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PROGRESS IN THE PROJECTIVE THEORY OF RELATIVITY

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FOREWORD

The publication of this book was delayed by unfortunate circumstances; it was made possible by its acceptance by the publisher Friedr. Wieweg and Son. Nevertheless, nothing needs to be added at the moment. A recently-appearing paper by O. HECKMANN, P. JORDAN, and W. FRICKE ([1]) represents only the first part of a discussion (which is still in a state of flux) of the solutions of the general field equations that are presented in this pamphlet.

In this presentation, a brief overview will be given of the recent results in the projective theory of relativity that have come about since the appearance of the beautiful summary of O. VEBLEN ([2]) in the year 1933. The essential step in this further development was taken by P. JORDAN in the year 1944.

The general affine (i.e., EINSTEINIAN) theory of relativity will be assumed to be known. The projective theory will then be developed from the ground up, but with the inclusion of JORDAN's extension. The reader will find a thorough overview in the initial introductory section of the matter that is contained in the results of some papers by P. JORDAN, Cl. MÜLLER, and the author, as well as some unpublished results.

I would like to thank Herrn P. JORDAN most warmly for providing the impetus for this investigation, for the great interest with which he followed the progress of the work, and for stimulating the posing of interesting problems by worthwhile discussions and inspiration.

It is the author's hope, as well as his wish, that he might introduce this book into discussion in such a way that its sphere of interest might be enlarged by some relevant cosmological problems. Its implications extend to recent results, and perhaps even into the structure of elementary particles, whose properties seem to be, in part, a mirror image of the ambient matter.

Berlin, in May 1951.

Günther Ludwig.

([1]) **O. Heckmann**, **P. Jordan**, and **W. Fricke**, Zeit. Astrophysik **28** (1951), 113.
([2]) **O. Veblen**, *Projektive Relativitätstheorie*, Berlin, 1933.

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CHAPTER I

UNIFIED FIELD THEORIES AND COSMOLOGY

The projective theory of relativity arose from EINSTEIN's general theory of relativity as an attempt to solve the problem of finding a unified theory that subsumes gravitation and electromagnetism. Among other things, the great achievement of EINSTEIN's theory consisted of interpreting the gravitational field as the geometric structure of the universal continuum – i.e., the four-dimensional space-time manifold. However, it raised the question of whether it might not be possible to describe gravitation and electromagnetism together as a geometric structure of the universe.

H. WEYL was the first to discover a theory along those lines in the year 1918 with the mathematically beautiful theory that he presented in his book *Raum, Zeit, Materie* (*Space, Time and Matter*). Since that theory was perceived to be not entirely satisfying from a physical standpoint, further attempts at presenting a unified field theory had to be undertaken. One path, which we will not describe in our little book, was pursued by A. S. EDDINGTON in the year 1923, then by EINSTEIN, and now, in more recent times, by E. SCHRÖDINGER. It consists of an extension of WEYL's geometry, in which one starts with only affine geometry in order to find field quantities and field laws, and only later identifies certain field quantities with the metric. We will go into the connection between SCHRÖDINGER's theory and the one that is proposed here at the conclusion of our report.

In connection with our theory, we will be especially interested in an attempt to unify gravitation and electromagnetism that T. KALUZA made in the year 1921. KALUZA introduced a five-dimensional continuum and obtained the field equations for gravitation and electromagnetism from the five-dimensional metric. After O. KLEIN simplified that theory, it was soon examined many times, although the question of just what one meant by the fifth dimension was initially left open.

The explanation for it was then given by the work of O. VEBLEN, who could interpret the five-dimensional theory as four-dimensional projective relativity. In the year 1933, O. VEBLEN summarized the state of the theory thoroughly in his book *Projektive Relativitätstheorie*. One will also find a thorough bibliography in this book of the work that has appeared up to now, such that we will list only the recent works at the end of the present report, especially since direct reference to only those recent works will be made in the following presentation.

In the same year 1933, the thus-completed theory appeared in an especially symmetric and elegant representation in a paper by W. PAULI, in which the DIRAC equation was also presented in terms of the projective theory. The theory, thus-presented, was then applied many times, and in particular, by A. PAIS. He showed in that way that the theory of meson fields could likewise be represented elegantly in projective form, which was especially close to the "mixed" field theories (vector + scalar fields).

The theory experienced an entirely new extension by the ideas of P. JORDAN, by which one proposes an invariant $J = g_{\mu\nu} X^{\mu} X^{\nu}$ as a field function [cf., no. **5** ([1]) in the present report], which had been somewhat artificially set to 1 in the past. Those ideas

([1]) Boldfaced numbers will refer to the sections of Chapters II and III.

arose in connection with an inductive argument that was concerned with dimensional analysis and cosmology. Since that inductive theory appeared in a beautiful summary form in P. JORDAN's book *Die Herkunft der Sterne* (*The Birth of Stars*), we would like to go into it only briefly here.

If one introduces atomic units (or so-called "natural" units), instead of C.G.S. units, such that the speed of light $c = 1$, PLANCK's constant $\hbar$ $(= h / 2\pi) = 1$, and the elementary length l $(\approx 2 \times 10^{-13}$ cm$) = 1$, then one will get the gravitational constant κ $(= 8\pi f / c^2$, f = constant in Newton's law of gravitation), the radius of the universe R, the age of the universe A, and the total mass of the universe M, in orders of magnitude:

$$\left.\begin{aligned} \kappa &\simeq \gamma^{-1}, \\ R &\simeq \gamma, \\ A &\simeq \gamma, \\ M &\simeq \gamma^{-1}, \end{aligned}\right\} \quad \text{with} \quad \gamma = 10^{40}. \tag{I.1}$$

Due to these order of magnitude relations, one might regard γ as the age of the universe, which would also imply that κ and M are not constant, but time-varying. Empirically, the mass of a star proves to be:

$$M_{st} \simeq \gamma^{3/2}, \tag{I.2}$$

which leads up to the assumption that the mass increase that is established by (I.1) will be compensated by the creation of new stars of mass (I.2).

In order to look for a theory in which κ, as a result of (I.1), was not constant, but a field function, P. JORDAN chose the quantity above $J = g_{\mu\nu} X^\mu X^\nu$ in the projective theory, since it proved to be essentially equal to just κ.

The working-out of the new extended theory was inaugurated by some papers of P. JORDAN, and then, together with Cl. MÜLLER, it was propelled onward to the presentation of possible field equations in vacuum and a discussion of them. A continuation of it in the direction of the representation of matter fields and the ultimate formulation of the field equations, in particular, was given by the author in some papers in which he arrived at a basis for the relations (I.1) and (I.2) deductively.

The theory that will be presented in what follows is very formal, as any such geometric theory would be, so it is precisely our intention that we shall go beyond the formal mathematics, as much as possible, and to call upon physical experiments only to test the results or to decide between several equally-justified possibilities. Therefore, the first half of our presentation is dedicated entirely to establishing the mathematical foundations of the theory, without the previously-established physical concepts playing a role. It is then in the second half that the physical results and consequences will come to the foreground.

Let us now give a brief overview of the contents of the following sections:

Chapter II of this report contains the mathematical form of the theory. In it, we have trod a somewhat different path from the one that was chosen in W. PAULI's presentation and transferred to P. JORDAN's new theory, as well as a somewhat different path from the presentation that was given by P. JORDAN in a recent paper, which is also essentially more concise than the first presentation. We have done that in order to subsume spinors in the most natural way. No. **2** contains the basis for and deeper analysis of the isomorphism theorem (2.19) that P. JORDAN found, in which special attention will be given to the connection between gauge transformations and the homogeneity properties of the field functions. Nos. **4**, **5**, **6** give the path that the author pursued in order to represent the theory. No. **7** gives a brief introduction to spinors and recalls many of the ideas in the paper by W. PAULI that was cited above. In no. **8**, the symbol for infinitesimal transformations that will be used later will be given, and the representation of the rotation groups will be sketched out in no. **9**. The theory of parallel translation and differentiation of measurement that will be given in no. **10** will differ from the presentation that was given in the PAULI paper by the addition of an extra term for tensors that are not normal (as it will be defined in no. **2**) and by the calculation of the affine splitting using the methods that are presented in nos. **4** to **6**. The affine decomposition of the curvature tensor will be calculated by the same method in no. **11**, in which the relations (11.15) and (11.16), and (11.19) were already presented by P. JORDAN in the first paper. Nos. **12** and **13** contain the essential general theorems upon which the entire theory that follows will rest, and the derivation of equations (13.44) and (13.55) in those numbers goes back to the investigations of A. PAIS, as modified by the introduction of the normal domain as in no. **4**. No. **13** gives a deeper meaning to the isomorphism theorem that was explained in no. **2**.

Chapter III then gives the actual physical applications. In nos. **14** to **18**, the field equations for gravitation and electromagnetism are presented explicitly and solved for some cosmological models, which yields a basis for (I.1) and (I.2). Ansätze for $\mathfrak{G}$, according to (14.1), were considered by P. JORDAN and Cl. MÜLLER with $V(J) = 0$, $W(J) = 0$, and $U(J) = J^{\alpha}$, and were discussed for the case $\alpha = 0$ in the absence of matter fields, in particular. As will be shown in no. **17**, we believe that one must set $\alpha = 1/2$ and $W(J) = -\lambda J^{-2}$. In no. **19**, the meaning of the energy-impulse tensor, the charge-current vector, and the matter invariants will be examined in the simplest case of scalar matter fields, as well as the influence of a variable gravitational constant. The last section shows the beauty (but, at the same time, the limits) of the projective theory, in which electrons, nucleons, and meson fields can be represented elegantly. Those representations will be given for the theories with constant J of W. PAULI and A. PAIS.

CHAPTER II

MATHEMATICAL THEORY

1. Projective description of the world continuum. – The most important basic assumptions of the projective theory of relativity shall be summarized, without placing any value on axiomatic completeness.

Let the space-time manifold, in which physical events play out, and which will be briefly called "the world," be four-dimensional topological space. Its points can also be related to unique coordinate quadruples (x^1, x^2, x^3, x^4) then, and they will be called *affine coordinates*. Its points (viz., world points) can likewise be mapped to the rays λX^μ ($-\infty < l < +\infty$) of the five-dimensional manifold of quintuples (X^0, X^1, X^2, X^3, X^4). The X^μ then represent the projective coordinates of the world points.

We briefly denote the space of affine coordinates by *W*, and the space of projective coordinates by *V*. *W* will then be four-dimensional, while *V* is five-dimensional. In what follows, Latin indices will run from 1 to 4, while Greek ones will run from 0 to 4.

Any world point *P* is represented by x^k, as well as by X^μ. The x^k must then be functions of the X^μ, and indeed, in such a way that x^k must be unchanging along a ray λX^μ ($-\infty < l < +\infty$):

$$x^k = f^k(X^0, X^1, \dots, X^4) = f^k(\lambda X^0, \lambda X^1, \dots, \lambda X^4). \tag{1.1}$$

The x^k are homogeneous functions of degree zero then. One will then have ([1]):

$$x_{|\nu} X^\nu = 0. \tag{1.2}$$

In what follows, the continuity of all of the functions that appear and their derivatives (to the extent that they are used) will be assumed.

2. Transformations groups. – The choice of coordinates x^k, as well as X^μ, is arbitrary, to a certain extent. One can then use other coordinates x'^k in place of the x^k, which will be functions of the x^k, and conversely:

$$x'^k = x'^k(x^1, x^2, x^3, x^4), \qquad x^k = x^k(x'^1, x'^2, x'^3, x'^4). \tag{2.1}$$

All of these coordinate transformations define a group, which we would like to denote by $\mathfrak{G}_4$. Whether or not a physical system is described in the x^k coordinate system or that of the x'^k changes nothing in its intrinsic structure. We therefore demand that the description of physical objects should be invariant under the group $\mathfrak{G}_4$.

([1]) In what follows, for any function $f(X^0, \dots, X^4)$, one will have $f_{|\mu} = \partial f / \partial X^\mu$, and for any function $g(x^1, \dots, x^4)$, one will have $g_{|k} = \partial g / \partial x^k$.

We impose the requirement that the degree of homogeneous functions of X^μ should remain unchanged under another choice of coordinates for V. We shall thus consider only those transformations:

$$X'^\mu = X'^\mu (X^0, \ldots, X^4), \tag{2.2}$$

that are homogeneous of degree one the X^μ. The group of those transformations shall be denoted briefly by $\mathfrak{H}_5$. The new requirement of the projective theory of relativity, when compared to the affine theory of relativity, is then the invariance of the description of physical objects under the group $\mathfrak{H}_5$.

One can apply the transformations of $\mathfrak{H}_5$ and $\mathfrak{G}_4$ independently of each other, but the functional connection (1.1) between affine and projective coordinates will generally change under that. However, one can also couple the transformations of $\mathfrak{H}_5$ and $\mathfrak{G}_4$ with each other in such a way that a certain functional connection (1.1) will remain preserved; one will have:

$$x^k = f^k (X^0, \ldots, X^4) \qquad \text{and also} \qquad x'^k = f'^k (X'^0, \ldots, X'^4), \tag{2.3}$$

with the same functions f^k. Any transformation of $\mathfrak{H}_5$ will then generate precisely one transformation in $\mathfrak{G}_4$ such that $\mathfrak{H}_5$ will be mapped homomorphically onto $\mathfrak{G}_4$ in that way, as one easily establishes. As a result, $\mathfrak{H}_5$ must be isomorphic to the factor group of $\mathfrak{G}_4$ by the normal subgroup $\mathfrak{N}$ that consists of all transformations of $\mathfrak{H}_5$ that generate the identity transformation in $\mathfrak{G}_4$. Since the f^k are homogeneous of degree zero, they are the transformations:

$$X'^\mu = \lambda (X^0, \ldots, X^4)\, X^\mu, \tag{2.4}$$

in which λ is a homogeneous function of degree zero. One then has:

$$\mathfrak{G}_4 \cong \mathfrak{H}_5 / \mathfrak{N}. \tag{2.5}$$

If a function of five variables is given:

$$F (X^0, \ldots, X^4)$$

then we would like to define a transformation $\mathfrak{T}_\rho$ in such a way that we set:

$$\mathfrak{T}_\rho F (X^0, \ldots, X^4) = F (\rho X^0, \ldots, \rho X^4), \tag{2.6}$$

in which $\rho (X^0, \ldots, X^4)$ is a homogeneous function of degree zero in the X^ν. The $\mathfrak{T}_\rho$ define an Abelian group that is briefly called $\mathfrak{P}$. For a homogeneous function of degree n, one will have:

$$\mathfrak{T}_\rho F = \rho^n F. \tag{2.7}$$

From EULER's theorem, one will then also have ([1]):

$$F_{|\mu} X^{\mu} = n F. \tag{2.8}$$

In particular, one will then have:

$$\left.\begin{array}{ll} \mathfrak{T}_\rho x^k = x^k, & x^k_{|\nu} = 0, \\ \mathfrak{T}_\rho X'^\mu = \rho X'^\mu, & X'^\mu_{|\nu} X^\nu = X'^\mu. \end{array}\right\} \tag{2.9}$$

The important concepts of "scalars" and "invariants" are defined by saying that φ is called a *scalar* when φ is invariant under the transformations of $\mathfrak{H}_5$, and ψ is an *invariant* when ψ is a scalar that is also invariant under $\mathfrak{P}$.

Let K be the field of real numbers. Furthermore, one might be given a K-module $\mathfrak{M}$ ([2]) with elements α, β, γ, ... We refer to a system of quantities α^ν in M that transform under the transformation (2.3) of $\mathfrak{H}_5$ by way of:

$$\alpha'^\nu = \alpha^\mu X'^\nu_{\ |\mu}, \tag{2.10}$$

as a *contravariant vector* at a point Q in V. Correspondingly, we refer to a *covariant vector* by way of β_ν with:

$$\beta'_\nu = \beta_\mu X^\mu_{\ |\nu'}, \qquad [\beta_\nu = \beta'_\mu X'^\mu_{\ |\nu'}, \text{ resp.}]. \tag{2.11}$$

Later on, we shall be concerned with the following special examples of $\mathfrak{M}$, among others: $\mathfrak{M} = K$ or $\mathfrak{M} = K(i)$, which is the field of complex numbers.

The contravariant, as well as the covariant, vectors each define a five-dimensional vector space that depends upon the point Q, which we would like to call *concomitant vector spaces*.

A *t-fold contravariant* and *r-fold contravariant tensor* of *rank* $(r + t)$ is given by quantities $\alpha^{\nu_1\cdots\nu_t}_{\mu_1\cdots\mu_r}$ that transform like the formal products of t contravariant and r covariant vectors. A scalar will also be called a *tensor of rank zero*. One can reduce the rank of a tensor by 2 by *contraction*. One understands a contracted tensor to mean a tensor whose components arise by summing over an identical pair of contravariant and covariant indices:

([1]) In what follows, identical indices will always be summed over.

([2]) The elements of $\mathfrak{M}$ define an additive Abelian group; i.e., $\alpha + \beta = \beta + \alpha$ is again an element of $\mathfrak{M}$, one has $(\alpha + \beta) + \gamma = \beta + (\alpha + \gamma)$, there is a null element 0 with $\alpha + 0 = \alpha$, and any α has an inverse $(-\alpha)$, such that $\alpha + (-\alpha) = 0$. Above and beyond that, if $a, b, c, \ldots$ are real numbers then αa will be an element of $\mathfrak{M}$ and $(\alpha + \beta)\, a = \alpha a + \beta a$, $\alpha (a + b) = \alpha a + \alpha b$, and $\alpha (ab) = (\alpha a)\, b$. For the first part of this report, it will suffice to define $\mathfrak{M}$ under the fields of complex or real numbers. The general formulation was chosen although we will use it in only Chapter II.

For the concept of a module, cf., e.g., van der WAERDEN, *Moderne Algebra*, 2nd ed., (v. 2), Berlin, 1940, pp. 98, *et seq.*

$$\beta^{\nu_1\cdots\nu_{\sigma-1}\nu_{\sigma+1}\cdots\nu_t}_{\mu_1\cdots\mu_{\rho-1}\mu_{\rho+1}\cdots\mu_r} = \alpha^{\nu_1\cdots\nu_{\sigma-1}\eta\nu_{\sigma+1}\cdots\nu_t}_{\mu_1\cdots\mu_{\rho-1}\eta\mu_{\rho+1}\cdots\mu_r} . \tag{2.12}$$

The contraction of a second-rank tensor is therefore a scalar. Addition of tensors of the same type will again give a tensor of the same type. The multiplication of two tensors $\alpha^{\nu_1\cdots\nu_t}_{\mu_1\cdots\mu_r}$ and $\beta^{\rho_1\cdots\rho_s}_{\sigma_1\cdots\sigma_p}$ will yield an $(t + s)$-fold contravariant and $(r + p)$-fold covariant tensor:

$$\alpha^{\nu_1\cdots\nu_t}_{\mu_1\cdots\mu_r}\,\beta^{\rho_1\cdots\rho_s}_{\sigma_1\cdots\sigma_p} = \gamma^{\nu_1\cdots\nu_t\,\rho_1\cdots\rho_s}_{\mu_1\cdots\mu_r\,\sigma_1\cdots\sigma_p} .$$

[In this, it is generally assumed that the product of the two quantities α and β is defined. For example, α and β can be real or complex numbers, or α can come from a module $\mathfrak{M}$, while β is a real number, or α and β are both from a module $\mathfrak{M}$, and $\alpha\,\beta$ can be a real number, such as the inner product of two vectors, or $\alpha\,\beta$ can be a formal product, which will then belong to the product module $\mathfrak{M} \times \mathfrak{M}$ ([1]).]

If a tensor field $\alpha^{\cdots}_{\cdots}$ is defined on V then the application of the operator $\mathfrak{T}_\rho$ to the tensor $\alpha^{\cdots}_{\cdots}$ will also be defined. We write it in the form:

$$\mathfrak{T}_\rho\,\alpha^{\nu_1\cdots\nu_t}_{\mu_1\cdots\mu_r} = \rho^{t-r} H_\sigma \alpha^{\nu_1\cdots\nu_t}_{\mu_1\cdots\mu_r} , \tag{2.13}$$

in which H_σ is an operator that is defined by that. H_σ is then invariant under the operation of contraction, as one easily confirms. Furthermore, the operators H_σ define a representation of the group $\mathfrak{P}$, since $H_{\rho_1\rho_2} = H_{\rho_1} H_{\rho_2}$. We define the infinitesimal transformation H by way of the equation:

$$H_{1+\varepsilon\lambda} = 1 + \varepsilon\,\mathfrak{P}\,\lambda, \tag{2.14}$$

which is true up to terms of order one in ε, which means that (2.14) is linear in λ. Since $\mathfrak{P}$ is Abelian, one can write:

$$H_\rho = e^{\Pi \ln \rho}. \tag{2.15}$$

Two important examples of this are:

1. $H_\rho = 1$. In this case, we call the tensors *normal tensors.* A normal tensor of rank one is also called a *normal vector*, and a normal tensor of rank zero is identical with the concept of an invariant.

2. $\mathfrak{M} = K(i)$ and $H_\rho = e^{i\,l \ln \rho}$, with l real. In this case, one then has $\Pi = i\,l$.

([1]) See remark ([2]) on page 6.

Any tensor that is not normal can be normalized. One chooses an arbitrary function η $(X^0, \dots, X^4)$ that is homogeneous of degree one. One then defines the tensor:

$$\alpha^{\cdots}_{\cdots} = H_\eta^{-1}\alpha^{\cdots}_{\cdots} \qquad [\alpha^{\cdots}_{\cdots} = H_\eta\, \underline{\alpha}^{\cdots}_{\cdots}\text{, resp.}] \tag{2.16}$$

with $H^{\eta-1} = H_\eta^{-1} = e^{-\Pi \ln \eta}$.

$$\begin{aligned}\mathfrak{T}_\rho\, \underline{\alpha}^{\cdots}_{\cdots} &= (\mathfrak{T}_\rho\, H_{\eta^{-1}})(\mathfrak{T}_\rho\, \alpha^{\cdots}_{\cdots}) = H_{\rho^{-1}\eta^{-1}}(\mathfrak{T}_\rho\, \alpha^{\cdots}_{\cdots}) \\ &= \rho^{t-r} H_{\rho^{-1}\eta^{-1}} H_\rho\, \alpha^{\cdots}_{\cdots} = \rho^{t-r} H_{\eta^{-1}}\, \alpha^{\cdots}_{\cdots} = \rho^{t-r}\, \underline{\alpha}^{\cdots}_{\cdots},\end{aligned}$$

such that $\underline{\alpha}^{\cdots}_{\cdots}$ will then be a normal tensor. However, the normalization $\underline{\alpha}$ of α is not defined uniquely, since η is arbitrary; it is undetermined up to a factor that is homogeneous of degree zero. If we then go from η to η' by way of:

$$\eta' = \sigma\,\eta \tag{2.17}$$

then $\underline{\alpha}$ will go to $\underline{\alpha}'$:

$$\underline{\alpha}' = H_{\eta'^{-1}}\, \alpha = H_{\sigma^{-1}}\, H_{\eta^{-1}} \alpha = H_\sigma^{-1}\, \underline{\alpha}. \tag{2.18}$$

We call the transformations (2.17), (2.18) the *gauge group* $\mathfrak{E}$. The coupling of (2.17), (2.18) yields an isomorphic map of $\mathfrak{P}$ onto $\mathfrak{E}$. $\mathfrak{E}$ itself is an affine group, since σ (which is homogeneous of degree zero in the X^ρ) depends upon only the affine coordinates. The two groups $\mathfrak{G}_4$ and $\mathfrak{E}$ together generate a group of affine transformations $(\mathfrak{G}_4, \mathfrak{E})$, which we can also regard as the transformation group of variables x^k, η. One has:

$$(\mathfrak{G}_4, \mathfrak{E}) \cong \mathfrak{H}_5\,. \tag{2.19}$$

In order to exhibit the isomorphic map of $\mathfrak{H}_5$ onto $(\mathfrak{G}_4, \mathfrak{E})$, we start with the homomorphic map of $\mathfrak{H}_5$ onto $\mathfrak{G}_4$ that was defined already in the beginning by (2.3). To the demand of the preservation of the functional connection (2.3), we add the preservation of the functional connection for the function η:

$$\eta = f(X^0, \dots, X^4) \qquad \text{and} \qquad \eta' = f(X'^0, \dots, X'^4), \tag{2.20}$$

by which the transformations of $\mathfrak{H}_5$ also transform η. Since the equations (2.3), (2.20) can be solved for the X^ν uniquely, $\mathfrak{H}_5$ will then be mapped isomorphically to a certain transformation group of the variables x^k, η that is precisely the group $(\mathfrak{G}_4, \mathfrak{E})$, as we shall soon show. If we consider the transformations of $\mathfrak{H}_5$ that remain invariant under that isomorphic map η then they will define a subgroup of $\mathfrak{H}_5$ that we would like to call $\mathfrak{H}_4$, and which is therefore mapped isomorphically to $\mathfrak{G}_4$. The transformations of the normal subgroup $\mathfrak{N} \subset \mathfrak{H}_5$ will be mapped isomorphically to $\mathfrak{E}$. Hence $(\mathfrak{H}_4, \mathfrak{N})$ will be mapped isomorphically to $(\mathfrak{G}_4, \mathfrak{E})$. However, one has $(\mathfrak{H}_4, \mathfrak{N}) = \mathfrak{H}_5$, so each element of $\mathfrak{H}_5$ will belong to a coset $t\,\mathfrak{N}$, in which t can be chosen from $\mathfrak{H}_4$, because every coset contains one (and only one) element of $\mathfrak{H}_4$, since all of the elements of $t\,\mathfrak{N}$ will be mapped to the same transformation of $\mathfrak{G}_4$, and since $\mathfrak{H}_4 \cong \mathfrak{G}_4$, every element of $\mathfrak{G}_4$ will correspond to one and only element of $\mathfrak{H}_4$.

With that, the structure of the group $(\mathfrak{G}_4, \mathfrak{E})$ is revealed, and linked with the projective groups $\mathfrak{H}_5$ and $\mathfrak{P}$.

To conclude this section, let us cite two examples. It follows from (2.11) and (2.9) that the X^μ define a contravariant normal vector. (The affine coordinates x^k do not define an affine vector!) The differentials dX^ν transform under $\mathfrak{H}_5$ according to:

$$dX'^\nu = X'^\mu{}_{|\nu}\, dX^\nu, \tag{2.21}$$

and thus define a vector. However, it is not normal, since one will have:

$$\mathfrak{T}_\rho\, dX^\mu = d(\rho X^\mu) = d\rho\, X^\mu + \rho\, dX^\mu = \rho\,(dX^\mu + X^\mu (\ln \rho)_{|\nu}\, dX^\nu),$$

under $\mathfrak{P}$, such that one will have for dX^μ that:

$$H_\rho\, dX^\mu = [\,\delta^\mu_\nu + X^\mu (\ln \rho)_{|\nu}]\, dX^\nu. \tag{2.22}$$

From (2.14), one has:

$$\Pi\,(\lambda\, dX^\mu) = X^\mu\, \lambda_{|\nu}\, dX^\nu.$$

3. Transition from the projective description to the affine one. – Covariant and contravariant vectors were defined in the previous section. Likewise, one can also define covariant and contravariant affine vectors α_k (β^k, resp.) by demanding that they should transformation under $\mathfrak{G}_4$ like:

$$\alpha_i = \alpha'_k\, x'^k_{|i}, \qquad \beta'^k = \beta^i\, x'^k{}_{|i}. \tag{3.1}$$

An affine vector can also be represented by projective components:

$$\alpha_\nu = \alpha_k\, x^k{}_{|\nu}. \tag{3.2}$$

We would like to regard α_ν and α_k as different components of the same vector. To abbreviate, we set:

$$x^k{}_{|\nu} = g^k_\nu. \tag{3.3}$$

The set of all vectors of the form (3.2) defines a four-dimensional subspace of the five-dimensional vector space of covariant projective vectors. In order to find a corresponding coupling of contravariant vectors, we set:

$$\beta^\mu = \beta^k g_k^\mu \tag{3.4}$$

for a contravariant affine vector β^k and unknown coefficients and demand that:

$$\alpha_k \beta^k = \alpha_\mu \beta^\mu = \alpha_k g_\mu^k \beta^l g_l^\mu .$$

It will then follow from this that:

$$g_\mu^k g_l^\mu = \delta_l^k . \tag{3.5}$$

The g_k^ν can be determined from this, up to a multiple of the solutions to the homogeneous equations:

$$g_k^\nu = \overset{\circ}{g}{}_k^\nu + \lambda_k X^\nu . \tag{3.6}$$

The factor can be established with no arbitrariness only later by an additional demand.

With the given definitions, the four-dimensional affine concomitant vector space defines a subspace of the five-dimensional projective concomitant spaces. Any vector in that four-dimensional subspace can be represented by affine components, as well as projective ones, if equations (3.2) and (3.4) can be solved for α_ν (β^μ, resp.) on the basis of the relation (3.5):

$$\alpha_k = \alpha_\nu g_k^\nu , \qquad\qquad \beta^k = \beta^\mu g_\mu^k . \tag{3.7}$$

For an arbitrary projective normal vectors α_ν (β^μ, resp.), one can likewise define the vectors α_k (β^k, resp.) from (3.7). However, they are not identical with the vectors α_ν (β^μ, resp.) that were given originally, but rather one has that $\alpha_\nu - \alpha_k g_\mu^k$ ($\beta^\mu - \beta^k g_k^\nu$, resp.) is equal to zero only when α_ν (β^μ, resp.) lies in the four-dimensional affine subspace. For arbitrary vectors, one calls the vectors that are defined by (3.7) *affine reductions.* The projective components of the affine reductions of α_ν (β^μ, resp.) are then:

$$\bar{\alpha}_\nu = \alpha_\mu g_k^\mu g_\nu^k \qquad \text{and} \qquad \bar{\beta}^\mu = \beta^\nu g_\nu^k g_k^\mu . \tag{3.8}$$

To abbreviate, we set:

$$d_\nu^\mu = g_k^\mu g_\nu^k . \tag{3.9}$$

The tensor d_ν^μ then represents the projection of the five-dimensional vector space onto its four-dimensional affine subspace by way of (3.8).

One has $X^\nu g_\nu^k = 0$ for the vector X^ν, such that X^ν has no component in the affine subspace. One also has $\bar{X}^\mu = X^\nu d_\nu^\mu = 0$ then.

One can define the reduction of normal tensors as one does for normal vectors; e.g.:

$$\alpha_i^k = \alpha_\nu^\mu \, g_i^\nu \, g_\mu^k$$

is the reduction of α_ν^μ. However, if a non-normal tensor is given – e.g., β_μ^ν – then the expression $\beta_\mu^\nu \, g_k^\mu g_\nu^i$ will not be homogeneous of degree zero in the X^ν, and it will therefore not be a function of only the affine x_k. Nevertheless, one can also get an affine tensor from β_μ^ν as its reduction when one first normalizes β_μ^ν and then reduces it as above. One will then obtain affine, gauge-invariant tensors from projective normal tensors, but one will get affine tensors that are not gauge-invariant from projective tensors that are not normal.

As an example, we consider the reduction of dX^ν. From (2.22), dX^ν is not a normal vector. The normalization of dX^ν is:

$$\underline{dX}^\mu = dX^\mu - X^\mu (\ln \eta)_{|\nu} \, dX^\nu. \tag{3.10}$$

One then obtains the reduction as:

$$(\underline{dX})^k = \underline{dX}^\mu \, g_\mu^k = dX^\mu \, g_\mu^k = dX^\mu \, x_{|\mu}^k = dx^k. \tag{3.11}$$

The projective components of dx^k are then:

$$\overline{dX}^\mu = d_\nu^\mu \, \underline{dX}^\nu = dx^k g_k^\nu . \tag{3.12}$$

We must then distinguish between the following three vectors: dX^μ, $\underline{dX}^k$, and $\overline{dX}^\mu$. The first of them is not normal, while the second and third ones are normal. The third one is the projection of the second one onto the four-dimensional affine subspace. The third one can then be represented by its projective components $\overline{dX}^\mu$, as well as by its affine ones dx^k.

4. Five-dimensional and four-dimensional integrals. – A point set in W that possesses a three-dimensional hypersurface with a well-defined normal direction as its boundary will be referred to as a *world-domain*, or briefly a *domain*. One can likewise define domains in V. However, we would like to consider only entirely special domains in V that we will call *normal domains*. A normal domain contains either no points at all of each ray λX^μ ($-\infty < \lambda < +\infty$) or exactly those points of an interval $\lambda_1 \leq \lambda \leq \lambda_2$ with $\ln(\lambda_1 / \lambda_2) = 1$ (i.e., $\lambda_2 = e\,\lambda_1$), and only those points. Any normal domain determines a unique world-domain. The points of a ray in V always correspond to a world-point then. An additive measure of a point set that is invariant under $\mathfrak{H}_5$ and $\mathfrak{P}$ is defined along a ray of V by $m(\lambda_1 / \lambda_2) = \ln(\lambda_1 / \lambda_2)$ as a measure of the interval from $\lambda_1 X^\mu$ to $\lambda_2 X^\mu$. When one establishes a normal domain, the measure of the points in the normal domain that corresponds to a single point of W will be equal to unity.

The volume element in V is defined by:

$$d\tau = dX^0\, dX^1\, dX^2\, dX^3\, dX^4. \tag{4.1}$$

For $\mathfrak{H}_5$, one has:

$$d\tau' = \| X'^{\mu} | \nu \| \, d\tau. \tag{4.2}$$

For $\mathfrak{P}$, one has:

$$\mathfrak{T}_\rho\, d\tau = \| \rho\, \delta^\mu_\nu + \rho_{|\nu} X^\mu \| \, d\tau.$$

The determinant can be calculated on the boundary:

$$\| \rho\, \delta^\mu_\nu + \rho_{|\nu} X^\mu \| = \rho^5 \| \delta^\mu_\nu + \rho^{-1} \rho_{|\nu} X^\mu \| = \rho^5 \begin{Vmatrix} (\delta^\mu_\nu + \rho^{-1}\rho_{|\nu} X^\mu) & X^\mu \\ 0 & 1 \end{Vmatrix}$$

$$= \rho^5 \begin{Vmatrix} \delta^\mu_\nu & X^\mu \\ -\rho^{-1}\rho_{|\nu} & 1 \end{Vmatrix} = \rho^5 \begin{Vmatrix} \delta^\mu_\nu & X^\mu \\ -\rho^{-1}\rho_{|\nu} & 1 + \rho^{-1}\rho_{|\nu} X^\nu \end{Vmatrix} = \rho^5 (1 + \rho^{-1} \rho_{|\nu} X^\nu).$$

Since ρ is homogeneous of degree zero, it follows that:

$$\mathfrak{T}_\rho\, d\tau = \rho^5\, d\tau, \tag{4.3}$$

such that $d\tau$ transforms like a normal tensor of rank five that is contravariant and antisymmetric in all indices.

One refers to $\mathfrak{t}^{\cdots}_{\cdots}$ as a tensor density when $\mathfrak{t}^{\cdots}_{\cdots}\, d\tau$ is a tensor. $\mathfrak{L}\, d\tau$ is then an invariant for an invariant density $\mathfrak{L}$. One then has that when

$$W = \int \mathfrak{L}\, d\tau \tag{4.4}$$

is taken over a normal domain in V, it will be an absolute invariant.

If one introduces the affine coordinates x^k into (4.4) as the new integration variables and the homogeneous function of the X^ν of degree one η that was employed in no. **2** then (4.4) will go to:

$$W = \int \mathfrak{L} \frac{1}{\Delta} d\bar{\overset{4}{\tau}}\, d\eta, \tag{4.5}$$

in which $d\bar{\overset{4}{\tau}} = dx^1\ dx^2\ dx^3\ dx^4$ is the volume element in the world-domain that corresponds to the normal domain, and Δ is equal to the functional determinant:

$$\Delta = \frac{\partial(\eta, x^1, \ldots, x^4)}{\partial(X^0, \ldots, X^4)} = \begin{Vmatrix} (\eta_{|\mu}) \\ (g^k_\mu) \end{Vmatrix}. \tag{4.6}$$

If one adds suitable multiples of the last four rows to the first one then one will get:

$$\Delta = \frac{1}{\Delta}\begin{Vmatrix} \eta_{|0}X^0 & \eta_{|1}\cdots\eta_{|4} \\ (g_0^k X^0 & g_1^k \cdots g_4^k) \end{Vmatrix} = \frac{1}{X^0}\begin{Vmatrix} \eta_{|\nu}X^\nu & \eta_{|1}\cdots\eta_{|4} \\ (g_\nu^k X^\nu & g_1^k \cdots g_4^k) \end{Vmatrix} = \frac{\eta}{X^0}\left\Vert g_1^k \cdots g_4^k \right\Vert. \tag{4.7}$$

One will then have $g_\nu^k X^\nu = x^k_{|\nu} X^\nu = 0$ and $\eta_{|\nu} X^\nu = \eta$. If one denotes the minors of $\eta_{,\nu}$ in (4.6) by Δ_ν then since the zeroth row is not distinguished from the other ones, one will have:

$$\Delta = \eta\,\frac{\Delta_0}{X^0} = \eta\,\frac{\Delta_1}{X^1} = \ldots = \eta\frac{\Delta_4}{X^4} = \eta\,\sigma, \tag{4.8}$$

in which σ transforms under $\mathfrak{G}_4$ like an antisymmetric, contravariant tensor of rank four – i.e., like $d\overset{\tilde{4}}{\tau}$ – and like an antisymmetric, contravariant, normal tensor of rank five under $\mathfrak{H}_5$ and $\mathfrak{P}$. σ is therefore homogeneous of degree – 5. Substituting (4.8) in (4.5) will yield:

$$W = \int \frac{\mathfrak{L}}{\sigma}\, d\overset{\tilde{4}}{\tau}\, d\ln\eta. \tag{4.9}$$

$\mathfrak{L} / \sigma$ is then homogeneous of degree zero, and therefore a function of only the x^k, but not η, such that integration over η in (4.9) can be performed. However, since the integral $\int d\ln\eta$ for any normal domain is equal to exactly the measure of the points along a ray that belong to that normal domain, and that will be equal to unity, it will follow that:

$$\left.\begin{aligned} W &= \int_{B_5} \mathfrak{L}\,d\tau = \int_{B_4} \frac{\mathfrak{L}}{\sigma}\,d\tau, \\ \Delta_\nu &= \sigma X^\nu; \quad \Delta_\nu = (-1)^\nu \left\Vert g_0^k \cdots g_{\nu-1}^k g_{\nu+1}^k \cdots g_4^k \right\Vert \end{aligned}\right\} \tag{4.10}$$

In this, B_4 is the four-dimensional world-domain that corresponds to the normal domain B_5. It follows from (4.7) that:

$$\sigma = \frac{1}{\eta}\Delta = \frac{1}{\eta}\begin{Vmatrix} (\eta_{|\nu}) \\ (g_\nu^k) \end{Vmatrix}, \tag{4.11}$$

and that the right-hand side of (4.11) is independent of the choice of the homogeneous function of degree one η; σ is gauge-invariant.

