_1 \Phi_1 + \cdots + \lambda_l \Phi_l \right\} dt$$

must be made equal to zero, while one draws upon the l equations (4) for fixed initial and final values of the $3q$ coordinates. The multipliers $\lambda_1, \ldots, \lambda_l$ to be determined prove to be pure functions of time t. Due to the rules of the calculus of variations, that problem will produce the equation:

(6) $$\sum_e m_e \left(\frac{d^2 x_e}{dt^2} \delta x_e + \frac{d^2 y_e}{dt^2} \delta y_e + \frac{d^2 z_e}{dt^2} \delta z_e \right) = \delta U + \lambda_1 \, \delta\Phi_1 + \lambda_2 \, \delta\Phi_2 + \ldots \lambda_l \, \delta\Phi_l,$$

which must be fulfilled independently of the $3q$ variations δx_ε, δy_ε, δz_ε, and in that way, conclude the system of differential equations of the mechanical problem in its own right. The variations δx_ε, δy_ε, δz_ε in equation (6) can be replaced with final differences $x_\varepsilon - a_\varepsilon$, $y_\varepsilon - b_\varepsilon$, $z_\varepsilon - c_\varepsilon$, respectively (*). The left-hand side will then go to an expression that is connected with the function G by the *characteristic relation:*

(7) $$\sum_e m_e \left[\frac{d^2 x_e}{dt^2}(x_e - a_e) + \frac{d^2 y_e}{dt^2}(y_e - b_e) + \frac{d^2 z_e}{dt^2}(z_e - c_e) \right]$$
$$= \frac{d^2 G}{dt^2} - \sum_e m_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2 \right],$$

(*) **Jacobi**, *Vorlesungen über Dynamik*, 4th Lecture, pp. 21.

and on the right-hand side, the variation δU will be converted into the expression:

$$\sum_e m_e \left[\frac{\partial U}{\partial x_e}(x_e - a_e) + \frac{\partial U}{\partial y_e}(y_e - b_e) + \frac{\partial U}{\partial z_e}(z_e - c_e)\right], \tag{8}$$

while the variation $\delta\Phi_\gamma$ is converted into an expression that is formed analogously. That implies the equation:

$$\begin{aligned} \frac{d^2 G}{dt^2} - \sum_e m_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2\right] \\ = \sum_e m_e \left[\frac{\partial U}{\partial x_e}(x_e - a_e) + \frac{\partial U}{\partial y_e}(y_e - b_e) + \frac{\partial U}{\partial z_e}(z_e - c_e)\right] \\ + \sum_\gamma \sum_e \lambda_\gamma \left[\frac{\partial \Phi_\gamma}{\partial x_e}(x_e - a_e) + \frac{\partial \Phi_\gamma}{\partial y_e}(y_e - b_e) + \frac{\partial \Phi_\gamma}{\partial z_e}(z_e - c_e)\right]. \end{aligned} \tag{9}$$

Under the assumptions that Φ_1 , Φ_2 , …, Φ_l *are homogeneous functions of the combinations* $x_e - a_e$, $y_e - b_e$, $z_e - c_e$ *and that the constant values that are prescribed in* (4) *are all zero, the double sum on the right-hand side of the last equation will vanish*, and that equation will be identical to equation (6) in the cited treatise.

If one would like to draw upon those cases of the motion of a system of material points for which the condition equations (4) are true, but the components along the x, y, z axes of the forces X_e , Y_e , Z_e that act upon the individual mass points m_e cannot be derived from a force function, then it is known that the differential equations of the problem can be summarized in *one equation* that will emerge from equation (6) above when one substitutes the expression:

$$\sum_e (X_e\,\delta x_e + Z_e\,\delta y_e + Z_e\,\delta z_e) \tag{10}$$

for the variation δU (*). For that reason, all of the conclusions that we inferred will remain in full force, as long as one uses the expression:

$$\sum_e [X_e\,(x_e - a_e) + Z_e\,(y_e - b_e) + Z_e\,(z_e - c_e)] \tag{8*}$$

in place of the expression (8) above.

The existence of a force function for the applied forces is not assumed in the treatise by **Clausius**: "Über einen auf die Wärme anwendbaren mechanische Satz," (Sitzungsberichte der Niederrheinishcn Gesellschaft in Bonn, June 1870, and Comptes

(*) **Lagrange**, *Mécanique analytique*, Part two, Section IV, arts. 10 and 11.

rendu de l'Academie des Sciences Paris, T. LXX, 20 June 1870), and in the treatise of **Ivan Villarceaux** "Sur un nouveau théorème de mécanique générale" (C. R. Acad. Sci., t. LXXV, no. 5, 29 July 1872). The relationship between those two treatises, which start from purely-mechanical considerations and apply the results that are found to the *mechanical theory of heat*, and the paper that was just cited, along with the earlier works of **Jacobi** in regard to it in (Crelle's Journal, Bd. 17, pp. 97), and *Vorlesungen über Dynamik*, pp. 21, are discussed in the Bulletin des sciences mathématiques et astronomiques, which is edited by **G. Darboux** and **J. Houel**, tome III, November 1872, pp. 349. Another paper by **Clausius** belongs with them, namely "Über die Beziehungen zwischen den bei Centralbewegungen vorkommenden charakteristischen Grösse," (Nachrichten v. d. K. G. d. Wiss. zu Göttingen, 25 December 1872, pp. 600). **De Gasparis** gave applications of the function that was just denoted by $2G$ to the *theory of attraction* in "Lettre sur un nouveau théorème de mécanique, communiquée par M. Ivan Villarceaux" (C. R. Acad. Sci. tone LXXV, no. 9, August 1872), along with **S. Newcomb** "Note sur un théorème de mécanique céleste" (C. R. Acad. Sci., tome LXXV, no. 26, 23 December 1872). **F. Lucas** gave applications to the *theory of small oscillations* in "Théorèmes généraux sur l'équilibre et le mouvement des systèmes matérials" (C. R. Acad. Sci., tome LXXV, no. 23, 2 December 1872), as well as a report by **de Saint-Venant** and "Partage de la force vive, due à un mouvement vibratoire composé, en celles qui seraient dues aux mouvement pendulaires simples et isochrones composants, de diverses périodes et amplitudes. Partage du travail dû su même mouvement composé, entre deux instants quelconques, en ceux qui seraient dus aux mouvements composants" (C. R. Acad. Sci., *ibid.*)

The oft-cited article "Über einen algebraisdchen Typus der Bedingungsgleichungen eines bewegten Massensystems" contains applications to two types of mechanical problems, where one type is concerned with *small oscillations of a system of material points*, while the other is concerned with *the attraction of a material point to a fixed center*. I think that I will treat some further applications of the function $2G$ to the problems of theoretical physics on a later occasion.

The present study mainly has to do with the assumption that the system of points m_e is not driven by any accelerating forces and is subject to *only one condition equation*. Accordingly, the force function U will be equal to zero in the problem that was just referred to, and the l conditions (4) will reduce to the *one:*

$$\Phi_1 = \text{const.} \tag{4*}$$

As long as *only one mass point* m_1 *is present*, that condition will mean that the mass-point cannot leave the *surface* Φ_1 = const., and that the function $2G$ will be equal to the product of the mass m_1 with the square of the distance from the position (x_1, y_1, z_1) to the position (a_1, b_1, c_1). If one now imagines that the position (x_1, y_1, z_1) on the surface Φ_1 = const. is given arbitrarily at a certain time t, along with the advance of the point on the surface during the next time-element dt, then the element of the path of the point in question will be associated with a certain normal section of the surface and the point (a_1, b_1, c_1) can be determined as the *center of curvature for that normal section*. As a result of this, the function $2G$ can be connected with the associated *radius of curvature* ρ by the equation:

(11) $$2G = m_1\,\rho^2.$$

When the equation (6) above corresponds to the simple assumption that was spoken of, the expression $\lambda_1\ \delta\Phi_1$ that appears on the right-hand side will represent *the moment of the pressure that the motion of the point m_1 exerts upon the surface* Φ_1 = const. The known relation between the value of the pressure and the radius of curvature ρ can then be expressed as follows by means of the relation (11):

(12) $$\frac{-1}{\sqrt{2G}} = \frac{\lambda_1\sqrt{\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\Phi_1}{\partial y_1}\right)^2 + \left(\frac{\partial\Phi_1}{\partial z_1}\right)^2}}{\sqrt{m_1^3\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2\right]}}.$$

It will now be shown how one can determine a system of q associated points (a_e , b_e , c_e) from a system of q points (x_e, y_e, z_e) by the demand that the first and second complete differentials of the function G in question differ from the first and second complete differentials, respectively, of the given function Φ_1 only by a finite factor. The system of q points thus-defined will represent a generalization of the concept of the center of curvature of a surface. There is then a complete analogy to equations (2) above that exists between the corresponding values of the function $2G$ and the values λ_1 that enter into equation (6), for which the function U equals zero and the number λ is assumed to be equal to unity. It will generalize the previously-cited theorem that when a point that is free of the influence of an accelerating force moves on a given surface, the reciprocal value of the radius of curvature is proportional to the pressure that acts upon the surface.

After I have established that fact, I will discuss the place that the associated concepts of mechanics and geometry assume from the standpoint that leads to the investigations that were published in Borchardt's Journal für Mathematik (Bd. 70, pp. 71-102 and Bd. 72, pp. 1-56) with the title "Untersuchungen in Betreff der ganzen homogenen Functionen von *n* Differentialen" and in (*ibid*., Bd. 74, pp. 116-149) with the title "Entwicklung einiger Eigenschaften der quadratischen Formen von *n* Variationsrechnung, in welchem das Problem der Mechanik enthalten ist." Some parts of the theory that have been separate up to now will be connected with each other by that consideration and will make one aware of a new confirmation of the fact that the assumptions of mechanics and geometry that are, in fact, valid are distinguished from some other closely-related assumptions in a characteristic way.

1.

When one juxtaposes the first complete differential of the function G and that of the function Φ_1 :

(13) $$dG = \sum_e m_e\,[(x_e - a_e)\,dx_e + (y_e - b_e)\,dy_e + (z_e - c_e)\,dz_e]$$

(14) $$d\Phi_1 = \sum_e \left(\frac{\partial \Phi_1}{\partial x_e} dx_e + \frac{\partial \Phi_1}{\partial y_e} dy_e + \frac{\partial \Phi_1}{\partial z_e} dz_e \right),$$

and demands that the differential dG should be equal to the differential $d\Phi_1$, up to a finite factor for given values of the $3q$ variables x_e, y_e, z_e *independently of the values of the differentials* dx_e, dy_e, dz_e, one can then determine the combinations:

$$m_e\,(x_e - a_e), \qquad m_e\,(y_e - b_e), \qquad m_e\,(z_e - c_e)$$

for the $3q$ quantities a_e, b_e, c_e, which must be equal to the partial differential quotients:

$$\frac{\partial \Phi_1}{\partial x_e}, \quad \frac{\partial \Phi_1}{\partial y_e}, \quad \frac{\partial \Phi_1}{\partial z_e},$$

respectively, up to a factor that coincides throughout. By means of the expression:

(15) $$(1, 1) = \sum_e \frac{1}{m_e} \left[\left(\frac{\partial \Phi_1}{\partial x_e} \right)^2 + \left(\frac{\partial \Phi_1}{\partial y_e} \right)^2 + \left(\frac{\partial \Phi_1}{\partial z_e} \right)^2 \right],$$

that state of affairs can be represented by the equations:

(16) $$\left\{ \begin{aligned} \frac{m_e(x_e - a_e)}{\sqrt{2G}} &= \frac{\dfrac{\partial \Phi_1}{\partial x_e}}{\sqrt{(1,1)}}, \\ \frac{m_e(y_e - b_e)}{\sqrt{2G}} &= \frac{\dfrac{\partial \Phi_1}{\partial y_e}}{\sqrt{(1,1)}}, \\ \frac{m_e(z_e - c_e)}{\sqrt{2G}} &= \frac{\dfrac{\partial \Phi_1}{\partial z_e}}{\sqrt{(1,1)}}. \end{aligned} \right.$$

At the same time, one has the relation between the differentials dG and $d\Phi_1$ that has the prescribed behavior:

(17) $$\frac{dG}{\sqrt{2G}} = \frac{d\Phi_1}{\sqrt{(1,1)}}.$$

We shall now define the second complete differentials of the function G and the function Φ_1:

(18) $$d^2 G = \sum_e m_e \, [(x_e - a_e)\, d^2 x_e + (y_e - b_e)\, d^2 y_e + (z_e - c_e)\, d^2 z_e] + \sum_e m_e (dx_e^2 + dy_e^2 + dz_e^2)\,,$$