If F is any function of the X^μ that is homogeneous of degree – 4 then $F_{,\mu}$ will be homogeneous of degree – 5, and $F_{|\mu}\, d\tau$ will be homogeneous degree zero. When the integral $\int F_{|\mu}\, d\tau$ is taken over a normal domain, it can be converted into a boundary integral. The boundary of a normal domain B can be decomposed into three parts in the following sense: The intersection of a ray λX^μ with the boundary of B (when it is not empty) consists of either just two points with the coordinates $\underline{X}^\mu$ and $\bar{X}^\mu = e\,\underline{X}^\mu$ or an entire interval Λ of measure unity. All points of the boundary of B are either points like $\underline{X}^\mu$ or $\bar{X}^\mu$ or points of Λ. The corresponding parts of the boundary might be denoted by

$\underline{R}$, $\overline{R}$, and R_Λ. The boundary components $\underline{R}$ and $\overline{R}$ are mapped to each other in a one-to-one correspondence by way of the rays of *V*. The normal directions to $\underline{R}$ and $\overline{R}$ at two corresponding points are the same, up to sign, and the magnitudes of surface elements of $\overline{R}$ are greater by a factor of e^4 than the corresponding ones on $\underline{R}$, although *F* is e^4-times smaller on $\overline{R}$ than it is on $\underline{R}$. The boundary integrals over $\underline{R}$ and $\overline{R}$ that come about by the conversion of $\int F_{|\mu}\, d\tau$ using Gauss's theorem will then cancel each other, such that $\int F_{|\mu}\, d\tau$ will be equivalent to a boundary integral over just R_Λ.

Under the map of B_5 onto the affine domain B_4, the points of R_Λ correspond to just the points of the boundary R_4 of B_4 such that a boundary integral over R_Λ will go to a boundary integral over R_4.

5. Projective and affine metric. – The metric in *V* is defined by a symmetric, normal tensor field $g_{\mu\nu}$ with $g = \| g_{\mu\nu} \| \neq 0$, in which we agree that the covariant components α_μ that are constructed from the contravariant components α^ν by:

$$\alpha_\mu = g_{\mu\nu}\, \alpha^\nu \tag{5.1a}$$

should be regarded as different components of *the same* vectors. The solution of equations (5.1a) reads:

$$\alpha^\nu = g^{\mu\nu}\, \alpha_\mu . \tag{5.1.b}$$

It follows from this that $g^{\mu\nu}$ is a contravariant normal tensor, and that:

$$g^{\mu\nu}\, g_{\nu\rho} = \delta^\mu_\rho . \tag{5.2}$$

Just as one does for vectors, one can also use (5.1) to *lower* and *raise* the indices of tensors with the help of $g_{\mu\nu}$ and $g^{\mu\nu}$, resp.; e.g.:

$$\alpha^{\mu\nu} = \alpha^\mu_\sigma\, g^{\sigma\nu}, \qquad \alpha_{\rho\sigma} = \alpha^{\mu\nu}\, g_{\mu\rho}\, g_{\nu\sigma}, \qquad \text{etc.} \tag{5.3}$$

(5.2) then also means that the metric tensor is identical to the identity tensor δ^μ_ρ; i.e., that, e.g., the $g^{\mu\nu}$ are the doubly-contravariant components of the identity tensor, and the $g_{\mu\nu}$ are the doubly-covariant ones.

The operator H_σ, as it was defined in no. **2**, is invariant under the raising and lowering of indices.

We refer to the invariant $\alpha^\nu\, \alpha_\nu$ briefly as the *length-squared* of the vector α^ν. The length-squared of X^ν is:

$$J = X^\nu X_\nu = X^\nu g_{\nu\mu} X^\mu . \tag{5.4}$$

The arbitrariness in the definition of the quantities g^ν_k in (3.6) can be eliminated by the requirement that the reduced vector should be independent of whether one starts with covariant or contravariant components. The reduction of X^ν will be equal to zero, so one must also demand that:

$$X_\nu \, g_k^\nu = 0$$

then. It will then follow, with (3.6) and (5.4), that:

$$\overset{\circ}{g}{}_k^\nu X_\nu + \lambda_k J = 0,$$

from which the λ_k, and therefore the g_k^ν, will be determined.

Since the reduction of X^ν is zero, and since the relation $\bar{\alpha}_\nu X^\nu = 0$ follows, with $\bar{\alpha}_\nu = \alpha_k \, g_\nu^k$, the operation of reduction is nothing but orthogonal projection in the direction of the vector X^ν onto the subspace that perpendicular to X^ν. We would like to refer to the projective five-dimensional concomitant vector space briefly as $\mathfrak{R}$ here, and to the four-dimensional affine subspace as $\overset{4}{\mathfrak{R}}$. $\overset{4}{\mathfrak{R}}$ is then the subspace of $\mathfrak{R}$ that is perpendicular to X^ν. It then follows from this that for the tensor d_ν^μ that is defined in (3.9) that implements the projection onto $\overset{4}{\mathfrak{R}}$:

$$\delta_\nu^\mu = d_\nu^\mu + J^{-1} X_\nu X^\mu. \tag{5.5}$$

For the vectors of $\overset{4}{\mathfrak{R}}$, one has:

$$\alpha_k = \alpha_\nu \, g_k^\nu = \alpha^\mu g_{\mu\nu} \, g_k^\nu = \alpha^l g_l^\mu g_{\mu\nu} \, g_k^\nu , \qquad \beta^k = \beta_l \, g_\nu^l \, g^{\mu\nu} \, g_\mu^k . \tag{5.6}$$

Due to that relation:

$$g_{ik} = g_i^\mu \, g_{\mu\nu} \, g_k^\nu \tag{5.7}$$

is the affine metric tensor. One can also read (5.5) as:

$$g_{\mu\nu} = g_\mu^i \, g_{ij} \, g_\nu^j + J^{-1} X_\mu X_\nu . \tag{5.8}$$

The metric tensor then splits affinely into the affine metric tensor and the invariant J.

Any tensor can be split into its affine parts with the help of the decomposition of the metric tensor as in (5.5) or (5.8); e.g., it will follow for a tensor of rank two that:

$$\alpha_{\nu\mu} = g_\nu^k \, g_\mu^l \, g_{kl} + J^{-1} \left(g_\nu^k X_\mu X^\sigma \alpha_{k\sigma} + g_\mu^l X_\nu X^\sigma \alpha_{\rho l} \right) + J^{-1} X_\nu X_\mu \, X^\rho X^\sigma \alpha_{\rho\sigma} .$$

It then splits affinely into a tensor (viz., its reduction):

$$\alpha_{kl} = g_k^\nu \, g_l^\mu \, \alpha_{\mu\nu} ,$$

two affine vectors:

$$\alpha_{k\mu}X^\mu = g_k^\nu \; X^\mu \; \alpha_{\nu\mu}\,, \quad \alpha_{\nu l}\,X^\nu = g_l^\mu X^\nu \alpha_{\nu\mu}\,,$$

and an affine scalar:

$$\alpha_{\mu\nu}X^\mu X^\nu.$$

A projective tensor equation then subsumes several affine tensor equations. The special elegance and harmony of the projective theory of relativity rests upon that fact.

One can introduce a basis of five normal vectors $g_\rho^{(\nu)}$ [in which (ν) enumerates the vectors] in the vector space $\mathfrak{R}$ such that those vectors are orthogonal and orthonormal; i.e.:

$$g_\rho^{(\nu)} g^{(\mu)\rho} = g_\rho^{(\nu)} g^{\rho\sigma} g_\sigma^{(\mu)} = g^{(\nu)(\mu)}, \tag{5.9}$$

in which $g^{(\nu)(\mu)} = 0$ when $(\nu) \neq (\mu)$ and is equal to $\pm$ 1 when $(\nu) = (\mu)$. The number of negative $g^{(\nu)(\mu)}$ (which is called the *index defect*) is independent of the choice of the $g_\rho^{(\nu)}$. We assume that the index defect is equal to unity and set:

$$g^{(\nu)(\mu)} = \left\{ \begin{array}{rll} 0 & \text{for} & (\nu) \neq (\mu) \\ 1 & \text{for} & (\nu) = (\mu) = 0,1,2,3 \\ -1 & \text{for} & (\nu) = (\mu) = (4). \end{array} \right\} = g_{(\nu)(\mu)}\,. \tag{5.10}$$

One can refer vectors and tensors to this new basis:

$$\alpha^\nu = \alpha^{(\sigma)} \, g_{(\sigma)}^\nu\,, \qquad \alpha_\nu = \alpha_{(\sigma)}\, g_\nu^{(\sigma)}\,, \qquad \alpha^{(\sigma)} = \; g_\nu^{(\sigma)}\, \alpha^\nu, \text{ etc.} \tag{5.11}$$

Thus, $g_{\nu\mu}$, $g^{\nu\mu}$, δ_μ^ν, $g_\rho^{(\nu)}$, $g_\nu^{(\rho)}$, $g^{(\nu)(\mu)}$, $g_{(\nu)(\mu)}$, $\delta_{(\mu)}^{(\nu)}$ are also components of the same metric tensor.

The vector and tensor components with indices in parentheses are invariant under $\mathfrak{H}_5$. They will transform under the group $\mathfrak{P}$ by way of the operator H_ρ in no. **2**. Nevertheless, those components are not scalars, since the choice of the basis $g_\nu^{(\sigma)}$ is not unique. One obtains all possible other bases from the basis $g_\nu^{(\sigma)}$ by *five-dimensional rotations:*

$$g_\nu'^{(\sigma)} = \Theta^{(\sigma)}{}_{(\rho)}\, g_\nu^{(\rho)}\,. \tag{5.12}$$

Since (5.9) must remain true under this, one will have:

$$\Theta^{(\sigma)}{}_{(\rho)}\, g_\nu^{(\rho)}\, g^{\nu\mu}\, \Theta^{(\tau)}{}_{(\eta)}\, g_\mu^{(\eta)} \; = g^{(\sigma)(\tau)},$$

or

$$\Theta^{(\sigma)}{}_{(\rho)}\, \Theta_{(\tau)}{}^{(\rho)} = \delta_{(\tau)}^{(\sigma)} \qquad \text{and} \qquad \Theta_{(\tau)}{}^{(\rho)}\, \Theta^{(\tau)}{}_{(\sigma)} = \delta_{(\sigma)}^{(\rho)}\,. \tag{5.13}$$

The $\Theta_{(\rho)}^{(\sigma)}$ in this are homogeneous functions of degree zero in the X^ν. We denote the group of these rotations by $\mathfrak{D}_5$ .The vector and tensor components with indices in parentheses transform under $\mathfrak{D}_5$ like:

$$\alpha'^{(\nu)} = \Theta^{(\nu)}{}_{(\mu)}\,\alpha^{(\mu)}, \qquad \alpha'_{(\nu)} = \Theta_{(\nu)}{}^{(\mu)}\,\alpha_{(\mu)}, \tag{5.14}$$

and correspondingly for tensors.

One also refers to the $g_\nu^{(\sigma)}$ as "fünfbeins" and the representation of tensors by their components with indices in parentheses as the "fünfbein representation."

One can choose the fünfbeins, in particular, such that one them falls in the direction of X^ν:

$$g_{(0)}^\mu = J^{-1/2}\,X^\mu . \tag{5.15}$$

Let the remaining four of them be $g_{(k)}^\mu$, with $(k) = 1, 2, 3, 4$. If a tensor is characterized by the indices (ρ) in what follows then it must be based upon a general fünfbein, but if it is described by splitting the indices into (0), (ν) then the it must be based upon the special choice (5.15).

Since the $g_{(k)}^\mu$ are perpendicular to $g_{(0)}^\mu$, those four vectors will lie in $\overset{4}{\mathfrak{R}}$ then, and will therefore be identical with their affine reductions; i.e., e.g.:

$$g_{(k)}^\mu = g_{(k)}^\nu\, d_\nu^\mu = g_{(k)}^\nu\, g_\nu^l\, g_l^\mu .$$

The affine components of these vectors are then:

$$g_{(k)}^l = g_{(k)}^\mu\, g_\mu^l . \tag{5.16}$$

They therefore also define a vierbein relative to the affine metric g_{ij} , such that, e.g.:

$$g_{(k)}^l\, g_{lm}\, g_{(r)}^m = g_{(k)(r)} ,$$

so the affine metric will be identical with the projective metric in the subspace $\overset{4}{\mathfrak{R}}$.

If the vector $g_{(0)}^\mu$ is established by (5.15) then the group $\mathfrak{D}_5$ will be restricted to the group $\mathfrak{D}_4$ of all rotations leave $g_{(0)}^\mu$ invariant. $\mathfrak{D}_4$ is the Lorentz group.

The theorem follows from all of this:

If a projective tensor is given in the fünfbein representation then one will find its affine splitting in such a way that one will denote the indices by (0), (r), *instead of* (ρ),

or, in a formulation that will be extended to arbitrary spinors later:

The transition from the projective to the affine description is equivalent to the reduction of the five-dimensional rotation group $\mathfrak{D}_5$ (or its representations) to $\mathfrak{D}_4 \subset \mathfrak{D}_5$ when one fixes the vector $g^{\mu}_{(0)} = J^{-1/2} X^{\mu}$.

For example, if $\alpha_{(\rho)}$ is a projective vector then $\alpha_{(\rho)}$ will decompose into the affine vector $\alpha_{(k)}$ and the affine scalar $\alpha_{(0)}$. In the language of groups: $\alpha_{(\rho)}$ transforms like a vector under $\mathfrak{D}_5$, but $\alpha_{(0)}$ will remain invariant under $\mathfrak{D}_4$ (it then feels the identity representation, and is therefore an affine scalar), and $\alpha_{(k)}$ will transform like an affine vector under $\mathfrak{D}_4$. A tensor $\beta_{(\rho)(\sigma)}$ splits in a corresponding way into an affine tensor $\beta_{(r)(s)}$, two affine vectors $\beta_{(r)(0)}$ and $\beta_{(0)(s)}$, and an affine scalar $\beta_{(0)(0)}$.

One will then have, e.g.:

$$\alpha^k = g^{k}_{(r)} \alpha^{(r)}, \text{ etc.}$$

with the previous notations.

The rotations of $\mathfrak{D}_5$ then act upon only indices in parentheses, since they correspond to a change of the basis $g^{(\rho)}_{\mu}$. Along with the group $\mathfrak{D}_5$, we also introduce the group $\bar{\mathfrak{D}}_5$ of length-preserving automorphisms of $\mathfrak{R}$. If α^{ν} is a vector in $\mathfrak{R}$, and F is an element of $\bar{\mathfrak{D}}_5$ (for fixed basis vectors!) then we will have:

$$\alpha'^{\nu} = \Phi^{\nu}{}_{\mu}\, \alpha^{\mu} \tag{5.17}$$

for the vector α'^{ν} that F maps α^{ν} to, or in a different component representation:

$$\left.\begin{aligned} \alpha'^{(\nu)} &= F\, \alpha^{(\nu)} = \Phi^{(\nu)}{}_{(\mu)} \alpha^{(\mu)}, \\ \alpha'_{\nu} &= F\, \alpha_{\nu} \;\; = \Phi^{\mu}{}_{\nu} \alpha_{\mu}, \text{etc.} \end{aligned}\right\} \tag{5.18}$$

The relation then follows:

$$\Phi^{\rho}{}_{\nu}\, \Phi_{\rho}{}^{\mu} = \delta^{\mu}_{\nu} = \Phi^{\mu}{}_{\rho}\, \Phi_{\nu}{}^{\rho}, \tag{5.19}$$

which is equivalent to (5.13). $\Phi_{\mu\nu}$ is a tensor. The rotations in $\bar{\mathfrak{D}}_5$ that leave the vector X^{ν} invariant define a subgroup that we will denote by $\bar{\mathfrak{D}}_4$. We introduce:

$$\left.\begin{aligned} d\Sigma &= m\, d\tau, \\ \text{with} \quad m &= \left\| g^{(\nu)}_{\mu} \right\| \end{aligned}\right\} \tag{5.20}$$

as the volume of the volume element $d\tau$. $d\Sigma$ is an invariant, due to the previously-presented transformation properties of $d\tau$. In $\mathfrak{D}_5$, one multiplies m by the determinant $\left\| \Theta_{(\nu)}{}^{(\sigma)} \right\|$. By squaring the determinant, it will follow from (5.13) that the square has the value unity, so it will have the value 1 or – 1. Now, $\mathfrak{D}_5$ is the direct product $\mathfrak{D}^{+}_5 \times \mathfrak{B}$ of the group $\mathfrak{D}_5^+$ of proper rotations with positive determinants and the reflection group $\mathfrak{B}$ that consists of the two elements:

$$g'^{(\sigma)}_{\mu} = g^{(\sigma)}_{\mu}, \quad g''^{(\sigma)}_{\mu} = -g^{(\sigma)}_{\mu}.$$

m is then invariant under $\mathfrak{D}_5^+$, while under the rotations of $\mathfrak{D}_5$ that do not belong to $\mathfrak{D}_5^+$, m will be multiplied by – 1.

When one introduces the relation $g_{\mu\nu} = g^{(\sigma)}_{\mu} g^{(\rho)}_{\nu} g_{(\sigma)(\rho)}$, the determinant $g = \| g_{\mu\nu} \|$ will be:

$$g = m^2 \| g_{(\sigma)(\rho)} \| = -m^2, \tag{5.21}$$

such one will also have:

$$d\Sigma = \pm\sqrt{-g}\, d\tau. \tag{5.22}$$

If L is an invariant then $\mathfrak{L} = L$ is an invariant density. (4.4) will then assume the form:

$$W = \int L\, d\Sigma = \int L\, m\, d\tau. \tag{5.23}$$

(4.10) will then go over to:

$$W = \int L \frac{m}{\sigma} d\overset{4}{\tau}. \tag{5.24}$$

However, one has:

$$g^{(i)}_{\mu} = g^{(i)}_{k} g^{k}_{\mu}, \qquad g^{(0)}_{\mu} = J^{-1/2} X_{\mu},$$

such that:

$$m = J^{-1/2} \begin{Vmatrix} (X_{\mu}) \\ (g^{(i)}_{k} g^{k}_{\mu}) \end{Vmatrix} = J^{-1/2} \begin{Vmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (g^{(i)}_{k}) & \\ 0 & & & \end{Vmatrix} \cdot \begin{Vmatrix} (X_{\mu}) \\ (g^{k}_{\mu}) \end{Vmatrix},$$

$$m = J^{-1/2} \overset{4}{m} X_{\mu} \Delta_{\mu} \qquad \text{with} \qquad \overset{4}{m} = \| g^{(i)}_{k} \|.$$

It follows from $\Delta_{\mu} = \sigma X_{\mu}$ upon multiplying by X_{μ} that:

$$X_{\mu} \Delta_{\mu} = \sigma J,$$

and therefore:

$$m = J^{1/2} \overset{4}{m} \sigma.$$

(5.24) ultimately goes to:

$$W = \int L\, J^{1/2} \overset{4}{m}\, d\overset{4}{\tau} = \int L\, J^{1/2} d\overset{4}{\Sigma}, \tag{5.25}$$

in which $d\overset{4}{\Sigma} = \overset{4}{m}\, d\overset{4}{\tau} = \pm\sqrt{-\overset{4}{g}}\, d\overset{4}{\tau}$ is the four-dimensional volume.

6. Metric field components. – Although the metric tensor $g_{\mu\nu}$ splits affinely into only a tensor g_{ik} and an invariant J, the derivatives $g_{\mu\nu|\sigma}$ of $g_{\mu\nu}$ cannot be represented in terms of only $g_{ik|l}$ and $J_{|k}$. In what follows, we shall employ mostly the $g_\nu^{(\mu)}$ and not the $g_{\mu\nu}$. The quantities $g^{(\mu)}{}_{\rho\nu}$ that are defined by:

$$g^{(\mu)}{}_{\rho\nu} = - g^{(\mu)}{}_{\rho\nu} = g^{(\mu)}{}_{\nu|\rho} - g^{(\mu)}{}_{\rho|\nu} \,, \tag{6.1}$$

indeed behave like normal tensor components under $\mathfrak{H}_5$ and $\mathfrak{P}$, but not under $\mathfrak{D}_5$. For that reason, one must observe that the $g^{(\mu)}{}_{\rho\nu}$ are not tensor components. While we would like to regard the components $g_\nu^{(\mu)}$ of the metric tensor as potentials of the metric field, we will speak of the $g^{(\mu)}{}_{\rho\nu}$ as the field strengths of the metric field. That terminology will later prove to be useful in the physical interpretation of the theory.

The affine quantities:

$$\overset{A}{g}{}^{(k)}{}_{ml} = g^{(k)}_{l|m} - g^{(k)}_{m|l} \,, \tag{6.2}$$

are defined in analogy with (6.1). Since $g_{(\mu)(\nu)}$ and $g_{(i)(k)}$ are constant, one will have:

$$\left.\begin{aligned} g_{(\mu)\rho\nu} &= g_{(\mu)(\sigma)}\, g^{(\sigma)}{}_{\rho\nu} = g_{(\mu)\nu|\rho} - g_{(\mu)\rho|\nu} \,, \\ \overset{4}{g}_{(k)ml} &= g_{(k)(j)}\, \overset{4}{g}{}^{(j)}{}_{ml} = g_{(k)l|m} - g_{(k)m|l} \,. \end{aligned}\right\} \tag{6.3}$$

Since:

$$g_\nu^{(l)} = g_\nu^k g_k^{(l)} \,, \tag{6.4}$$

it will follow that:

$$g_{\nu|\rho}^{(l)} = g_{\nu|\rho}^k g_k^{(l)} + g_\nu^k g_{k|\rho}^{(l)} = x^k{}_{|\nu|\rho} g_k^{(l)} + g_\nu^k g_{k|\rho}^{(l)} \,.$$

Subtracting the equation that arises by switching ν and ρ will yield:

$$g^{(l)}{}_{\rho\nu} = g_\nu^k\, g^{(l)}{}_{k|\rho} - g_\rho^k\, g^{(l)}{}_{k|\nu} = g_\nu^k\, g^{(l)}{}_{k|m}\, g_\rho^m - g_\rho^k\, g^{(l)}{}_{k|\nu}\, g_\nu^m \,,$$

or

$$g_{(l)\rho\nu} = g_\rho^m\, g_\nu^k\, \overset{4}{g}_{(l)mk} \,.$$

For $g_{(l)(\rho)(\nu)} = g_{(l)\sigma\nu}\, g_{(\rho)}^\sigma\, g_{(\nu)}^\mu$, one gets from this that:

$$\left.\begin{aligned} g_{(l)(m)(n)} &= \overset{4}{g}_{(l)(m)(n)} , \\ g_{(l)(0)(n)} &= \overset{4}{g}_{(l)(m)(0)} = 0. \end{aligned}\right\} \tag{6.5}$$

We introduce the vector:

$$Y^\nu = J^{-1} X^\nu. \tag{6.6}$$

The antisymmetric tensor:

$$F_{\nu\mu} = Y_{\mu\,|\,\nu} - Y_{\nu\,|\,\mu} \tag{6.7}$$

can be derived from it. It will then follow that:

$$F_{\nu\mu}\, X^\mu = Y_{\mu\,|\,\nu}\, X^\mu - Y_{\nu\,|\,\mu}\, X^\mu = Y_{\mu\,|\,\nu}\, X^\mu + Y_\nu\,.$$

Since $Y_\mu\, X^\mu = 1$, one will have:

$$Y_{\mu\,|\,\nu}\, X^\mu + Y_\mu\, \delta^\mu_\nu = 0,$$

such that:

$$F_{\nu\mu}\, X^\mu = 0. \tag{6.8}$$

The tensor $F_{\nu\mu}$ is the identical with its affine reduction, such that:

$$F_{\nu\mu} = g^k_\nu\, g^l_\mu\, F_{kl}\,, \qquad\qquad F_{kl} = g^\mu_k\, g^\nu_l\, F_{\mu\nu}\,. \tag{6.9}$$

With the help of the function $\eta\,(X^\nu)$, which is homogeneous of degree one and has been employed several times already, we define:

$$\varphi_\nu = Y_\nu - (\ln\,\eta)_{|\nu}\,. \tag{6.10}$$

The φ_ν are then the components of a normal vector in $\overset{4}{\mathfrak{R}}$, so $(\ln\,\eta)_{|\nu}$ are normal vector components that are invariant $\mathfrak{H}_5$ and $\mathfrak{P}$, and one will have $j_\nu\, X^\nu = 0$. Therefore, one will have: $\varphi_\nu = g^k_\nu\, \varphi_k$, $\varphi_k = g^\nu_k\, \varphi_\nu$. It will then follow from (6.10) by differentiation that:

$$\varphi_{\nu\,|\,\mu} = g^k_{\nu|\mu}\, \varphi_k + g^k_\nu\, \varphi_{k|l}\, g^l_\mu = Y_{\nu\,|\,\mu} + (\ln\,\eta)_{|\nu\,|\,\mu}\;.$$

If one infers the equation that results from this by switching the indices ν and μ then that will yield:

$$(\varphi_{k\,|\,l} - \varphi_{l\,|\,k})\;\, g^k_\nu\, g^l_\mu = Y_{\nu\,|\,\mu} - Y_{\mu\,|\,\nu} = F_{\mu\nu}\,.$$

With (6.9), that will yield:

$$F_{lk} = \varphi_{k\,|\,l} - \varphi_{l\,|\,k}\,. \tag{6.11}$$

Since η was an arbitrary function that was homogeneous degree one, the vector φ_ν will not be gauge-invariant then. Under the gauge transformation (2.17):

$$\eta' = \sigma\,\eta, \qquad \ln\,\eta' = \ln\,\eta + \ln\,\sigma, \tag{6.12}$$

φ_ν will change by a gradient:

$$\varphi'_\nu = \varphi_\nu - (\ln\,\sigma)_{|\nu}\,, \qquad\qquad \varphi'_k = \varphi_k - (\ln\,\sigma)_{|\,k}\,. \tag{6.13}$$

Under an infinitesimal gauge transformation $\sigma = 1 + \varepsilon\,\lambda$, one then has:

$$\delta\varphi_\nu = -\,\varepsilon\,\lambda_{|\nu}\,, \qquad\qquad \delta\varphi_k = -\,\varepsilon\,\lambda_{|k}\,. \tag{6.13a}$$

With $g_{(0)\nu} = J^{-1/2} X_\nu = J^{1/2} Y_\nu$, one then has:

$$g_{(0)\nu\,|\,\rho} = J^{1/2}\, Y_{\nu\,|\,\rho} + \tfrac{1}{2}\, J^{-1/2}\, J_{|\,\rho}\, Y_\nu\,.$$

If one infers the equation that arises from this by switching ν and ρ then it will follow that:

$$g_{(0)\nu\rho} = J^{1/2}\, F_{\rho\nu} + \tfrac{1}{2} J^{-1/2}\, (J_{|\,\rho}\, X_\nu - J_{|\,\nu}\, X_\rho)\,. \tag{6.14}$$

That will yield, in particular:

$$g_{(0)(m)(n)} = J^{1/2}\, F_{(m)(n)} \tag{6.15}$$

and

$$g_{(0)(0)\,\nu} = \tfrac{1}{2}\,(Y_\nu - J^{-1}\, J_{|\,\nu})\,.$$

With $J_{|\,(l)} = J_{|k}\, g^{k}_{(l)}$, it follows that:

$$g_{(0)(0)(l)} = -\, g_{(0)(l)(0)} = -\, J^{-1}\, J_{|\,(l)}\,. \tag{6.16}$$

Due to the antisymmetry of $g_{(\mu)\nu\rho}$ in the last two indices, all components with equal indices – such as $g_{(l)(m)(m)}$, $g_{(0)(m)(m)}$, $g_{(l)(0)(0)}$, $g_{(0)(0)(0)}$ – will be equal to zero.

7. Spinors. – We start with the four DIRAC matrices α_i (i is not a tensor index!) with the relations:

$$\tfrac{1}{2}\,(\alpha_i\, \alpha_k + \alpha_k\, \alpha_i) = \delta_{ik}\,. \tag{7.1}$$

All matrix representations of (7.1) can be reduced to four-rowed ones that are all equivalent to each other. If one irreducible solution of (7.1) is given then one will get all other irreducible solutions in the form:

$$S^{-1}\alpha_k\, S. \tag{7.2}$$

Since the α_k are an irreducible system, one will have that any matrix C that commutes with all α_k will be a multiple of the identity matrix. One constructs:

$$\alpha_0 = \alpha_1\, \alpha_2\, \alpha_3\, \alpha_4 \tag{7.3}$$

from α_1, α_2, α_3, α_4, so it follows that:

$$\alpha_0^2 = 1, \qquad \alpha_0\, \alpha_k + \alpha_k\, \alpha_0 = 0, \tag{7.4}$$

such that α_μ (μ = 0, 1, 2, 3, 4 is not a tensor index!) are five anti-commuting roots of unity:

$$\tfrac{1}{2}\,(\alpha_\mu\, \alpha_\nu + \alpha_\nu\, \alpha_\mu) = \delta_{\mu\nu}\,. \tag{7.5}$$

If we now set:

$$\gamma_{(\mu)} = \sqrt{g_{(\mu)(\mu)}}\ \alpha_\mu \qquad \text{[no summation over } (\mu)\text{!]} \tag{7.6}$$

then we will get:

$$\tfrac{1}{2}[\gamma_{(\mu)}\,\gamma_{(\nu)} + \gamma_{(\nu)}\,\gamma_{(\mu)}] = \gamma_{\mu\nu}\,. \tag{7.7}$$

(μ) is again a tensor index in this. The product $\gamma_{(0)}\,\gamma_{(1)}\,\gamma_{(2)}\,\gamma_{(3)}\,\gamma_{(4)}$ is the single linearly-independent component of a fifth-rank tensor that is antisymmetric in all indices (i.e., a *pseudo-scalar*). One calculates from (7.7) that:

$$[\gamma_{(0)}\,\gamma_{(1)}\,\gamma_{(2)}\,\gamma_{(3)}\,\gamma_{(4)}]^2 = \|\, g_{(\mu)(\nu)} \,\| = -1. \tag{7.8}$$

Furthermore, $\gamma_{(0)}\,\gamma_{(1)}\,\gamma_{(2)}\,\gamma_{(3)}\,\gamma_{(4)}$ commutes with all $\gamma_{(\nu)}$, so it will be a multiple of the identity matrix:

$$\gamma_{(0)}\,\gamma_{(1)}\,\gamma_{(2)}\,\gamma_{(3)}\,\gamma_{(4)} = \varepsilon\sqrt{\|\, g_{(\mu)(\nu)} \,\|} \qquad \text{with} \quad \varepsilon = \pm 1. \tag{7.9}$$

While (7.9) follows from just (7.7), one will get $\varepsilon = +1$ by the special choice of (7.6) and (7.3). If another solution $\gamma'_{(\nu)}$ of (7.7) is given then one will set:

$$\alpha'_\mu = \frac{\varepsilon}{\sqrt{g_{(\mu)(\nu)}}}\,\gamma'_{(\mu)} \qquad \text{[no summation over } (\mu)\text{!].}$$

One will then have:

$$\tfrac{1}{2}(\alpha'_\mu\,\alpha'_\nu + \alpha'_\nu\,\alpha'_\mu) = \delta_{\mu\nu}$$

and

$$\gamma'_{(0)}\,\gamma'_{(1)}\,\gamma'_{(2)}\,\gamma'_{(3)}\,\gamma'_{(4)} = \varepsilon\,\|\, g_{(k)(l)} \,\|^{1/2}\, g_{(0)(0)}^{-1/2}\,\gamma'_{(0)} = \|\, g_{(k)(l)} \,\|^{1/2}\,\alpha'_0\,.$$

Hence, one will also have:

$$\alpha'_1\,\alpha'_2\,\alpha'_3\,\alpha'_4 = \alpha'_0\,.$$

There will then exist a matrix S such that $\alpha'_\mu = S^{-1}\alpha_\mu\,S$, i.e.:

$$\gamma'_{(\mu)} = \varepsilon\,S^{-1}\gamma_{(\mu)}\,S. \tag{7.10}$$

If (7.10) were also fulfilled for a matrix T then one would need to have:

$$S^{-1}\gamma_{(\mu)}\,S = T^{-1}\gamma_{(\mu)}\,T,$$

from which:

$$T\,S^{-1}\gamma_{(\mu)} = \gamma_{(\mu)}\,T\,S^{-1},$$

and therefore the commutability of $T\,S^{-1}$ with all $\gamma_{(\mu)}$ would follow, such that:

$$T\,S^{-1} = c\,1, \qquad T = c\,S.$$

However $c\,S$ likewise fulfills (7.10).

Since the Hermitian conjugates $\gamma^*_{(\mu)}$ of $\gamma_{(\mu)}$ satisfy the equations:

$$\left.\begin{aligned}\tfrac{1}{2}[\gamma^*_{(\mu)}\,\gamma^*_{(\nu)} + \gamma^*_{(\nu)}\,\gamma^*_{(\mu)}] &= g_{(\mu)(\nu)}, \\ \gamma^*_{(0)}\,\gamma^*_{(1)}\gamma^*_{(2)}\,\gamma^*_{(3)}\gamma^*_{(4)} &= -\sqrt{\| g_{(\mu)(\nu)}},\end{aligned}\right\} \tag{7.11}$$

there will be a matrix β such that:

$$\gamma^*_{(\mu)} = -\beta\,\gamma_{(\mu)}\,\beta^{-1}. \tag{7.12}$$

That implies:

$$\gamma_{(\mu)} = -\beta^{*-1}\,\gamma^*_{(\mu)}\,\beta^* = \beta^{*-1}\beta\,\gamma^*_{(\mu)}\;\beta^{-1}\beta^*.$$

$\beta^{-1}\beta^*$ will then commute with all $\gamma_{(\mu)}$, so it will be a multiple of the identity matrix:

$$\beta^{-1}\beta^* = c1, \quad \beta^* = c\,\beta, \qquad \beta = \bar{c}\,\beta = c\,\bar{c}\,\beta, \tag{7.13}$$

and therefore:

$$c\,\bar{c} = 1. \tag{7.14}$$

Now, c can be chosen freely, to some degree, since along with β, $\beta' = \rho\,e^{i\varphi/2}\beta$ also satisfies equation (7.12).

$$\beta'^* = \rho\,e^{-i\varphi/2}\beta^* = \rho\,e^{-i\varphi/2}c\beta = e^{-i\varphi/2}c\beta' = c'\,\beta', \qquad \text{with} \quad c' = e^{-i\varphi\,c}.$$

We choose $c = -1$. It will then follow from (7.12) and (7.13) that:

$$(\beta\,\gamma_{(\mu)})^* = \beta\,\gamma_{(\mu)}, \qquad \beta^* = -\beta, \qquad (i\,\beta)^* = i\beta. \tag{7.15}$$

We associate the concomitant vector space $\mathfrak{R}$ with a four-dimensional spin space $\mathfrak{R}_s$. By its very definition, spin space is a four-dimensional module of linear forms:

$$\mathfrak{R}_s\;(U_1, U_2, U_3, U_4). \tag{7.16}$$

In what follows, spinor indices will always be characterized by upper-case Latin letters. The choice of basis is arbitrary, and we will refer to the transition from one basis to another as an *S-transformation* and the associated group as $\mathfrak{S}$. A spinor ψ in $\mathfrak{R}_s$ can be represented as:

$$\psi = U_K\,\psi^K, \tag{7.17}$$

in which ψ^K are complex numbers [or, more generally, elements of a module $K\,(i)\,\mathfrak{M}$, as in no. **2**, in which $K\,(i)$ is the field of complex numbers].

We now regard the $\gamma_{(\mu)}$ as automorphisms of $\mathfrak{R}_s$, in which $\psi \to \psi' = \gamma_{(\mu)}\,\gamma$, with

$$U'_K = U_L\;\gamma_{(\mu)}{}^L{}_K\,. \tag{7.18}$$

Under the transition to another basis V_K by way of:

$$V = (U_1, U_2, U_3, U_4)\,\Sigma, \qquad \text{i.e., } V_K = U_L\,\Sigma^L{}_K\,, \tag{7.19}$$

the matrix $\gamma_{(\mu)} = (\gamma_{(\mu)}{}^L{}_K)$ will go to:

$$\gamma'_{(\mu)} = \Sigma^{-1}\gamma_{(\mu)}\,\Sigma. \tag{7.20}$$

One will then obtain another representation of the $\gamma_{(\mu)}$, and by changing the basis as in (7.20), all other representations, up to the factor $\varepsilon = \pm 1$ in:

$$\gamma_{(0)}\,\gamma_{(1)}\,\gamma_{(2)}\,\gamma_{(3)}\,\gamma_{(4)} = \varepsilon\, i. \tag{7.21}$$

From (7.17), ψ can be written:

$$\psi = V_K\,\psi'^K = U_L\,\Sigma^L{}_K\,\psi'^K = U_L\,\psi^L$$

in the new basis (7.19); i.e.:

$$\Sigma^L{}_K\,\psi'^K = \psi^L, \qquad \psi'^K = (\Sigma^{-1})^L{}_K\,\psi^L. \tag{7.22}$$

With (7.20), it will then follow from (7.15) that:

$$(\beta\,\Sigma\,\gamma'_{(\mu)}\,\Sigma^{-1})^* = \beta\,\Sigma\,\gamma'_{(\mu)}\,\Sigma^{-1}$$

or

$$(\Sigma\,\beta\,\gamma'_{(\mu)}\,\Sigma^{-1})^* = \Sigma\,\beta\,\gamma'_{(\mu)}\,\Sigma^{-1}, \tag{7.23}$$

and

$$(\Sigma\,\beta\,\Sigma^{-1})^* = \Sigma\,\beta^*\,\Sigma = -\,\Sigma^*\,\beta\,\Sigma, \tag{7.24}$$

$$\beta' = \Sigma^*\,\beta\,\Sigma \tag{7.25}$$

will fulfill the same conditions for $\gamma'_{(\mu)}$ that β does for $\gamma_{(\mu)}$. From (7.25) and (7.22), the quantities:

$$\psi^+_L = i\,\bar{\psi}^{\dot{K}}\,\beta_{\dot{K}L} \qquad \text{with} \quad \beta = (\beta_{\dot{K}L}), \tag{7.26}$$

in which $\bar{\psi}^{\dot{K}}$ is the complex conjugate of ψ^K, transform contragrediently to the ψ^L. A Hermitian form is then defined upon the spin space $\mathfrak{R}_s$ by $i\,\beta$ (which is invariant under $\mathfrak{S}$), for which one will have:

$$\left.\begin{array}{ll} (\psi,\varphi) = \overline{(\psi,\varphi)}, & (U_K, U_L) = i\,\beta_{\dot{K}L}, \\ (\varphi,\psi) = i\,\bar{\varphi}^{\dot{K}}\,\beta_{\dot{K}L}\psi^L, & (\psi,\psi) = i\,\bar{\psi}^{\dot{K}}\,\beta_{\dot{K}L}\psi^L, \\ \multicolumn{2}{l}{\dfrac{1}{l}(\psi,\gamma_{(\mu)}\psi) = \bar{\psi}^{\dot{K}}\,\beta_{\dot{K}L}\,\gamma_{(\mu)}{}^L{}_M\psi^M = \text{real number.}} \end{array}\right\} \tag{7.27}$$

If one goes to another basis in the vector space $\mathfrak{R}$ by way of a rotation in $\mathfrak{D}_5$ using (5.12) then the relations (7.7) will remain unchanged, from which, the existence of a matrix S with $\| S \| = 1$ and:

$$\gamma'_{(\mu)} = \Theta_{(\mu)}{}^{(\sigma)}\,\gamma_{(\sigma)} = \varepsilon\,S\,\gamma_{(\mu)}\,S^{-1} \tag{7.28}$$

will follow from (7.10). Since:

$$\gamma'_{(0)}\,\gamma'_{(1)}\,\gamma'_{(2)}\,\gamma'_{(3)}\,\gamma'_{(4)} = \|\,\Theta_{(\mu)}{}^{(\sigma)}\,\|\,\gamma_{(0)}\,\gamma_{(1)}\,\gamma_{(2)}\,\gamma_{(3)}\,\gamma_{(4)}\,,$$

one will have $\varepsilon = +1$ for the elements of $\mathfrak{D}_5^+$ and $\varepsilon = -1$ for the other ones. The transformations S then define a representation of the five-dimensional rotation group, which we would like to denote by $D^5_{(1/2,1/2)}$. If one performs a change of basis in $\mathfrak{R}_s$ at the same time as a change of basis in $\mathfrak{R}$ using (7.19) and (7.20), with $\Sigma = S$, then the matrices $\gamma_{(\mu)}{}^K{}_L$ will remain invariant under the rotations in $\mathfrak{D}_5^+$, and will be multiplied by – 1 for the rotations with negative determinants. By contrast, the elements of the matrices $\gamma_\mu = g_\mu^{(\nu)}\,\gamma_{(\nu)}$ are not invariant under $\mathfrak{D}_5^+$, since the $g_\mu^{(\nu)}$ will be changed. Those couplings of the transformations in $\mathfrak{R}$ and $\mathfrak{R}_s$ shall always be assumed from now on.

Since the expressions (7.27) are invariant under arbitrary basis changes in $\mathfrak{R}_s$, (φ, ψ) will be invariant under $\mathfrak{D}_5$ with the coupling that was established, while $(\varphi, \gamma_{(\mu)}\,\psi)$ are the components of a vector, $(\varphi, \gamma_{(\mu)}\,\gamma_{(\nu)}\psi)$ are the components of a tensor, etc.

If we perform a rotation F of the $\gamma_{(\mu)}$ like the components of a vector using (5.18), then it will follow, precisely as it did above, that there is a matrix T with $\|\,T\,\| = 1$, such that:

$$\Phi_{(\rho)}{}^{(\sigma)}\,\gamma_{(\mu)} = \varepsilon\, T^{-1}\gamma_{(\mu)}\,T \qquad \text{with} \qquad \varepsilon = \begin{cases} +1 & \text{for } T \in \bar{\mathfrak{D}}_5^+, \\ -1 & \text{otherwise.}\end{cases} \tag{7.29}$$

If we now ultimately define the application of the operator F on the $\gamma_{(\mu)}$ by:

$$F\,\gamma_{(\rho)} = \Phi_{(\rho)}{}^{(\sigma)}\,T\,\gamma_{(\sigma)}\,T^{-1} \tag{7.30}$$

then it will follow that:

$$F\,\gamma_{(\sigma)} = \varepsilon\,\gamma_{(\sigma)}\,. \tag{7.31}$$

The matrices $\gamma_{(\sigma)}$ then remain invariant under the rotations of $\mathfrak{D}_5^+$. By contrast, from (7.29), under the application of the operator F to the spinor ψ, it will be subjected to the transformation:

$$F\,\psi^K = T^K{}_L\,\psi^L. \tag{7.32}$$

The quantities $(\varphi, \gamma_\mu\,\psi)$, as a vector, will go to another vector $(\varphi, \Phi_\nu{}^\mu\,\gamma_\mu\,\psi)$ under F, the tensor $(\varphi, \gamma_\mu\,\gamma_\nu\,\psi)$ will transform similarly, etc.

The transformations (7.32) define a representation $\bar{D}^5_{(1/2,1/2)}$ of $\bar{\mathfrak{D}}_5$.

Like any vector, $\gamma_{(\mu)}$ can also be decomposed into an affine vector $\gamma_{(k)}$ and a component in the direction of X^μ that has the form $\gamma_{(0)}$. For the transformations of $\mathfrak{D}_4$, from (7.28), one will then have:

$$\gamma'_{(0)} = \gamma_{(0)} = \varepsilon\, S\,\gamma_{(0)}\,S^{-1}, \qquad \gamma_{(0)}\,S = \varepsilon\,S\,\gamma^{(0)}.$$

The representation of the proper Lorentz group $\mathfrak{D}_4^+$ (determinant equal to + 1) that is given by $D^5_{(1/2,1/2)}$ is reducible then, since $\gamma_{(0)}$ commutes with all of the S that are represented. That will be explained once more in no. **9**.

We assume that the $\gamma_{(\mu)}$ that are invariant under $\mathfrak{D}_5$ are absolute constants; i.e., independent of X^ν, such that $\gamma_{(\mu)\,|\nu} = 0$. With that, we then establish that the same $\gamma_{(\mu)}$ must be chosen at all points of V, and imagine the basis in spin space as being established accordingly. The spinor components ψ^K will be functions of X^ν, in general. We correspondingly set:

$$\mathfrak{T}_\rho\, \psi^K = H_\rho\, \psi^K. \tag{7.33}$$

If $H_\rho = 1$ then we will call ψ a *normal spinor*. Normalization is performed precisely as it was for tensors in no. **2**, and therefore needs no further clarification.

8. Infinitesimal transformations. – The most important tool for the investigation of the groups $\mathfrak{H}_5$, $\mathfrak{G}_4$, etc., is their infinitesimal transformations. The infinitesimal transformations of the group $\mathfrak{H}_5$ are given by:

$$X'^\mu = X^\mu + \varepsilon\, \xi^\mu, \tag{8.1}$$

in which ξ^μ are the contravariant components of a vector, and ε is a small quantity whose powers higher than one can be neglected. Since ξ^μ is a normal vector, one will then have:

$$\xi^\mu{}_{|\nu}\, X^\nu = \xi^\mu. \tag{8.2}$$

A vector α^μ will then have the new components:

$$\alpha'^\mu = \alpha^\mu + \varepsilon\, \xi^\mu{}_{|\rho}\, \alpha^\rho, \qquad \delta\alpha^\mu = \alpha'^\mu - \alpha^\mu = \varepsilon\, \xi^\mu{}_{|\rho}\, \alpha^\rho. \tag{8.3}$$

We define the operator $\mathfrak{O}_\mu{}^\nu$ by:

$$\mathfrak{O}_\mu{}^\nu\, \alpha^\rho = \alpha^\nu\, \delta^\rho_\mu\,, \qquad \mathfrak{O}_\mu{}^\nu\, \alpha_\rho = -\, \alpha_\mu\, \delta^\nu_\rho\,. \tag{8.4}$$

(8.3) can then be written as:

$$\delta\alpha^\rho = \varepsilon\, \xi^\mu{}_{|\nu}\, \mathfrak{O}_\mu{}^\nu\, \alpha^\rho. \tag{8.5}$$

For a product of vector components, one has:

$$\delta(\alpha^\rho\, \beta_\sigma \ldots) = (\delta\alpha^\rho)\beta_\sigma \ldots + \alpha^\rho\, (\delta\beta_\sigma) \ldots + \ldots,$$

and we correspondingly define:

$$\mathfrak{D}_\mu^{\ \nu}(\alpha^\rho \beta_\sigma \ldots) = (\mathfrak{D}_\mu^{\ \nu}\alpha^\rho)\beta_\sigma \ldots + \alpha^\rho(\mathfrak{D}_\mu^{\ \nu}\beta_\sigma) + \ldots, \tag{8.6}$$

such that the operator $\mathfrak{D}_\mu^{\ \nu}$ will also be defined for arbitrary tensor components by (8.6). One will then have:

$$\delta\left(t^{\mu_1\cdots}_{\nu_1\cdots}\right) = \varepsilon\,\xi^\mu{}_{|\nu}\,\mathfrak{D}_\mu^{\ \nu}\,t^{\mu_1\cdots}_{\nu_1\cdots} \tag{8.7}$$

under the transformation (8.1). Furthermore, the change in the volume element $d\tau$ under the infinitesimal transformation (8.1) is:

$$\delta\,d\tau = \varepsilon\,d\tau\,\xi^\nu{}_{|\nu}\,. \tag{8.8}$$

If $\mathfrak{L}$ is an invariant density then $\mathfrak{L}\,d\tau$ will be invariant under $\mathfrak{H}_5$, so $\delta(\mathfrak{L}\,d\tau) = 0$, such that:

$$\delta\,\mathfrak{L} = -\,\varepsilon\,\mathfrak{L}\,\xi^\nu{}_{|\nu}\,. \tag{8.9}$$

Since m is an invariant density, from (5.20), one will have:

$$\delta m = -\,\varepsilon\,m\,\xi^\nu{}_{|\nu}\,. \tag{8.10}$$

The operation δ in this always means the change in quantities at the same point Q under changes in the coordinates, or, when written out explicitly for any quantity F (X'^μ are the new coordinates of the point Q, which has the coordinates X^μ in the old coordinate system):

$$\delta F = F'(X'^\mu) - F(X^\mu). \tag{8.11}$$

However, one can also write the change under the transformation (8.1) in such a way that one considers the change in F, not at the same point Q, but for fixed coordinate values, and thus defines:

$$\delta^* F = F'(X^\mu) - F(X^\mu). \tag{8.12}$$

It will then follow directly that:

$$\delta F = \delta^* F + \varepsilon\,F_{|\mu}\,\xi^\mu\,. \tag{8.13}$$

An infinitesimal transformation of $\mathfrak{G}_4$ is given by:

$$x'^k = x^k + \varepsilon\,\xi^k. \tag{8.14}$$

The following formulas:

$$\left.\begin{aligned}
&\overset{4}{\mathfrak{D}}{}_l^k \alpha^m = \alpha^k \delta_l^m , \quad \overset{4}{\mathfrak{D}}{}_l^k \alpha_m = -\alpha_l \delta_m^k ,\\
&\delta\left(t_{l\cdots}^{k\cdots}\right) = \varepsilon\, \xi^m{}_{|n} \overset{4}{\mathfrak{D}}{}_m^n \left(t_{l\cdots}^{k\cdots}\right),\\
&\delta\, d\overset{4}{\tau} = \varepsilon\, d\overset{4}{\tau}\, \xi^k{}_{|k} ,\\
&\delta\overset{4}{m} = -\varepsilon\, \overset{4}{m}\, \xi^k{}_{|k} ,\\
&\delta F = \delta^* F + \varepsilon\, F_{|k}\, \xi^k .
\end{aligned}\right\} \tag{8.15}$$

are then understandable with no further explanation.

From (5.17), an infinitesimal transformation of $\bar{\mathfrak{D}}_5$ is given by a tensor:

$$\Phi^\mu{}_\nu = \delta_\nu^\mu + \varepsilon\, \varphi^\mu{}_\nu . \tag{8.16}$$

It will then follow from (5.19) that:

$$(\delta_\nu^\rho + \varepsilon\,\varphi^\rho{}_\nu)(\delta_\rho^\mu + \varepsilon\,\varphi^\mu{}_\rho) = \delta_\nu^\mu + \varepsilon\,(\varphi^\mu{}_\nu + \varphi_\nu{}^\mu) = \delta_\nu^\mu ,$$

such that $\varphi_{\mu\nu}$ will be a skew-symmetric tensor. If we set $F = 1 + \varepsilon f$ in (5.17) or (5.18) then it will follow for a vector α^ν that:

$$\delta\alpha^\nu = \varepsilon f\, \alpha^\nu, \tag{8.17}$$

in which:

$$f\, \alpha^\nu = \varphi^\nu{}_\mu\, \alpha^\mu . \tag{8.18}$$

f can be decomposed into:

$$f = \tfrac{1}{2}\, \varphi^\mu{}_\nu\, \mathrm{P}_\mu{}^\nu, \tag{8.19}$$

in which the $\mathrm{P}_\mu{}^\nu$ are the special infinitesimal transformations:

$$\mathrm{P}_\mu{}^\nu\, \alpha^\rho = \delta_\mu^\rho \alpha^\nu - \alpha_\mu\, g^{\nu\rho}. \tag{8.20}$$

They satisfy the commutation relations:

$$[\mathrm{P}^{\alpha\beta}, \mathrm{P}^{\sigma\rho}] = \mathrm{P}^{\alpha\beta}\,\mathrm{P}^{\sigma\rho} - \mathrm{P}^{\sigma\rho}\,\mathrm{P}^{\alpha\beta} = g^{\beta\sigma}\,\mathrm{P}^{\alpha\rho} + g^{\beta\sigma}\,\mathrm{P}^{\alpha\rho} - g^{\alpha\sigma}\,\mathrm{P}^{\beta\rho} - g^{\beta\rho}\,\mathrm{P}^{\alpha\sigma}. \tag{8.21}$$

If one imposes the condition upon the transformations of $\bar{\mathfrak{D}}_5$ that they must leave the vector X^ν invariant then $\bar{\mathfrak{D}}_5$ will reduce to the subgroup $\bar{\mathfrak{D}}_4$. One will then get all infinitesimal transformations of $\bar{\mathfrak{D}}_4$ when one sets all $\varphi^{(\mu)}{}_{(\nu)}$ that have one index (0) equal to zero, and considers only the $\mathrm{P}_{(\alpha)}{}^{(\beta)}$ for which both the (α) and (β) are non-zero.

From (5.12), the infinitesimal transformations of $\mathfrak{D}_5$ are given by:

$$\Theta_{(\nu)}{}^{(\sigma)} = \delta_{(\nu)}^{(\mu)} + \varepsilon\, \vartheta_{(\nu)}{}^{(\sigma)}. \tag{8.22}$$

(5.13) implies that $\vartheta_{(\nu)(\sigma)} = -\vartheta_{(\sigma)(\nu)}$, and is thus skew-symmetric. One will then have:

$$\delta\alpha^{(\mu)} = \varepsilon\, \vartheta_{(\sigma)}{}^{(\mu)}\, \alpha^{(\sigma)}, \quad \delta\alpha_{(\mu)} = \varepsilon\, \vartheta^{(\sigma)}{}_{(\mu)}\, \alpha_{(\sigma)}\,. \tag{8.23}$$

We define the operator $P_{(\nu)}{}^{(\sigma)}$, which acts upon only indices in parentheses ([1]):

$$P_{(\nu)}{}^{(\sigma)}\, \alpha^{(\mu)} = \alpha^{(\sigma)}\delta^{(\mu)}_{(\nu)} - \alpha_{(\sigma)}\, g^{(\sigma)(\mu)}. \tag{8.24}$$

Hence, (8.22) can also be written:

$$\delta\alpha^{(\mu)} = \tfrac{1}{2}\,\varepsilon\, \vartheta^{(\nu)}{}_{(\sigma)}\, P_{(\nu)}{}^{(\sigma)}\, \alpha^{(\mu)}. \tag{8.25}$$

It can be verified from this that:

$$P_{(\nu)}{}^{(\sigma)}\, \alpha^{(\mu)} = 0 = P^{\nu\sigma}\, \alpha^{\mu}\,,$$

since α^{μ} has no indices in parentheses. One has $\mathrm{P}_{(\nu)}{}^{(\sigma)}\, \alpha^{(\mu)} = P_{(\nu)}{}^{(\sigma)}\, \alpha^{(\mu)}$, but $\mathrm{P}_{\nu}{}^{\sigma}\, \alpha^{\mu} \neq P_{\nu}{}^{\sigma}\, \alpha^{\mu}$!

9. Representations of the rotation groups. – Representations were as good as completely examined already by CARTAN ([2]) and WEYL ([3]), and were applied to physical problems by LUBANSKI ([4]). Due to the restricted scope of this report, we shall therefore refer only briefly to the results.

Instead of the infinitesimal transformations $\mathrm{P}^{(\mu)(\nu)}$, we introduce:

$$\left.\begin{aligned} R^{(\mu)(\nu)} &= \mathrm{P}^{(\mu)(\nu)} \qquad \text{for } (\mu) \neq (4), (\nu) \neq (4), \\ R^{(4)(\nu)} &= i\, \mathrm{P}^{(4)(\nu)}. \end{aligned}\right\} \tag{9.1}$$

From (8.21), one then has the commutation relations:

$$\left.\begin{aligned} [R^{(\alpha)(\beta)}, R^{(\sigma)(\rho)}] = {} & \delta^{(\beta)(\sigma)} R^{(\alpha)(\rho)} + \delta^{(\rho)(\alpha)} R^{(\beta)(\sigma)} \\ & - \delta^{(\alpha)(\sigma)} R^{(\beta)(\rho)} - \delta^{(\beta)(\rho)} R^{(\alpha)(\sigma)}. \end{aligned}\right\} \tag{9.2}$$

Furthermore, one replaces $R^{(\mu)(\nu)}$ with:

([1]) One must observe the difference between the *P* that is defined here and the P that was defined above!

([2]) E. CARTAN, Bull. Soc. Math. de France, **11** (1913), 53.

([3]) H. WEYL, Math. Zeit. **23** (1925), 271; *ibid.*, **24** (1925), 328.

([4]) J. K. LUBANSKI, Physica **9** (1942), 310-324, 325-338.

$$\left.\begin{aligned}
A_z &= -\frac{1}{2i}(R^{(1)(2)} + R^{(3)(4)}), \\
B_z &= -\frac{1}{2i}(R^{(1)(2)} - R^{(3)(4)}), \\
A_p &= \frac{1}{2i}[R^{(3)(2)} + R^{(4)(1)} + i(R^{(3)(1)} - R^{(4)(1)})], \\
B_p &= \frac{1}{2i}[R^{(3)(2)} - R^{(2)(1)} - i(R^{(3)(1)} - R^{(4)(2)})], \\
A_q &= \frac{1}{2i}[R^{(3)(2)} + R^{(4)(1)} - i(R^{(3)(1)} - R^{(4)(2)})], \\
B_q &= \frac{1}{2i}[R^{(3)(2)} - R^{(2)(1)} - i(R^{(3)(1)} + R^{(4)(2)})], \\
C_p &= \frac{1}{i\sqrt{2}}(R^{(0)(2)} + i\,R^{(0)(1)}), \\
D_p &= \frac{1}{i\sqrt{2}}(R^{(4)(0)} + i\,R^{(3)(0)}), \\
C_q &= \frac{1}{i\sqrt{2}}(R^{(0)(2)} - i\,R^{(0)(1)}), \\
D_q &= \frac{1}{i\sqrt{2}}(R^{(4)(0)} - i\,R^{(3)(0)}).
\end{aligned}\right\} \tag{9.3}$$

The A_m and B_n are then the infinitesimal transformations of the subgroup $\bar{\mathfrak{D}}_4$ for which the vector X^ν remains fixed. They are identical with the corresponding quantities in van der WAERDEN, *Die Gruppentheoretische Methode in der Quantenmechanik*, in which their commutation relations are also given.

Since A_z commutes with B_z, the two can both be simultaneously brought into diagonal form. An irreducible representation of $\bar{\mathfrak{D}}_5$ is determined uniquely by the greatest eigenvalue r of $(A_z + B_z)$, and the greatest possible eigenvalue s of $(A_z - B_z)$ when the eigenvalue r of $(A_z + B_z)$ is fixed. We would like to denote this representation by $\bar{D}^5_{(r,s)}$ (r, s). r and s are greater than or equal to zero in this, and both of them are either integer or half-integer, and $r \geq s$. The degree of this representation is:

$$N_{(r,\,s)} = \tfrac{1}{6}(2r + 3)(2s + 1)(r + s + 2)(r - s + 1). \tag{9.4}$$

The representation $\bar{D}^5_{(0,0)}$ is the identity representation of degree one. Except for that trivial representation, the representation with the smallest degree is $\bar{D}^5_{(1/2,1/2)}$ with $N_{(1/2,\,1/2)} = 4$. It must then be identical with the representation that was given already in no. **7** by the spin space $\mathfrak{R}_s$. We would like to give the infinitesimal transformations $\mathrm{P}^{(\alpha)(\beta)}$ for this representation explicitly. We assert that:

$$\mathrm{P}^{(\alpha)(\beta)}\,(\psi^K) = \tfrac{1}{2}\,\gamma^{(\alpha)(\beta)\,K}{}_L\psi^L = \tfrac{1}{4}[\,\gamma^{(\alpha)}\,\gamma^{(\beta)} - \gamma^{(\alpha)}\,\gamma^{(\beta)}]^K{}_L\psi^L\,. \tag{9.5}$$

In order to show that, we start with (7.29) and set $T = 1 + \varepsilon\, t$, corresponding to (8.16) and (8.19):

$$\gamma_{(\rho)} + \varepsilon\,\varphi_{(\rho)}{}^{(\sigma)}\,\gamma_{(\sigma)} = (1 - \varepsilon\, t)\,\gamma_{(\rho)}\,(1 + \varepsilon\, t)\,;$$

i.e.:

$$\varphi_{(\rho)}{}^{(\sigma)}\,\gamma_{(\sigma)} = \gamma_{(\rho)}\,t - t\,\gamma_{(\rho)}\,, \tag{9.6}$$

or, with (9.5):

$$\varphi_{(\rho)(\sigma)}\,\gamma^{(\sigma)} = \tfrac{1}{4}\,\varphi_{(\mu)(\nu)}\,[\gamma_{(\sigma)}, \gamma^{(\mu)(\nu)}]. \tag{9.7}$$

We must then prove (9.7). Since one must assume that $(\mu) \neq (\nu)$, one will have:

$$\gamma^{(\mu)(\nu)} = \gamma^{(\mu)}\,\gamma^{(\nu)} \qquad \text{for} \qquad (\mu) \neq (\nu),$$

such that

$$\begin{aligned}[\gamma_{(\sigma)}, \gamma^{(\mu)(\nu)}] &= \gamma_{(\sigma)}\,\gamma^{(\mu)}\,\gamma^{(\nu)} - \gamma_{(\sigma)}\,\gamma^{(\mu)}\,\gamma^{(\nu)},\\ &= 2\delta^{(\mu)}_{(\rho)}\,\gamma^{(\nu)} - \gamma^{(\mu)}\,[\gamma_{(\sigma)}\,\gamma^{(\nu)} + \gamma^{(\nu)}\,\gamma_{(\sigma)}] = 2[\delta^{(\mu)}_{(\rho)}\,\gamma^{(\nu)} - \gamma^{(\mu)}\delta^{(\nu)}_{(\rho)}].\end{aligned}$$

(9.7) is then proved.

When we restrict the transformations of $\bar{\mathfrak{D}}_5$ to the subgroup $\bar{\mathfrak{D}}_4$ by demanding that the vector X^ν should be unvarying, the representation $\bar{D}^5_{(r,s)}$ will decompose into irreducible representations of the Lorentz group:

$$\bar{D}^5_{(r,s)} = \sum_{p=s}^{r}\sum_{q=-s}^{s}\bar{D}^4_{(p,q)}\,. \tag{9.8}$$

$\bar{D}^4_{(p,q)}$ are irreducible representations of the group $\bar{\mathfrak{D}}_4^+$ (which one gets from, e.g., the symmetric spinors $c_{\mu_1\cdots\mu_k\dot{\nu}_1\cdots\dot{\nu}_l}$, as in van der WAERDEN). $\bar{D}^4_{(p,q)}$ and $\bar{D}^4_{(p,-q)}$ together define an irreducible representation of the complete Lorentz group.

From no. **5**, one will obtain the reduction of any spinor field, which might be represented by quantities ψ that transform under, e.g., $\bar{D}^5_{(r,s)}$, by complete normalization through the reduction of $\bar{D}^5_{(r,s)}$ as in (9.8). For the simple spinors in the spin space $\mathfrak{R}_s$, one has:

$$\bar{D}^5_{(1/2,1/2)} = (\bar{D}^4_{(1/2,-1/2)} + \bar{D}^4_{(1/2,1/2)})\,. \tag{9.9}$$

$\bar{D}^5_{(1/2,-1/2)}$ is then irreducible under the complete Lorentz group. $\bar{D}^5_{(1,0)}$ is then identical with the group $\bar{\mathfrak{D}}_5$ itself. One has the vector space $\mathfrak{R}$ itself as the space of representation. From (9.8), one has:

$$\bar{D}^5_{(1,0)} = \bar{D}^4_{(1,0)} + \bar{D}^4_{(1,0)}\,. \tag{9.10}$$

However, this decomposition is identical to the decomposition of the vectors in $\mathfrak{R}$ by means of the affine splitting in no. **5**, page 15.

One can obtain all the representations $\bar{D}^5_{(r,s)}$ as the tensor images of the spinor space $\mathfrak{R}_s$, since the representation that is induced in the product space $(\mathfrak{R}_s)^n$ is $(\bar{D}^5_{(1/2,-1/2)})^n$, and when it is reduced, it will read:

$$(\bar{D}^5_{(1/2,-1/2)})^n = \sum_{r=0 \text{ or } 1/2}^{n/2} \sum_{s=0 \text{ or } 1/2}^{r} c^n_{rs} \, \bar{D}^5_{(r,s)} , \tag{9.11}$$

with positive integers c^n_{rs}. In particular, for $r > s > 0$, one will have:

$$\bar{D}^5_{(r,s)} \times \bar{D}^5_{(1/2,1/2)} = \bar{D}^5_{(r+1/2,s+1/2)} + \bar{D}^5_{(r+1/2,s-1/2)} + \bar{D}^5_{(r-1/2,s+1/2)} + \bar{D}^5_{(r-1/2,s-1/2)} ,$$

and for $r = s > 0$:

$$\bar{D}^5_{(r,r)} \times \bar{D}^5_{(1/2,1/2)} = \bar{D}^5_{(r+1/2,\,r+1/2)} + \bar{D}^5_{(r+1/2,\,r-1/2)} + \bar{D}^5_{(r-1/2,\,r-1/2)} ,$$

and for $r > 0$:

$$\bar{D}^5_{(r,0)} \times \bar{D}^5_{(1/2,1/2)} = \bar{D}^5_{(r+1/2,\,1/2)} + \bar{D}^5_{(r-1/2,\,1/2)} ,$$

and

$$\bar{D}^5_{(0,0)} \times \bar{D}^5_{(1/2,1/2)} = \bar{D}^5_{(1/2,\,1/2)} .$$

The group $\mathfrak{D}_5$, with its subgroup $\mathfrak{D}_4$, as a group that is isomorphic to $\bar{\mathfrak{D}}_5$ ($\bar{\mathfrak{D}}_4$, resp.), possesses the same representations as $\bar{\mathfrak{D}}_5$ and $\bar{\mathfrak{D}}_4$, but with the different meanings that were given in no. **5**. We denote those representations by $D^5_{(r,\ s)}$. From no. **8**, the infinitesimal rotations $S_{(\alpha)}^{(\beta)}$ act upon only indices in parentheses and spinor indices, but not upon the indices μ, k. For a simple spinor – e.g., ψ^K – one also has (9.5) for $P^{(\alpha)(\beta)}$, instead of $\mathrm{P}^{(\alpha)(\beta)}$.

10. Parallel displacement and differentiation of measure. – Let Q' be a point in V that is close to Q. Let the vector $\overline{QQ'}$ be given by the components $\varepsilon\,\xi^\mu$ (ε is a small number). We call an infinitesimal homomorphism of $\mathfrak{R}$ (and the associated $\mathfrak{R}_s$) at the point Q to $\mathfrak{R}$ ($\mathfrak{R}_s$, resp.) at the point Q' a *parallel displacement* of a vector $\alpha_{(\rho)}$ when one has, from no. **2**:

$$\Pi\,\lambda\,\alpha_{(\rho)} = \lambda\,\Pi\alpha_{(\rho)} .$$

Under the group $\mathfrak{P}$, the square of its length $\alpha_{(\rho)}\,\alpha^{(\rho)}$ will not change when $\xi^\mu X_\mu = 0$, and it will change by $\varepsilon\,\lambda\,\Pi\,(\alpha_{(\rho)}\,\alpha^{(\rho)})$ when $\xi^\mu X_\mu = \lambda\,X^\mu$. While the square of the length does not change for a normal vector then, for a non-normal vector, it will change under parallel displacement in the direction of the vector X^μ precisely as it must, based upon its homogeneity properties. The parallel displacement thus-defined will then be an extension of the usual concept of the equality of two vectors at different points. If follows from the requirement that was imposed that one will have:

$$\delta_\| \, \alpha_{(\rho)} = - \varepsilon \, \xi^\mu \, (\tfrac{1}{2} \, \omega_\mu{}^{(\lambda)}{}_{(\nu)} \, \mathrm{P}_{(\lambda)}{}^{(\nu)} - Y_\mu \, \Pi) \, \alpha_{(\rho)} \tag{10.1}$$

for the change of the components $\alpha_{(\rho)}$, with still-undetermined coefficients $\omega_\mu{}^{(\lambda)}{}_{(\nu)}$.

Under the transition from the bracketed indices in (10.1) to the unbracketed ones, one should note that it will follow from $\alpha_{(\rho)}(Q) \to \alpha'_{(\rho)}(Q') = \alpha_{(\rho)}(Q) + \delta_\| \, \alpha_{(\rho)}$ that:

$$\alpha_\nu(Q) = g_\nu^{(\rho)}(Q) \, \alpha_{(\rho)}(Q) \to g_\nu^{(\rho)}(Q) \, \alpha'_{(\rho)}(Q) \, .$$

Now, one has:

$$g_\nu^{(\rho)}(Q) = g_\nu^{(\rho)}(Q') - \varepsilon \, \xi^\mu \, g_{\nu|\mu}^{(\rho)}(Q') ,$$

such that one will get:

$$\delta_\| \, \alpha_\nu = -\tfrac{1}{2} \, \varepsilon \, \xi^\mu \, \omega_{\mu\lambda\eta} \, \mathrm{P}^{\lambda\eta} \, \alpha_\nu - \varepsilon \, \xi^\mu \, g_{\nu|\mu}^{(\rho)} \, g^\sigma{}_{(\rho)} \, \alpha_\sigma + \varepsilon \, \xi^\mu \, Y_\mu \, \Pi \, \alpha_\nu \, . \tag{10.2}$$

Since the $\omega_{\mu\lambda\nu}$ are antisymmetric in λ and η, one will have:

$$\tfrac{1}{2} \, \omega_{\mu\lambda\nu} \, \mathrm{P}^{\lambda\eta} \, \alpha_\nu = \omega_{\mu\lambda\nu} \, \mathfrak{Q}^{\lambda\eta} \, \alpha_\nu \, ,$$

and thus, from (10.2):

$$\delta_\| \, \alpha_\nu = - \varepsilon \, \xi^\mu \, [(\omega_\mu{}^\lambda{}_\nu + g_{\eta|\mu}^{(\rho)} \, g^\lambda{}_{(\rho)}) \, \mathfrak{Q}_\lambda{}^\eta - Y_\mu \, \Pi] \, \alpha_\nu \, , \tag{10.3}$$

which can also be written:

$$\delta_\| \, \alpha_\nu = -\varepsilon \, \xi^\mu \, [\, \Gamma_{\mu\eta}^\lambda \, \mathfrak{Q}_\lambda{}^\eta - Y_\mu \, \Pi] \, \alpha_\nu \, , \tag{10.4}$$

with:

$$\Gamma_{\mu\eta}^\lambda = \omega_\mu{}^\lambda{}_\nu + g_{\nu|\mu}^{(\rho)} \, g^\lambda{}_{(\rho)} \, . \tag{10.5}$$

With $g_{\nu\lambda} \, \Gamma_{\mu\eta}^\lambda = \Gamma_{\nu,\,\mu\eta}$, the last equation will imply that:

$$\Gamma_{\nu,\,\mu\eta} = \omega_{\nu\mu\eta} + g_{\eta|\mu}^{(\rho)} \, g_{\nu\,(\rho)} \, .$$

Due to the antisymmetry of $\omega_{\nu\mu\eta}$ in ν and η, it will then follow from this that:

$$\Gamma_{\nu,\,\mu\eta} + \Gamma_{\eta,\,\mu\nu} = g_{\eta|\mu}^{(\rho)} \, g_{\nu\,(\rho)} + g_{\nu|\mu}^{(\rho)} \, g_{\eta\,(\rho)} = g_{\eta\nu\,|\,\mu} \, . \tag{10.6}$$

For an arbitrary tensor or spinor $\alpha_{\cdots}^{\cdots}$, from (10.1) and (10.2), one will have, in full generality:

$$\delta_\| \, \alpha_{\cdots}^{\cdots} = \varepsilon \, \xi^\mu (\Gamma_\mu - Y_\mu \Pi) \, \alpha_{\cdots}^{\cdots} \, , \tag{10.7}$$

with:

$$\Gamma_\mu = \Gamma^\lambda_{\mu\eta}\,\mathfrak{D}^\eta_\lambda + \tfrac{1}{2}\,\omega_\mu{}^{(\lambda)}{}_{(\eta)} P_{(\lambda)}^{(\eta)} . \tag{10.8}$$

We make the following further assumption about parallel displacement: If one has yet another neighboring point Q'' (let the infinitesimal vector $\overline{QQ''}$ be given by $\varepsilon'\eta^\mu$), along with Q', then when $\varepsilon'\eta^\mu$ is parallel-displaced from Q to Q', it shall determine the same point Q''' ($\overline{QQ'''}$ is the parallel displacement of $\varepsilon'\eta^\mu$) that $\varepsilon\,\xi^\mu$ determines when it is parallel-displaced from Q to Q''. That means that:

$$\varepsilon\,\varepsilon'\,\xi^\mu\,\Gamma^\lambda_{\mu\nu}\,\eta^\nu = \varepsilon\,\varepsilon'\,\eta^\mu\,\Gamma^\lambda_{\mu\nu}\,\xi^\nu,$$

and therefore:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \tag{10.9}$$

is symmetric in μ and ν.

The $\Gamma^\lambda_{\mu\nu}$ and $\omega_{\mu\lambda\nu}$ do not define tensor components, since they represent the relationship between two vectors at two different points. It then follows from (10.9) and (10.6) that:

$$\Gamma_{\nu,\,\sigma\rho} = \tfrac{1}{2}(g_{\nu\sigma\,|\,\rho} + g_{\nu\rho\,|\,\sigma} - g_{\sigma\rho\,|\,\nu}). \tag{10.10}$$

It will follow from (10.5) when one subtracts the equation with μ and η switched that:

$$\omega_{\mu\lambda\nu} - \omega_{\eta\lambda\mu} + (g^{(\rho)}_{\eta\,|\mu} + g^{(\rho)}_{\mu\,|\eta})\,g_{\lambda(\rho)} = 0;$$

or, with (6.1):

$$\omega_{\mu\lambda\nu} - \omega_{\eta\lambda\mu} + g^{(\rho)}{}_{\eta\mu}\,g_{\lambda(\rho)} = g_{\lambda\eta\mu} . \tag{10.11}$$

One ultimately obtains from this that:

$$\omega_{\nu\lambda\mu} = \tfrac{1}{2}(g_{\lambda\mu\nu} + g_{\mu\nu\lambda} - g_{\nu\lambda\mu}). \tag{10.12}$$

Parallel displacement is linked uniquely with the metric by that.

Parallel displacement in the four-dimensional world W will be defined in a manner that is precisely analogous to the way that it is defined in V, except that the terms with the operator Π will not appear. All of the formulas above will remain true with a corresponding reinterpretation, except that the quantities in W will be denoted by $\overset{4}{\Gamma}_{k\mu}$, $\overset{4}{\omega}_{klm}$, etc., to distinguish them. (One does *not* have $\overset{4}{\Gamma}_k = g^\mu_k\,\Gamma_\mu$, etc., then, but $\Gamma_k = g^\mu_k\,\Gamma_\mu$ must be distinguished from $\overset{4}{\Gamma}_k$.)