(19) $$d^2 \Phi_1 = \sum_e \left(\frac{\partial \Phi_1}{\partial x_e} d^2 x_e + \frac{\partial \Phi_1}{\partial y_e} d^2 y_e + \frac{\partial \Phi_1}{\partial z_e} d^2 z_e \right) + \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e \, \partial x_{e'}} dx_e \, dx_{e'} + \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e \, \partial y_{e'}} dx_e \, dy_{e'} + \cdots + \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial z_e \, \partial z_{e'}} dz_e \, dz_{e'}\,,$$

and then express the requirement that the differential $d^2 G$ must be equal to the differential $d^2 \Phi_1$, up to a finite factor, *for arbitrarily-varying values of the second differentials* $d^2 x_e$, $d^2 y_e$, $d^2 z_e$, *and arbitrary, but fixed, values of the first differentials* dx_e , dy_e , dz_e . Equation (18) is then converted into the *characteristic relation* (7) by dividing by dt^2. Equations (16) have the consequences that the first component of $d^2 G$ and first component of $d^2 \Phi_1$, which include the second differentials, have the desired character and that the former of them has the same relationship to the latter that the expression $\sqrt{2G}$ has to the expression $\sqrt{(1,1)}$. For that reason, the equation:

(20) $$\frac{d^2 G}{\sqrt{2G}} = \frac{d^2 \Phi_1}{\sqrt{(1,1)}}$$

must be true.

In order for our requirement to be fulfilled, it is necessary and sufficient that the second components of $d^2 G$ and the second component of $d^2 \Phi_1$, which are equal to quadratic forms in the $3q$ differentials dx_e , dy_e , dz_e , should likewise have that relationship. The equation thus arises:

(21) $$\frac{\sum_e m_e \, (dx_e^2 + dy_e^2 + dz_e^2)}{\sqrt{2G}} = \frac{\sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e \, \partial x_{e'}} dx_e \, dx_{e'} + \frac{\partial^2 \Phi_1}{\partial x_e \, \partial y_{e'}} dx_e \, dy_{e'} + \cdots + \frac{\partial^2 \Phi_1}{\partial z_e \, \partial z_{e'}} dz_e \, dz_{e'}}{\sqrt{(1,1)}}\,.$$

The $3q$ *quantities* a_e , b_e , c_e *will be determined completely by that equation, in conjunction with equations* (16), *for given values of the variables* x_e , y_e , z_e *and the differentials* dx_e , dy_e , dz_e , *so the system of values* (a_e , b_e , c_e) *that emerges will represent a generalization of the center of curvature.*

In the mechanical problem for which the force function U vanishes, and *only one* condition equation (4^*) is given, the equation (6) above will become the following one:

(22) $$\sum_e m_e \left(\frac{d^2 x_e}{dt^2} \delta x_e + \frac{d^2 y_e}{dt^2} \delta y_e + \frac{d^2 z_e}{dt^2} \delta z_e \right) = \lambda_1 \, \delta\Phi_1 \, .$$

Therefore, the expression $\lambda_1 \, \delta\Phi_1$ represents *the sum of the moments of all pressures* that are produced by the condition equation (4*). One can infer the following consequences from that equation:

(23) $$\frac{d\Phi_1}{dt} = 0, \qquad \frac{d^2\Phi_1}{dt^2} = 0 \, .$$

The choice of the differential quotients $\frac{dx_e}{dt}, \frac{dy_e}{dt}, \frac{dz_e}{dt}$ will be restricted by the first of those systems. One can employ the second one to represent the expression λ_1. When one divides the second differential $d^2\Phi_1$ that is written out explicitly in (19) by the quantity dt^2, the quantities $\frac{d^2 x_e}{dt^2}, \frac{d^2 y_e}{dt^2}, \frac{d^2 z_e}{dt^2}$ will contain the factors $\frac{\partial\Phi_1}{\partial x_e}, \frac{\partial\Phi_1}{\partial y_e}, \frac{\partial\Phi_1}{\partial z_e}$, while the corresponding quantities on the left-hand side of (22) will exhibit the factors $m_e \, \delta x_e$, $m_e \, \delta y_e$, $m_e \, \delta z_e$. As soon as those expressions are replaced with the expressions $\frac{\partial\Phi_1}{\partial x_e}, \frac{\partial\Phi_1}{\partial y_e}, \frac{\partial\Phi_1}{\partial z_e}$ in (22), the components in question will coincide, and at the same time, the expression:

$$\delta\Phi_1 = \sum_e \left(\frac{\partial\Phi_1}{\partial x_e} \delta x_e + \frac{\partial\Phi_1}{\partial y_e} \delta y_e + \frac{\partial\Phi_1}{\partial z_e} \delta z_e \right)$$

will be converted into the expression:

$$\sum_e \frac{1}{m_e} \left[\left(\frac{\partial\Phi_1}{\partial x_e} \right)^2 + \left(\frac{\partial\Phi_1}{\partial y_e} \right)^2 + \left(\frac{\partial\Phi_1}{\partial z_e} \right)^2 \right] = (1, 1) \, .$$

In that way, the determination of the quantity λ_1 will follow from (19) and (22):

(24) $$\lambda_1 \, (1, 1) = \sum \left(\frac{d^2 x_e}{dt^2} \frac{\partial\Phi_1}{\partial x_e} + \frac{d^2 y_e}{dt^2} \frac{\partial\Phi_1}{\partial y_e} + \frac{d^2 z_e}{dt^2} \frac{\partial\Phi_1}{\partial z_e} \right) - \frac{d^2\Phi_1}{dt^2} \, ,$$

which one can also give the form:

(24*) $$\lambda_1 \, (1, 1) = \sum_{e,e'} \frac{\partial^2\Phi_1}{\partial x_e \, \partial x_{e'}} \frac{dx_e}{dt} \frac{dx_{e'}}{dt} - \sum_{e,e'} \frac{\partial^2\Phi_1}{\partial x_e \, \partial y_{e'}} \frac{dx_e}{dt} \frac{dy_{e'}}{dt} - \ldots - \sum_{e,e'} \frac{\partial^2\Phi_1}{\partial z_e \, \partial z_{e'}} \frac{dz_e}{dt} \frac{dz_{e'}}{dt}$$

by another application of (19). A comparison of that result with the one in (21) will then produce the equation:

$$\frac{-1}{\sqrt{2G}} = \frac{\lambda_1 \sqrt{(1,1)}}{\sum_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2 \right]} . \tag{25}$$

That includes equation (12) *within it, and describes the connection between the function* $\sqrt{2G}$ *and the function* $\lambda_1 \sqrt{(1,1)}$ *, the first of which represents a generalization of the radius of curvature, while the second one represents a generalization of the concept of pressure.*

It can be noted in passing that when one chooses the function G itself, instead of the function Φ_1, in the variational problem that leads to equation (22), the relevant system of differential equations will belong to the category that was integrated completely in Journal f. Math., Bd. 72, pp. 38. At the same time, it will follow from what was done there that the form:

$$\sum m_e (dx_e^2 + dy_e^2 + dz_e^2) ,$$

which consists of $3q = n$ positive squares, will go to a form in $(n - 1)$ differentials when one variable is eliminated by using the equation $G = 1 / 2\alpha$, in place of (4^*), and that form will represent the square of the line element for the manifold of $(n - 1)$ remaining variables and with constant positive curvature α.

2.

From the ideas that were presented earlier, the concepts in mechanics and geometry that were just spoken of can be extended as follows: Let $x_{\mathfrak{a}}$ be a system of n variable quantities, in which the symbol $\mathfrak{a}$, like $\mathfrak{b}$, $\mathfrak{c}$, …, as well later on, runs through the numbers 1, 2, …, n. Let $f(dx)$ mean an essentially-positive form of degree p in the differentials $dx_{\mathfrak{a}}$, in which the coefficients depend upon the variables $x_{\mathfrak{a}}$ arbitrarily. Let the determinant of the second derivatives $\frac{\partial^2 f(dx)}{\partial dx_{\mathfrak{a}}\, \partial dx_{\mathfrak{b}}}$ be not equal to zero identically, and let U and $\Phi_1, \Phi_2, \ldots, \Phi_l$ be functions of the only variables $x_{\mathfrak{a}}$. One now demands that the variables $x_{\mathfrak{a}}$ should be made to depend upon an independent variable t in such a way that the first variation of the integral:

$$\int \left[f\left(\frac{dx}{dt}\right) + U + \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \cdots + \lambda_l \Phi_l \right] dt \tag{26}$$

will vanish for fixed initial and final values of the variables $x_{\mathfrak{a}}$, while the l equations:

(27) $$\Phi_\alpha = \text{const.}$$

must be fulfilled. That problem will be converted into the variational problem for the integral (5) when the n variables $x_\mathfrak{a}$ go to the $3q$ coordinates x_e , y_e , z_e , and the form $f(dx)$ goes to the form $\frac{1}{2}\sum m_e(dx_e^2 + dy_e^2 + dz_e^2)$. The functions U, Φ_1, …, Φ_l, and the multipliers to be determined λ_1, λ_2, …, λ_l are denoted in the same way in both cases.

We shall next consider the integral (26) under the assumptions that no conditions (27) are present, and that the function U is equal to zero. The demand that was just expressed will coincide with the other demand that the first variation of the integral:

(28) $$R = \int \sqrt{p\, f(dx)}$$

must be zero (*). The integration values $x_\mathfrak{a}$ that satisfy that requirement and are determined by the conditions that they must satisfy the equations $x_\mathfrak{a} = x_\mathfrak{a}(0)$ and $x'_\mathfrak{a} = x'_\mathfrak{a}(0)$ for a value $t = t_0$, in which the addition of a prime suggests differentiation with respect to the variable t and the constants $x_\mathfrak{a}(0)$ and $x'_\mathfrak{a}(0)$ are given arbitrarily, will be *functions of only the quantities* $x_\mathfrak{a}(0)$ *and the combinations* (**):

(29) $$x'_\mathfrak{a}(0)\,(t - t_0) = u_\mathfrak{a}$$

in this case. When the quantities $x_\mathfrak{a}(0)$ are constant and the combinations $u_\mathfrak{a}$ are variable, the latter will represent a *system of normal variables for the form* $f(dx)$ (***). The associated value of the integral R, when extended from the system $x_\mathfrak{a}(0)$ to the system $x_\mathfrak{a}$, will then be expressed by the equation (†):

(30) $$R^p = p\, f_0\,(u).$$

$f_0\,(u)$ emerges from the form $f(dx)$, when the relevant values $x_\mathfrak{a}(0)$ are substituted for the variables $x_\mathfrak{a}$, and the relevant values $u_\mathfrak{b}$ are substituted for the differentials $dx_\mathfrak{b}$. When one introduces the variables $u_\mathfrak{b}$ into the form $f(dx)$ that will yield the transformation equation:

(31) $$f(dx) = \varphi(du).$$

The resulting form of degree p in the differentials $du_\mathfrak{a}$, $\varphi(du)$ will be called a *normal type* for the form $f(dx)$.

(*) Journal f. Mathematik, Bd. 74, pp. 120, *et seq.*
(**) Journal f. Mathematik, Bd. 70, pp. 86, *et seq.*
(***) Journal f. Mathematik, Bd. 72, pp. 1, *et seq.*
(†) Journal f. Mathematik, Bd. 74, pp. 126.

When one introduces the normal variables u_α into the functions U, Φ_1, …, Φ_l will convert the more general integral to be varied (26) into the integral:

$$(26^*) \qquad \int \left[\varphi\left(\frac{dx}{dt}\right) + U + \lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \cdots + \lambda_l \Phi_l \right] dt\,.$$

Under the aforementioned special assumption that the coordinates x_e , y_e , z_e should enter in place of the variables x_α , the form:

$$\tfrac{1}{2} \sum m_e (dx_e^2 + dy_e^2 + dz_e^2)$$

will appear in place of $f(dx)$, and we imagine that the coordinates a_e , b_e , c_e will enter in place of the initial values $x_\alpha(0)$. Now, since the variational problem for the integral (2) will enter into the variational problem for the integral (28), and since the variational problem for the integral (2) will be solved by the advance of each mass-point m_e along a straight line with uniform velocity, and therefore by the equations:

$$\begin{aligned} x_e &= a_e + x_e'(0)\,(t - t_0), \\ y_e &= b_e + y_e'(0)\,(t - t_0), \\ z_e &= c_e + z_e'(0)\,(t - t_0), \end{aligned}$$

under the prevailing relationships, the normal variables u_α *will be nothing but the coordinate differences:*

$$x_e - a_e\,, \qquad y_e - b_e\,, \qquad z_e - c_e\,.$$

For that reason, the normal type $\varphi(dx)$ *will be equal to the given form:*

$$\tfrac{1}{2} \sum m_e (dx_e^2 + dy_e^2 + dz_e^2)$$

identically, and the function $2 f_0(u)$ *will coincide with the function:*

$$2G = \sum m_e \, [(x_e - a_e)^2 + (y_e - b_e)^2 + (z_e - c_e)^2]\,.$$

The variational problem for the integral (26^*) will generally imply the equation:

$$(32) \qquad \sum_\alpha \left[\frac{d}{dt} \frac{\partial \varphi(u')}{\partial u_\alpha'} - \frac{\partial \varphi(u')}{\partial u_\alpha} \right] \delta u_\alpha = \delta U + \lambda_1\, \delta\Phi_1 + \ldots + \lambda_l\, \delta\Phi_l\,,$$

which must be satisfied independently of the values of the variations δu_α . *We replace the variations* δu_α *with the normal variables themselves* u_α , *respectively, and obtain the equation:*