If a vector field or tensor field $\alpha(Q)$ (from which, we suppress the indices) is given then one can define the difference between $\alpha(Q')$ at the point Q' and its parallel-translate $\alpha'(Q')$ from Q to the point Q', which must be a vector or tensor again. We define the operator ∇_μ by:

$$\alpha(Q') - \alpha'(Q') = \varepsilon\,\xi^\mu\,\nabla_\mu\,\alpha .$$

One refers to it as *differentiation of measure*, to distinguish it from ordinary differentiation $\partial / \partial X^{\mu}$. While $\partial / \partial X^{\mu}$ is *not* a tensor operation, since it compares tensor components from $\mathfrak{R}$ at the point Q and the $\mathfrak{R}$ at the distinct point Q', nevertheless, ∇_μ is a tensor operation; i.e., $\nabla_\mu \alpha$ is a tensor with rank that is higher than that of α by one, which will also be called the *gradient* of α. Just as we abbreviate $\partial\alpha / \partial X^{\mu}$ by $\alpha_{|\mu}$, we will shorten differentiation of measure $\nabla_\mu \alpha$ to $\alpha_{\|\mu}$. With (10.7), it will then follow that:

$$\alpha_{\|\mu} = \alpha_{|\mu} + (\Gamma_\mu - Y_\mu \Pi)\, \alpha; \tag{10.13}$$

e.g.:

$$\alpha^{\lambda}{}_{\|\mu} = \alpha^{\lambda}{}_{|\mu} + \Gamma^{\lambda}_{\mu\nu}\alpha^{\nu} - Y_\mu \Pi \alpha^{\lambda}; \quad \alpha^{(\lambda)}{}_{\|\mu} = \alpha^{(\lambda)}{}_{|\mu} + \omega_{\mu}{}^{(\lambda)}{}_{(\nu)}\, \alpha^{\nu} - Y_\mu \Pi \alpha^{(\lambda)}. \tag{10.14}$$

The operator $\overset{4}{\nabla}_k$ of affine differentiation of measure is defined correspondingly:

$$\alpha^{k}{}_{\|l} = \alpha^{k}{}_{|l} + \overset{4}{\Gamma}{}^{k}_{lm}\, \alpha^{m}, \qquad \alpha^{(k)}{}_{\|l} = \alpha^{(k)}{}_{|l} + \overset{4}{\omega}{}_{l}{}^{(k)}{}_{(m)}\, \alpha^{m}. \tag{10.15}$$

This is identical with equation (10.6), since $g_{\eta\nu\|\mu} = 0$. Due to the tensor character of the differentiation of measure, one will then have, in general, that:

$$\left.\begin{array}{llll} g^{\lambda\mu}{}_{\|\nu} = 0, & g_{\lambda\mu\|\nu} = 0, & g^{\lambda}_{\mu\|\nu} = 0, & g_{(\lambda)(\mu)\|\nu} = 0, \\ \gamma^{(\sigma)}{}_{\|\nu} = 0, & \gamma^{\sigma}{}_{\|\nu} = 0, & \gamma_{\sigma\|\sigma} = 0, & \beta_{\|\nu} = 0. \end{array}\right\} \tag{10.16}$$

In order to find the connection between $\overset{4}{\nabla}_k$ and ∇_μ, we define $\Gamma_{(\nu)} = g^{\mu}_{(\nu)}\, \Gamma_\mu$. It will then follow from (10.8) for tensors and spinors in the fünfbein representation that:

$$\Gamma_{(\nu)} = \tfrac{1}{2}\, \omega_{(\nu)(\lambda)(\mu)}\, P^{(\lambda)(\mu)}. \tag{10.17}$$

With the use of (10.12) and (6.5), (6.15), (6.16), it will then follow that:

$$\left.\begin{array}{l} \omega_{(l)(m)(n)} = \overset{4}{\omega}{}_{(l)(m)(n)}, \\ \omega_{(0)(m)(n)} = -\tfrac{1}{2} J^{1/2} F_{(m)(n)}, \\ \omega_{(l)(0)(n)} = -\tfrac{1}{2} J^{1/2} F_{(l)(n)}, \\ \omega_{(0)(0)(n)} = \tfrac{1}{2} J^{-1} J_{|(n)}. \end{array}\right\} \tag{10.18}$$

With that, it will follow from (10.17) that:

$$\Gamma_{(0)} = -\tfrac{1}{4} J^{1/2}\, F_{(l)(m)}\, P^{(l)(m)} + \tfrac{1}{2}\, J^{-1}\, J_{|(m)}\, P^{(0)(m)} \tag{10.19}$$

and

$$\Gamma_{(n)} = \overset{4}{\Gamma}{}_{(n)} + \tfrac{1}{2} J^{1/2}\, F_{(m)(n)}\, P^{(0)(m)}. \tag{10.20}$$

Despite several later examples, the reduction of the differentiation in measure of a vector shall be carried out explicitly. We introduce the normalized vector with the components $\underline{\alpha}^\nu = H_{\eta-1}\,\alpha^\nu$. It will then follow step-wise with $H_\eta = e^{\ln\eta\,\Pi}$ that:

$$\begin{aligned}\alpha^{(\lambda)}{}_{\|\mu} &= (e^{\ln\eta\,\Pi}\,\alpha^{(\lambda)})_{|\mu} + \omega_\mu{}^{(\lambda)}{}_{(\nu)}\,e^{\ln\eta\,\Pi}\,\alpha^{(\lambda)} - Y_\mu\,e^{\ln\eta\,\Pi}\,\Pi\,\underline{\alpha}^\nu \\ &= (\ln\eta)_{|\mu}\,\Pi\,H_\eta\,\underline{\alpha}^\nu + H_\eta\,\underline{\alpha}^{(\lambda)}{}_{|\mu} + \omega_\mu{}^{(\lambda)}{}_{(\nu)}\,H_\eta\underline{\alpha}^\nu - Y_\mu\,H_\eta\,\Pi\,\underline{\alpha}^\nu,\end{aligned}$$

or

$$\alpha^{(\lambda)}{}_{\|(\mu)} = H_\eta\,[(\ln\eta)_{|(\mu)} - Y_\mu\,\Pi\,\underline{\alpha}^\nu + \omega_\mu{}^{(\lambda)}{}_{(\nu)}\,\underline{\alpha}^\nu]. \tag{10.21}$$

It will follow from this, with (6.10), that:

$$\alpha^{(\lambda)}{}_{\|(\mu)} = H_\eta\,[\varphi_{(\mu)}\,\Pi\,\underline{\alpha}^\nu + \underline{\alpha}^{(\lambda)}{}_{|(\mu)} + \omega_{(\mu)}{}^{(\lambda)}{}_{(\nu)}\,\underline{\alpha}^\nu]. \tag{10.22}$$

Now, $\underline{\alpha}^{(\lambda)}{}_{|(0)} = \underline{\alpha}^{(\lambda)}{}_{|\nu}\,g^{\nu}_{(0)} = J^{-1/2}\,\underline{\alpha}^{(\lambda)}{}_{|\nu}\,X^\nu = 0$, since $\underline{\alpha}^{(\lambda)}$ is homogeneous of degree zero. One has $\varphi_{(0)} = 0$, moreover.

It will then follow from this, with (10.18), that:

$$\left.\begin{aligned}\alpha^{(0)}{}_{\|(0)} &= H_\eta\,\tfrac{1}{2}\,J^{-1}J_{|n}\,\underline{\alpha}^n, \\ \alpha^{(0)}{}_{\|m} &= H_\eta\,[-\varphi_m\,\Pi\,\underline{\alpha}^{(0)} + \alpha^{(0)}{}_{|m} + \tfrac{1}{2}\,J^{1/2}F_{nm}\,\underline{\alpha}^n], \\ \alpha^{l}{}_{\|(0)} &= H_\eta\,[-\tfrac{1}{2}\,J^{1/2}F^{l}{}_{n}\,\underline{\alpha}^n - \tfrac{1}{2}\,J^{1/2}J'^{l}\,\underline{\alpha}^{(0)}], \\ \alpha^{l}{}_{\|m} &= H_\eta\,[-\varphi_m\,\Pi\,\underline{\alpha}^l + \alpha^{l}{}_{|_4 m} + \tfrac{1}{2}\,J^{1/2}F_m{}^{l}\,\underline{\alpha}^{(0)}].\end{aligned}\right\} \tag{10.23}$$

One cares to refer to the operation:

$$(\beta^{\cdots}_{\cdots})_{\|\_k} = (\beta^{\cdots}_{\cdots})_{\|_4 k} - \varphi_k\,\Pi\,(\beta^{\cdots}_{\cdots}) \tag{10.24}$$

as *gauge differentiation in measure*, since Π represents the infinitesimal gauge transformation of $\beta^{\cdots}_{\cdots}$. The last equation in (10.23) can then be written:

$$\alpha^{l}{}_{\|m} = H_\eta\,[\underline{\alpha}^{l}{}_{\|\_m} + \tfrac{1}{2}\,J^{1/2}F_m{}^{l}\underline{\alpha}^{(0)}]. \tag{10.25}$$

Likewise, the second equation in (10.23) can be written:

$$\alpha^{(0)}{}_{\|m} = H_\eta\,[\underline{\alpha}^{(0)}{}_{\|\_m} + \tfrac{1}{2}\,J^{1/2}F_{nm}\,\underline{\alpha}^{(0)}]. \tag{10.26}$$

If one forms the expression $(m\,\alpha^\mu)_{\|\mu}$ (α^μ is a vector) from (5.20) with m then one should note that m contains only factors whose measure derivatives vanish, such that $m_{\|\mu} = 0$. It will then follow that:

$$(m\,\alpha^\mu)_{\|\mu} = m\,\alpha^\mu{}_{\|\mu} + m_{\|\mu}\,\alpha^\mu = m\,\alpha^\mu{}_{\|\mu}. \tag{10.27}$$

If we set $\alpha^\mu = m^{-1}\,\mathfrak{w}^\mu$ (in which $\mathfrak{w}^\mu$ is a tensor density) then that will imply:

$$\mathfrak{w}^\mu{}_{\|\mu} = \mathfrak{w}^\mu{}_{|\mu} + (\Gamma^\lambda_{\lambda\mu} - m^{-1}\,m_{|\mu})\,\mathfrak{w}^\mu - Y_\mu\,\Pi\,\mathfrak{w}^\mu.$$

Since $m\,\Gamma^\lambda_{\lambda\mu} = m_{|\mu}$, as one checks, it will then follow that:

$$\mathfrak{w}^\mu{}_{\|\mu} = \mathfrak{w}^\mu{}_{|\mu} - Y_\mu\,\Pi\,\mathfrak{w}^\mu. \tag{10.28}$$

In order to define the measure derivative of an arbitrary tensor density, we first calculate $\mathfrak{w}_{\|\mu}$; i.e., the derivative of a scalar density. It follows from (α^μ is a normal vector in this):

$$(\mathfrak{w}\;\alpha^\mu)_{\|\mu} = (\mathfrak{w}\,\alpha^\mu)_{|\mu} - Y_\mu\;\alpha^\mu\,\Pi\,\mathfrak{w}$$

that one has the relation:

$$\mathfrak{w}_{|\mu}\,\alpha^\mu + \mathfrak{w}\;\alpha^\mu{}_{|\mu} - Y_\mu\,\alpha^\mu\,\Pi\,\mathfrak{w} = \mathfrak{w}_{\|\mu}\,\alpha^\mu + \mathfrak{w}\;\alpha^\mu{}_{|\mu} + \mathfrak{w}\,\Gamma^\lambda_{\lambda\mu}\,\alpha^\mu,$$

or, since α^μ is arbitrary:

$$\mathfrak{w}_{\|\mu} = \mathfrak{w}_{|\mu} - \Gamma^\lambda_{\lambda\mu}\,\mathfrak{w} - Y_\mu\,\Pi\,\mathfrak{w} = \mathfrak{w}_{|\mu} - m^{-1}\,m_{|\mu}\,\mathfrak{w} - Y_\mu\,\Pi\,\mathfrak{w}\,. \tag{10.29}$$

It will then follow from this that for an arbitrary tensor density:

$$\mathfrak{t}^{\cdots}_{\cdots\|\mu} = \mathfrak{t}^{\cdots}_{\cdots|\mu} - \Gamma^\lambda_{\lambda\mu}\mathfrak{t}^{\cdots}_{\cdots} + \Gamma_\mu\mathfrak{t}^{\cdots}_{\cdots} - Y_\mu\,\Pi\,\mathfrak{w}\,, \tag{10.30}$$

in which Γ_μ is an operator that acts upon the indices of $\mathfrak{t}^{\cdots}_{\cdots}$ as if $\mathfrak{t}^{\cdots}_{\cdots}$ were a tensor.

11. Curvature. – Parallel displacement of a vector along a closed path will not lead back to its initial position, in general. For that reason, a circuit around an infinitesimal surface element shall be examined. Along with the point Q, let two neighboring points Q' and Q'' be given by way of the infinitesimal vectors $\overrightarrow{QQ'} = (\xi^\mu)$ and $\overrightarrow{QQ''} = (\eta^\mu)$. A fourth point Q''', together with Q, Q', Q'' will define an infinitesimal rectangle. One introduce the notation $\overrightarrow{Q'Q'''} = (\eta^\mu + \delta\eta^\mu)$ and $\overrightarrow{Q''Q'''} = (\xi^\mu + \delta\xi^\mu)$. One will then have $\overrightarrow{QQ'''} = \overrightarrow{QQ''} + \overrightarrow{Q''Q'''} = \overrightarrow{QQ'} + \overrightarrow{Q'Q'''} = (\eta^\mu + \xi^\mu + \delta\xi^\mu) = (\xi^\mu + \eta^\mu + \delta\eta^\mu)$ relative Q, such that one will have $\delta\eta^\mu = \delta\xi^\mu$.

For a vector or tensor α (the indices might be suppressed once more), parallel displacement of Q to Q' means the application of the operator $1 - \xi^\mu\,(\Gamma_\mu - Y_\mu\,\Pi)$:

$$\alpha \to \alpha' = [1 - \xi^\mu\,\Gamma_\mu\,(Q) - \xi^\mu\,Y_\mu\,(Q)\,\Pi(Q)]\;\alpha\,.$$

Parallel displacement of Q' to Q'' will yield:

$$\alpha' \to \overset{1}{\alpha'''} = [1 - (\eta^\mu + \delta\eta^\mu)(\Gamma_\mu(Q') - Y_\mu(Q')\,\Pi(Q'))]\,\alpha',$$

such that ultimately, after parallel displacement around the path $Q \to Q' \to Q''$:

$$\begin{aligned}\overset{1}{\alpha'''} &= [1 - (\eta^\nu + \delta\eta^\nu)(\Gamma_\nu(Q') - Y_\nu(Q')\,\Pi(Q'))] \\ &\quad \times [1 - \xi^\mu(\Gamma_\mu(Q) - Y_\mu(Q)\,\Pi(Q))]\,\alpha.\end{aligned} \tag{11.1}$$

Likewise, the path $Q \to Q'' \to Q'''$ will give:

$$\begin{aligned}\overset{2}{\alpha'''} &= [1 - (\xi^\mu + \delta\xi^\mu)(\Gamma_\mu(Q'') - Y_\mu(Q'')\,\Pi(Q''))] \\ &\quad \times [1 - \eta^\nu(\Gamma_\nu(Q) - Y_\nu(Q)\,\Pi(Q))]\,\alpha.\end{aligned} \tag{11.2}$$

Since the difference between (11.1) and (11.2) is the difference of two tensors at the same point Q''', it will be a tensor:

$$\eta^\nu \xi^\mu\,[\Gamma_{\mu|\nu} - \Gamma_{\nu|\mu} + (Y_\nu\,\Pi)_{|\mu} - (Y_\mu\,\Pi)_{|\nu} + \Gamma_\nu\,\Gamma_\mu - \Gamma_\mu\,\Gamma_\nu]\,\alpha\,. \tag{11.3}$$

Since the length of a normal vector is invariant under parallel displacement, the operator that appears in (11.3), namely:

$$\eta^\nu \xi^\mu\,(\Gamma_{\mu|\nu} - \Gamma_{\nu|\mu} + [\Gamma_\nu, \Gamma_\mu]) = \tfrac{1}{2}(\xi^\mu\,\eta^\nu - \xi^\mu\,\eta^\nu)(\Gamma_{\mu|\nu} - \Gamma_{\nu|\mu} + [\Gamma_\nu, \Gamma_\mu]),$$

must be an infinitesimal rotation. If one denotes the surface element $\xi^\mu\,\eta^\nu - \xi^\mu\,\eta^\nu$ by $d\sigma^{\nu\mu}$ then the parallel displacement around $d\sigma^{\nu\mu}$ will be equivalent to the infinitesimal transformation:

$$\begin{aligned}&\tfrac{1}{4}\,d\sigma^{\nu\mu}\,R_{\nu\mu\lambda\rho}\,\mathrm{P}^{\lambda\rho} - \tfrac{1}{2}F_{\nu\mu}\,\Pi + \tfrac{1}{2}(Y_\nu\,\Pi_{|\mu} - Y_\mu\,\Pi_{|\nu});\\ &R_{\nu\mu\lambda\rho}\,\mathrm{P}^{\lambda\rho} = 2\,(\Gamma_{\mu|\nu} - \Gamma_{\nu|\mu} + [\Gamma_\nu, \Gamma_\mu]).\end{aligned} \tag{11.4}$$

One refers to $R_{\nu\mu\lambda\rho}$ as the *curvature tensor*. From its very definition, it is antisymmetric in ν, μ and λ, ρ:

$$R_{\mu\nu\lambda\rho} = -\,R_{\nu\mu\lambda\rho} = -\,R_{\nu\mu\rho\lambda}\,. \tag{11.5}$$

One then has the further symmetry properties:

$$R_{\mu\nu\lambda\rho} + R_{\nu\lambda\mu\rho} + R_{\lambda\mu\nu\rho} = 0 \tag{11.6}$$

and

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}\,. \tag{11.7}$$

(11.7) is a result of (11.6) and (11.5). In order to show that, we consider the equations that are obtained from (11.6) by cyclic permutation of μ, ν, λ, ρ:

$$R_{\nu\lambda\rho\mu} + R_{\lambda\rho\nu\mu} + R_{\rho\nu\lambda\mu} = 0, \tag{11.6a}$$

$$R_{\lambda\rho\mu\nu} + R_{\rho\mu\lambda\nu} + R_{\mu\lambda\rho\nu} = 0, \tag{11.6b}$$

$$R_{\rho\mu\nu\lambda} + R_{\mu\nu\rho\lambda} + R_{\nu\rho\mu\lambda} = 0. \tag{11.6c}$$

If one adds (11.6) and (11.6a) and then subtracts (11.6b) and (11.6c) then it follow, with (11.5), that:

$$2\, R_{\mu\nu\lambda\rho} - 2\, R_{\lambda\rho\mu\nu} = 0.$$

In order to prove (11.6), we consider the following parallelepiped: Let ξ^μ, η^μ, ζ^μ be three infinitesimal vectors that determine the neighboring points Q_{100}, Q_{010}, Q_{001} from the point Q_{000} . From the demands that were imposed upon parallel displacement, when ξ^μ is parallel-displaced along η^μ, and η^μ is parallel-displaced along ξ^μ, that will determine the same point Q_{110} ; the points Q_{101} and Q_{011} are defined analogously. If one now imagines that ξ^μ is parallel-displaced along η^μ then one will get the vector $\overrightarrow{Q_{010}\,Q_{110}}$. If one now further parallel-displaces $\overrightarrow{Q_{010}\,Q_{110}}$ along the segment $\overrightarrow{Q_{010}\,Q_{011}}$ then one will obtain a vector that is determined from a point Q'' to Q_{011} . However, one will also obtain the same point Q'' when one parallel-displaces the vector ζ^μ along η^μ and then along $\overrightarrow{Q_{010}\,Q_{110}}$. One will find two more points Q' and Q''' correspondingly when one performs the same process with the other vectors. The three points Q', Q'', Q''' define a triangle. However, the three sides of the triangle are the vectors $R_{\mu\nu\lambda}{}^{\rho}\,\xi^\mu\,\eta^\nu\,\zeta^\lambda$, $R_{\mu\nu\lambda}{}^{\rho}\,\eta^\mu\,\zeta^\nu\,\xi^\lambda$, $R_{\mu\nu\lambda}{}^{\rho}\,\zeta^\mu\,\xi^\nu\,\eta^\lambda$. Their vector sum is zero then, and (11.6) will follow from that directly.

From (10.13), the operator ∇_μ of measure differentiation reads:

$$\nabla_\mu = \frac{\partial}{\partial X^\mu} + \Gamma_\mu - Y_\mu\,\Pi.$$

That will then imply the commutation relations:

$$[\nabla_\mu, \nabla_\nu] = \tfrac{1}{2} R_{\mu\nu\lambda\rho}\,\mathrm{P}^{\lambda\rho} - F_{\mu\nu}\,\Pi + Y_\mu\,\Pi_{|\,\nu} - Y_\nu\,\Pi_{|\,\mu}\,. \tag{11.8}$$

The commutability of measure differentiation and the path-independence of parallel-displacement are then equivalent to each other. Since equation (11.8) is a tensor equation, one also have:

$$[\nabla_{(\mu)}, \nabla_{(\nu)}] = \tfrac{1}{2} R_{(\mu)(\nu)(\lambda)(\rho)}\,\mathrm{P}^{(\lambda)(\rho)} - F_{(\mu)(\nu)}\,\Pi + Y_{(\mu)}\,\Pi_{|\,(\nu)} - Y_{(\nu)}\,\Pi_{|\,(\mu)}\,. \tag{11.9}$$

One can pose relations in the affine world W that are entirely analogous to the ones in V, which do not need to be referred to as "extra," since one needs only to replace the Greek indices with Latin ones, ∇_μ, with $\overset{4}{\nabla}_k$, $R_{\mu\nu\lambda\rho}$, with $\overset{4}{R}_{mnlr}$, etc., while terms with the operator Π will not appear.

The parts of the curvature tensor into which it splits affinely can be read off easily from (11.9). Similar to (10.23), one obtains for a normal vector $\alpha^{(\lambda)}$:

$$\alpha^{(l)}{}_{\|\,(n)\,\|\,(m)} = (\alpha^{(l)}{}_{\|(m)})_{\|_4(n)} + \omega_{(m)}{}^{(l)}{}_{(0)}\,\alpha^{(0)}{}_{\|\,(n)} + \omega_{(m)(n)(0)}\,\alpha^{(l)}{}_{\|\,(0)}\,.$$

Since, from (11.9), one has:

$$\alpha^{(l)}{}_{\|(n)\|(m)} - \alpha^{(l)}{}_{\|(m)\|(n)} = R_{(m)(n)}{}^{(l)}{}_{(r)}\,\alpha^{(l)} + R_{(m)(n)}{}^{(l)}{}_{(0)}\,\alpha^{(0)},$$

one can read off the following two equations, along with (10.18) and (10.23):

$$R_{mnlr} = \overset{4}{R}_{mnlr} + \tfrac{1}{4} J\,(F_{ml}\,F_{rn} + F_{nl}\,F_{mr} + 2\,F_{mn}\,F_{lr}), \tag{11.10}$$

$$R_{(0)nlr} = \tfrac{1}{2} J^{1/2} F_{lr\|_4 n} + \tfrac{1}{4} J\,(J_{|l}\,F_{nr} + J_{|r}\,F_{ln} + 2\,J_{|n}\,F_{lr}). \tag{11.11}$$

On the other hand, for a normal vector $\alpha^{(\rho)}$ whose affine part is $\alpha^{(r)} = 0$, it will follow that:

$$\alpha^{(l)}{}_{\|(0)\|m} = (\alpha^{(l)}{}_{\|(0)})_{\|_4 m} + \omega_{(m)(0)}{}^{(s)}\,a^{(l)}{}_{\|(s)},$$

$$\alpha^{(l)}{}_{\|(m)\|(0)} = \omega_{(0)}{}^{(l)}{}_{(s)}\,a^{(s)}{}_{\|(m)} + \omega_{(0)}{}^{(l)}{}_{(0)}\,a^{(l)}{}_{\|(m)} + \omega_{(0)(m)}{}^{(s)}\,a^{(l)}{}_{\|(s)} + \omega_{(0)(m)(0)}\,a^{(l)}{}_{\|(0)},$$

and

$$\alpha^{(l)}{}_{\|(0)\|(m)} - \alpha^{(l)}{}_{\|(m)\|(0)} = R_{(m)(0)}{}^{(l)}{}_{(0)}\,\alpha^{(0)},$$

and with the same equations are above, one will get the result that:

$$R_{(0)n(l)r} = -\tfrac{1}{2} J^{1/2} J_{|r\|_4 n} + \tfrac{1}{4} J\,F_{pn}\,F^{p}{}_{r} + \tfrac{1}{4} J^{-2}\,J_{|n}\,J_{|r}\,. \tag{11.12}$$

No further components can be calculated, due to the symmetry properties of the curvature tensor. (11.10) to (11.12) then determine the tensor $R_{\mu\nu\lambda\rho}$ completely.

Now, one sets:

$$R_{\mu\rho} = R_{\mu}{}^{\nu}{}_{\nu\rho} = R_{\mu\nu\lambda\rho}\,g^{\nu\lambda} = R_{\rho\mu}\,. \tag{11.13}$$

One will then have the relation:

$$R_{(\mu)(\rho)} = R_{(\mu)}{}^{(n)}{}_{(n)(\rho)} + R_{(\mu)(0)(0)(\rho)}\,. \tag{11.14}$$

That implies that:

$$R_{mr} = \overset{4}{R}_{mr} + \tfrac{1}{2} J\,F_{mn}\,F_{r}^{\;n} + \tfrac{1}{2} J^{1/2} J_{|r\|_4 m} - \tfrac{1}{4} J^{-2}\,J_{|m}\,J_{|r} \tag{11.15}$$

and

$$R_{(0)r} = \tfrac{1}{2} J^{1/2} F^{n}{}_{|r\|_4 n} + \tfrac{1}{4} J^{-1/2}\,J_{|n}\,F^{n}{}_{r}\,. \tag{11.16}$$

and [1]:

$$R_{(0)(0)} = \tfrac{1}{2} J^{-1} J^{|r}{}_{\|_4 r} - \tfrac{1}{2} J\,F_{pr}\,F^{pr} - \tfrac{1}{4} J^{-2}\,J_{|r}\,J^{|r}\,. \tag{11.17}$$

The contraction of $R_{\mu\rho}$ is the curvature scalar:

$$R = R^{\mu}{}_{\mu} = R^{m}{}_{m} + R_{(0)(0)}\,. \tag{11.18}$$

Hence, from (11.15) and (11.17):

([1]) We always mean: $F^{\|\mu} = F_{\|\nu}\,g^{\nu\mu}$, $F_{\|\mu} = F^{\|\nu}\,g_{\nu\mu}$, $F^{\|k} = F_{\|i}\,g^{ik}$, etc.

$$R = \overset{4}{R} + \tfrac{1}{4} J\, F_{mn}\, F^{mn} + J^{-1} J^{|m}{}_{\|_4 m} - \tfrac{1}{2} J^{-2}\, J_{|m}\, J^{|m}\,. \tag{11.19}$$

12. Variational principles and field equations. – The metric in the space V is given by the metric field $g_\mu^{(\nu)}$. Later, we will also have to deal with matter fields $\psi_{(M)}$, where the $\psi_{(M)}$ can be tensors and spinors. [(M) symbolically represents the various matter fields and their various associated indices. If (M) appears twice then one must sum over them, as with any other indices.] We seek to arrive at the field equations for all fields from a variational principle:

$$\delta \int_B (\mathfrak{G} + \mathfrak{L})\, d\tau = 0, \tag{12.1}$$

in which B is a normal domain in V, $\mathfrak{G}$, $\mathfrak{L}$ are invariant densities that are yet to be described in detail, and the variations $\delta\psi_{(M)}$ and $\delta g_\mu^{(\nu)}$ are set to zero on the boundary R_Λ. For $\mathfrak{G}$, we would like to focus on only those functions that depend upon only $g_\mu^{(\nu)}$, $g_{\mu|\rho}^{(\nu)}$, X^ν, and $g_{\mu|\rho|\sigma}^{(\nu)}$. The $g_{\mu|\rho|\sigma}^{(\nu)}$ might appear only linearly in it with coefficients that are free of the $g_{\mu|\rho}^{(\nu)}$. For $\mathfrak{L}$, we allow only functions of $\psi_{(M)}$, $\psi_{(M)\,|\,\nu}$, X^ν, $g_\mu^{(\nu)}$, and $g_{\mu|\rho}^{(\nu)}$.

Since $\mathfrak{G}$ depends upon the $g_{\mu|\rho|\sigma}^{(\nu)}$ linearly with coefficients that are free of the $g_{\mu|\rho}^{(\nu)}$, one can use GAUSS's law and partial integration to convert:

$$\int_B \mathfrak{G}\, d\tau = \int_B \mathfrak{K}\, d\tau + \int_{R_\Lambda} \cdots, \tag{12.2}$$

in which $\mathfrak{K}$ depends upon only $g_\mu^{(\nu)}$, $g_{\mu|\rho}^{(\nu)}$, and X^ν. When one sets:

$$\frac{\delta \mathfrak{K}}{\delta g_\mu^{(\sigma)}} = \frac{\partial \mathfrak{K}}{\partial g_\mu^{(\sigma)}} - \left(\frac{\partial \mathfrak{K}}{\partial g_{\mu|\nu}^{(\sigma)}} \right)_{|\nu}, \qquad \frac{\delta \mathfrak{L}}{\delta \psi_{(M)}} = \frac{\partial \mathfrak{L}}{\partial \psi_{(M)}} - \left(\frac{\partial \mathfrak{L}}{\partial \psi_{(M)|\mu}} \right)_{|\mu}, \text{ etc.,} \tag{12.3}$$

the variation in (12.1) will yield:

$$\delta \int_B (\mathfrak{G} + \mathfrak{L})\, d\tau = \int \left[\left(\frac{\delta \mathfrak{K}}{\delta g_\mu^{(\sigma)}} + \frac{\delta \mathfrak{L}}{\delta g_\mu^{(\sigma)}} \right) \delta g_\mu^{(\sigma)} + \frac{\delta \mathfrak{L}}{\delta \psi_{(M)}} \delta \psi_{(M)} \right] d\tau + \int_{R_\Lambda} \cdots \tag{12.4}$$

That variation must be equal to zero when the changes δ in the field quantities vanish on the boundary. [That requirement can be fulfilled for the boundary R_Λ, due to its peculiar nature, and despite the homogeneity of the $\delta g_\mu^{(\sigma)}$ (!), but it would not be fulfilled, e.g., for $\underline{R}$ and $\overline{R}$ when the changes δ do not vanish identically.] Since the $\delta g_\mu^{(\sigma)}$ can be chosen arbitrarily (but homogenously of degree – 1), one will then obtain the field equations:

$$\frac{\delta \mathfrak{K}}{\delta g_\mu^{(\sigma)}} + \frac{\delta \mathfrak{L}}{\delta g_\mu^{(\sigma)}} = 0 \tag{12.5}$$

for the metric field. The variational derivatives $\delta \mathfrak{K} / \delta g_\mu^{(\sigma)}$, etc., are components of a tensor density; that will follow directly from (12.4). If one subjects (12.4) to a transformation from $\mathfrak{H}_5$ or $\mathfrak{D}_5$ then the boundary integral will remain zero if it was zero before, and the integrand of the volume integral must then be a scalar density. Since the $\delta g_\mu^{(\sigma)}$ and $\delta\psi_{(M)}$ are components of tensors, the same thing will be true of their variational derivatives. We set:

$$\frac{\delta \mathfrak{K}}{\delta g_\mu^{(\sigma)}} = \mathfrak{K}_{(\sigma)}{}^{\mu}, \qquad \frac{\delta \mathfrak{L}}{\delta g_\mu^{(\sigma)}} = \mathfrak{S}_{(\sigma)}{}^{\mu}, \tag{12.6}$$

and

$$\frac{\delta \mathfrak{L}}{\delta \psi_{(M)}} = \mathfrak{L}^{(M)}. \tag{12.7}$$

The last equation can be interpreted as saying that $L^{(M)} = (1/m)\ \mathfrak{L}^{(M)}$ transforms contravariantly to $\psi_{(M)}$. Let the tensors that correspond to the tensor densities (12.6) be:

$$K_{(\sigma)}{}^{\mu} = m^{-1}\ \mathfrak{K}_{(\sigma)}{}^{\mu}, \qquad S_{(\sigma)}{}^{\mu} = m^{-1}\ \mathfrak{S}_{(\sigma)}{}^{\mu}. \tag{12.8}$$

With the notations in (12.7), the field equations for the matter field will read:

$$\mathfrak{L}^{(M)} = 0. \tag{12.9}$$

If we set $m^{-1}\ \mathfrak{L} = L$ and $m^{-1}\ \mathfrak{G} = G$ then, from (5.25), we will have:

$$\int (\mathfrak{G} + \mathfrak{L})\, d\tau = \int (G + L)\, m\, d\tau = \int (G + L)\, J^{1/2}\, \overset{4}{m}\, d\overset{4}{\tau}\ . \tag{12.10}$$

If we now denote:

$$\left.\begin{aligned} \overset{4}{\mathfrak{G}} &= G\, J^{1/2}\, \overset{4}{m} = J^{1/2} \mathfrak{G}\, m^{-1}\, \overset{4}{m}, \\ \overset{4}{\mathfrak{L}} &= L\, J^{1/2}\, \overset{4}{m} = J^{1/2} \mathfrak{L}\, m^{-1}\, \overset{4}{m} \end{aligned}\right\} \tag{12.11}$$

then the field equations can also be derived from the affine variational principle:

$$\delta \int (\overset{4}{\mathfrak{G}} + \overset{4}{\mathfrak{L}})\, d\overset{4}{\tau} = 0. \tag{12.12}$$

One must then employ the affine variables $g_k^{(i)}$, φ_k in place of the field variables $g_\mu^{(\sigma)}$ and J, resp., and the $\psi_{(M_4)}$ that one gets from an affine splitting in place of the $\psi_{(M)}$. For example, if $\psi_{(M)}$ is a vector ψ_μ then $\psi_{(M_4)}$ will consist of an affine vector ψ_k and an affine scalar $\psi_{(0)}$.

As before, further arguments will carry over directly to the affine case, when one correspondingly defines the tensors:

$$\frac{\delta \overset{4}{\mathfrak{K}}}{\delta g_k^{(i)}} = \overset{4}{\mathfrak{K}}{}_{(i)}{}^{k}, \qquad \frac{\delta \overset{4}{\mathfrak{L}}}{\delta g_k^{(i)}} = \overset{4}{\mathfrak{S}}{}_{(i)}{}^{k}, \tag{12.13}$$

and

$$\frac{\delta \overset{4}{\mathfrak{L}}}{\delta \psi_{(M_4)}} = \overset{4}{\mathfrak{L}}{}^{(M_4)}. \tag{12.14}$$

Under the transition from five-dimensional to four-dimensional integrals, one must observe that the boundary R_Λ will go to the boundary R of the four-dimensional world-domain, such that the integral $\int_{R_\Lambda} \cdots$ will go to the boundary integral $\int_R \cdots$. That fact allows one to rewrite equation (12.4) as an affine one directly:

$$\begin{aligned}\int (\mathfrak{G} + \mathfrak{L})\, d\tau &= \delta \int (\overset{4}{\mathfrak{G}} + \overset{4}{\mathfrak{L}})\, d\overset{4}{\tau} \\ &= \int [(\mathfrak{K}_{(\sigma)}{}^{\mu} + \mathfrak{S}_{(\sigma)}{}^{\mu})\, \delta g_\mu^{(\sigma)} + \mathfrak{L}^{(M)} \delta\psi_{(M)}]\, J^{1/2} \frac{\overset{4}{m}}{m} d\overset{4}{\tau} + \int_R \cdots\end{aligned} \tag{12.15}$$

It follows from the equations:

$$g_\mu^{(0)} = J^{-1/2}\, Y_\mu, \qquad g_\mu^{(i)} = g_k^{(i)} g_\mu^k \tag{12.16}$$

that:

$$\delta g_\mu^{(0)} = J^{-1/2}\, Y_\mu\, \delta J + J^{1/2}\, \delta Y_\mu = \tfrac{1}{2} J^{-1} g_\mu^{(0)}\, \delta J + J^{1/2}\, \delta\varphi_\mu \tag{12.17}$$

[the last one is true from (6.10)], and:

$$\delta g_\mu^{(i)} = \delta g_k^{(i)} \cdot g_\mu^k. \tag{12.18}$$

(12.15) will then go to:

$$\begin{aligned}&\delta \int (\overset{4}{\mathfrak{G}} + \overset{4}{\mathfrak{L}})\, d\overset{4}{\tau} \\ &\quad= \int [(\mathfrak{K}_{(\sigma)}{}^{(\mu)} + \mathfrak{S}_{(\sigma)}{}^{(\mu)})\, \delta g_\mu^{(0)} + (\mathfrak{K}_{(i)}{}^{\mu} + \mathfrak{S}_{(i)}{}^{\mu})\, \delta g_\mu^{(i)} + \mathfrak{L}^{(M)} \delta\psi_{(M)}]\, J^{1/2} \frac{\overset{4}{m}}{m} d\overset{4}{\tau} + \int_R \cdots \\ &\quad= \int [\tfrac{1}{2} J^{-1} (\mathfrak{K}_{(0)(0)} + \mathfrak{S}_{(0)(0)})\, \delta g_k^{(i)} + J^{1/2} (\mathfrak{K}_{(0)}{}^{k} + \mathfrak{S}_{(0)}{}^{k})\, \delta\varphi_k\end{aligned} \tag{12.19}$$

$$+(\mathfrak{K}_{(i)}{}^{k}+\mathfrak{S}_{(i)}{}^{k})\,\delta g_{k}^{(i)}+\mathfrak{L}^{(M)}\delta\psi_{(M)}]\,J^{1/2}\,\frac{\overset{4}{m}}{m}\,d\overset{4}{\tau}+\int_{R}\cdots$$

On the other hand, one has:

$$\delta\int(\overset{4}{\mathfrak{G}}+\overset{4}{\mathfrak{L}})\,d\overset{4}{\tau}$$

$$=\int[(\overset{4}{\mathfrak{K}}{}_{(i)}{}^{k}+\overset{4}{\mathfrak{S}}{}_{(i)}{}^{k})\,\delta g_{k}^{(i)}+(\mathfrak{k}^{r}+\mathfrak{t}^{r})\,\delta\varphi_{r}+(\mathfrak{a}+\mathfrak{b})\;\delta J+\overset{4}{\mathfrak{L}}{}^{(M)}\delta\psi_{(M_4)}]\,d\overset{4}{\tau}+\int_{R}\cdots, \quad (12.20)$$

in which, along (12.13), one has:

$$\left.\begin{aligned}
&\mathfrak{k}^{r}=\frac{\delta\overset{4}{\mathfrak{K}}}{\delta\varphi_{r}}, \quad \mathfrak{t}^{r}=\frac{\delta\overset{4}{\mathfrak{L}}}{\delta\varphi_{r}},\\
&\mathfrak{a}=\frac{\delta\overset{4}{\mathfrak{K}}}{\delta J}, \quad \mathfrak{b}=\frac{\delta\overset{4}{\mathfrak{L}}}{\delta J}.
\end{aligned}\right\} \quad (12.21)$$

A comparison of (12.19) with (12.20) will yield the relations:

$$\left.\begin{aligned}
&\overset{4}{K}{}_{(i)}{}^{k}=J^{1/2}K_{(i)}{}^{k},\\
&\overset{4}{S}{}_{(i)}{}^{k}=J^{1/2}S_{(i)}{}^{k},\\
&k^{r}\;=J\,K_{(0)}{}^{r},\\
&t^{r}\;=J\,S_{(0)}{}^{r},\\
&a\;=\tfrac{1}{2}J^{-1/2}K_{(0)(0)},\\
&b\;=\tfrac{1}{2}J^{-1/2}S_{(0)(0)}.
\end{aligned}\right\} \quad (12.22)$$

We refer to $S^{\mu\nu}$ as the *matter tensor*, $\overset{4}{S}{}^{ik}$ as the *four-matter tensor*, t^{k} as the *four-matter vector*, and b as the *matter invariant*. With an affine splitting, equations (12.5) will then read:

$$\overset{4}{K}{}_{(i)}{}^{k}+\overset{4}{S}{}_{(i)}{}^{k}=0, \quad (12.23a)$$

$$k^{r}+t^{r}=0, \quad (12.23b)$$

$$a+b=0. \quad (12.23c)$$

13. Matter tensor and conservation laws. – Before we examine the field equation in detail, we shall first derive the general theorems and consequences that the form of the action principle implies.