(33) $$\sum_\alpha \left[\frac{d}{dt}\frac{\partial\varphi(u')}{\partial u'_\alpha} - \frac{\partial\varphi(u')}{\partial u_\alpha}\right] u_\alpha = \sum_\alpha \frac{\partial U}{\partial u_\alpha} u_\alpha + \sum_{\alpha,\gamma} \lambda_\gamma \frac{\partial \Phi_\gamma}{\partial u_\alpha} u_\alpha \,.$$

Everything now comes down to showing how the expression that is found on the left-hand side of this equation is connected with the function $f_0\,(u)$. If one once more introduce the differentials du_α in place of the differential quotients u'_α in the normal type $\varphi\,(u')$ then one will define the identity relation:

(34) $$\sum_\alpha d\,\frac{\partial\varphi(du)}{\partial\, du_\alpha}\delta u_\alpha = d\sum_\alpha \left[\frac{\partial\varphi(du)}{\partial\, du_\alpha}\delta u_\alpha - \frac{\partial\varphi(\delta u)}{\partial\,\delta u_\alpha} du_\alpha\right]$$
$$+ \; d\sum_\alpha \frac{\partial\varphi(\delta u)}{\partial\,\delta u_\alpha} du_\alpha - \sum_\alpha \frac{\partial\varphi(du)}{\partial\, du_\alpha} d\delta u_\alpha \,,$$

which is valid for all systems du_α and δu_α. Now, the complete differential $df_0\,(u)$ can be represented in the following way, in which the substitution of the quantities u_α for the δu_α is suggested by enclosing the expression in question with square brackets (*):

(35) $$df_0\,(u) = \sum_\alpha \left[\frac{\partial\varphi(\delta u)}{\partial\,\delta u_\alpha}\right] du_\alpha \;.$$

The fact that $\varphi\,(du)$ is a homogeneous function of degree p in the differentials du_α further implies that:

(36) $$p\;\varphi\,(du) = \sum_\alpha \frac{\partial\varphi(du)}{\partial\, du_\alpha} du_\alpha \,.$$

When one substitutes u_α for δu_α in (34), that will yield the equation:

(37) $$\sum_\alpha d\,\frac{\partial\varphi(du)}{\partial\, du_\alpha} u_\alpha = d\left\{\sum_\alpha \left[\frac{\partial\varphi(du)}{\partial\, du_\alpha}\delta u_\alpha - \frac{\partial\varphi(\delta u)}{\partial\,\delta u_\alpha} du_\alpha\right]\right\} - d^2 f_0\,(u) - p\;\varphi\,(du),$$

which reveals the desired connection completely. As long as the number $p = 2$, the expression:

$$\sum_\alpha \left[\frac{\partial\varphi(du)}{\partial\, du_\alpha}\delta u_\alpha - \frac{\partial\varphi(\delta u)}{\partial\,\delta u_\alpha} du_\alpha\right]$$

will be equal to zero, from a basic property of quadratic forms. *Therefore, under the assumption that* $p = 2$, *one will have the characteristic relation:*

(*) Journal f. Mathematik, Bd. 72, pp. 8.

(38) $$\sum_{\mathfrak{a}} d\,\frac{\partial\varphi(du)}{\partial\, du_{\mathfrak{a}}}\,u_{\mathfrak{a}} = d^2 f_0(u) - 2\,\varphi(du)\,.$$

From now on, we will assume that this assumption has been made. *Hence, equation* (33) *will take on the definitive form:*

(39) $$\frac{d^2 f_0(u)}{dt^2} - 2\varphi\left(\frac{du}{dt}\right) = \sum_{\mathfrak{a}} \frac{\partial}{\partial u_{\mathfrak{a}}}\left[\varphi\left(\frac{du}{dt}\right) + U\right] u_{\mathfrak{a}} + \sum_{\mathfrak{a},\gamma} \lambda_\gamma \frac{\partial \Phi_\gamma}{\partial u_{\mathfrak{a}}}\, u_{\mathfrak{a}}\,,$$

when we apply (38). *The equation* (9) *above is included in this equation as a special case*, and it can be linked with considerations that are similar to the ones that were made in regard to that equation.

3.

In the case where the function U is equal to zero and one again sets the number $p = 2$, the variational problem of the integral (26) will become the one that was presented in (Journal f. Mathematik, Bd. 71, pp. 275). In agreement with the notations that are used there, let:

(40) $$f(dx) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}}\,,$$

and further:

(41) $$|\, a_{\mathfrak{a},\mathfrak{b}}\,| = \Delta, \qquad \frac{\partial\Delta}{\partial a_{\mathfrak{a},\mathfrak{b}}} = A_{\mathfrak{a},\mathfrak{b}}\,.$$

The functions $y_1, y_2, \ldots, y_l$, which were set equal to constant in that article, presently correspond to the functions $\Phi_1, \Phi_2, \ldots, \Phi_l$. The multipliers $\lambda_1, \lambda_2, \ldots, \lambda_l$ will be denoted by the same signs. We shall ponder the assumption that *only one* function Φ_1 is present. The vanishing of the first variation of the given integral (26) will then yield the equation:

(42) $$\sum_{\mathfrak{a}}\left(\frac{d}{dt}\frac{\partial f(x')}{\partial x'_{\mathfrak{a}}} - \frac{\partial f(x')}{\partial x_{\mathfrak{a}}}\right)\delta x_{\mathfrak{a}} = \lambda_1\,\delta\Phi_1\,.$$

In order to determine the expression λ_1 by means of the equations:

(43) $$\begin{cases} \dfrac{d\Phi_1}{dt} = \displaystyle\sum_{\mathfrak{c}} \frac{\partial\Phi_1}{\partial x_{\mathfrak{c}}}\frac{dx_{\mathfrak{c}}}{dt} = 0, \\[2ex] \dfrac{d^2\Phi_1}{dt^2} = \displaystyle\sum_{\mathfrak{c}} \frac{\partial\Phi_1}{\partial x_{\mathfrak{c}}}\frac{d^2 x_{\mathfrak{c}}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2\Phi_1}{\partial x_{\mathfrak{a}}\,\partial x_{\mathfrak{b}}}\frac{dx_{\mathfrak{a}}}{dt}\frac{dx_{\mathfrak{b}}}{dt} = 0, \end{cases}$$

which will be true from now on, one can introduce expressions in place of the variations $\delta x_{\mathfrak{a}}$ in (42) such that the factor of $\frac{d^2 x_{\mathfrak{c}}}{dt^2}$ on the left-hand side of that equation coincides with $\frac{\partial \Phi_1}{\partial x_{\mathfrak{c}}}$, which is the corresponding factor in the expression for $\frac{d^2 \Phi_1}{dt^2}$. Since the coefficient $a_{\mathfrak{a},\mathfrak{c}}$ of the form $2f(dx)$ appears on the left-hand side of (42) as a factor of the combination $\frac{d^2 x_{\mathfrak{a}}}{dt^2}\,\delta x_{\mathfrak{a}}$, the variation $\delta x_{\mathfrak{a}}$ must be replaced with the expression:

(44) $$\sum_{\mathfrak{k}} \frac{A_{\mathfrak{a},\mathfrak{k}}}{\Delta} \frac{\partial \Phi_1}{\partial x_{\mathfrak{k}}}$$

for the given purpose. With that substitution, the complete variation $\delta\Phi_1 = \sum_{\mathfrak{a}} \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} \delta x_{\mathfrak{a}}$ will go to the expression:

(45) $$\sum_{\mathfrak{a},\mathfrak{c}} \frac{A_{\mathfrak{a},\mathfrak{c}}}{\Delta} \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} \frac{\partial \Phi_1}{\partial x_{\mathfrak{c}}} = (1, 1),$$

and that will imply the following representation for λ_1 :

(46) $$\lambda_1\,(1, 1) = \sum_{\mathfrak{a}} \left(\frac{d}{dt} \frac{\partial f(x')}{\partial x'_{\mathfrak{a}}} - \frac{\partial f(x')}{\partial x_{\mathfrak{a}}} \right) \sum_{\mathfrak{k}} \frac{A_{\mathfrak{a},\mathfrak{k}}}{\Delta} \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} - \frac{d^2 \Phi_1}{dt^2},$$

which coincides with the representation that was given in the cited reference. At this point, it should be emphasized that when an arbitrary system of new independent variables is introduced in place of the system of variables $x_{\mathfrak{a}}$ in the relevant variational problem, the expression (1, 1), as well as the expression λ_1, will go to corresponding expressions that are constructed from the new elements (*). Therefore, if the previously-defined system of *normal variables* $u_{\mathfrak{a}}$ is introduced and one appeals to the notations:

(47) $$\begin{cases} \varphi(du) = \frac{1}{2} \sum_{\mathfrak{a},\mathfrak{b}} p_{\mathfrak{a},\mathfrak{b}}\, du_{\mathfrak{a}}\, du_{\mathfrak{b}}, \\ |\, p_{\mathfrak{a},\mathfrak{b}} \,| = \Pi, \quad \frac{\partial \Pi}{\partial p_{\mathfrak{a},\mathfrak{b}}} = P_{\mathfrak{a},\mathfrak{b}}, \end{cases}$$

for the *normal type* $\varphi(du)$, then (45) and (46) will imply the new equations:

(*) When the function Φ_1, which is to be set to a constant, is replaced with function of that function, which is to be set to a constant, the product $\lambda_1 \sqrt{(1,1)}$, which represents the generalization of the concept of pressure, will still remain invariant.

(48)
$$\sum_{\mathfrak{a},\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}} \frac{\partial \Phi_1}{\partial u_{\mathfrak{k}}} = (1, 1),$$

(49)
$$\lambda_1\,(1, 1) = \sum_{\mathfrak{a}} \left(\frac{d}{dt} \frac{\partial \varphi(u')}{\partial u'_{\mathfrak{a}}} - \frac{\partial \varphi(u')}{\partial u_{\mathfrak{a}}} \right) \sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}} - \frac{d^2 \Phi_1}{dt^2}.$$

The associated system of quantities $x_{\mathfrak{a}}$ (0) has a decisive meaning for the system of normal variables $u_{\mathfrak{a}}$. From the definition that is given in (29), all of the normal variables $u_{\mathfrak{a}}$ will vanish whenever the system of values $x_{\mathfrak{a}}$ satisfies the equations $x_{\mathfrak{a}} = x_{\mathfrak{a}}$ (0). Moreover, a manifold of first order that starts from the system of values $u_{\mathfrak{a}} = 0$ and for which the ratios of the variables $u_{\mathfrak{a}}$ to each other remain unchanged will satisfy the variational problem of the integral (28). The fact that, under the assumptions of article **1**, this first-order manifold represents nothing but the advance of each mass-point m_e from the position (a_e , b_e , c_e) along a straight line with uniform velocity has been mentioned numerous times. Just as the system of values a_e , b_e , c_e was determined before, we will now determine the system of values $x_{\mathfrak{a}}$ (0) by certain requirements, and indeed the explicit expression of the variables $x_{\mathfrak{a}}$ in terms of the variables $u_{\mathfrak{a}}$ and the fixed values $x_{\mathfrak{a}}(0)$ will not be required for that.