Since $\mathfrak{G}$ was assumed to be invariant under $\mathfrak{D}_5$, under an infinitesimal rotation in $\mathfrak{D}_5$, one will have:

$$0 = \delta\int \mathfrak{G}\, d\tau = \delta\int \mathfrak{K}\, d\tau + \int_{R_\Lambda} \cdots = \int \frac{\delta\mathfrak{K}}{\delta g_\mu^{(\sigma)}} \delta g_\mu^{(\sigma)}\, d\tau + \int_{R_\Lambda} \cdots \tag{13.1}$$

In this, one has, from (8.22)

$$\delta g_\mu^{(\sigma)} = \varepsilon\, \vartheta^{(\sigma)}{}_{(\rho)}\, g_\mu^{(\rho)} = \varepsilon\, \vartheta^{(\sigma)}{}_\mu\,. \tag{13.2}$$

When one lets the $\vartheta^{(\sigma)}{}_\mu$ be zero on R_Λ, so the integral over R_Λ will vanish, it will then follow from (13.1) that:

$$\mathfrak{K}_{(\sigma)}{}^\mu\, \vartheta^{(\sigma)}{}_\mu = 0,$$

or

$$\mathfrak{K}^{\mu\nu}\, \vartheta_{\mu\nu} = 0. \tag{13.3}$$

Since $\vartheta_{\nu\mu}$ is antisymmetric, but otherwise arbitrary, if will follow from (13.3) that:

$$\mathfrak{K}^{\nu\mu} = \mathfrak{K}^{\mu\nu}, \tag{13.4}$$

is a symmetric tensor.

If we now consider infinitesimal transformations of $\mathfrak{H}_5$ then, since $\int \mathfrak{G}\, d\tau$ is invariant under $\mathfrak{H}_5$, we will have:

$$0 = \delta\int \mathfrak{G}\, d\tau = \delta\int \mathfrak{K}\, d\tau + \int_{R_\Lambda} \cdots$$

With (8.8), it will then follow that:

$$0 = \int (\delta\mathfrak{K} + \varepsilon\, \mathfrak{K}\, \xi^\nu{}_{|\nu})\, d\tau + \int_{R_\Lambda} \cdots$$

If one employs (8.13) then one will get:

$$0 = \int \delta^* \mathfrak{K}\, d\tau + \int_{R_\Lambda} \cdots \tag{13.5}$$

from partial integration. Since the possible explicit dependency of the quantity $\mathfrak{K}$ upon the X^ν plays no role in the definition of δ^*, it will follow from (13.5) that:

$$0 = \int \mathfrak{K}_{(\sigma)}{}^\mu\, \delta^* g_\mu^{(\sigma)} d\tau + \int_{R_\Lambda} \cdots \tag{13.6}$$

Since:

$$\delta^* g_\mu^{(\sigma)} = \varepsilon\, \xi^a{}_{|\beta}\, \mathfrak{O}_a{}^\beta\, g_\mu^{(\sigma)} - \varepsilon\, g_{\mu|\nu}^{(\sigma)}\, \xi^\nu, \tag{13.7}$$

(13.6) will imply that:

$$0 = \varepsilon \int [-(\mathfrak{K}_{(\sigma)}^{\ \mu}\, \mathfrak{D}_a^{\beta}\, g_\mu^{(\sigma)})_{|\beta} - \mathfrak{K}_{(\sigma)}^{\ \mu}\, g_{\mu|a}^{(\sigma)}]\xi^\mu d\tau + \int_{R_\Lambda} \cdots \tag{13.8}$$

If we now choose ξ^a to behave on the boundary in such a way that $\int_{R_\Lambda} \cdots$ vanishes then:

$$(\mathfrak{K}_{(\sigma)}^{\ \mu}\, \mathfrak{D}_a^{\beta}\, g_\mu^{(\sigma)})_{|\beta} + \mathfrak{K}_{(\sigma)}^{\ \mu}\, g_{\mu|a}^{(\sigma)} = 0. \tag{13.9}$$

We have the relation:

$$\mathfrak{K}_a^{\ \beta} = -\,\mathfrak{K}_{(\sigma)}^{\ \mu}\, \mathfrak{D}_a^{\beta}\, g_\mu^{(\sigma)}, \tag{13.10}$$

so

$$\mathfrak{D}_a^{\beta}\, g_\mu^{(\sigma)} = -\,g_\mu^{(\sigma)} \delta_\mu^\beta .$$

It will then follow from (13.9) that:

$$\mathfrak{K}_a^{\ \beta}{}_{|\beta} - \mathfrak{K}_{(\sigma)}^{\ \mu}\, g_{\mu|a}^{(\sigma)} = 0.$$

Since $\mathfrak{K}^{\mu\nu}$ is symmetric, it will follow that:

$$\mathfrak{K}_{(\sigma)}^{\ \mu}\, g_{\mu|a}^{(\sigma)} = \mathfrak{K}^{\mu\nu}\, g_{(\sigma)\nu}\, g_{\mu|a}^{(\sigma)} = \tfrac{1}{2}\mathfrak{K}^{\mu\nu}\, g_{\nu\mu|a} ,$$

such that finally:

$$\mathfrak{K}_a^{\ \beta}{}_{|\beta} - \tfrac{1}{2}\mathfrak{K}^{\mu\nu}\, g_{\nu\mu|a} = \mathfrak{K}_a^{\ \beta}{}_{\|\beta} = 0, \tag{13.11}$$

and therefore, we will have:

$$K^{a\beta}{}_{\|\beta} = 0 \tag{13.12}$$

for the tensor $K^{a\beta}$.

Now, let us apply the same process to the integral $\int \mathfrak{L}\, d\tau$! Before we do that, note that the relation (13.10) for $\mathfrak{L}$ has the equation:

$$\mathfrak{S}_a^{\ \beta} = -\,\mathfrak{S}_{(\sigma)}^{\ \mu}\, \mathfrak{D}_a^{\ \beta}\, g_\mu^{(\sigma)} \tag{13.13}$$

as a consequence. Under an infinitesimal transformation of $\mathfrak{H}_5$, one will then have:

$$0 = \delta \int \mathfrak{L}\, d\tau = \int [\delta^* \mathfrak{L} + \varepsilon (\mathfrak{L}\, \xi^\nu)_{|\nu}]\, d\tau . \tag{13.14}$$

It will then follow step-wise that:

$$\delta^* \mathfrak{L} = \mathfrak{S}_{(\sigma)}^{\ \mu}\, \delta g_\mu^{(\sigma)} + \left(\frac{\partial \mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}} \delta^* g_{\mu|\nu}^{(\sigma)} \right)_{|\nu} + \mathfrak{L}^{(M)}\, \delta^* \psi_{(M)} + \left(\frac{\partial \mathfrak{L}}{\partial \psi_{(M)|\nu}} \delta^* \psi_{(M)} \right)_{|\nu} . \tag{13.15}$$

Once more, the possible dependency of the function $\mathfrak{L}$ on X^μ plays no role here in the definition of $\delta^* \mathfrak{L}$. Now, one has:

$$\delta^* \psi_{(M)} = \varepsilon\, \xi^\mu{}_{|\nu}\, \mathfrak{D}_\mu^{\ \nu}\, \psi_{(M)} - \varepsilon\, \psi_{(M)\,|\,\nu}\, \xi^\nu . \tag{13.16}$$

With (13.7) and (13.13), one will get:

$$\begin{aligned}\mathfrak{S}_{(\sigma)}{}^{\mu}\,\delta^{*}g_{\mu}^{(\sigma)} &= \varepsilon\,\mathfrak{S}_{(\sigma)}{}^{\mu}\,\xi^{\rho}{}_{|\mu}\,\mathfrak{D}_{\rho}{}^{\nu}\,g_{\mu}^{(\sigma)} - \varepsilon\,\mathfrak{S}_{(\sigma)}{}^{\mu}\,g_{\mu}^{(\sigma)}\,\xi^{\nu} \\ &= \varepsilon\,[-\mathfrak{S}_{\rho}{}^{\mu}\,\xi^{\rho}{}_{|\nu} - \mathfrak{S}_{(\sigma)}{}^{\mu}\,g_{\mu|\nu}^{(\sigma)}\,\xi^{\nu}] \\ &= \varepsilon\,[\mathfrak{S}_{\rho}{}^{\mu}{}_{|\mu} - \mathfrak{S}_{(\sigma)}{}^{\mu}\,g_{\mu|\nu}^{(\sigma)}]\,\xi^{\nu} - \varepsilon\,(\mathfrak{S}_{\rho}{}^{\nu}\,\xi^{\rho})_{|\nu}\,.\end{aligned}$$

It likewise follows that:

$$\begin{aligned}\mathfrak{L}^{(M)}\,\delta^{*}\psi_{(M)} &= \varepsilon\,\mathfrak{L}^{(M)}\,\xi^{\mu}{}_{|\nu}\,\mathfrak{D}_{\rho}{}^{\nu}\,\psi_{(M)} - \varepsilon\,\psi_{(M)\,|\,\nu}\,\xi^{\nu}\,\mathfrak{L}^{(M)}, \\ &= \varepsilon\,[\mathfrak{L}^{(M)}\,\mathfrak{D}_{\nu}{}^{\mu}\,\psi_{(M)\,|\,\mu} - \varepsilon\,\psi_{(M)\,|\,\nu}\,\mathfrak{L}^{(M)}]\,\xi^{\nu} + \varepsilon\,(\mathfrak{L}^{(M)}\,\xi^{\mu}\,\mathfrak{D}_{\mu}{}^{\nu}\,\psi_{(M)})_{\,|\,\nu}\,.\end{aligned}$$

With that, (13.14) will go to:

$$\int [\mathfrak{X}_{\nu}\,\xi^{\nu} + (\mathfrak{Y}^{\nu}{}_{\mu}\,\xi^{\mu} + \mathfrak{Z}_{\mu}^{\nu\lambda}\,\xi^{\mu}{}_{\,|\,\lambda})_{|\,\nu}]\,d\tau = 0, \tag{13.17}$$

with

$$\left.\begin{aligned}\mathfrak{X}_{\nu} &= \mathfrak{S}_{\nu\,|\mu}^{\mu} - \mathfrak{S}_{(\sigma)}^{\mu}\,g_{\nu\;\,|\mu}^{(\sigma)} - (\mathfrak{L}^{(M)}\mathfrak{D}_{\nu}^{\mu}\psi_{(M)})_{|\mu} - \mathfrak{L}^{(M)}\psi_{(M)|\nu}, \\ \mathfrak{Y}^{\nu}_{\mu} &= \mathfrak{L}\,\delta^{\nu}_{\mu} - \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\psi_{(M)|\mu} - \mathfrak{S}_{\mu}^{\nu} + \mathfrak{L}^{(M)}\mathfrak{D}_{\mu}^{\nu}\psi_{(M)}, \\ \mathfrak{Z}_{\mu}^{\nu\lambda} &= \frac{\partial\mathfrak{L}}{\partial g_{\rho|\nu}^{(\sigma)}}\mathfrak{D}_{\mu}^{\lambda}g_{\rho}^{(\sigma)} + \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\mathfrak{D}_{\mu}^{\lambda}\psi_{(M)}.\end{aligned}\right\} \tag{13.18}$$

The second part of the integral (13.17) can be converted into an integral over the boundary R_{Λ} . If we initially choose ξ^{ν} in such a way that this boundary integral vanishes then it will follow that:

$$\mathfrak{X}_{\nu} = 0. \tag{13.19}$$

However, it will then follow from (13.17) that:

$$\int (\mathfrak{Y}^{\nu}{}_{\mu}\,\xi^{\mu} + \mathfrak{Z}_{\mu}^{\nu\lambda}\,\xi^{\mu}{}_{\,|\,\lambda})_{|\,\nu}\,d\tau = 0. \tag{13.20}$$

We now choose:

$$\xi^{\mu} = a^{\mu}{}_{\lambda}\,X^{\lambda}, \tag{13.21}$$

with constant $a^{\mu}{}_{\lambda}$ in particular. (The ξ^{μ} themselves cannot be chosen to be constant, since they must be homogeneous of degree one in the X^{λ}.) It will then follow that:

$$(\mathfrak{Y}^{\nu}{}_{\mu}\,X^{\lambda} + \mathfrak{Z}_{\mu}^{\nu\lambda})_{|\,\nu} = 0; \tag{13.22}$$

i.e.:

$$\mathfrak{Y}^{\nu}{}_{\mu\,|\,\nu}\,X^{\lambda} + \mathfrak{Y}^{\lambda}{}_{\mu} + \mathfrak{Z}_{\mu|\nu}^{\nu\lambda} = 0. \tag{13.23}$$

With (13.23), (13.20) will go to:

$$\int (\mathfrak{Y}^\nu{}_\mu\, \xi^\mu{}_{|\lambda}\, X^\lambda + \mathfrak{Z}^{\nu\lambda}_\mu\, \xi^\mu{}_{|\lambda})_{|\nu}\, d\tau = \int (\mathfrak{Y}^\nu{}_\mu\, X^\lambda + \mathfrak{Z}^{\nu\lambda}_\mu)\, \xi^\mu{}_{|\lambda|\nu}\, d\tau = 0. \tag{13.24}$$

Since $\xi^\mu{}_{|\nu}$ is homogeneous of degree zero, one will have:

$$\xi^\mu{}_{|\lambda|\nu}\, X^\lambda = \xi^\mu{}_{|\nu|\lambda}\, X^\lambda = 0, \tag{13.25}$$

such that it will follow from (13.24) that:

$$\mathfrak{Z}^{\nu\lambda}_\mu\, \xi^\mu{}_{|\lambda|\nu} = 0. \tag{13.26}$$

Due to the condition (13.25), it will follow from this that:

$$\mathfrak{Z}^{\nu\lambda}_\mu + \mathfrak{Z}^{\lambda\nu}_\mu = \mathfrak{A}_\mu{}^\nu\, X^\lambda + \mathfrak{A}_\mu{}^\nu\, X^\lambda\,. \tag{13.27}$$

Due to the homogeneity of the field variables, the $\mathfrak{A}_\mu{}^\nu$ remain largely undetermined. If one adds an expression to $\mathfrak{L}$ that is, e.g., zero identically:

$$\mathfrak{L}' = \mathfrak{L} + (g^{(\sigma)}_{\rho|\nu}\, X^\nu + g^{(\sigma)}_\rho)\; \mathfrak{F}_{(\sigma)}{}^\rho,$$

in which $\mathfrak{F}_{(\sigma)}{}^\rho$ can be arbitrary functions of $g^{(\sigma)}_\rho$, $g^{(\sigma)}_{\rho|\nu}$, $\psi_{(M)}$, $\psi_{(M)|\nu}$, and X^μ, then one will get:

$$\frac{\partial \mathfrak{L}'}{\partial g^{(\sigma)}_{\rho|\nu}} = \frac{\partial \mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\nu}} + X^\nu\, \mathfrak{F}_{(\sigma)}{}^\rho.$$

Despite the fact that $\mathfrak{L}' = \mathfrak{L}$, the derivatives will no longer agree. It will then follow that:

$$\mathfrak{Z}'^{\nu\lambda}_\mu + \mathfrak{Z}'^{\lambda\nu}_\mu = \mathfrak{Z}^{\nu\lambda}_\mu + \mathfrak{Z}^{\lambda\nu}_\mu + X^\nu\, \mathfrak{F}^\rho_{(\sigma)}\, \mathfrak{D}_\mu{}^\lambda\, g^{(\sigma)}_\rho + X^\lambda\, \mathfrak{F}^\rho_{(\sigma)}\, \mathfrak{D}_\mu{}^\nu\, g^{(\sigma)}_\rho\,.$$

$\mathfrak{A}_\mu{}^\lambda$ can be calculated from (13.27), when one considers that from (13.18), $\mathfrak{Z}^{\nu\lambda}_\mu$ cannot contain a factor X^λ, but at most, a factor X^ν, since $\mathfrak{D}_\mu{}^\lambda\, g^{(\sigma)}_\rho$ and $\mathfrak{D}_\mu{}^\lambda\, \psi_{(M)}$ do not contain the factor X^λ. Therefore:

$$\mathfrak{A}_\mu{}^\lambda = \mathfrak{A}'^\rho_{(\sigma)}\, \mathfrak{D}_\mu{}^\lambda\, g^{(\sigma)}_\rho + \mathfrak{A}''^{(M)}\, \mathfrak{D}_\mu{}^\lambda\, \psi_{(M)}\,,$$

with

$$\mathfrak{A}'^\rho_{(\sigma)} = \frac{\partial}{\partial X^\nu} \frac{\partial \mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\nu}}\,, \qquad \mathfrak{A}''^{(M)} = \frac{\partial}{\partial X^\nu} \frac{\partial \mathfrak{L}}{\partial \psi_{(M)|\nu}}\,,$$

in which ν is not summed over, and $\partial / \partial X^\nu$ is the derivative with respect to the X^ν that appears explicitly. One can now put $\mathfrak{L}$ into another form, such that:

$$\mathfrak{A}'^{\rho}_{(\sigma)} = 0, \qquad \mathfrak{A}''^{(M)} = 0$$

without changing the value of $\mathfrak{L}$. We will always use that form as a basis in what follows. However, $\mathfrak{Z}_{\mu}^{\nu\lambda}$ will be antisymmetric in ν and λ then.

Equation (13.23) can be simplified further. It follows from (13.18) that:

$$\mathfrak{Y}^{\nu}{}_{\mu|\nu} = \mathfrak{Y}_{|\mu} - \left(\frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\nu}} g^{(\sigma)}_{\rho|\mu}\right)_{|\nu} - \left(\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\psi_{(M)|\mu}\right)_{|\nu} - \mathfrak{S}_{\mu}{}^{\nu}{}_{|\nu} + (\mathfrak{L}^{(M)}\mathfrak{O}_{\mu}{}^{\nu}\psi_{(M)})_{|\nu} .$$

If one again denotes the partial derivative of $\mathfrak{L}$ with respect to X^{μ} when the field variables are held constant by $\partial\mathfrak{L} / \partial X^{\mu}$ (one then has $\partial\mathfrak{L} / \partial X^{\mu} \neq \mathfrak{L}_{|\mu}$) then it will follow for $\mathfrak{L}_{|\mu}$ that:

$$\mathfrak{L}_{|\mu} = \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho}} g^{(\sigma)}_{\rho|\mu} + \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\nu}} g^{(\sigma)}_{\rho|\nu|\mu} + \frac{\partial\mathfrak{L}}{\partial\psi_{(M)}}\psi_{(M)|\mu} + \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\psi_{(M)|\nu|\mu} + \frac{\partial\mathfrak{L}}{\partial X^{\mu}}$$

$$= \mathfrak{S}_{(\sigma)}^{\rho} g^{(\sigma)}_{\rho|\mu} + \left(\frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\nu}} g^{(\sigma)}_{\rho|\mu}\right)_{|\nu} + \mathfrak{L}^{(M)}\psi_{(M)|\mu} + \left(\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\psi_{(M)|\mu}\right)_{|\nu} + \frac{\partial\mathfrak{L}}{\partial X^{\mu}} .$$

Thus, one finally has:

$$\mathfrak{Y}^{\nu}{}_{\mu|\nu} = \mathfrak{S}_{(\sigma)}^{\rho} g^{(\sigma)}_{\rho|\mu} - \mathfrak{S}_{(\sigma)|\nu}^{\rho} + \mathfrak{L}^{(M)}\psi_{(M)|\mu} + (\mathfrak{L}^{(M)}\mathfrak{O}_{\mu}^{\nu}\psi_{(M)})_{|\mu} + \frac{\partial\mathfrak{L}}{\partial X^{\mu}} .$$

Together with (13.19), it follows that:

$$\mathfrak{Y}^{\nu}{}_{\mu|\nu} = \frac{\partial\mathfrak{L}}{\partial X^{\mu}} . \tag{13.28}$$

In affine geometry, all arguments will proceed in parallel with the help of the groups $\mathfrak{G}_4$, $\mathfrak{D}_4$, except that one will immediately obtain the simpler equations:

$$\overset{4}{\mathfrak{Z}}{}^{nl}_{m} + \overset{4}{\mathfrak{Z}}{}^{ln}_{m} = 0, \tag{13.27a}$$

$$\overset{4}{\mathfrak{Y}}{}^{n}_{m|n} = 0, \tag{13.28a}$$

in place of (13.27) and (13.28). When (13.28) is substituted in (13.23), that will give:

$$\mathfrak{Y}^{\nu}{}_{\mu} + \frac{\partial\mathfrak{L}}{\partial X^{\mu}} X^{\lambda} + \mathfrak{Z}^{\nu\lambda}_{\mu|\nu} = 0 . \tag{13.29}$$

From the invariance under $\mathfrak{D}_4$, it will follow that for an infinitesimal transformation:

$$\delta\mathfrak{L} = \mathfrak{S}_{(\sigma)}^{\ \rho}\,\delta g_{\mu}^{(\sigma)} + \left(\frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}\delta g_{\mu}^{(\sigma)}\right)_{|\nu} + \mathfrak{L}^{(M)}\delta\psi_{(M)} + \left(\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\delta\psi_{(M)}\right)_{|\nu}, \tag{13.30}$$

in which $\delta g_{\mu}^{(\sigma)}$ is given by (13.2), and:

$$\delta\psi_{(M)} = \varepsilon\,\vartheta_{(a)(\beta)}\,P^{(a)(\beta)}\,\psi_{(M)}. \tag{13.31}$$

It follows from this that:

$$\left.\begin{aligned} 0 = \delta\int\mathfrak{L}\,d\tau = \varepsilon\int\Big[(\mathfrak{S}^{(a)(\beta)}\vartheta_{(a)(\beta)} + \tfrac{1}{2}\mathfrak{L}^{(M)}\vartheta_{(a)(\beta)}P^{(a)(\beta)}\psi_{(M)}) \\ + \left(\frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}g^{(a)(\beta)}g_{\mu}^{(\beta)}\vartheta_{(a)(\beta)} + \tfrac{1}{2}\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\vartheta_{(a)(\beta)}P^{(a)(\beta)}\psi_{(M)}\right)_{|\nu}\Big]d\tau. \end{aligned}\right\} \tag{13.32}$$

Since $\vartheta_{(a)(\beta)}$ can be chosen arbitrarily, except for its antisymmetry, we initially choose it so that the second part of the integral (13.32) (which can be converted into an integral over the boundary) vanishes, and obtain:

$$\boxed{\mathfrak{S}^{(a)(\beta)} - \mathfrak{S}^{(\beta)(a)} + \mathfrak{L}^{(M)}P^{(a)(\beta)}\psi_{(M)} = 0.} \tag{13.33}$$

It will then follow from (13.32) that:

$$\left(\frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}g^{(a)(\beta)}g_{\mu}^{(\beta)}\vartheta_{(a)(\beta)} + \tfrac{1}{2}\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}\vartheta_{(a)(\beta)}P^{(a)(\beta)}\psi_{(M)}\right)_{|\nu} = 0. \tag{13.34}$$

If we assume that $\vartheta_{(a)(\beta)}$ are constants then:

$$\left(\frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}g^{(a)(\beta)}g_{\mu}^{(\beta)} - \frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}g^{(\beta)(\sigma)}g_{\mu}^{(a)} + \tfrac{1}{2}\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}P^{(a)(\beta)}\psi_{(M)}\right)_{|\nu} = 0. \tag{13.35}$$

It will follow from that and (13.34), moreover, that:

$$\left(\frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}g^{(a)(\beta)}g_{\mu}^{(\beta)} - \frac{\partial\mathfrak{L}}{\partial g_{\mu|\nu}^{(\sigma)}}g^{(\beta)(\sigma)}g_{\mu}^{(a)} + \tfrac{1}{2}\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\nu}}P^{(a)(\beta)}\psi_{(M)}\right)_{|\nu}\vartheta_{(a)(\beta)|\nu} = 0. \tag{13.36}$$

Since $\vartheta_{(a)(\beta)}$ is an arbitrary (antisymmetric) homogeneous function of degree zero, one will have $\vartheta_{(a)(\beta)\,|\,\nu}\,X^{\nu} = 0$. It will then follow from (13.36) that the term in parentheses is equal to $\mathfrak{C}^{(a)(\beta)}\,X^{\nu}$. If we then employ the condition on $\mathfrak{L}$ that was established on page 50 then it will follow that the term in parentheses cannot have the form $\mathfrak{C}^{(\alpha)(\beta)} X^{\nu}$, since one must have $\mathfrak{A}'^{\rho}_{(\sigma)} = 0$, $\mathfrak{A}''^{(M)} = 0$, from page 49. One therefore ultimately has:

$$\frac{\partial \mathfrak{L}}{\partial g^{(\sigma)}_{\mu|\nu}}\left(g^{(\alpha)(\sigma)} g^{(\beta)}_{\mu} - g^{(\sigma)(\sigma)} g^{(\alpha)}_{\mu}\right) + \frac{\partial \mathfrak{L}}{\partial \psi_{(M)|\nu}} P^{(\alpha)(\beta)} \psi_{(M)} = 0. \tag{13.37}$$

We will now evaluate equations (13.19), (13.29), and (13.33) in detail. For a normal tensor density $\mathfrak{S}_{\nu}{}^{\mu}$, one will have:

$$\mathfrak{S}_{\nu}{}^{\mu}{}_{\|\mu} = \mathfrak{S}_{\nu}{}^{\mu}{}_{|\mu} - \mathfrak{S}_{\nu}{}^{\mu}{}_{|\mu} - \tfrac{1}{2}(g^{(\sigma)}_{\lambda|\nu} g_{(\sigma)\mu} + g_{\lambda\nu|\mu})(\mathfrak{S}^{\lambda\mu} - \mathfrak{S}^{\mu\lambda}),$$

in general, or:

$$\mathfrak{S}_{\nu}{}^{\mu}{}_{\|\mu} = \mathfrak{S}_{\nu}{}^{\mu}{}_{|\mu} - \mathfrak{S}_{(\sigma)}{}^{\mu}\, g^{(\sigma)}_{\lambda|\nu} - \omega_{\nu\lambda\mu}\, \mathfrak{S}^{\lambda\mu}. \tag{13.38}$$

The same thing will also be true for the normal tensor density $\mathfrak{L}^{(M)}\, \mathfrak{D}_{\nu}{}^{\mu}\, \psi_{(M)}$:

$$(\mathfrak{L}^{(M)}\, \mathfrak{D}_{\nu}{}^{\mu}\, \psi_{(M)})_{\|\mu} = (\mathfrak{L}^{(M)}\, \mathfrak{D}_{\nu}{}^{\mu}\, \psi_{(M)})_{|\mu} - \mathfrak{L}^{(M)}\, \Gamma^{\lambda}_{\nu\mu}\, \mathfrak{D}_{\lambda}{}^{\mu}\, \psi_{(M)}\,. \tag{13.39}$$

On the other hand, one generally has:

$$\psi_{(M)\,\|\,\mu} = \psi_{(M)\,|\,\mu} + \Gamma^{\lambda}_{\nu\mu}\, \mathfrak{D}_{\lambda}{}^{\mu}\, \psi_{(M)} + \tfrac{1}{2}\, \omega_{\nu}{}^{(\lambda)}{}_{(\mu)}\, P_{(\lambda)}{}^{(\mu)} \psi_{(M)} - Y_{\nu}\, \Pi\, \psi_{(M)}\,. \tag{13.40}$$

It will then follow, with (13.33), that:

$$\mathfrak{L}^{(M)} \psi_{(M)\,\|\,\mu} = \mathfrak{L}^{(M)} \psi_{(M)\,|\,\mu} + \mathfrak{L}^{(M)}\, \Gamma^{\lambda}_{\nu\mu}\, \mathfrak{D}_{\lambda}{}^{\mu}\, \psi_{(M)} - \omega_{\nu\lambda\mu}\, \mathfrak{S}^{\lambda\mu} - Y_{\nu}\, \mathfrak{L}^{(M)}\, \Pi\, \psi_{(M)}\,. \tag{13.41}$$

Addition of (13.39) and (13.41) yields:

$$\begin{aligned}(\mathfrak{L}^{(M)}\, \mathfrak{D}_{\nu}{}^{\mu}\, \psi_{(M)})_{\|\mu} &+ \mathfrak{L}^{(M)} \psi_{(M)\,\|\,\nu} \\ &= (\mathfrak{L}^{(M)}\, \mathfrak{D}_{\nu}{}^{\mu}\, \psi_{(M)})_{|\mu} + \mathfrak{L}^{(M)} \psi_{(M)\,|\,\nu} - \omega_{\nu\lambda\mu}\, \mathfrak{S}^{\lambda\mu} - Y_{\nu}\, \mathfrak{L}^{(M)}\, \Pi\, \psi_{(M)}\,.\end{aligned} \tag{13.42}$$

If one subtracts (13.42) from (13.38) then it will follow, with (13.18), that:

$$\mathfrak{X}_{\nu} = \mathfrak{S}_{\nu}{}^{\mu}{}_{\|\mu} - (\mathfrak{L}^{(M)}\, \mathfrak{D}_{\nu}{}^{\mu}\, \psi_{(M)})_{\|\mu} - \mathfrak{L}^{(M)}\, \psi_{(M)\,\|\,\nu} - Y_{\nu}\, \mathfrak{L}^{(M)}\, \Pi\, \psi_{(M)}\,. \tag{13.43}$$

(13.19) will then imply the identity:

$$\boxed{\mathfrak{S}^{\mu}_{\nu\|\mu} = (\mathfrak{L}^{(M)} \mathfrak{D}_{\nu}{}^{\mu} \psi_{(M)})_{\|\mu} + Y_{\nu}\, \mathfrak{L}^{(M)} \Pi\, \psi_{(M)} + \mathfrak{L}^{(M)} \Pi\, \psi_{(M)|\nu}.} \tag{13.44}$$

It follows from (13.29), with (13.18), that:

$$\mathfrak{S}_\nu^{\ \mu} = \mathfrak{L}\,\delta_\nu^\mu + \mathfrak{L}^{(M)}\,\mathfrak{O}_\nu^{\ \mu}\,\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\mu}}\,g^{(\sigma)}_{\rho|\nu} - \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\psi_{(M)|\nu} + \frac{\partial\mathfrak{L}}{\partial X^\nu}X^\mu + \mathfrak{Z}^{\lambda\mu}_{\nu|\lambda}. \tag{13.45}$$

With (13.40) and (13.37), one will get:

$$\left.\begin{aligned}\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\psi_{(M)\|\nu} = \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\psi_{(M)|\nu} + \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\Gamma^\lambda_{\nu\rho}\mathfrak{O}_\lambda^{\ \rho}\,\psi_{(M)|\nu} - \omega_\nu^{(\sigma)}{}_\rho\,\frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\mu}}\\ -\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}Y_\nu\,\Pi\,\psi_{(M)|\nu}.\end{aligned}\right\} \tag{13.46}$$

When this is substituted in (13.45), with (10.6), one will get:

$$\left.\begin{aligned}\mathfrak{S}_\nu^{\ \mu} = \mathfrak{L}\,\delta_\nu^\mu + \mathfrak{L}^{(M)}\mathfrak{O}_\nu^{\ \mu}\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\psi_{(M)\|\nu} - \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}Y_\nu\Pi\,\psi_{(M)|} + \frac{\partial\mathfrak{L}}{\partial X^\nu}X^\mu\\ +\frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\Gamma^\lambda_{\nu\rho}\mathfrak{O}_\lambda^{\ \rho}\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\mu}}\Gamma^\lambda_{\nu\rho}g^{(\sigma)}_\lambda + \mathfrak{Z}^{\lambda\mu}_{\nu|\lambda}.\end{aligned}\right\} \tag{13.47}$$

From (13.18), one has:

$$\mathfrak{Z}^{\mu\rho}_\lambda = \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\mathfrak{O}_\lambda^{\ \rho}\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\rho|\mu}}\,g^{(\sigma)}_\lambda, \tag{13.48}$$

such that:

$$\left.\begin{aligned}\mathfrak{S}_\nu^{\ \mu} = \mathfrak{L}\,\delta_\nu^\mu + \frac{\partial\mathfrak{L}}{\partial X^\nu}X^\mu + \mathfrak{L}^{(M)}\mathfrak{O}_\nu^{\ \mu}\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\psi_{(M)\|\nu} - \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}Y_\nu\Pi\,\psi_{(M)|}\\ +\mathfrak{Z}^{\lambda\mu}_{\nu|\lambda} + \Gamma^\lambda_{\nu\rho}\mathfrak{Z}^{\mu\rho}_\lambda.\end{aligned}\right\} \tag{13.39}$$

It follows from (13.48) that:

$$\begin{aligned}\mathfrak{Z}^{\mu\rho\lambda} = \mathfrak{Z}^{\mu\rho}_\sigma g^{\sigma\lambda} &= \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\mathfrak{O}^{\lambda\rho}\,\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\eta|\mu}}\,g^{(\sigma)\lambda}\,\delta^\rho_\eta,\\ \mathfrak{Z}^{\mu\lambda\rho} &= \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}\mathfrak{O}^{\rho\lambda}\,\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\eta|\mu}}\,g^{(\sigma)\rho}\,\delta^\lambda_\eta.\end{aligned}$$

Subtracting these will yield:

$$\left.\begin{aligned}\mathfrak{W}^{\mu\rho\lambda} &= \mathfrak{Z}^{\mu\rho\lambda} - \mathfrak{Z}^{\mu\lambda\rho}\\ &= \frac{\partial\mathfrak{L}}{\partial\psi_{(M)|\mu}}(\mathfrak{O}^{\lambda\rho} - \mathfrak{O}^{\rho\lambda})\psi_{(M)} - \frac{\partial\mathfrak{L}}{\partial g^{(\sigma)}_{\eta|\mu}}(g^{(\sigma)\lambda}\,\delta^\rho_\eta - g^{(\sigma)\rho}\,\delta^\lambda_\eta).\end{aligned}\right\} \tag{13.50}$$

From (13.37), if one takes (13.50) into account then (since $\mathrm{P}^{\lambda\rho} = \mathfrak{O}^{\lambda\rho} - \mathfrak{O}^{\rho\lambda} + P^{\lambda\rho}$):

$$\mathfrak{W}^{\mu\rho\lambda} = \frac{\partial \mathfrak{L}}{\partial \psi_{(M)|\mu}} \mathrm{P}^{\lambda\rho} \, \psi_{(M)} \, . \tag{13.51}$$

Since $\mathfrak{W}^{\mu\rho\lambda}$ is antisymmetric in μ and ρ, one can, inversely, calculate the $\mathfrak{Z}^{\mu\rho\lambda}$ from the $\mathfrak{W}^{\mu\rho\lambda}$ by way of:

$$\mathfrak{Z}^{\mu\rho\lambda} = \tfrac{1}{2}(\mathfrak{W}^{\mu\rho\lambda} + \mathfrak{W}^{\rho\lambda\mu} - \mathfrak{W}^{\lambda\mu\rho}), \tag{13.52}$$

such that the expression for $\mathfrak{Z}^{\mu\rho\lambda}$ will then contain only operations on the matter field quantities explicitly. Since $\psi_{(M)\,\|\,\mu}$ differs from $\psi_{(M)\,|\,\mu}$ only by summands, $\partial\mathfrak{L} / \partial\psi_{(M)\,|\,\mu}$ will be a tensor density:

$$\mathfrak{L}^{(M)\,\mu} = \frac{\partial \mathfrak{L}}{\partial \psi_{(M)|\mu}} . \tag{13.53}$$