The first of those requirements points to the fact that the differential $df_0\,(u)$ must be equal to the differential $\delta\,\Phi_1$ for arbitrary values of the differentials $du_{\mathfrak{a}}$, up to a finite factor. Therefore, due to the relation (35), the quantities $\left[\frac{\partial \varphi(\delta u)}{\partial\,\delta u_{\mathfrak{a}}}\right]$ must have just the same relationship to the corresponding quantities $\frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}}$. Since, from (47), one has:

$$\frac{\partial \varphi(\delta u)}{\partial\,\delta u_{\mathfrak{a}}} = p_{\mathfrak{a},1}\ \delta u_1 + p_{\mathfrak{a},2}\ \delta u_2 + \ldots + p_{\mathfrak{a},\mathfrak{n}}\ \delta u_{\mathfrak{n}}\,,$$

the quantities $[\delta u_{\mathfrak{a}}] = u_{\mathfrak{a}}$ must have same relationship to the combination $\sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}}$, and the expression $\sqrt{[2\varphi(\delta u)]}$, which is equal to the expression $\sqrt{2\,f_0(u)}$ (*), must have the some relationship to the expression $\sqrt{(1,1)}$, which is defined by equation (48). On those grounds, the equations:

(50)
$$\frac{u_{\mathfrak{a}}}{\sqrt{2\,f_0(u)}} = \frac{\sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}}}{\sqrt{(1,1)}}$$

and the equation:

(*) Journal f. Mathematik, Bd. 72, pp. 7, formula (13).

(51) $$\frac{df_0(u)}{\sqrt{2 f_0(u)}} = \frac{d\Phi_1}{\sqrt{(1,1)}}$$

must be true. The second requirement, under which, the second differentials $d^2 u_\mathfrak{a}$ must vary independently, but the first differentials are regarded as chosen to be fixed, will be represented by the equation:

(52) $$\frac{d^2 f_0(u) - \sum_\mathfrak{a} \frac{\partial \varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a}}{\sqrt{2 f_0(u)}} = \frac{d^2\Phi_1}{\sqrt{(1,1)}}.$$

The same equation will be satisfied by the second differentials, since equation (51) is true. *The characteristic relation* (38), *which was appealed to for the intended conversion of* (33) *into* (39), *will also effect the intended conversion of* (52) *into the equation:*

(53) $$\frac{\sum_\mathfrak{a} d\,\frac{\partial \varphi(du)}{\partial\, du_\mathfrak{a}} u_\mathfrak{a} - \sum_\mathfrak{a} \frac{\partial \varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a} + 2\varphi(du)}{\sqrt{2 f_0(u)}} = \frac{d^2\Phi_1}{\sqrt{(1,1)}}.$$

An application of (50) will lead to the representation:

(54) $$\frac{2\varphi(du)}{\sqrt{2 f_0(u)}} = -\sum_\mathfrak{a}\left(d\,\frac{\partial \varphi(du)}{\partial\, du_\mathfrak{a}} - \frac{\partial \varphi(du)}{\partial u_\mathfrak{a}}\right) \frac{\sum_\mathfrak{k} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_\mathfrak{k}}}{\sqrt{(1,1)}} + \frac{d^2\Phi_1}{\sqrt{(1,1)}}.$$

Now, since the right-hand side of this equation, when divided by dt^2, *coincides with the right-hand side of* (49), *up to sign, that will yield the result that:*

(55) $$\frac{-1}{\sqrt{2 f_0(u)}} = \frac{\lambda_1 \sqrt{(1,1)}}{2\varphi(u')},$$

which includes equation (25) *as a special case.*

4.

The n quantities $x_\mathfrak{a}$ (0) that belong to the system of normal variables, are determined indirectly by equation (51), which represents a system of (n – 1) independent equation, due to the independence of the differentials $du_\mathfrak{a}$ and equation (55) that we just obtained. In order to get a direction determination, we turn our attention on the aforementioned first-order manifold that solved the variational problem for the integral (28) and extended from the system of values $u_\alpha = 0$ to the given system of values u_α that satisfies the equation Φ_1 = const. When that first-order manifold is referred to the variables x_α, it will extend from the system of values x_α (0) to the system of values x_α that satisfies the equations Φ_1 = const. and corresponds to the system of values u_α. Since $p = 2$, the associated values of the integral R in terms of normal variables u_α will be expressed by the equation:

$$R = \sqrt{2 f_0(u)}\,. \tag{56}$$

When the variables take on the increments Dx_α as one advances along the first-order manifold that was spoken of, the expression $\dfrac{df_0(u)}{\sqrt{2 f_0(u)}}$ will admit the following representation in terms of the variables x_α (*):

$$\frac{df_0(u)}{\sqrt{2 f_0(u)}} = \frac{\sum_\alpha \dfrac{\partial f(Dx)}{\partial Dx_\alpha} dx_\alpha}{\sqrt{2 f(Dx)}}\,. \tag{57}$$

The combination (1, 1) is expressed in terms of the variables x_α by equation (45). *We can therefore replace* (51) *with the equation:*

$$\frac{\sum_\alpha \dfrac{\partial f(Dx)}{\partial Dx_\alpha} dx_\alpha}{\sqrt{2 f(Dx)}} = \frac{d\Phi_1}{\sqrt{(1,1)}}\,. \tag{58}$$

The ratios of the differentials Dx_α will be determined by them; i.e., *the final element of the indicated first-order manifold will be determined in such a way that the final element* Dx_α *is normal to the manifold of order* $(n - 1)$ Φ_1 = const. *when one recalls the form* $2f(Dx)$, which is discussed in (Journal f. Math., Bd. 74, pp. 144).

The form $2\varphi(u')$ in equation (55) can also be replaced by the form $2 f(x')$ by means of (31), and will represent a combination in the system of values x_α and dx_α / dt by means of equations (45) and (46). *That will yield the equation:*

$$\frac{-1}{R} = \frac{\lambda_1 \sqrt{(1,1)}}{2 f(x')}\,, \tag{59}$$

which will imply the value of the integral R in the system of values x_α *and* dx_α / dt.

(*) Journal f. Mathematik, Bd. 74, pp. 128.

Equations (58) and (59) then determine the system of values $x_\mathfrak{a}$ (0) by the conditions that the first-order manifold that starts from it and makes the first variation of the integral (28) vanish will emerge from a given system of values $x_\mathfrak{a}$ that belongs to the manifold of order (n – 1), while the final element $Dx_\mathfrak{a}$ will be normal to that manifold of order (n – 1) relative to the form $2f(Dx)$, and that the associated integral R must assume the prescribed value. Conversely, if one imagines that the first-order manifold that is spoken of starts from the system of values $x_\mathfrak{a}$ then its evolution will be determined completely by that system and the element $Dx_\mathfrak{a}$, and the prescribed value of the integral R ultimately determine the system of values $x_\mathfrak{a}$ (0) in question (*). The possibility of that determination assumed in that. One easily recognizes that equations (58) and (59) have the property that when a new system of independent variables are introduced in place of the variables $x_\mathfrak{a}$ and also when the function Φ_1 that is to be set to a constant is replaced with a function of that function, those equations will go to *equations that are formed analogously from the new elements. The given determination is therefore completely independent of the choice of the form of the function* Φ_1 . As soon as only one mass-point is assumed in the considerations of article 1, the indicated first-order manifold will be the straight line that starts from the point (x_1, y_1, z_1) and points normally to the surface Φ_1 = const., and which cuts out the length of the radius of curvature that is determined from the point (x_1, y_1, z_1) to the center of curvature (a_1, b_1, c_1).