$\mathfrak{Z}^{\mu\rho\lambda}$ is therefore likewise a tensor density, such that:

$$\mathfrak{Z}^{\lambda\mu}_{\nu\|\lambda} = \mathfrak{Z}^{\lambda\mu}_{\nu|\lambda} - \Gamma^{\sigma}_{\lambda\nu}\mathfrak{Z}^{\lambda\mu}_{\sigma} + \Gamma^{\mu}_{\lambda\sigma}\mathfrak{Z}^{\lambda\sigma}_{\nu} .$$

Since $\mathfrak{Z}^{\lambda\mu}_{\nu}$ is antisymmetric in $\lambda\mu$, the last term will vanish, such that:

$$\mathfrak{Z}^{\lambda\mu}_{\nu\|\lambda} = \mathfrak{Z}^{\lambda\mu}_{\nu|\lambda} + \Gamma^{\sigma}_{\lambda\nu}\mathfrak{Z}^{\mu\lambda}_{\sigma} . \tag{13.54}$$

Along with (13.53) and (13.54), (13.49) will go to:

$$\boxed{\mathfrak{S}^{\ \mu}_{\nu} = \mathfrak{L}\,\delta^{\mu}_{\nu} + \frac{\partial \mathfrak{L}}{\partial X^{\nu}} X^{\mu} + \mathfrak{L}^{(M)}\mathfrak{O}^{\ \mu}_{\nu}\psi_{(M)} - \mathfrak{L}^{(M)\mu}(\psi_{(M)\|\nu} + Y_{\nu}\,\Pi\,\psi_{(M)}) + \mathfrak{Z}^{\lambda\mu}_{\nu\|\lambda} .} \tag{13.55}$$

Now, if the matter field equations (12.9) are fulfilled by $\psi_{(M)}$ then equations (13.33), (13.44), and (13.55) can be simplified essentially even further. It follows from (13.33) that:

$$\boxed{\mathfrak{S}^{\nu\mu} = \mathfrak{S}^{\mu\nu};} \tag{13.56}$$

i.e., the matter tensor is symmetric, on the basis of the field equations. It follows from (13.44) that:

$$\boxed{\mathfrak{S}^{\ \mu}_{\nu\|\mu} = 0.} \tag{13.57}$$

(13.55) implies:

$$\boxed{\mathfrak{S}_\nu^{\ \mu} = \mathfrak{L}\,\delta_\nu^\mu + \frac{\partial\mathfrak{L}}{\partial X^\nu}\,X^\mu - \mathfrak{L}^{(M)\mu}(\psi_{(M)\|\nu} + Y_\nu\,\Pi\,\psi_{(M)}) + \mathfrak{Z}^{\lambda\mu}_{\nu\|\lambda}.} \tag{13.58}$$

Since $\mathfrak{S}^{\nu\mu}$ is symmetric, one can replace $\mathfrak{Z}^{\lambda\mu\nu}$ with $\frac{1}{2}(\mathfrak{Z}^{\lambda\mu\nu} + \mathfrak{Z}^{\lambda\nu\mu})$ in (13.58), such that, with (13.51), one will have:

$$\left.\begin{aligned} \mathfrak{S}^{\nu\mu} = \mathfrak{L}\,g^{\nu\mu} + \tfrac{1}{2}\frac{\partial\mathfrak{L}}{\partial X^\sigma}(g^{\sigma\nu}X^\mu + g^{\sigma\nu}X^\mu) - \tfrac{1}{2}\mathfrak{L}^{(M)\mu}(\psi_{(M)}^{\ \ \|\nu} + Y^\nu\,\Pi\,\psi_{(M)}) \\ -\tfrac{1}{2}\mathfrak{L}^{(M)\nu}(\psi_{(M)}^{\ \ \|\mu} + Y^\mu\,\Pi\,\psi_{(M)}) + \tfrac{1}{2}(\mathfrak{W}^{\mu\nu\lambda} + \mathfrak{W}^{\nu\mu\lambda})_{\|\lambda} \end{aligned}\right\} \tag{13.59}$$

On the basis of the coupling (12.22), certain affine relations must be a consequence of the relations for the matter tensor $S_\nu^{\ \mu}$. The symmetry of $\overset{4}{\mathfrak{G}}_{ik}$ is an immediate consequence of the symmetry of $\mathfrak{S}_{\nu\mu}$. A brief calculation will succeed in shortening the tensor equation (13.57). One next has:

$$S^{(\nu)(\mu)}{}_{\|(\mu)} = S^{(\nu)(\mu)}{}_{|(\mu)} + \omega_{(\mu)}{}^{(\nu)}{}_{(\alpha)}\,S^{(\alpha)(\mu)} + \omega_{(\mu)}{}^{(\mu)}{}_{(\beta)}\,S^{(\nu)(\beta)}. \tag{13.60}$$

The affine splitting will yield two equations. First of all, it will follow, with (10.20), that:

$$S^{(0)(\mu)}{}_{\|(\mu)} = S^{(0)m}{}_{\|_4 m} + J^{-1}\,J_{|m}\,S^{(0)m}. \tag{13.61}$$

Secondly:

$$S^{(n)(\mu)}{}_{\|(\mu)} = S^{(n)m}{}_{\|_4 m} + J^{-1}\,J_{|m}\,S^{(0)m} + \tfrac{1}{2}J^{-1}\,J_{|(r)}\,S^{(n)(r)} - \tfrac{1}{2}J^{-1}\,J^{(n)}\,S^{(0)(0)}. \tag{13.62}$$

It will then follow from (13.12) that:

$$K^{(0)m}{}_{\|_4 m} + J^{-1}\,J_{|m}\,K^{(0)m} = 0 \tag{13.63}$$

and

$$K^{nm}{}_{\|_4 m} + J^{1/2}\,F_m{}^n\,K^{(0)m} + \tfrac{1}{2}J^{-1}\,J_{|m}\,K^{nm} - \tfrac{1}{2}J^{-1}\,J^m\,K^{(0)(0)} = 0\,. \tag{13.64}$$

With the notation (12.22), these equations will then read:

$$\boxed{k^m{}_{\|_4 m} = 0} \tag{13.65}$$

and

$$\boxed{\overset{4}{K}{}_n{}^m{}_{\|_4 m} + F_{mn}k^m - J_{|n}a = 0.} \tag{13.66}$$

It will likewise follow that:

$$\boxed{t^m{}_{\|_4 m} = 0,} \tag{13.67}$$

$$\boxed{\overset{4}{S}{}_{n}{}^{m}{}_{\|_4 m} + F_{mn} t^m - J_{|n} b = 0.} \tag{13.68}$$

From (13.67), the matter vector t^m has zero divergence. From (13.68), the divergence of the four-matter tensor is not equal to zero, on the basis of the field F_{mn} and the field J. (The physical meaning of that will become clearer later on.) Equations (13.67) and (13.68) will be true only when the matter field equations are fulfilled. By contrast, equations (13.65) and (13.66) are identities.

If one now applies the process of infinitesimal transformations that led to the mathematical identities for the tensors $K^{\mu\nu}$ and $S^{\mu\nu}$ to the affine case and the integral (12.12) then one must imagine that $\overset{4}{\mathfrak{G}}$ depends upon not only the $g_k^{(i)}$, but also on the φ_k and J, which has just the consequence that, in general, one has $\overset{4}{K}{}^{nm}{}_{\|_4 m} \neq 0$. However, everything can be carried over analogously for $\overset{4}{\mathfrak{L}}$ (when one imagines that $\overset{4}{\mathfrak{L}}$ depends upon φ_k and J, in addition to $\psi_{(M_4)}$), such that one will get the following identity from (13.33), when it is rewritten in affine form:

$$\overset{4}{\mathfrak{G}}{}^{(\alpha)(\beta)} - \overset{4}{\mathfrak{G}}{}^{(\beta)(\alpha)} + \overset{4}{\mathfrak{L}}{}^{(M_4)} P^{(\alpha)(\beta)} \psi_{(M_4)} = 0, \tag{13.69}$$

and from (13.44), one will get the corresponding affine equation:

$$\overset{4}{\mathfrak{G}}{}_{n}{}^{m}{}_{\|_4 m} = (\overset{4}{\mathfrak{L}}{}^{(M_4)} \overset{4}{\mathfrak{D}}{}_{n}{}^{m} \psi_{(M_4)} + t^k \overset{4}{\mathfrak{D}}{}_{n}{}^{m} \varphi_k) + \overset{4}{\mathfrak{L}}{}^{(M_4)} \psi_{(M_4)\|_4 n} + t^k \varphi_{k\|_4 n} + b\, J_{|n}\,. \tag{13.70}$$

If the matter field equations are fulfilled then, since:

$$\overset{4}{\mathfrak{D}}{}_{n}{}^{m} \varphi_k = -\varphi_m\, \delta_k^m\,,$$

that will imply the symmetry relation:

$$\overset{4}{\mathfrak{S}}{}^{nm} = \overset{4}{\mathfrak{S}}{}^{mn} \tag{13.71}$$

and the divergence equation:

$$\overset{4}{S}{}_{n}{}^{m}{}_{\|_4 m} = -(t^m \varphi_n)_{\|_4 m} + t^m \varphi_{m\|_4 n} + b\, J_{|n}\,, \tag{13.72}$$

or

$$\overset{4}{S}{}_{n}{}^{m}{}_{\|_4 m} + t^m{}_{\|_4 m}\, \varphi_n + t^m (\varphi_{n\,\|_4 m} - \varphi_{m\,\|_4 n}) - b\, J_{|n} = 0;$$

i.e.:

$$\overset{4}{S}{}_{n}{}^{m}{}_{\|_4 m} + t^m{}_{\|_4 m}\, \varphi_n + t^m F_{mn} - b\, J_{|n} = 0, \tag{13.73}$$

which will coincide with (13.68) when one uses (13.67). However, (13.67) is also a consequence of the gauge invariance of $\overset{4}{\mathfrak{L}}$. ($\overset{4}{\mathfrak{L}}$ is gauge-invariant, since $\mathfrak{L}$ is an invariant density.) If we perform the transformation (6.13a) and a corresponding infinitesimal gauge transformation on the $\psi_{(M_4)}$ then we will have:

$$0 = \delta \int \overset{4}{\mathfrak{L}}\, d\overset{4}{\tau} = \varepsilon \int \left[-\frac{\delta \overset{4}{\mathfrak{L}}}{\delta \varphi_k} \lambda_{|k} - \frac{\delta \overset{4}{\mathfrak{L}}}{\delta \psi_{(M_4)}} \lambda \Pi \psi_{(M_4)} + \left(\frac{\delta \mathfrak{L}}{\delta \varphi_{k|n}} \lambda_{|k} - \frac{\delta \mathfrak{L}}{\delta \psi_{(M_4)|n}} \lambda \Pi \psi_{(M_4)} \right)_{|n} \right] d\overset{4}{\tau}$$

or

$$0 = \varepsilon \int \left[\left(\mathfrak{t}^k{}_{|k} - \overset{4}{\mathfrak{L}}{}^{(M_4)} \Pi \psi_{(M_4)} \right) \lambda - \left(\mathfrak{t}^n \lambda + \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{k|n}} \lambda_k + \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|n}} \lambda \Pi \psi_{(M_4)} \right)_{|n} \right] d\overset{4}{\tau}. \tag{13.74}$$

If one chooses λ such that the second term (which can be converted in to a boundary integral) vanishes then it will follow that:

$$\boxed{\mathfrak{t}^k{}_{|k} - \overset{4}{\mathfrak{L}}{}^{(M_4)} \Pi \psi_{(M_4)} = 0.} \tag{13.75}$$

It will then follow from (13.74), moreover, that:

$$\left(\mathfrak{t}^n \lambda + \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{k|n}} \lambda_k + \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|n}} \lambda \Pi \psi_{(M_4)} \right)_{|n} = 0. \tag{13.76}$$

If one sets λ constant, in particular, then that will yield:

$$\mathfrak{t}^n{}_{|n} = -\left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|n}} \Pi \psi_{(M_4)} \right)_{|n}. \tag{13.77}$$

The vector density $(-\partial \overset{4}{\mathfrak{L}} / \partial \psi_{(M_4)|n}) \Pi \psi_{(M_4)}$ will then have the same divergence as the vector density $\mathfrak{t}^n$. If the matter field equations $\overset{4}{\mathfrak{L}}{}^{(M_4)} = 0$ are fulfilled then the following two divergences will be equal to zero:

$$\boxed{\mathfrak{t}^n{}_{|n} = 0,} \tag{13.78}$$

$$\left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|n}} \Pi \psi_{(M_4)} \right)_{|n} = 0. \tag{13.79}$$

With (13.77), (13.76) will imply the two identities:

$$\mathfrak{t}^k = -\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|k}} \Pi \psi_{(M_4)} - \left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{k|n}} \right)_{|n}, \tag{13.80}$$

$$\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{k|n}} + \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{n|k}} = 0. \tag{13.81}$$

The vector density $\mathfrak{t}^k$ then differs from the vector density $(-\partial \overset{4}{\mathfrak{L}} / \partial \psi_{(M_4)|n}) \Pi \psi_{(M_4)}$ by the divergence-free term $-\ (\partial \overset{4}{\mathfrak{L}} / \partial \varphi_{k|n})_{|n}$. It follows from the definition of $\mathfrak{t}^k$ in (12.22) and equation (13.80) that:

$$\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_k} = -\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|k}} \Pi \psi_{(M_4)}. \tag{13.82}$$

The derivation of the affine equation that corresponds to (13.58) will be carried out in no. **21** with (21.13) as its result.

Let us now go on to the integral over $\overset{4}{\mathfrak{G}}$! Since $\overset{4}{\mathfrak{G}}$ still contains second derivatives, they can be eliminated by partial integration:

$$\int \overset{4}{\mathfrak{G}} \, d\overset{4}{\tau} = \int \overset{4}{\mathfrak{K}} \, d\overset{4}{\tau} + \int_R \cdots \tag{13.83}$$

On the basis of that, the integral $\int \overset{4}{\mathfrak{K}} \, d\overset{4}{\tau}$ is not an invariant now under arbitrary transformations of $\mathfrak{G}_4$, but just the ones that vanish on the boundary such that variation of the boundary integral will give zero. If one goes over the arguments in pages 46 through 52 then one will see that equations (13.33) and (13.44), which we would now like to present in affine form, are already a consequence of that invariance in their own right. They will therefore also be true for $\overset{4}{\mathfrak{K}}$, such that (since $P^{(a)(b)} \varphi_k = P^{(a)(b)} J = 0$):

$$\overset{4}{\mathfrak{K}}{}^{nm} = \overset{4}{\mathfrak{K}}{}^{mn} \tag{13.84}$$

is a symmetric tensor, and:

$$\begin{aligned}\overset{4}{K}{}_n{}^m{}_{\|_4 m} &= (k^r \overset{4}{\mathfrak{D}}{}_n{}^m \varphi_r)_{\|_4 m} + k^m \varphi_{m\|_4 n} + a\, J_{|n} \\ &= -(k^m \varphi_n)_{\|_4 m} + k^m \varphi_{m\|_4 n} + a\, J_{|n}\end{aligned}$$

or

$$\overset{4}{K}{}_n{}^m{}_{\|_4 m} + k^m{}_{\|_4 m}\,\varphi_n + k^m F_{mn} - a\,J_{|n} = 0, \tag{13.85}$$

which agrees with (13.66), with the condition (13.65). However, (13.65) is once more a consequence of gauge invariance. Since $\overset{4}{\mathfrak{K}}$ does not depend upon the matter field quantities, the equation that corresponds to (13.75) will imply (13.65) directly.

The groups $\mathfrak{H}_5$, $\mathfrak{D}_5$ then lead to the same identities for the projective integrals that the groups $\mathfrak{G}_4$, $\mathfrak{D}_4$ lead to for the corresponding affine integrals. In general, the group ($\mathfrak{G}_4$, $\mathfrak{G}$) $\cong \mathfrak{H}_5$ will first take on its deeper meaning in the projective theory, and the projective integrals and field equations have greater mathematical simplicity and symmetry than the affine ones.

CHAPTER III

PHYSICAL APPLICATIONS

14. Field equations for the metric field. – Whereas in nos. **12** and **13**, the field equations were examined in regard to their mathematical structure, here, a special Ansatz shall be attempted for the action quantity $\mathfrak{G}$. The requirements on $\mathfrak{G}$ that were posed in no. **12** are satisfied by, e.g.:

$$\mathfrak{G} = U(J)\,[R + W(J)\,J_{|\mu}\,J^{|\mu} + V(J)]\,\sqrt{-g}\,, \tag{14.1}$$

in which R is the curvature scalar that was defined in (11.18), while $U(J)$, $W(J)$, and $V(J)$ are functions of J.

For the calculation of the variational derivatives $K_{\nu\mu}$, from (12.6) and (12.4), one must note that $\mathfrak{G}$, in the form (14.1), is a function of $g_{\mu\nu}$, $g_{\mu\nu|\sigma}$, $g_{\mu\nu|\sigma|\rho}$, X_ν, such that the fact that:

$$\delta g_{\mu\nu} = \delta g_\mu^{(\sigma)} g_{\nu(\sigma)} + g_{\mu(\sigma)}\delta g_\nu^{(\sigma)}$$

will imply the relation:

$$\mathfrak{K}_{\nu\mu} = \frac{\delta\mathfrak{K}}{\delta g_{\mu\nu}} + \frac{\delta\mathfrak{K}}{\delta g_{\nu\mu}}\,.$$

From (14.1), one gets:

$$\begin{aligned}\delta\mathfrak{G} = {} & [U'R + (U\,W)'\,J_{|\mu}\,J^{|\mu} + (U\,V)']\sqrt{-g}\;\;X^\nu X^\mu\;\delta g_{\nu\mu} \\ & + \tfrac{1}{2}g^{\nu\mu}\,U\,[R + W\,J_{|\rho}\,J^{|\rho} + V]\sqrt{-g}\;\delta g_{\nu\mu} + U\,W\,J_{|\nu}\,J_{|\mu}\;\;\delta g^{\nu\mu}, \\ & + U(J)\;\delta R\sqrt{-g} + 2\,U\,W\,J_{|\rho}\;\delta J^{|\rho}\;\sqrt{-g}\,.\end{aligned}$$

When the last term is integrated, that will yield:

$$\int 2\,U\,W\,J_{|\rho}\;\delta J^{|\rho}\;\sqrt{-g}\;d\tau = -\int 2\,(U\,W\,J^{|\rho})_{|\rho}\;\sqrt{-g}\;X^\nu X^\mu\;\delta g_{\nu\mu}\,d\tau + \int_{R_\Lambda}\cdots$$

It remains for us to evaluate the penultimate term further. It follows from $R = R_{\nu\mu}\,g^{\nu\mu}$ that:

$$\delta R = R_{\nu\mu}\;\delta g^{\nu\mu} + \delta R_{\nu\mu}\,g^{\nu\mu} = -R^{\nu\mu}\;\delta g_{\nu\mu} + g^{\nu\mu}\;\delta R_{\nu\mu}\,,$$

in which the relation $\delta g^{\nu\mu} = -g^{\nu\rho}\,g^{\nu\sigma}\;\delta g_{\rho\sigma}$ was employed. For the calculation of $\delta R_{\nu\mu}$, we remark that since $\delta\Gamma^\rho_{\sigma\eta}$ is a projective tensor, the difference at the point Q' will be a vector α^ν that is parallel-translated from Q to Q' ($\overline{QQ'} = \xi^\sigma$), first with the help of $\Gamma^\rho_{\sigma\eta}$ and then with the help of $\Gamma^\rho_{\sigma\eta} + \delta\Gamma^\rho_{\sigma\eta}$:

$$\delta\Gamma^\rho_{\sigma\eta}\;\alpha^\eta\;\zeta^\sigma,$$

and must be the difference of two vectors at the same point. It follows from (11.8) that for a normal vector α^ρ, one will have:

$$\alpha^\rho{}_{\|\nu\|\mu} - \alpha^\rho{}_{\|\mu\|\nu} = R_{\mu\nu}{}^\rho{}_\tau\,\alpha^\tau,$$

or when contracted with (11.13):

$$\alpha^\nu{}_{\|\nu\|\mu} - \alpha^\nu{}_{\|\mu\|\nu} = R_{\mu\tau}\,\alpha^\tau.$$

It follows from this that when one varies $g_{\mu\nu}$ by $\delta g_{\mu\nu}$, one will have:

$$\delta R_{\mu\nu}\,\alpha^\nu = \delta(\alpha^\nu{}_{\|\nu\|\mu}) - \delta(\alpha^\nu{}_{\|\mu\|\nu}).$$

Since $\alpha^\nu{}_{\|\rho} = \alpha^\nu{}_{|\rho} + \Gamma^\nu_{\rho\sigma}\alpha^\sigma$, it follows that $\delta\alpha^\nu{}_{\|\rho} = \delta\Gamma^\nu_{\rho\sigma}\alpha^\sigma$, and thus, one will ultimately have:

$$\begin{aligned}\delta(\alpha^\nu{}_{\|\rho\|\eta}) &= \delta(\alpha^\nu{}_{\|\rho\|\eta}) + \delta\Gamma^\nu_{\eta\tau}\alpha^\tau_{\|\rho} - \delta\Gamma^\alpha_{\eta\rho}\alpha^\nu_{\|\alpha} \\ &= (\delta\Gamma^\nu_{\eta\sigma})_{\|\eta}\alpha^\sigma + \delta\Gamma^\nu_{\rho\sigma}\alpha^\sigma_{\|\eta} + \delta\Gamma^\nu_{\eta\sigma}\alpha^\sigma_{\|\rho} - \delta\Gamma^\alpha_{\eta\rho}\alpha^\nu_{\|\alpha}.\end{aligned}$$

That implies directly that:

$$\delta R_{\mu\nu} = (\delta\Gamma^\rho_{\rho\nu})_{\|\eta} - (\delta\Gamma^\rho_{\nu\mu})_{\|\rho}.$$

It follows from that by partial integration that:

$$\int U\,g^{\nu\mu}\delta R_{\nu\mu}\sqrt{-g}\,d\tau = \int (U_{|\rho}\,g^{\nu\mu}\delta\Gamma^\rho_{\nu\mu} - U_{|\mu}\,g^{\nu\mu}\delta\Gamma^\rho_{\rho\nu})\sqrt{-g}\,d\tau + \int_{R_\Lambda}\cdots$$

One can derive from (10.10) that:

$$\delta\Gamma^\rho_{\nu\mu} = \tfrac{1}{2}g^{\rho\lambda}\,[(\delta g_{\lambda\nu})_{\|\mu} + (\delta g_{\lambda\mu})_{\|\nu} - (\delta g_{\nu\mu})_{\|\lambda}],$$

so one will get:

$$\int U\,g^{\nu\mu}\delta R_{\nu\mu}\sqrt{-g}\,d\tau = \int (U^\lambda_{\|\lambda}\,g^{\nu\mu}\delta g_{\mu\nu} - U^{|\mu\|\nu}\,\delta g^{\nu\mu})\sqrt{-g}\,d\tau + \int_{R_\Lambda}\cdots$$

by a further partial integration. If one substitutes everything into the initial formula for $\delta\mathfrak{G}$ then one can read off:

$$\begin{aligned}\tfrac{1}{2}K_{\nu\mu} &= [U'R - (U\,W)'\,J_{|\rho}\,J^{|\rho} - 2\,U\,W\,J^{|\rho}{}_{\|\rho} + (U\,V)']\,X_\nu\,X_\mu \\ &\quad + \tfrac{1}{2}g_{\nu\mu}\,U\,[R + W\,J_{|\rho}\,J^{|\rho} + V] - U\,W\,J_{|\nu|\mu} - U\,R_{\nu\mu} + U^{|\lambda}{}_{\|\lambda}\,g_{\nu\mu} - U_{\|\mu\|\nu}.\end{aligned}$$

One can rearrange this into:

$$\left.\begin{aligned}\tfrac{1}{2}K_{\nu\mu} &= [U'R - (UW)'J_{|\rho}J^{|\rho} - 2UW\,J^{|\rho}_{\ \ \|\rho} + (UV)']X_\nu X_\mu \\ &\quad + \tfrac{1}{2}g_{\nu\mu}[U\,R + (2U'' + UW)\,J_{|\rho}J^{|\rho} + 2U'\,J^{|\rho}_{\ \ \|\rho} + UV] \\ &\quad - U\,R_{\nu\mu} - (U'' + UW)\,J_{|\nu}J_{|\mu} - U'J_{|\mu\|\nu}.\end{aligned}\right\} \tag{14.3}$$

One can derive the affine splitting of $K_{\nu\mu}$ either directly from (14.13) with the help of nos. **10** and **11**, or get it from the affine variational principle with (12.22). Using the first way, it will follow, with the use of (10.23) for $J_{|\mu\|\nu}$, and with $J_{|(0)} = 0$, that:

$$\begin{aligned}\tfrac{1}{2}K_{nm} = \tfrac{1}{2}g_{nm}\,[U\,R + (2U'' + U\,V)\,J_{|r}\,J^{|r} + 2U'J^{|r}_{\ \ \|r} + U'J^{-1}\,J_{|r}\,J^{|r} + U\,V] \\ - U\,R_{nm} - (U'' + U\,W)\,J_{|n}\,J_{|m} - U'J_{|m}\,,J_{\|_4 n}\,.\end{aligned}$$

One substitutes the expressions for R and R_{nm} that were given in (11.15) to (11.19) into this, and it will follow that:

$$\left.\begin{aligned}\frac{1}{2U}K_{nm} &= -(\overset{4}{R}_{nm} - \tfrac{1}{2}g_{nm}\overset{4}{R}) - \frac{J}{2}(F_{nr}F_m^{\ r} - \tfrac{1}{4}F_{pr}F^{pr}g_{nm}) \\ &\quad - \left(\frac{U''}{U} + W - \tfrac{1}{4}J^{-2}\right)J_{|n}J_{|m} + \left(\frac{U''}{U} + W - \tfrac{1}{4}J^{-2} + \tfrac{1}{2}J^{-1}\frac{U'}{U}\right)J_{|r}J^{|r}g_{nm} \\ &\quad - \left(\frac{U'}{U} + \tfrac{1}{2}J^{-1}\right)\left(J_{|m\|_4 n} - J^{|r}_{\ \ \|_4 r}g_{nm}\right) + \tfrac{1}{2}g_{nm}V.\end{aligned}\right\} \tag{14.4}$$

It is simpler to calculate $K_{n(0)}$ with (10.23):

$$\tfrac{1}{2}K_{n(0)} = -U\,R_{n(0)} - \tfrac{1}{2}U'J^{1/2}\,F_{rn}\,J^{|r}.$$

If one substitutes (11.16) here then that will yield:

$$U^{-1}K_{n(0)} = -J^{1/2}F^{r}_{\ n\|_4 r} - (\tfrac{3}{2}J^{-1/2} + U'J^{1/2})\,F_{rn}\,J^{|r}. \tag{14.5}$$

It will then remain for one to calculate $K_{(0)(0)}$:

$$\begin{aligned}\tfrac{1}{2}K_{(0)(0)} = (J\,U' + \tfrac{1}{2}U)\,R - U\,R_{(0)(0)} + [U'' - \tfrac{1}{2}U\,W - J\,(U\,W)']\,J_{|r}\,J^{|r} \\ + (U' - 2\,J\,U\,W)\,J^{r}_{\ \|_4 r} + (U\,V)'\,J + \tfrac{1}{2}U\,V.\end{aligned}$$

With (11.17) and (11.19), that will yield:

$$\left.\begin{aligned}\tfrac{1}{2}U^{-1}K_{(0)(0)} &= \left(\tfrac{1}{2}+J\frac{U'}{U}\right)R+\tfrac{1}{4}\left(\tfrac{3}{2}+J\frac{U'}{U}\right)J\,F_{mn}F^{mn}\\ &\quad+2\left(\frac{U'}{U}-J\,W\right)J^{|n}_{\|n}+\left(\tfrac{1}{2}+J\frac{U'}{U}\right)V+J\,V'\\ &\quad+\left(\frac{U''}{U}-\tfrac{1}{2}J^{-1}\frac{U'}{U}-\tfrac{1}{2}W-J\,W'-J\frac{U'}{U}W\right)J_{|r}J^{|r}.\end{aligned}\right\}\qquad(14.6)$$

If one substitutes the expressions for K_{nm}, $K_{n(0)}$, $K_{(0)(0)}$ that were found in (12.23) then the field equations for the metric field will be exhibited in that way. In particular, if no matter is present then they will read simply:

$$K_{nm}=0,\qquad K_{n(0)}=0,\qquad K_{(0)(0)}=0.$$

The same field equations will also follow as in no. **12** from the affine variational principle applied to:

$$\overset{4}{\mathfrak{G}}=J^{1/2}\,U\,[R+W(J)\,J_{|m}\,J^{|m}+V(J)]\,\sqrt{-\overset{4}{g}}\,,$$

or, with (11.19):

$$\overset{4}{\mathfrak{G}}=J^{1/2}\,U(J)[\overset{4}{R}+\tfrac{1}{4}J\,F_{mn}F^{mn}+J^{-1}J^{|m}_{\|_4 m}+(W(J)-\tfrac{1}{2}J^{-2})J_{|m}J^{|m}+V(J)]\sqrt{-\overset{4}{g}}\,.\qquad(14.7)$$

The quantities $\overset{4}{K}_{ik}$, k_l, a can be calculated very easily from (12.13) and (12.21) when one applies the derivatives that lead to (14.3) *mutatis mutandis*; viz., when one sets $J^{1/2}\,U$ in place of U and replaces the Greek indices with Latin ones. In that way, one is spared the direct calculation of the affine splitting of (14.3), and one will obtain the quantities (14.4), (14.5), and (14.6) immediately on the basis of the identities (12.22).

15. Identification. – Up to now, we have made no assumptions about the physical meaning of the quantities that were introduced. We must do that now in order to be able to infer physical consequences of the theory. For this, it is best to appeal to known things. We then write down the total action quantity in affine form according to no. **12** and (14.7):

$$\left.\begin{aligned}\overset{4}{\mathfrak{G}}+\overset{4}{\mathfrak{L}}=\sqrt{-\overset{4}{g}}\,J^{1/2}U(J)\Big[&\overset{4}{R}+\tfrac{1}{4}J\,F_{mn}F^{mn}+J^{-1}J^{|m}_{\|_4 m}\\ &+(W(J)-\tfrac{1}{2}J^{-2})\,J_{|m}J^{|m}+V(J)+\frac{J}{2}\frac{2}{J\,U}L\Big].\end{aligned}\right\}\qquad(15.1)$$

If J is constant then (15.1) will take on the form of a variational principle:

$$\overset{4}{R}+\frac{J}{2}\tfrac{1}{2}F_{mn}F^{mn}+V(J)+\frac{J}{2}\frac{2}{JU}L\,. \tag{15.2}$$

However, a comparison with the general theory of relativity will immediately give the following interpretation:

g_{mn}	EINSTEIN's gravitational potentials
F_{mn}	Electromagnetic field strengths
$\tfrac{1}{2}J=\kappa$	Gravitational constant

This interpretation will also be confirmed for K_{mn} from (14.4), since K_{mn} will take on the form of EINSTEIN's gravitational field equations when it is set to zero. The energy-impulse tensor also has the usual form:

$$\overset{e}{T}_{ik}=F_{ir}F_k^{\;r}-\tfrac{1}{2}F_{mn}F^{mn}g_{ik}\,.$$

We will be in agreement with that when we define the energy-impulse tensor of matter by:

$$T_{(i)}^{(k)}=-\frac{1}{2\sqrt{-\overset{4}{g}}}\frac{\delta\left(2J^{-1}U^{-1}\sqrt{-\overset{4}{g}}\right)}{\delta g_k^{(i)}}, \tag{15.3}$$

according to (15.2). In this, $\delta(\ldots)/\delta g_k^{(i)}$ is EULER's variational derivative that was defined in no. **12**. When one refers to (12.13) and (12.22), it will follow from (15.3) that:

$$\left.\begin{aligned}\overset{4}{S}_{ik}&=-J^{1/2}U(J)T_{ik},\\ S_{ik}&=-J\,U(J)T_{ik}.\end{aligned}\right\} \tag{15.4}$$

The field equations (12.23) then read:

$$-U^{-1}K_{nm}=-\tfrac{1}{2}U^{-1}K_{nm}=\tfrac{1}{2}J^{-1/2}U^{-1}\overset{4}{S}_{nm}=-\frac{J}{2}T_{nm}\,, \tag{15.5}$$

which says that, from (14.4), T_{nm} enter in the same way for matter as $\overset{4}{T}_{nm}$ does for the electromagnetic field.

Once energy and impulse are identified, it only remains for us to account for the charge-current vector. It follows from (14.5), with (12.22), that:

$$F_{r\,\|_4 n}^{\;n}=-U^{-1}J^{-3/2}t_r+\left(\frac{U'}{U}+\tfrac{3}{2}J^{-1}\right)J_{|n}F^{n}_{\;r}\,. \tag{15.6}$$

We then refer to:

$$s_r = - U^{-1}\, J^{-3/2}\, t_r \tag{15.7}$$

as the *charge-current of matter* and:

$$v_r = \left(\frac{U'}{U} + \tfrac{3}{2} J^{-1}\right) J_{|n} F^n{}_r \tag{15.8}$$

as the *polarization current of the vacuum*.

16. Solutions of the field equations. – The field equations that were formulated in no. **14** and interpreted in no. **15** admit solutions when $V(J) = 0$. The electromagnetic field and matter field are equal to zero, J = const., and the gravitational field is a solution of the EINSTEIN equations:

$$\overset{4}{R}_{ik} = 0. \tag{16.1}$$

If $V(J) \neq 0$ then a solution J = const. will not be possible, since the gravitational equations $\overset{4}{R} + 2V = 0$ that follow from (14.4) by contraction would contradict the ones that would follow from (14.6), namely $\overset{4}{R} + V = 0$.

Up to now, there has been no deeper discussion of the interpretation of the term $V(J)$ in the field equations. For the sake of simplicity, we would like to always set $V(J)$ equal to zero in what follows. For that reason, we would like to make the quantity $W(J)$, which is coupled to J^{-2} as a summand in (14.4), proportional to J^{-2}:

$$W(J) = - \lambda\, J^{-2}, \tag{16.2}$$

into which we have introduced a dimensionless constant λ. In order to introduce no further constants, (14.4) suggest the Ansatz for $U(J)$:

$$U(J) = J^{\alpha}, \tag{16.3}$$

so U''/U will be likewise proportional to J^{-2}. We shall first fix the exponent α later on. The field equations will then read:

$$\left.\begin{aligned}
&\overset{4}{R}_{ik} - \tfrac{1}{2} g_{ik}\, \overset{4}{R} + \frac{J}{2}\left(F_{ir} F_k{}^r - \tfrac{1}{4} F_{pr} F^{pr} g_{ik}\right) - \left[\lambda + \tfrac{1}{4} + \alpha(1-\alpha)\right] J^{-2} J_{|i} J_{|k} \\
&+ \tfrac{1}{2}\left[\lambda + \tfrac{1}{2} + \alpha\left(\tfrac{1}{2} - \alpha\right)\right] J^{-2} J_{|r} J^{|r} g_{ik} + \left(\alpha + \tfrac{1}{2}\right) J^{-1}\left(J_{|i\|_4 r} - J^{|r}{}_{\|_4 r}\, g_{ik}\right) \\
&\quad = \tfrac{1}{2} J^{-\alpha-1/2}\, \overset{4}{S}_{ik}, \\
&F^n_{r\,\|_4 n} = -J^{-\alpha-3/2} t_r + \left(\alpha + \tfrac{3}{2}\right) J^{-1} J_{|n} F^n{}_r, \\
&\left(\tfrac{1}{2} + \alpha\right) \overset{4}{R} + \tfrac{1}{4}\left(\tfrac{3}{2} + \alpha\right) F_{mn} F^{mn} + 2(\alpha + \lambda) J^{-1} J^{|n}{}_{\|_4 n} - \left(\tfrac{3}{2} - \alpha\right)(\alpha + \lambda) J^{-2} J_{|r} J^{|r} + J^{1/2-\alpha} b = 0.
\end{aligned}\right\} \tag{16.4}$$

The fact that these equations possess the solutions J = const. when $F_{ik} = 0$ and $\overset{4}{S}_{ik} = 0$, $b = 0$, as mentioned above, implies the demand that one must still have J = const. approximately for weak gravitational fields, such that for the absolutely largest domain, equations (16.4) will go to MAXWELL's equations for electrodynamics and EINSTEIN's equations for gravitation. Solutions of the latter equations will also be approximate solutions to the equations (16.4) above then, as long as the fields are weak.

The variability of J can first manifest itself for cosmological dimensions or for extreme energy densities. Two problem statements then suggest themselves naturally: The cosmos as a complete entity and the creation of stars, as was inferred inductively by P. JORDAN (Section I). Solutions to (16.4) can be given for both problems, with certain idealizations.

17. A model for the cosmos and the creation of stars. – In order to be able to describe an expanding world in a purely kinematical way, we choose x^1, x^2, x^3 to be spatial coordinate and $t = x^4$ to be a time coordinate. Let the spatial part of the world (x^1, x^2, x^3) be a hypersphere of radius $\rho(t)$, and let x^1, x^2, x^3 be coordinates of the unit sphere. We can then write the line-element of the world as:

$$ds^2 = \rho^2\, ds^2 - dt^2, \tag{17.1}$$

in which ds^2 is the line element of the unit sphere. If we introduce the notation $g = \| g_{ik} \|$ (i, k = 1, 2, 3) then $\overset{3}{g} = -\overset{4}{g} = (\overset{4}{m})^2 = \rho^6\,(x^1, x^2, x^3)$, in which ξ does not depend upon time t.

It follows from (13.68) that:

$$\overset{4}{\mathfrak{S}}{}_{4\,\|_4 n}^{\;n} = \overset{4}{\mathfrak{S}}{}_{4\,|n}^{\;n} - \tfrac{1}{2}\, g_{rs|4}\, \overset{4}{\mathfrak{S}}{}^{rs} = J_{|4} - F_{m4}\, \mathfrak{t}^m .$$

If we now set:

$T_{44} = \varepsilon$ = energy density,
$T_{ik} = p\, g_{ik}$, (i, k = 1, 2, 3, p = pressure of matter)

in (15.4), under the assumption of an isotropic homogeneous distribution of matter, then we will get:

$$(J^{3/2}\, U\, \rho^3\, \varepsilon)^{\cdot} = \dot{J}\, b \rho^3 - J^{3/2} U\, P(\rho^3)^{\cdot} - F_{m4}\, t^m\, \rho^3. \tag{17.2}$$

The total material energy is then given by:

$$E = \int \varepsilon \sqrt{g}\, dx^1\, dx^2\, dx^3 = \varepsilon\, \rho^3 \int \sqrt{\xi}\, dx^1\, dx^2\, dx^3 = 2\pi^2\, \varepsilon\, \rho^3. \tag{17.3}$$

As a first application of (17.2), we consider the special case $p = b = 0$, and the electromagnetic field $F_{ik} = 0$. While the physical meaning of the assumptions $p = 0$, $F_{ik} = 0$ is immediately obvious, the meaning of $b = 0$ is not entirely clear. We will explain the meaning of b in no. **19**. With the assumptions that we have made, we will then get:

$$\left.\begin{aligned} \varepsilon &= J^{-3/2}U^{-1}\rho^{-3}\gamma = J^{-3/2-\alpha}\rho^{-3}\gamma, \\ E &= 2\pi^2 J^{-3/2}\gamma = 2\pi^2 J^{-3/2-\alpha}\gamma, \end{aligned}\right\} \tag{17.4}$$

in which γ is an undetermined constant, and the second formula follows from (16.3).

For the electromagnetic tensor:

$$\overset{4}{S}{}^{(e)}_{ir} = -J^{3/2}\,U\,\overset{e}{T}_{ik} = -J^{3/2}\,U\,(F_{in}\,F^{kn} - F_{mn}\,F^{mn}\,g_{ik}), \tag{17.5}$$

which, as we showed above, enters into the field equations in the same place for the electromagnetic field as the tensor $\overset{4}{S}_{ik}$ does for the matter field, one gets from (15.6) that:

$$\overset{4}{S}{}^{(e)k}_{i\ \ \|_4 k} = -J^{3/2}\,U\,F_{in\,\|\,k}\,F^{kn} - F_{in}\,t^n + \tfrac{1}{4}J^{3/2}U\,F_{mn\|_4 i}\,F^{mn} + \tfrac{1}{4}F_{mn}(J^{3/2}U\,F^{mn})_{\|_4 i}$$

$$= -\tfrac{1}{2}J^{3/2}U(F_{in\,\|_4 k} + F_{ki\,\|_4 n})F^{kn} - F_{in}t^n + \tfrac{1}{2}J^{3/2}U\,F_{kn\,\|_4 i}F^{kn} + \tfrac{1}{4}\left(\tfrac{3}{2} + \frac{U'}{U}J\right)J^{3/2}U\,F_{mn}F^{mn}J_{|i}\,,$$

or, since:

$$F_{in\|_4 k} + F_{nk\|_4 i} + F_{ki\,\|_4 n} = 0,$$

which is a consequence of (6.11), it will ultimately follow that:

$$\overset{4}{S}{}^{(e)k}_{i\ \ \|_4 k} = F_{ni}\,t^n + J_{|\,i}\,b^{(e)}, \tag{17.6}$$

with

$$b^{(e)} = \tfrac{1}{4}\left(\tfrac{3}{2} + \frac{U'}{U}J\right)J^{3/2}U\,F_{mn}F^{mn}. \tag{17.7}$$

However, $b^{(e)}$ corresponds completely in the field equation (16.4) to the b of matter. If one adds the relations for $\overset{4}{S}_{ik}$ and $\overset{4}{S}{}^{(e)}_{ik}$ then it will follow that:

$$(\overset{4}{S}{}^{(e)k}_{i} + \overset{4}{S}{}_{i}^{k})_{\|_4 k} = J_{|i}\,(b^{(e)} + b). \tag{17.8}$$

When applied to the expanding universe above, it will follow exactly that:

$$(J^{3/2}\,U\,\rho^3\,e)^{\cdot} = \dot{J}\,(b^{(e)} + b)\,\rho^3 - J^{3/2}\,U\,p\,(\rho^3)^{\cdot}, \tag{17.9}$$

in which ε and p are, however, the energy density (pressure, resp.) of matter + electromagnetic radiation field. If the temperature is high enough that we can set the pressure p in (17.9) equal to the highest-possible value $p = \tfrac{1}{3}\varepsilon$, and if $b + b^{(e)} = 0$, moreover, then we will get from (17.9) that:

$$\left.\begin{aligned} 3p = \varepsilon = J^{-3/2}U^{-1}\rho^{-4}\sigma = J^{-3/2-\alpha}\rho^{-4}\sigma, \\ E = 2\pi^2 J^{-3/2}U^{-1}\rho^{-1}\sigma = 2\pi^2 J^{-3/2-\alpha}\rho^{-1}\sigma, \end{aligned}\right\} \tag{17.10}$$

in which σ is an undetermined constant. The assumptions $p = \frac{1}{3}\varepsilon$ and $b + b^{(e)} = 0$ are fulfilled for arbitrary temperatures because in the event that only electromagnetic radiation without matter is present, from (17.2), $b^{(e)}$ will be proportional to $\mathfrak{E}^2 - \mathfrak{H}^2$ ($\mathfrak{E}$ = electric field strength, $\mathfrak{H}$ = magnetic field strength), which will vanish in the mean.

We can also write down the four-matter tensor directly from (15.4) for the two cases that were just cited:

$$\left.\begin{aligned} &1.\quad p = b = 0, \quad F_{ik} = 0 \quad (\text{i.e., } \overset{4}{S}{}^{(e)}_{ik} = 0); \\ &\overset{4}{S}_{44} = -\rho^{-3}\gamma, \quad \overset{4}{S}_{ik} = 0 \quad \text{for} \quad i \text{ or } k \neq 4. \end{aligned}\right\} \tag{17.11}$$

$$\left.\begin{aligned} &2.\quad p = \tfrac{1}{3}\varepsilon, \quad b + b^{(e)} = 0, \\ &\left(\overset{4}{S}_{44} + \overset{4}{S}{}^{(e)}_{44}\right) = -\rho^{-4}\sigma, \quad \left(\overset{4}{S}_{ik} + \overset{4}{S}{}^{(e)}_{ik}\right) = -\tfrac{1}{3}\rho^{-4}\sigma g_{ik}, \\ &\text{and}\left(\overset{4}{S}_{4i} + \overset{4}{S}{}^{(e)}_{4i}\right) = 0 \quad \text{for} \quad i, k = 1, 2, 3. \end{aligned}\right\} \tag{17.12}$$

When the field equations (16.4) are applied to the cosmological model in question, they will take on the form:

$$\left.\begin{aligned} &\overset{4}{R}_{ik} - \tfrac{1}{2} g_{ik}\overset{4}{R} - [\lambda + \tfrac{1}{4} + \alpha(1-\alpha)]\, J^{-2} J_{|i} J_{|k} + \tfrac{1}{2}[\lambda + \tfrac{1}{2} + \alpha(\tfrac{1}{2} - \alpha)]\, J^{-2} J_{|r} J^{|r} g_{ik} \\ &\qquad + (\alpha + \tfrac{1}{2}) J^{-1}(J_{\|_4 k} - J^{|r}_{\|_4 r} g_{ik}) = \tfrac{1}{2} J^{-\alpha-1/2}(\overset{4}{S}_{ik} + \overset{4}{S}{}^{(e)}_{ik}) = -\tfrac{1}{3}\rho^{-4}\sigma g_{ik} \\ &(\tfrac{1}{2} + \alpha) R + 2(\alpha + \lambda) J^{-1} J^{|n}_{\|_4 n} - (\tfrac{3}{2} - \alpha)(\alpha + \lambda) J^{-2} J_{|r} J^{|r} + J^{1/2-\alpha}(b + b^{(r)}) = 0, \end{aligned}\right\} \tag{17.13}$$

in which the values (17.11) [(17.12), resp.] have been substituted for $\overset{4}{S}_{ik} + \overset{4}{S}{}^{(e)}_{ik}$ and $b + b^{(e)}$, resp. In order to evaluate these equations explicitly, we must still calculate R_{ik} , R, etc.

In order to do that, we start from (17.1) and choose x^1, x^2, x^3 especially such that the unit sphere is given by:

$$(x^0)^2 + \sum_{i=1}^{3}(x^i)^2 = 1, \tag{17.14}$$

such that:

$$ds^2 = (dx^0)^2 + \sum_{i=1}^{3} (dx^i)^2 \,. \tag{17.15}$$

Due to the homogeneity of space, it suffices to calculate all quantities at just one point – e.g., for $x^1 = x^2 = x^3 = 0$ – and indeed to calculate the metric tensor up to second-order quantities in the x^1, x^2, x^3 and the Γ^i_{kl} up to first-order quantities in the x^1, x^2, x^3. We agree that the sign $\doteq$ means that equality is valid only at the point $x^1 = x^2 = x^3 = 0$.

It follows from (17.1), (17.14), and (17.15) that:

$$ds^2 \doteq \rho^2 \left(\sum_{i=1}^{3} (dx^i)^2 + \sum_{i=1}^{3} x^k x^l dx^k dx^l \right) - dt^2,$$

so

$$\left.\begin{aligned} &g_{ik} \doteq \rho^2(\delta_{ik} + x^i x^k) \quad i, k = 1, 2, 3, \\ &g_{i4} = 0, \quad i \neq 4, \\ &g_{44} = -1, \end{aligned}\right\} \tag{17.16a}$$

from which, it will follow immediately that:

$$\left.\begin{aligned} &g^{ik} \doteq \rho^{-2}(\delta_{ik} - x^i x^k) \quad i, k = 1, 2, 3, \\ &g^{i4} = 0, \quad i \neq 4, \\ &g^{44} = -1. \end{aligned}\right\} \tag{17.16b}$$

The three-index symbols then follow from this:

$$\left.\begin{aligned} &\Gamma^i_{rs} \doteq x^i \delta_{rs}, \qquad \Gamma^4_{4s} = \Gamma^s_{44} = \Gamma^4_{44} = 0, \\ &\Gamma^4_{rs} \doteq \dot\rho\,\rho\,\delta_{rs}, \quad \Gamma^i_{4s} \doteq \frac{\dot\rho}{\rho}\delta_{is}, \quad (i, r, s = 1, 2, 3), \end{aligned}\right\} \tag{17.17}$$

in which $\dfrac{\partial}{\partial x^4} = \dfrac{\partial}{\partial t}$ has been replaced with a dot. It follows from:

$$\alpha^i_{\|_4 i \|_4 k} - \alpha^i_{\|_4 k \|_4 i} = \overset{4}{R}_{ki}\, \alpha^i$$

that for a vector α^i with $\alpha^i{}_{|k} \doteq 0$, $\alpha^i{}_{|k|l} \doteq 0$, with (17.17):

$$(i, k \neq 4) \qquad \left\{ \begin{aligned} &\alpha^i_{\|_4 k} \doteq x^i \alpha^k + \frac{\dot\rho}{\rho}\delta_{ik}\alpha^4, \quad &&\alpha^i_{\|_4 k} \doteq \dot\rho\,\rho\,\alpha^k, \\ &\alpha^i_{\|_4 4} \doteq \alpha^i, &&\alpha^4_{\|_4 4} \doteq 0, \end{aligned}\right.$$

and therefore:

$$\alpha^{i}_{\|_4 i} = \sum_{k=1}^{3} \alpha^{k}_{\|_4 k} + \alpha^{4}_{\|_4 4} \doteq \sum_{k=1}^{3} x^k \alpha^k + 3\frac{\dot{\rho}}{\rho}\alpha^4 .$$

One will then have:

$$\alpha^{i}_{\|_4 i \|_4 k} = \left(\alpha^{i}_{\|_4 i}\right)_{|k} \doteq \alpha^k \text{ for } k \neq 4,$$

$$\alpha^{i}_{\|_4 i \|_4 4} = \left(\alpha^{i}_{\|_4 i}\right)_{|k} \doteq 3\left(\frac{\ddot{\rho}}{\rho} \rightarrow \frac{\dot{\rho}^2}{\rho^2}\right)\alpha^4 ,$$

moreover, and:

$$k \neq 4: \qquad \alpha^{i}_{\|_4 k \|_4 i} = \sum_{l=1}^{3} \alpha^{l}_{\|_4 k \|_4 l} + \alpha^{4}_{\|_4 k \|_4 4} \doteq 3\alpha^k + 2\dot{\rho}^2\alpha^k ,$$

$$\alpha^{i}_{\|_4 4 \|_4 i} = \sum_{l=1}^{3} \alpha^{l}_{\|_4 k \|_4 l} + \alpha^{4}_{\|_4 4 \|_4 4} \doteq 3\frac{\dot{\rho}^2}{\rho^2}\alpha^4 ,$$

such that:

$$\left.\begin{aligned} i, k \neq 4: \quad & \overset{4}{R}_{ki} \doteq -\delta_{ik}(2 + 2\dot{\rho}^2 + \ddot{\rho}\rho), \\ & \overset{4}{R}_{ki} \doteq 0, \quad \overset{4}{R}_{ki} \doteq 3\frac{\ddot{\rho}}{\rho}. \end{aligned}\right\} \tag{17.18}$$

Due to the homogeneity of space, it will then generally follow with (17.16a) that:

$$\left.\begin{aligned} i, k \neq 4: \quad & \overset{4}{R}_{ik} = -\rho^{-2} g_{ik}(2 + 2\dot{\rho}^2 + \ddot{\rho}\rho), \\ & \overset{4}{R}_{4k} = 0, \quad \overset{4}{R}_{44} = 3\frac{\ddot{\rho}}{\rho}, \quad \overset{4}{R} = -\frac{6}{\rho^2}(1 + 2\dot{\rho}^2 + \ddot{\rho}\rho), \end{aligned}\right\} \tag{17.19}$$

and finally (i, $k \neq 4$):

$$\overset{4}{R}_{ik} - \tfrac{1}{2} g_{ik} \overset{4}{R} = \rho^{-2} g_{ik}(1 + 2\dot{\rho}^2 + \ddot{\rho}\rho) , \qquad \overset{4}{R}_{44} - \tfrac{1}{2} g_{44} \overset{4}{R} = -3\rho^{-2}(1 + \dot{\rho}^2) . \tag{17.20}$$

Due to the homogeneity of space, the scalar J can depend upon only t, such that with (17.17), it will follow that:

$$\left.\begin{aligned} & J_{|i\|_4 k} = -\frac{\ddot{\rho}}{\rho} g_{ik} J, \qquad i, k \neq 4, \quad J^{|4}_{\|_4 4} = -\ddot{J}, \\ & J^{|l}_{\|_4 4} = J^{|4}_{\|_4 l} = 0, \qquad l \neq 4, \\ & J^{|l}_{\|_4 l} = -\left(\ddot{J} + 3\frac{\dot{\rho}}{\rho}\dot{J}\right). \end{aligned}\right\} \tag{17.21}$$

If one substitutes (17.20) and (17.21) into (17.18) then it will follow that:

$$\left.\begin{aligned}
&g_{ik}\left\{\frac{1}{\rho^2}+\left(\frac{\dot\rho}{\rho}\right)^2+2\frac{\dot\rho}{\rho}-\frac{1}{2}\left[\lambda+\frac{1}{2}+\alpha(1-\alpha)\right]\left(\frac{\dot J}{J}\right)^2-2\left(\alpha+\frac{1}{2}\right)\frac{\dot\rho\dot J}{\rho J}+\left(\alpha+\frac{1}{2}\right)\frac{\ddot J}{J}\right\}\\
&\qquad=\frac{1}{2}J^{-\alpha-1/2}(\overset{4}{S}_{ik}+\overset{4}{S}{}^{(e)}_{ik}),\\
&-\frac{3}{\rho^2}-3\left(\frac{\dot\rho}{\rho}\right)^2-\left[\frac{\lambda}{2}+\frac{3}{4}\alpha-\frac{1}{2}\alpha^2\right]\left(\frac{\dot J}{J}\right)^2-3\left(\alpha+\frac{1}{2}\right)\frac{\dot\rho\dot J}{\rho J}=\frac{1}{2}J^{-\alpha-1/2}(\overset{4}{S}_{44}+\overset{4}{S}{}^{(e)}_{44}),\\
&-6\left(\alpha+\frac{1}{2}\right)\frac{1}{\rho^2}-6\left(\alpha+\frac{1}{2}\right)\left(\frac{\dot\rho}{\rho}\right)^2-6\left(\alpha+\frac{1}{2}\right)\frac{\ddot\rho}{\rho}-2(\alpha+\lambda)\frac{\dot J}{J}-3(\alpha+\lambda)\frac{\dot\rho\dot J}{\rho J}\\
&\qquad+\left(\frac{3}{2}-\alpha\right)(\alpha+\lambda)\left(\frac{\dot J}{J}\right)^2+J^{1/2-\alpha}(b+b^{(e)})=0.
\end{aligned}\right\}\qquad(17.22)$$

Let us first consider the case of (17.11)! Finding a solution for arbitrary values of α seems pretty hopeless. However, if the inductive cosmological arguments of P. JORDAN that were sketched out in Chapter I are valid then one should expect that it is just the model (17.11) that must give a solution of the form $\rho = \rho_0\, t$, because stellar velocities that are close to the velocity of light will justify the Ansatz of setting $p \ll \varepsilon$ – i.e., $p \sim 0$ – in all cases. (The velocities of the stars are reckoned relative to the coordinate system of the x^1, x^2, x^3, such that the expansion velocity of the universe does not enter into it.) Another Ansatz $\rho = \rho_0\, t^\mu$, with $\mu \neq 1$ would not be able to lead to a simpler solution either, since $1 / \rho^2$ and $(\dot\rho / \rho)^2$ would not contain the same powers of t then. If we make the Ansatz $J = J_0\, t^\beta$ for J then it will follow from the second equation (17.22) that the terms in (17.20) will contain the power t^{-2} when $\beta = -1 / (\alpha + 1/2)$. Now, it is remarkable that there exist solutions of the simple form:

$$\rho = \rho_0\, t, \qquad J = J_0\, t^{-\frac{1}{\alpha+\frac{1}{2}}} \qquad (17.23)$$

for only two values of α. Substituting (17.23) into (17.22), with (17.11), will yield:

$$\left.\begin{aligned}
&\frac{1}{\rho_\alpha^2}+1-\frac{\lambda+1+\alpha(\frac{1}{2}-3\alpha)}{2(\alpha+\frac{1}{2})^2}=0,\\
&\frac{1}{\rho_\alpha^2}+1-\frac{\lambda+\alpha(\frac{3}{2}-3\alpha)}{6(\alpha+\frac{1}{2})^2}=\frac{1}{2}J_0^{-\alpha-\frac{1}{2}}\rho_\alpha^{-3}\gamma,\\
&\frac{1}{\rho_\alpha^2}+1-\frac{\lambda+\alpha}{2(\alpha+\frac{1}{2})^2}=0.
\end{aligned}\right\}\qquad(17.24)$$

In order for this simple solution to be possible, the first and last equation must coincide, which implies the quadratic equation for α:

$$\alpha^2 + \frac{1}{6}\alpha - \frac{1}{3} = 0$$

with the two roots $\alpha = 1/2$ and $\alpha = -2/3$. From (17.23), the value $\alpha = -2/3$ gives $J = J_0 t^6$; i.e., an increase by t^6, which cannot agree with experiment in any case, according to Chapter I. The best prospect for agreement with experiments is given by the value $\alpha = 1/2$. For that reason, we decide to fix the still-mathematically-arbitrary function $U(J)$ with (16.3) as:

$$U(J) = J^{1/2}. \tag{17.25}$$

(17.24) will then imply the two equations:

$$\left.\begin{aligned} &\frac{1}{\rho_\alpha^2} + 1 - \frac{1}{4} - \frac{\lambda}{2} = 0, \\ &\frac{1}{\rho_\alpha^2} + \frac{1}{12} - \frac{\lambda}{6} = \frac{1}{2} J_0^{-1} \rho_\alpha^{-3} \gamma. \end{aligned}\right\} \tag{17.26}$$

The value for ρ_α:

$$\rho_\alpha = \frac{2}{\sqrt{2\lambda - 3}}$$

follows from the first equation. The second equation then yields:

$$\gamma = 2 J_0\, \rho_\alpha (4 + \rho_\alpha^2).$$

With the introduction of a new constant β_0 in place of γ, with (17.4), one will have:

$$\left.\begin{aligned} &\rho = t\frac{2}{\sqrt{2\lambda - 3}}, \quad \tfrac{1}{2} J = \kappa = \frac{1}{\rho}\frac{8(\lambda - 1)}{2\lambda - 3}\frac{1}{\rho_0}, \\ &\varepsilon = \frac{\beta_0}{\rho}, \qquad\qquad E = 2\pi^2 \rho^2 \beta_0. \end{aligned}\right\} \tag{17.27}$$

We think of all quantities in (17.27) as being measured in natural units, as in Chapter I. We recognize immediately that the relations (17.27) will agree with the inductively-inferred order-of-magnitude relations (I.1) when λ is a number with order of magnitude unity, and β_0 is likewise roughly unity in natural units. The pure number λ that was introduced in (16.2) can be derived only from experiment.

For the other extreme case (17.12), in order to solve (17.22) for $\alpha = 1/2$, we make the Ansatz:

$$\rho = \rho_b\, t, \qquad J = J_1\, t^{-2},$$

and find that:

$$\frac{1}{\rho_b^2} + 3 - (1 + 2\lambda) = -\frac{1}{6}\frac{\sigma}{J_1\,\rho_b^4},$$
$$-\frac{3}{\rho_b^2} + 3 - (1 + 2\lambda) = -\frac{1}{2}\frac{\sigma}{J_1\,\rho_b^4},$$
$$-\frac{6}{\rho_b^2} - 6 + (1 + 2\lambda) = 0.$$

It follows from the third of these equations that:

$$1 + 2\lambda = 3 + \frac{3}{\rho_b^2}.$$

If one substitutes this into the other two equations then they will become identities and yield the relation:

$$\sigma = 12\ J_1^2\rho_b^2.$$

With the values (17.10) for the energy density ε and the total energy E, when one introduces the constant $J_1^{-2}\rho_b^4\sigma$ in place of σ, one will ultimately get the result that:

$$\left.\begin{aligned} &\rho = t\sqrt{\frac{3}{2}}\frac{1}{\sqrt{\lambda-1}}, \quad && \tfrac{1}{2}J = \kappa = \frac{1}{\rho^2}\frac{6}{\beta_1}, \\ &\varepsilon = \beta_1, && E = 2\pi^2\rho^3\beta_1. \end{aligned}\right\} \tag{17.28}$$

Just as the value of β_0 was not fixed in the first model, the value of β_1 is not fixed here, either. In natural units, we would have to expect a value for β_1 that would have order of magnitude unity in natural units (i.e., $\varepsilon = \beta_1$ atomic nuclear density). Generally, we must still specify what sort of experiment we can compare the latter model with. From an idea of P. JORDAN, this model seems to be suitable for giving a deductive foundation for the creation of stars.

In order to do that, we consider the following solutions of our field equations, which certainly exist (but are not calculated exactly): Along with the cosmos as a whole, a smaller stellar cosmos is created spontaneously that is completely separate from the cosmos as a whole, as might be suggested in Fig. 1 in exaggerated ratios of quantities, and in which the three spatial dimensions have been replaced with a single one. From (17.28), κ will decrease with the age of the star like t^{-2} in the small stellar cosmos. After a certain time, κ will have attained a value in the star that is the same as the value of κ in the cosmos as a whole, so it will be possible to continue the solution in such a way that the small stellar cosmos fuses with the cosmos as a whole. It is at that moment that the "new star" will first become visible, and indeed, the expansion velocity of the fusion will be around the speed of light initially. From (17.27) and (17.28), one will have:

$$\rho_{\text{star}} = (\rho_{\text{universe}})^{1/2} \left(\frac{\beta_0}{\beta_1}\right)^{1/2} \frac{\sqrt{3}}{2} \sqrt{\frac{2\lambda-3}{\lambda-1}},$$

at the time point of the fusion, so the order of magnitude in natural units will be:

$$\rho_{\text{star}} \sim (\rho_{\text{universe}})^{1/2}, \qquad t_{\text{star}} \sim (t_{\text{universe}})^{1/2}.$$

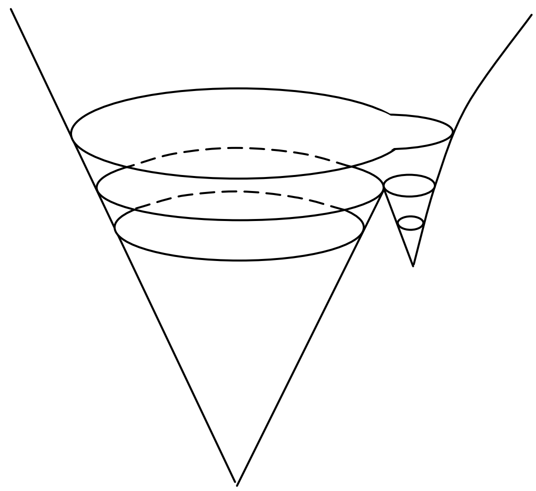

Figure 1.

The current age of the universe is around 10^{40} in natural units, such that for a star that is presently being created, the time of its existence up to its emergence in the cosmos will amount to around 10^{20} (~ 10^{-3} sec). The mass of the newly-created star will coincide with its mass at the time point that it emerges in the cosmos, up to order of magnitude, i.e.:

$$E_{\text{star}} \sim (\rho_{\text{star}})^3 \sim (\rho_{\text{universe}})^{3/2} \sim (t_{\text{universe}})^{3/2} \tag{17.29}$$

so for a star that is being presently created, E_{star} ~ (~ 50 solar masses). The radius at the moment when it emerges in the cosmos would be $\rho_{\text{star}} \sim 10^{20}$ (~ 200 km). At that moment, one must regard the star as a degenerate one that consists of matter that is almost all neutrons. The explosion that would result from the emergence of the star must probably be identified with the phenomenon of a supernova. For the consequences of this picture of stellar creation and its relationship with the problems of cosmic radiation and the creation of the elements, whose detailed discussion has still not been concluded, one would do best to read the two works of P. JORDAN *Die Herkunft der Sterne* and "Zur Theorie der Sternentstehung" (References [**9**] and [**10**], since these considerations go beyond the scope of the projective theory of relativity, and therefore, the scope of this booklet).

18. Current and charge. – One has the following divergence relation for the matter vector $\mathfrak{t}^m$ that was defined in (12.21):

$$\mathfrak{t}^m{}_{|m} = 0. \tag{18.1}$$

If we introduce the current vector s_r as in (13.7) then we will have:

$$\left.\begin{aligned} &(J^{3/2} U\, \mathfrak{s}^m)_{|m} = 0, \\ \text{or}\quad &\mathfrak{s}^m{}_{|m} = -\left(\frac{U'}{U} + \frac{3}{2} J^{-1}\right) J_{|m}\, \mathfrak{s}^m. \end{aligned}\right\} \tag{18.2}$$

It follows from (15.8) by differentiation that:

$$v^r_{\|_4 r} = \left(\frac{U'}{U} + \frac{3}{2} J^{-1}\right) J_{|m}\, F^{mr}{}_{\|_4 r}\,,$$

and with (15.6):

$$v^r_{\|_4 r} = \left(\frac{U'}{U} + \frac{3}{2} J^{-1}\right) J_{|m}\, s^m\,,$$

such that one will then have:

$$(\mathfrak{s}^m + \mathfrak{v}^m)_{|m} = 0, \qquad (s^m + v^m)_{\|_4 m} = 0. \tag{18.3}$$

The divergence of the charge-current of matter alone does not vanish, but only that of the combined matter and polarization current. The question then arises of how the total charge if matter will behave in an expanding universe. In the coordinates x^1, x^2, x^3, t of page 43, (18.2) will read:

$$\sum_{l=1}^{3} (J^{3/2} U \rho^3 s^l)_{|l} + (J^{3/2} U \rho^3 s^4)_{|4} = 0.$$

If:

$$Q = \int s^4 \rho^3 \sqrt{\zeta}\; dx^1\, dx^2\, dx^3 \tag{18.4}$$

is the total charge of matter then integrating the previous equation over all of space x^1, x^2, x^3 will give the relation:

$$(J^{3/2}\, U\, Q)^{\cdot} = 0,$$

or, with $U = J^{1/2}$:

$$J^2\, Q = \text{const.} \tag{18.5}$$

If the constant in (18.5) is not zero then the total charge of matter will not, in fact, remain constant, but will increase like J^{-2}. However, from experiments, the total charge Q of matter is probably equal to zero, and the constant in (18.5) must also be set equal to zero. From (18.5), the condition $Q = 0$ will then be compatible with the expansion of the universe and variable J.

19. Scalar matter field. – Whereas, up to now, we have introduced the matter field only indirectly by way of the matter tensor $S_{\mu\nu}$ with the use of the associated conservation law, in this section we would like to explicitly examine the simplest-possible matter field. We introduce a real invariant ψ as the field quantity; ψ is then homogeneous of degree zero. The physical meaning of ψ is given by the energy-impulse tensor and the charge-current vector. The simplest Ansatz for $\mathfrak{L}$ is then $\mathfrak{L} = L\, m$, with:

$$L = \tfrac{1}{2}[\alpha(J)\ \psi^{|\nu}\ \psi_{|\nu} + \beta(J)\ \psi^2]. \tag{19.1}$$

The matter field equations (12.9) then read:

$$(\alpha\,\psi^{|\nu})_{\|\,\nu} - \beta\,\psi = 0. \tag{19.2}$$

Since $\psi^{|\,\nu}\ \psi_{|\,\nu} = \psi^{|k}\ \psi_{|\,k}$, from (12.11), one will have:

$$\overset{4}{L} = \tfrac{1}{2} J^{1/2}\,(\alpha\,\psi^{|\nu}\ \psi_{|\nu} + \beta\,\psi^2), \tag{19.3}$$

from which, the field equations in affine form will imply that:

$$(J^{1/2}\alpha\psi^{|k})_{\|_4 k} - J^{1/2}\ \beta\,\psi = 0, \tag{19.4}$$

which one can also obtain naturally from (19.2) by direct calculation with the help of (10.23).

Since ψ is a scalar, from (13.51), one will have $\mathfrak{W}^{\mu\rho\lambda} = 0$ and therefore also $\mathfrak{Z}^{\lambda\mu}_{\nu} = 0$. The matter tensor density is then calculated from (13.58):

$$\mathfrak{S}_\nu{}^\mu = \mathfrak{L}\,\delta^\mu_\nu + \frac{\partial \mathfrak{L}}{\partial X^\nu} X^\mu - \frac{\partial \mathfrak{L}}{\partial \psi_{|\mu}} \psi_{|\nu}\,.$$

The matter tensor then reads:

$$S_\nu{}^\mu = \tfrac{1}{2}(\alpha\,\psi^{|\nu}\ \psi_{|\nu} + \beta\,\psi^2)\ \delta^\mu_\nu - \alpha\,\psi^{|\mu}\ \psi_{|\nu} + (\alpha'\,\psi^{|\lambda}\ \psi_{|\lambda} + \beta'\psi^2)\,X^\nu X^\mu, \tag{19.5}$$

in which α' and β' are the derivatives of α and β. That will imply the affine splitting:

$$\left.\begin{aligned} S^m_n &= \tfrac{1}{2}(\alpha\psi^{|k}\psi_{|k} + \beta\psi^2)\,\delta^m_n - \alpha\psi^{|m}\psi_{|n}, \\ S^{(0)m} &= 0, \\ S^{(0)(0)} &= J(\alpha'\psi^{|k}\psi_{|k} + \beta'\psi^2) + \tfrac{1}{2}(\alpha\psi^{|k}\psi_{|k} + \beta\psi^2). \end{aligned}\right\} \tag{19.6}$$

It follows from (12.22) that the four-matter tensor is:

$$\overset{4}{S}{}_n{}^m = \tfrac{1}{2}J^{1/2}\,(\alpha\,\psi^{|k}\ \psi_{|k} + \beta\,\psi^2)\ \delta^m_n - J^{1/2}\ \alpha\,\psi^{|\,m}\ \psi_{|\,n}\,. \tag{19.7}$$

The matter tensor is equal to zero:

$$t^m = 0, \tag{19.8}$$

and the matter invariant is equal to:

$$b = \tfrac{1}{2} J^{1/2} (\alpha' \psi^{|k} \psi_{|k} + \beta' \psi^2) + \tfrac{1}{4} J^{-1/2} (\alpha \psi^{|k} \psi_{|k} + \beta \psi^2). \tag{19.9}$$

Since $t^m = 0$, the ψ-field will represent uncharged matter. The energy-impulse follows from four-matter tensor:

$$T_i^{\ k} = -J^{-1} U^{-1} \left[\tfrac{1}{2}(\alpha \psi^{|l} \psi_{|l} + \beta\psi^2)\,\delta_i^k - \alpha\psi^{|k} \psi_{|i}\right]. \tag{19.10}$$

The field variable ψ can be just as well replaced with any multiple of ψ without changing anything in the physical meaning of energy and impulse. For the sake of simplicity, we set:

$$\left.\begin{aligned} \alpha(J) &= J\,U, \\ \beta(J) &= J\,U\,\mu^2(J). \end{aligned}\right\} \tag{19.11}$$

(19.10) will then assume the form:

$$T_i^{\ k} = -\tfrac{1}{2} J^{-1} U^{-1} (\psi^{|l} \psi_{|l} + \mu^2\psi^2)\,\delta_i^k + \alpha\psi^{|k} \psi_{|i}\,, \tag{19.12}$$

in which μ is, as is known, the mass (in natural units) of the corresponding particle under field quantization.