When one regards the quantities $x_\mathfrak{a}$ to be fixed and the quantities $dx_\mathfrak{a} / dt$ to be variable and, from (43), restricted by only the equation:

$$\frac{d\Phi_1}{dt} = 0,$$

and when one raises the question of *what system of values* $dx_\mathfrak{a} / dt$ *for the first differential in the expression that is defined in* (59) *will make* $1 / R$ *vanish*, one will have expressed *a maximum-minimum problem*, which emerges from *the general maximum-minimum problem* that is presented in (Journal f. Math., Bd. 71, pp. 277) and discussed under the assumption that $l = 1$. The results that were published in that reference for that problem are therefore applicable to the present problem with no further discussion. That sheds light upon the fact that the indicated problem will become the *problem of largest and smallest radii of curvature* in the aforementioned simplest case of article 1.

When one compares the more general results that were just found with the more specialized ones that were presented earlier, it must emerge that equations (39) and (52) contain one of the expressions:

$$\sum_\mathfrak{a} \frac{\partial \varphi\left(\frac{du}{dt}\right)}{\partial u_\mathfrak{a}} u_\mathfrak{a} \qquad \text{and} \qquad \sum_\mathfrak{a} \frac{\partial \varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a}$$

(*) Journal f. Mathematik, Bd. 74, pp. 130, *et seq.*

respectively, which do not enter into the corresponding equations (9) and (20). As was emphasized, *the form with constant coefficients* $\frac{1}{2}\sum_e m_e(dx_e^2 + dy_e^2 + dz_e^2)$ appears in the latter equations in place of the form $f(dx)$, so the *normal variables* u_α will go to *the differences* $(x_e - a_e)$, $(y_e - b_e)$, $(z_e - c_e)$, *the normal type* $\varphi(du)$ will coincide with the form $\frac{1}{2}\sum_e m_e(dx_e^2 + dy_e^2 + dz_e^2)$ *itself*, and *the function* $f_0(u)$ will become *the function G.* Under those circumstances, the *normal type* $\varphi(du)$ *will be a form with constant coefficients*, and whenever $\varphi(du)$ becomes a form with constant coefficients, the expression $\sum_\alpha \frac{\partial\varphi(du)}{\partial u_\alpha} u_\alpha$ must obviously vanish. However, it is also proved in (Journal f. Math., Bd. 70, pp. 92, *et seq.*) that *when the form* $f(dx)$ *can be transformed into a form with constant coefficients, the normal type* $\varphi(du)$ *will represent such a form, and that the expression* $\sum_\alpha \frac{\partial\varphi(du)}{\partial u_\alpha} u_\alpha$ *can vanish only when the form* $f(dx)$ *can be transformed into a form with constant coefficients.* Namely, the left-hand side of the equation that was denoted by (59) on (*loc. cit.*, pp. 94) will go to the expression $\sum_\alpha \frac{\partial\varphi(\delta u)}{\partial u_\alpha} u_\alpha$ when it is multiplied by $(t - t_0)$. *The necessary and sufficient condition for the vanishing of the expression* $\sum_\alpha \frac{\partial\varphi(du)}{\partial u_\alpha} u_\alpha$ *then consists of saying that the form* $f(dx)$ *can be transformed into a form with constant coefficients* (*). *Since it was initially demanded that the quadratic form* $f(dx)$ *must be essentially-positive and have a non-vanishing determinant, the normal type* $\varphi(du)$ *must have the same property, and when the form* $f(dx)$ *can be transformed into a form with constant coefficients, that normal type must necessarily equal an aggregate of squares of n differentials*. On those grounds, the assumption that are actually true in mechanics, which were founded in article 1, represent the most general situation under which an essentially-positive quadratic form $f(dx)$ is compatible with the vanishing of the expression $\sum_\alpha \frac{\partial\varphi(du)}{\partial u_\alpha} u_\alpha$.

When one lets the variables x_α coincide with the combinations $\sqrt{m_e}\,(x_e - a_e)$, $\sqrt{m_e}\,(y_e - b_e)$, $\sqrt{m_e}\,(z_e - c_e)$, the form $\frac{1}{2}\sum_e m_e(dx_e^2 + dy_e^2 + dz_e^2)$ will coincide with the form $\frac{1}{2}\sum_\alpha dx_\alpha^2$. It was already set down in (Journal f. Math., Bd. 71, pp. 284) how, under the assumption that $f(dx) = \frac{1}{2}\sum_\alpha dx_\alpha^2$, the theory of the function $\lambda_1\sqrt{(1, 1)}$ is very closely connected with the *extension of the theory of curvature* that **Kronecker** gave (Monatscbericht der Berliner Akademie, August 1869). In fact, the quantity that

(*) A *direct criterion* for the form $f(dx)$ to have that character is presented and proved in Journal f. Math., Bd. 70, pp. 94, *et seq.*

Kronecker called ρ coincides with the one that was called $\sqrt{2G}$ or R above, and the extension that we gave in article 1 for the concept of center of curvature differs from the one that **Kronecker** developed only by its connection with the mechanical representation and the choice of the steps that would lead to that objective. Moreover, the *maximum-minimum problem* that was suggested corresponds precisely to the one that **Kronecker** treated in the aforementioned place. However, in order to explain *why admission to those investigations is even possible when one starts in mechanics and geometry, and why the results of mechanics that are contained in equations* (9) *and* (39) *and the results of geometry that are based upon equations* (20) *and* (52) *can depend upon the same algorithms*, I would like to recall something that **Gauss** said in the paper "Beiträge zur Theorie der algebraischen Gleichungen" in regard to the manner by which one proves fundamental theorems about algebraic equations that nonetheless takes on a much broader sense. He said:

> "However, at its basis, the actual content of all argumentation belongs to a higher realm in which one studies general, abstract quantities that are independent of spatial ones, and in which one addresses those combinations of quantities that are connected with continuity, which is a realm that has been explored only slightly at this time, and in which one also cannot move without having a language that is borrowed from spatial structures."

Bonn, 16 February 1873.