In order to explain the meaning of the matter invariant b, we would like to investigate the field equations in the coordinates system x^1, x^2, x^3, t that was used as a basis in no. **17**. If $\sqrt{-\overset{4}{g}} = \sqrt{\overset{3}{g}} = \rho^3\,\zeta^{1/2}$, with the notations that were used there, then the field equations (19.4) will read:

$$(\rho^3\,\zeta^{1/2}\,J\,\alpha\,\psi^{|k})_{|k} - \rho^3\,\zeta^{1/2}\,J^{1/2}\,\beta\,\psi = 0. \tag{19.13}$$

If we set $J^{1/2}\,\alpha = \sigma$ and denote differentiation with respect to t by a dot, to abbreviate, then it will follow that:

$$\ddot{\psi} + \mu^2\psi + \left(\frac{\dot{\sigma}}{\sigma} + 3\frac{\dot{\rho}}{\rho}\right)\dot{\psi} - \rho^{-2}\zeta^{-1/2}\sum_{l,k=1}^{3}(\zeta\,\psi_{|k}\gamma^{kl})_{|l} = 0,$$

in which we have set $g^{kl} = \rho^{-2}\,\gamma^{kl}$. We solve these equations by way of the Ansatz:

$$\psi = Y(t)\,Z\,(x^1, x^2, x^3).$$

It will then follow that:

$$\left.\begin{aligned} &\ddot{Y}+\left(\frac{\dot{\sigma}}{\sigma}+3\frac{\dot{\rho}}{\rho}\right)\dot{Y}+\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)Y=0,\\ &\sum_{k,l=1}^{3}(\zeta\, Z_{|k}\gamma^{kl})_{|l}+\zeta\, l^2 Z=0. \end{aligned}\right\} \tag{19.14}$$

The solutions to the second of these two equations are spherical functions, which we would not like to give explicitly, however. l is capable of taking on only discrete values. However, if ρ is very large then l / ρ will be practically continuous in the first equation (19.14). Since we would like to investigate the macroscopic behavior of the solutions of (19.14) in the mean, the second equation will have no essential significance for that purpose. We make the Ansatz:

$$Y = \sigma^{-1/2}\,\rho^{-3/2}\,\varphi$$

for Y. We will then get the following equation for φ, which is free of $\dot{\varphi}$:

$$\ddot{\psi}+\left[\frac{1}{4}\left(\frac{\dot{\sigma}}{\sigma}\right)^2-\frac{1}{2}\frac{\ddot{\sigma}}{\sigma}-\frac{1}{2}\frac{\dot{\sigma}}{\sigma}\frac{\dot{\rho}}{\rho}+\frac{3}{2}\left(\frac{\dot{\rho}}{\rho}\right)^2-\frac{3}{2}\frac{\ddot{\rho}}{\rho}+\mu^2+\left(\frac{l}{\rho}\right)^2\right]\varphi=0.$$

We now consider the state of the universe, moreover, where ρ, $\rho/\dot{\rho}$, $\sqrt{\rho/\ddot{\rho}}$, like $\sigma/\dot{\sigma}$, $\sqrt{\sigma/\ddot{\sigma}}$, and also $\mu/\dot{\mu}$, $\sqrt{\mu/\ddot{\mu}}$ are large in comparison to the elementary length. Since $1 / \mu$ has the order of magnitude of the elementary length (because μ is the natural unit of mass for elementary particles), we can simplify the equation for φ above to:

$$\ddot{\psi}+\left[\mu^2+\left(\frac{l}{\rho}\right)^2\right]\varphi = 0.$$

With the conditions above, we can easily solve this equation (the real part of it, resp.) approximately by way of the Ansatz:

$$\varphi = u\,e^{iv}\,.$$

The two equations follow for u and v:

$$\begin{aligned} \ddot{u}-u\,\dot{v}^2+\left[\mu^2+\left(\frac{l}{\rho}\right)^2\right]u &= 0,\\ 2\dot{u}\,\dot{v}+u\,\ddot{v} &= 0. \end{aligned}$$

From the assumptions above, we can neglect $\ddot{u}$ in the first of these two equations, such that:

$$\dot{v} = \left[\mu^2 + \left(\frac{l}{\rho}\right)^2\right]^{1/2}, \qquad v = \int^t \left[\mu^2 + \left(\frac{l}{\rho}\right)^2\right]^{1/2} dt, \text{ resp.}$$

One will then get:

$$\frac{\dot{u}}{u} = -\frac{1}{2}\frac{\ddot{v}}{v}, \qquad u = (\dot{v})^{-1/2} = \left[\mu^2 + \left(\frac{l}{\rho}\right)^2\right]^{-1/4}$$

from the second equation. One will then have, to a sufficient approximation ([1]):

$$\psi = \psi_0\, Z_l\, J^{-1/4}\, \alpha^{-1/2}\, \rho^{-3/2} \left[\mu^2 + \left(\frac{l}{\rho}\right)^2\right]^{-1/4} \cos\left\{\int^t \left[\mu^2 + \left(\frac{l}{\rho}\right)^2\right]^{1/2} dt\right\}. \tag{19.15}$$

We would like to use this solution to calculate the spatial and temporal means of the tensor $\overset{4}{S}{}_i^k$ and the matter invariant b over several periods of ψ. If follows from (19.7), with (19.11), that:

$$\overset{4}{S}{}_i^k = \tfrac{1}{2} J^{1/2}\alpha\left(\mu^2\psi^2 - \dot{\psi}^2 + \sum_{n=1}^{3}\psi^{|n}\psi_{|n}\right)\delta_i^k - J^{1/2}\,\alpha\,\psi^{|k}\,\psi_{|i}\,, \tag{19.16}$$

and from (19.9):

$$b = \tfrac{1}{2} J^{-1/2}\alpha\left[\left(1 + 2J\frac{\alpha'}{\alpha}\right)\left(\mu^2\psi^2 - \dot{\psi}^2 + \sum_{n=1}^{3}\psi^{|n}\psi_{|n}\right) + 4J\mu\,\mu'\psi^2\right]. \tag{19.17}$$

We would next like to show that the mean value of:

$$\mu^2\psi^2 - \dot{\psi}^2 + \sum_{n=1}^{3}\psi^{|n}\psi_{|n}$$

is equal to zero. When ρ is large, only those values of l that are likewise large will play a role. However, if we normalize Z_l in such a way that the mean of Z_l^2 is equal to unity then in the mean over all spherical functions that belong to the eigenvalue l^2 in (19.14) we will have:

([1]) The asymptotic integration of (19.14) that is presented here leads, e.g., to the asymptotic representation of the BESSEL functions by way of the equation $\ddot{y} + \frac{1}{t}\dot{y} + a\,y = 0$. Cf., e.g., COURANT-HILBERT, *Methoden der mathemschen Physik*, v. 1, Berlin 1931, page 285 *et seq.*

$$\overline{\psi^{|m}\psi_{|n}} = \psi_0^2 J^{-1/2}\alpha^{-1}\rho^{-3}\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{-1/2}\frac{1}{6}\left(\frac{l}{\rho}\right)^2\delta_n^m \qquad \text{for } m, n \neq 4.$$

The temporal mean of $\cos^2(\ldots)$ has been set to 1/2 in this. We will then have:

$$\overline{\mu^2\psi^2 - \dot{\psi}^2 + \sum_{k=1}^{3}\psi^{|k}\psi_{|k}} = \psi_0^2 J^{-1/2}\alpha^{-1}\rho^{-3}\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{-1/2}\frac{1}{2}\left[\mu^2-\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)+\left(\frac{l}{\rho}\right)^2\right]$$
$$= 0$$

On the other hand:

$$\overline{\psi^{|4}\psi_{|4}} = -\psi_0^2 J^{-1/2}\alpha^{-1}\rho^{-3}\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{-1/2}\frac{1}{2}.$$

Thus, one ultimately has:

$$\left.\begin{aligned} m, n \neq 4: \quad & \overline{\overset{4}{S}{}_i^k} = -\frac{1}{6}\psi_0^2\rho^{-3}\left(\frac{l}{\rho}\right)^2\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{1/2}\delta_i^k, \quad \overline{\overset{4}{S}{}_4^n} = 0, \\ & \overline{\overset{4}{S}{}_4^4} = \tfrac{1}{2}\psi_0^2\rho^{-3}\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{1/2}, \end{aligned}\right\} \tag{19.18}$$

and

$$\bar{b} = \tfrac{1}{2}\mu'\psi_0^2\rho^{-3}\mu\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{-1/2}. \tag{19.19}$$

One gets the energy density ε and pressure p from (19.18):

$$\left.\begin{aligned} \varepsilon &= \tfrac{1}{2}J^{-3/2}U^{-1}\rho^{-3}\psi_0^2\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{1/2}, \\ p &= \tfrac{1}{6}J^{-3/2}U^{-1}\rho^{-3}\psi_0^2\left(\frac{l}{\rho}\right)^2\left(\mu^2+\left(\frac{l}{\rho}\right)^2\right)^{-1/2}. \end{aligned}\right\} \tag{19.20}$$

One easily verifies that, in fact, equation (17.2) is fulfilled with (19.19) and (19.20) and $t^m = 0$. One obtains the two extreme cases that were discussed in no. **17** for $(l / \rho) \ll \mu$ [$(l / \rho) \gg \mu$, resp.]. In the former case:

$$\left.\begin{aligned} \varepsilon &\simeq \tfrac{1}{2}J^{-3/2}U^{-1}\rho^{-3}\psi_0^2\mu^2, \\ p &\simeq 0, \end{aligned}\right\} \tag{19.21}$$

and in the second case:

$$\left.\begin{aligned} \varepsilon &\simeq \tfrac{1}{2} J^{-3/2} U^{-1} \rho^{-3} \psi_0^{\,2} \frac{l}{\rho}, \\ p &\simeq \tfrac{1}{2} J^{-3/2} U^{-1} \rho^{-3} \psi_0^{\,2} \frac{l}{\rho}. \end{aligned}\right\} \tag{19.22}$$

Formulas (19.20) also admit the corpuscular interpretation of n particles per unit volume of rest mass μ and impulse l / ρ, with which, one will then have:

$$n = \tfrac{1}{2} J^{-3/2} U^{-1} \rho^{-3} \psi_0^{\,2} .$$

With this example, we have explicitly established the formula for the matter tensor that was employed in no. **17**. However, the assertion that we made in the section above that we must set $b = 0$, which still has yet to be justified, also receives some degree of explanation. $\bar{b}$ is equal to zero when the mass μ of the elementary particle that corresponds to the ψ field does not depend upon J. One can learn nothing about whether μ does or does not depend upon J purely on the basis of the theory that was presented here. However, since μ^{-1} proves to have an order of magnitude of the elementary length experimentally, that would give much support to the assumption that μ is a constant, and therefore independent of J. However, $\bar{b} = 0$ then.

A basis for having $\bar{b} = 0$ can given under more general assumptions. If $\overset{4}{L}$ has the form:

$$\overset{4}{L} = J^{3/2}\, U(J)\, M\left(\psi_{(M_4)}, \psi_{(M_4)|k}\right),$$

in which M might no longer depend upon J and $J_{|k}$, so from (12.21):

$$b = (J^{\,3/2}\, U)'\, M = \overset{4}{L} \frac{d}{dJ} \log\,(J^{\,3/2}\, U),$$

then it will be proportional to $\overset{4}{L}$. Now, if $\overset{4}{L}$ (and therefore M) is a homogeneous function of degree n in $\psi_{(M_4)}$, $\psi_{(M_4)|k}$, moreover; i.e., if:

$$\overset{4}{L}\left(\lambda\psi_{(M_4)}, \lambda\psi_{(M_4)|k}\right) = \lambda^n\, \overset{4}{L}\left(\psi_{(M_4)}, \psi_{(M_4)|k}\right),$$

then one will have:

$$n\overset{4}{\mathfrak{L}} = \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)}} \psi_{(M_4)} + \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|k}} \psi_{(M_4)|k} = \frac{\delta \overset{4}{\mathfrak{L}}}{\delta \psi_{(M_4)}} \psi_{(M_4)} + \left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|k}} \psi_{(M_4)} \right)_{|k} .$$

Now, if the matter field equations are fulfilled then:

$$n\overset{4}{\mathfrak{L}} = \left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|k}} \psi_{(M_4)} \right)_{|k} .$$

If we integrate this equation over the space x^1, x^2, x^3 of the expanding cosmos then it will follow that:

$$n\int \overset{4}{\mathfrak{L}} \, d\,x^1 \, dx^2 \, dx^3 = \left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|k}} \psi_{(M_4)} \right)^{\cdot} .$$

Now, if the solution of the field equations is temporally approximated by periodic functions with frequencies that are large in comparison to the variability of J then the temporal mean over several periods of the latter expression will be equal to zero, and therefore when one takes means over space and time, one will ultimately have:

$$\overline{\overset{4}{\mathfrak{L}}} = 0 \quad \text{and} \qquad \overline{b} = 0.$$

As far as the physical meaning of the individual assumptions is concerned, it should be noted that the first assumption about the dependency of the action quantity $\overset{4}{\mathfrak{L}}$ on J represents just an extension of the assumption that $\mu' = 0$. From (15.3), the expression M that was introduced above will then represent just the quantity from which the energy-impulse tensor can be calculated directly with no further assumptions from factors that depend upon J. The first assumption is then equivalent to saying that energy and impulse do not depend upon J explicitly. In the next section, we will learn of cases in which that is not the case exactly, but only approximately, since the extra terms diminish along with J, and therefore play no role for $t \gg 1$ (i.e., the age of the universe is large compared to an elementary time unit). The experimental fact that all elementary particle masses and atomic masses have the order of magnitude unity in natural units and have no abnormally large values is suggests precisely that LAGRANGE functions that do not fulfill the first assumption (at least approximately) seem to play no role in nature.

The second assumption of homogeneity is fulfilled by all of the LAGRANGE functions with degree 2 that have be examined up to now and are important in practice when one ignores the interaction of the fields with each other. One can also formulate this in such a way that the second assumption is, to some extent, equivalent with the assumption of an ideal gas; that assumption cannot be justified in general then. However, we can justify that assumption (approximately) in the two cases that were considered in no. **17**:

In the case of the universe as a whole, we can regard the individual matter components (even when they are themselves composite) as components with constant mass and practically negligible interaction (ideal gas as the matter content of the universe).

In the case of stellar creation, the temperature of degenerate neutron matter must be regarded as being so high that the interactions will likewise play no role in comparison to the kinetic energies.

It then seems to me that the assumptions about p and b that were made in no. **17** are the most natural ones for the macroscopic treatment of the two problems that were posed there.

The field of an invariant ψ that was considered up to now proves to be uncharged. A charged matter field can already be described by a (single) scalar field Ψ if we let Ψ be complex, and introduce the demand for the group $\mathfrak{P}$ that:

$$\mathfrak{T}_r\,\Psi = e^{il\ln\rho}\,\Psi. \tag{19.23}$$

With a complex Ψ, we have basically introduced two field functions Ψ and Ψ^* (complex conjugate of Ψ) or also the real and imaginary parts of Ψ. The simplest invariant for a LAGRANGE function is:

$$L = \tfrac{1}{2}[\alpha(J)\,\Psi^{*\|\nu}\,\Psi_{\|\nu} + \beta(J)\,\Psi^*\,\Psi]. \tag{19.24}$$

In the case of (19.23), the infinitesimal transformation Π of the group $\mathfrak{P}$ is given by:

$$\Pi\,\Psi = i\,l\,\Psi, \tag{19.25}$$

such that, from (10.24):

$$\Psi_{\|\nu} = \Psi_{|\nu} - i\,l\,Y_\nu\,\Psi. \tag{19.26}$$

(19.24) then reads:

$$L = \tfrac{1}{2}[\alpha\,\Psi^{*|\nu}\,\Psi_{|\nu} + i\,l\,\alpha\,Y^\nu(\Psi_{|\nu}\,\Psi^* - \Psi^*_{|\nu}\,\Psi) + (l^2\,J^{-1}\,\alpha + \beta)\,\Psi^*\,\Psi].$$

The middle term in that expression does not have the normal form for L that was required in no. **13** in order to be able to calculate the matter tensor $S_{\nu\mu}$. It follows from (19.23) that:

$$\Psi_{|\nu}\,X^\nu = i\,l\,\Psi, \qquad \text{i.e.,} \qquad \Psi_{|\nu}\,Y^\nu = i\,l\,J^{-1}\,\Psi,$$

such that one can also write L in the form:

$$L = \tfrac{1}{2}[\alpha\,\Psi^{*|\nu}\,\Psi_{|\nu} + (\beta - l^2\,J^{-1}\alpha)\,\Psi^*\,\Psi]. \tag{19.27}$$

The field equations follow most simply from (19.24):

$$(\alpha\,\Psi^{|\nu})_{|\nu} - \beta\,\Psi = 0. \tag{19.28}$$

For the calculation of the matter tensor from (19.27) according to (13.58), one should note that here, as above, $\mathfrak{Z}_\nu^{\lambda\mu} = 0$. It will then follow that:

$$S_\nu{}^\mu = L\,\delta_\nu^\mu - \tfrac{1}{2}\alpha\,(\Psi^{*|\nu}\,\Psi_{|\nu} + \Psi^{|\nu}\,\Psi^*{}_{|\nu}) - [\alpha\,\Psi^{*|\rho}\,\Psi_{|\rho} + (\beta - l^2\,J^{-1}\alpha)\,\Psi^*\,\Psi]\,X_\nu\,X^\mu,$$

or:

$$\left.\begin{aligned} S_\nu^{\ \mu} &= L\,\delta_\nu^\mu - \tfrac{1}{2}\alpha(\Psi^{*\|\mu}\,\Psi_{\|\nu} + \Psi^{\|\mu}\,\Psi^*_{\|\nu}) \\ &\quad - \tfrac{1}{2} il\,J^{-1}(\Psi^{*\|\mu}X_\nu\Psi - \Psi_{\|\nu}X^\mu\Psi^* - \Psi^{\|\mu}X_\nu\Psi^* + \Psi^*_{\|\nu}X^\mu\Psi) \\ &\quad + [\alpha'\,\Psi^{*\|\rho}\,\Psi_{\|\rho} + \beta'\,\Psi^*\Psi]X_\nu X^\mu. \end{aligned}\right\} \tag{19.29}$$

For the calculation of the affine quantities, the scalar Ψ must be normalized in the way that was described on page 8. Since $H_\rho = e^{il\ln\rho}$, the normalized invariant that is associated with Ψ will be:

$$\psi = e^{-il\ln\rho}\,\Psi, \qquad \Psi = e^{il\ln\rho}\,\psi. \tag{19.30}$$

With (10.24), one calculates from this that:

$$\Psi_{\|(0)} = 0, \qquad \Psi_{\|(0)} = e^{il\ln\rho}\,\psi_{|\_k}\,, \tag{19.31}$$

in which:

$$\psi_{|\_k} = \psi_{|k} - i\,l\,\varphi_k\,\psi.$$

The field equation (19.28) can be rewritten in affine form immediately with the help of (19.31) and formulas (10.23) and (10.25):

$$(J^{1/2}\,\alpha\,\psi^{|\_k})_{|\_k} - \beta\,J^{1/2}\,\psi = 0. \tag{19.32}$$

These field equations are naturally also a consequence of the affine variational principle for:

$$L = \tfrac{1}{2}\,J^{1/2}\,(\alpha\,\psi^{|\_k}\,\psi_{|\_k} + \beta\,\psi^*\,\psi). \tag{19.33}$$

The affine splitting of the matter tensor can be calculated from (19.29) and (19.31):

$$\left.\begin{aligned} S_n^{\ m} &= \tfrac{1}{2}(\alpha\psi^{*|\_k}\psi_{|\_k} + \beta\psi^*\psi)\delta_n^m - \tfrac{1}{2}\alpha(\psi^{*|\_m}\psi_{|\_n} + \psi^{|\_m}\psi^*_{|\_n}), \\ S^{(0)m} &= -\tfrac{1}{2}i\,l\,J^{-1/2}\alpha(\psi^{*|\_m}\psi - \psi^{|\_m}\psi^*), \\ S^{(0)(0)} &= \tfrac{1}{2}(\alpha\psi^{*|\_k}\psi_{|\_k} + \beta\psi^*\psi)\delta_n^m + J(\alpha'\psi^{*|\_m}\psi_{|\_n} + \beta'\psi^*\psi). \end{aligned}\right\} \tag{19.34}$$

One gets the energy-impulse tensor, charge-current vector, and matter invariant from this using (12.22), (15.4), and (15.7), resp.:

$$\left.\begin{aligned} T_i^{\ k} &= -J^{-1}U^{-1}\alpha\left[\tfrac{1}{2}(\psi^{*|\_l}\,\psi_{|\_l} + \beta\psi^*\psi)\,\delta_i^k - \tfrac{1}{2}(\psi^{*|\_k}\,\psi_{|\_i} + \psi^{|\_k}\,\psi^*_{|\_i})\right], \\ s_r &= \tfrac{1}{2}i\,l\,J^{-1}U^{-1}\alpha(\psi^*_{|\_r}\psi - \psi_{|\_r} + \psi^*), \\ b &= \tfrac{1}{4}J^{-1/2}\alpha(\psi^{*|\_k}\,\psi_{|\_k} + \mu^2\psi^*\psi) + \tfrac{1}{2}J^{1/2}(\alpha'\,\psi^{*|\_k}\,\psi_{|\_k} + \beta'\,\psi^*\psi). \end{aligned}\right\} \tag{19.35}$$

Since $s_r \neq 0$, the Ψ field represents charged matter. If we employ (19.11) then we will get the usual form for $T_i^{\ k}$ and s_r , such that under quantization, μ, as well as l, will yield the charge of the particle. This interpretation of μ and l is likewise implied by the discussion of the motion of the Ψ-wave field for the case in which the "wave length" is small in comparison to the change in the metric field. In full mathematical exactitude, this means that in the Ansatz:

$$\Psi = A\ e^{i\varphi(X^\nu)} \tag{19.36}$$

for the eikonal or phase function, one has $|\ \varphi_{\|\nu}\ | \gg |\ \alpha' J_{|\nu}\ |, |\ \varphi^{\|\nu}{}_{\|\nu}\ |$. (19.36) implies that:

$$\Psi_{\|\nu} = A\ i\ \varphi_{\|\nu}\ e^{i\varphi}, \tag{19.37}$$

with:

$$\varphi_{\|\nu} = \varphi_{|\nu} - l\ Y_\nu\ . \tag{19.38}$$

φ is not an invariant, but only a scalar, since it follows from (19.23) that:

$$\mathfrak{T}_\rho\ \varphi = \varphi + l \ln \rho,$$

such that for the infinitesimal transformation of the group $\mathfrak{P}$:

$$\Pi\ \varphi = l, \tag{19.39}$$

with which, (19.38) will follow from (10.24). It will then follow further from (19.37) that:

$$(\alpha \Psi^{\|\nu})_{\|\nu} = A\ [-\ \alpha\ \varphi^{\|\nu}\ \varphi_{\|\nu} + i\ (\alpha\ \varphi^{\|\nu})_{\|\nu}]\ e^{i\varphi},$$

and with the assumptions that were made:

$$(\alpha \Psi^{\|\nu})_{\|\nu} \simeq -\ A\ \alpha\ \varphi^{\|\nu}\ \varphi_{\|\nu}\ e^{i\varphi}.$$

The first-order partial differential equation for φ then follows from field equation (19.28):

$$\alpha\ \varphi^{\|\nu}\ \varphi_{\|\nu} + \beta = 0, \tag{19.40}$$

which corresponds to the eikonal equation of geometric optics. The "rays" of the Ψ-field are given by the characteristics of (19.40):

$$\left.\begin{aligned} &F(X^\nu, p_\nu) = \alpha\ p^\nu p_\nu - \alpha l^2 J^{-1} + \beta, && \frac{dX^\nu}{ds} = 2\alpha p^\nu, \\ &\frac{d\varphi}{ds} = 2\alpha p_\nu p^\nu, && \frac{dp_\nu}{ds} = -(\alpha' p_\nu p^\nu + \beta' - \alpha' l^2 J^{-1} + \alpha l^2 J^{-2})\ J_{|\nu} \\ & && \qquad - \alpha\ p_\sigma g^{\sigma\mu}{}_{|\nu}\ p_\mu . \end{aligned}\right\} \tag{19.41}$$

As is known, $F(X^\nu, p_\nu)$ will then be an integral of the characteristic equation. If $p_\nu = \varphi_{|\nu}$ and $F(X^\nu, p_\nu) = 0$ for the initial manifold then it will be true for the entire solution of

(10.40); i.e., for the entire manifold that is spanned by the characteristics that go through the initial manifold. Since, from (19.39), one will have:

$$\varphi_{|\nu} X^{\nu} = l = p_{\nu} X^{\nu}, \tag{19.42}$$

for the initial values of (19.41), that condition will always be fulfilled; i.e., $dl / ds = 0$, which one can also calculate explicitly. We can put the last equation in (19.41) into an invariant form. Next, p_ν is a normal vector, since it follows from $\varphi_{|\mu} X^{\mu} = l$ by differentiation that $\varphi_{|\mu\|\nu} X^{\nu} + \varphi_{|\nu} = 0$, and therefore $p_{\nu|\mu} X^{\mu} + p_{\nu} = 0$. It will then follow from (19.41) that:

$$\frac{d_{\|} p_\nu}{ds} = p_{\nu\|\mu} \frac{dX^\mu}{ds} = -(\alpha' p_\mu p^\mu + \beta' + \alpha l^2 J^{-2} - \alpha' l^2 J^{-1}) J_{|\nu}. \tag{19.41a}$$

The rays (19.41) characterize the classical paths of the particles that correspond to the field. The path is given projectively by (19.41a), in which one considers (19.42) to be the initial value. If we set $\beta = \alpha\mu^2$, in turn, then we can also write (19.41a) as:

$$2\alpha^2 p_{\nu\|\mu} p^\mu = -\alpha'(\alpha p^\mu p_\mu + \beta - \alpha l^2 J^{-1}) J_{|\nu} - 2\alpha^2 \mu\mu' J_{|\nu} - \alpha^2 l^2 J^{-2} J_{|\nu}.$$

Due to (19.40), $F(X^\nu, p_\nu) = 0$ in (19.41), such that:

$$p_{\nu\|\mu} p^\mu = \tfrac{1}{2} J_{|\nu} (l^2 J^{-2} + 2\mu\mu'). \tag{19.43}$$

In order to find the affine representation of the path, we next remark that, with (19.41) and (19.42), we will have:

$$p_{(0)} = J^{-1/2} l, \qquad p_n = \frac{1}{2\alpha} \frac{dx^n}{ds}. \tag{19.44}$$

The two equations follow from (19.43), with (10.23):

$$\begin{aligned} 0 &= p_{(0)\|(m)} p^{(m)} + p_{(0)\|(0)} p^{(0)} \\ &= (J^{-1/2} l)_{|m} \frac{1}{2\alpha} \frac{dx^m}{ds} + \frac{1}{2} J^{-1} J_{|m} \frac{dx^m}{ds} J^{-1/2} l = J^{-1/2} \frac{1}{2\alpha} \frac{dl}{ds}, \end{aligned}$$

$$\begin{aligned} -\tfrac{1}{2} J_{|\nu} (l^2 J^{-2} + 2\mu\mu') \\ = p_{n\|_4 m} \frac{1}{2\alpha} \frac{dx^m}{ds} + \tfrac{1}{2} J^{1/2} F_{mn} J^{-1/2} l \frac{1}{2\alpha} \frac{dx^m}{ds} - \tfrac{1}{2} J^{1/2} F_{mn} l \frac{1}{2\alpha} \frac{dx^m}{ds} J^{-1/2} - \tfrac{1}{2} J^{-1} J_{|n} J^{-1} l^2. \end{aligned}$$

The first equation gives the aforementioned fact $dl / ds = 0$ once more. The second equation can also be written:

$$p_{n\|_4 m} \frac{dx^m}{ds} = l\, F_{mn} \frac{dx^m}{ds} - 2\alpha\mu\mu' J_{|n}. \tag{19.45}$$

In order to introduce the proper time $d\tau = \sqrt{-dx^k dx_k}$ in place of *ds*, we substitute the value into:

$$d\tau^2 = -4\,\alpha^2\,p^k\,p_k\,ds^2$$

that (19.41) yields when $F(X^\nu, p_\nu) = 0$:

$$p^k\,p_k = -\frac{\beta}{\alpha} = -\mu^2,$$

which will give:

$$d\tau = 2\,\alpha\,\mu\,ds.$$

If we denote the four-velocity by:

$$u^k = \frac{dx^k}{d\tau},$$

then it will follow from (19.45):

$$\frac{d_{\|_4}(\mu u_n)}{d\tau} = (\mu u_n)_{\|_4 m} u^m = l\,F_{nm}\,u^m - \mu\,J_{|n} = l\,F_{nm}\,u^m - \mu_{|n}\,. \tag{19.46}$$

However, in the event that $\mu'(J) = 0$ – i.e., in the event that μ is independent of J – this is precisely the known equation of motion for a mass point of mass μ and charge l. If the mass μ is not independent of J then a new, supplementary term $\mu' J_{|n}$ will appear in the equation of motion. However, from the argument above, it would be plausible to assume that μ = const. For uncharged particles ($l = 0$, μ = const.), one will get projective geodetic lines $p_{\nu\|\mu}\,p^\mu = 0$ from (19.43), which will yield geodetic lines $u_{n\|_4 m}\,u^m = 0$ in the affine case. If $l \neq 0$ then, from (19.43), the projective geodetic lines $p_{\nu\|\mu}\,p^\mu = 0$ will give the paths of particles whose mass would be $\mu = \sqrt{l^2 J^{-1} + \text{const.}}$. This very large, but never observed, mass seems to have no meaning in nature.

20. Spinor field. Electron wave field. – In these next, final sections, we would like to show that the known field laws for electron, neutron, and meson fields can also be represented in the projective theory of relativity in an elegant way.

We can describe the electron wave field, as in no. **7**, by a spinor Ψ^K with complex components, for which one will have, under $\mathfrak{P}$:

$$\mathfrak{T}_\rho\,\Psi^K = e^{i\,l\ln\rho}\,\Psi^K. \tag{20.1}$$

A simple invariant that can be employed as a LAGRANGE function is:

$$L = \mathfrak{Re}\,\frac{1}{i}\left[\rho(J)\bar{\Psi}^{\dot{K}}(\beta\gamma^\mu)_{\dot{K}M}\Psi^M_{\;\|\mu} + \sigma(J)\bar{\Psi}^{\dot{K}}\beta_{\dot{K}M}\Psi^M\right], \tag{20.2}$$

in which $\mathfrak{Re}$ is the symbol for the real part, and $\rho(J)$, $\sigma(J)$ are real functions of J. Since, from no. **7**, $\beta\gamma^\mu$ and $(1/i)\,\beta$ are Hermitian matrices, one can also write (20.2) as:

$$L = \frac{\rho}{2i}\left[\bar{\Psi}^{\dot{K}}(\beta\gamma^\mu)_{\dot{K}M}\,\Psi^M_{\|\mu} - \bar{\Psi}^{\dot{K}}(\beta\gamma^\mu)_{\dot{K}M}\,\Psi^M\right] + \frac{\sigma}{i}\bar{\Psi}^{\dot{K}}\beta_{\dot{K}M}\Psi^M . \qquad (20.3)$$

When one sets $\sigma = \rho\,\mu$, that will imply the following matter field equations:

$$\gamma^{\mu L}{}_M\left[\Psi^M_{\|\mu} + \frac{1}{2}\frac{\rho'}{\rho}J_{|\mu}\Psi^M\right] + \mu\,\Psi^L = 0. \qquad (20.4)$$

(20.3) has the normal form that was prescribed in no. **13**, since no terms like $X^\mu\,\Psi_{|\mu}$ enter into it, so we can calculate the matter tensor from (20.3), using (13.58). One constructs the tensor:

$$W^{\mu\rho\lambda} = \frac{\partial\mathfrak{L}}{\partial\Psi^M_{|\mu}}\mathrm{P}^{\lambda\rho}\,\Psi^M + \frac{\partial\mathfrak{L}}{\partial\bar{\Psi}^{\dot{M}}_{|\mu}}\mathrm{P}^{\lambda\rho}\,\bar{\Psi}^M$$

from (13.51). From (9.5), one has:

$$\mathrm{P}^{\lambda\rho}\,\Psi^M = \tfrac{1}{2}\,\gamma^{\lambda\rho M}{}_L\,\Psi^M, \qquad \mathrm{P}^{\lambda\rho}\,\bar{\Psi}^{\dot{M}} = \tfrac{1}{2}\,\bar{\Psi}^{\dot{L}}\,\bar{\gamma}^{\lambda\rho}{}_{\dot{L}}{}^{\dot{M}},$$

in which $\bar{\gamma}^{\lambda\rho}{}_{\dot{L}}{}^{\dot{M}}$ is the matrix that is the Hermitian conjugate of $\gamma^{\lambda\rho M}{}_L$. Thus, one has:

$$W^{\mu\rho\lambda} = \frac{\rho}{4i}\left[\bar{\Psi}^{\dot{K}}(\beta\gamma^\mu\gamma^{\lambda\rho})_{\dot{K}N}\,\Psi^N - \bar{\Psi}^{\dot{K}}(\bar{\gamma}^{\lambda\rho}\beta\gamma^\mu)_{\dot{N}K}\,\Psi^K\right],$$

i.e.:

$$W^{\mu\rho\lambda} = \mathfrak{Re}\left[\frac{\rho}{2i}\bar{\Psi}^{\dot{K}}(\beta\gamma^\mu\gamma^{\lambda\rho})_{\dot{K}M}\,\Psi^M\right],$$

because $\beta\gamma^\mu$ is a Hermitian matrix. It follows from this that:

$$W^{\mu\nu\lambda} + W^{\nu\mu\lambda} = \mathfrak{Re}\left[\frac{\rho}{2i}\bar{\Psi}^{\dot{K}}\beta_{\dot{K}L}\,(\gamma^\mu\gamma^{\lambda\nu} + \gamma^\nu\gamma^{\lambda\mu})^L{}_M\,\Psi^M\right]. \qquad (20.5)$$

With $\gamma^{\lambda\nu} = \tfrac{1}{2}(\gamma^\lambda\gamma^\nu - \gamma^\nu\gamma^\lambda)$ and the relations $\tfrac{1}{2}(\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda) = g^{\nu\lambda}\,1$, it will follow that:

$$\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda = g^{\mu\lambda}\,\gamma^\nu + g^{\nu\lambda}\,\gamma^\mu - 2\,g^{\mu\nu}\,\gamma^\lambda .$$

However, since $\beta\,\gamma^\nu$ are Hermitian, the expression in brackets in (20.5) will be pure imaginary, and thus:

$$W^{\mu\nu\lambda} + W^{\nu\mu\lambda} = 0 . \qquad (20.6)$$

With $\Pi\,\Psi^L = i\,l\,\Psi^L$, from (13.64), one will have:

$$S^{\nu\mu} = L\,g^{\nu\mu} + \mathfrak{Re}\left\{-\frac{\rho}{2i}\bar{\Psi}^{\dot{K}}(\beta\gamma^{\mu})_{\dot{K}M}\Psi^{M\|\nu} + \left(\frac{\sigma}{2}J^{-1/2} - \rho l\,J^{-1}\right)\bar{\Psi}^{\dot{K}}(\beta\gamma^{\mu})_{\dot{K}M}\Psi^{M}X^{\nu}\right.$$
$$\left. + 2\left[\frac{\rho'}{i}\bar{\Psi}^{\dot{K}}(\beta\gamma^{\rho})_{\dot{K}M}\Psi^{M}_{\|\rho} + \left(\sigma' + \rho l\,J^{-3/2} - \frac{\sigma}{2}J^{-1}\right)\bar{\Psi}^{\dot{K}}(\beta\gamma_{(0)})_{\dot{K}M}\Psi^{M}\right]X^{\nu}X^{\mu}\right\}.$$

If we employ the field equation (20.4) and then the fact that:

$$\mathfrak{Re}\left(\frac{\rho'}{i}J^{|\lambda}\bar{\Psi}^{\dot{K}}(\beta\gamma_{\lambda})_{\dot{K}M}\Psi^{M}\right) = 0,$$

then it will follow from (20.2) that $L = 0$, and therefore:

$$\left.\begin{aligned} S^{\nu\mu} = \mathfrak{Re}\Big\{&-\tfrac{1}{2}\bar{\Psi}^{\dot{K}}(\beta\gamma^{\mu})_{\dot{K}M}\left[\frac{\rho}{i}\Psi^{M\|\nu} + 2\rho l\,J^{-1}X^{\nu}\Psi^{M}\right] + (\nu\leftrightarrow\mu) \\ &+ \frac{2\rho\mu'}{J}X^{\nu}X^{\mu}\,\bar{\Psi}^{\dot{K}}\beta_{\dot{K}L}\Psi^{L}\Big\}. \end{aligned}\right\} \tag{20.7}$$

In order to calculate the affine splitting, it is necessary to first calculate the affine splitting of $\Psi^L{}_{\|\mu}$. Since Ψ^L is not a normal spinor, we introduce the normalization ψ^L of Ψ^L by way of:

$$\Psi^L = e^{i\,l\ln\eta}\,\psi^L. \tag{20.8}$$

Since the representation $D^5_{(1/4,1/2)}$ is also irreducible as a representation of $\mathfrak{D}_4$, ψ^L is already the affine splitting of Ψ^L. One will then have:

$$\Psi^L{}_{\|\mu} = e^{i\ln\eta}\left\{\psi^L_{|\mu} + \tfrac{1}{2}\omega_{\mu\rho\lambda}\mathrm{P}^{\rho\lambda}\psi^L + i\,l[(\ln\eta)_{|\mu} - Y_{\mu}]\psi^L\right\},$$

or, with (6.10):

$$\Psi^L{}_{\|\mu} = e^{i\ln\eta}\left(\psi^L_{|\mu} + \tfrac{1}{2}\omega_{\mu\rho\lambda}\mathrm{P}^{\rho\lambda}\psi^L + i\,l\,\varphi_{\mu}\,\psi^L\right).$$

The splitting is then:

$$\Psi^L{}_{\|(0)} = e^{i\ln\eta}\;\omega_{(0)(\rho)(\lambda)}\;\gamma^{(\rho)(\lambda)L}{}_M\;\psi^M,$$

$$\Psi^L{}_{\|(m)} = e^{i\ln\eta}\,(\psi^M{}_{\|\_(m)} + \tfrac{1}{2}\,\omega_{(m)(0)(l)}\;\gamma^{(0)(l)L}{}_M\;\psi^M),$$

in which:

$$\psi^L{}_{\|\_(m)} = \psi^L_{\|_4 m} - i\,l\,\varphi_m\,\psi^L.$$

With (10.18), it follows that:

$$\left.\begin{aligned}\Psi^L_{\|(0)} &= \tfrac{1}{4}e^{il\ln\eta}\left(-\tfrac{1}{2}J^{1/2}F_{rl}\,\gamma^{rlL}{}_M + J^{-1}J_{|(l)}\,\gamma^{(0)(0)L}{}_M\right)\psi^M,\\ \Psi^L_{\|m} &= \tfrac{1}{4}e^{il\ln\eta}\left(\gamma^L_{\|\_m} + \tfrac{1}{4}J^{1/2}F_{(l)m}\,\gamma^{(0)(l)L}{}_M\,\psi^M\right).\end{aligned}\right\}\tag{20.9}$$

With that, one calculates the affine form of the field equations (20.4):

$$\gamma^{mL}{}_M\left[\psi^M{}_{\|\_m} + \tfrac{1}{2}\left(\frac{\rho'}{\rho} + \tfrac{1}{2}J^{-1}\right)J_{|m}\,\psi^M\right] + \gamma^{(0)\,L}{}_M\left[-\tfrac{1}{8}J^{1/2}F_{rl}\,\gamma^{rlM}{}_N\psi^N + \tfrac{1}{4}J^{-1}J_{|\,l}\,\gamma^{(0)lM}{}_N\,\psi^M\right] + \mu\,\psi^L = 0.$$

If one considers that:

$$\gamma^m\,\gamma^{(0)\,l} = \gamma^{(0)}\,\gamma^{lm}, \qquad \gamma^{(0)}\,\gamma^{(0)\,l} = \gamma^l$$

then one will get:

$$\gamma^{mL}_{\;\;M}\left[\gamma^{mL}_{\;\;\|\_m} + \tfrac{1}{2}\left(\frac{\rho'}{\rho} + \frac{1}{2}J^{-1}\right)J_{|m}\psi^M\right] + \tfrac{1}{8}J^{1/2}F_{rl}\,(\gamma^{(0)}\gamma^{rl})^L_{\;M}\,\psi^M + \mu\,\psi^L = 0. \tag{20.10}$$

That agrees with the DIRAC equation of an electron with charge l and mass μ, up to the supplementary terms … $J_{|\,m}$ … and … F_{rl} … Once we have presented the energy-impulse tensor, the charge-current vector, and the matter invariant, we will go into the meaning of the supplementary terms. It will follow from (20.7) that:

$$S^{\,nm} = \mathfrak{Re}\left[-\frac{\rho}{2i}\bar{\psi}^{\dot{K}}(\beta\,\gamma^m)_{\dot{K}M}\,\psi^{M\|\_n}\right] - \frac{\rho}{8i}J^{1/2}F_l^{\;n}\bar{\psi}^{\dot{K}}(\gamma\gamma^{(0)}\gamma^{ml})_{\dot{K}M}\,\psi^M + (m\leftrightarrow n) \tag{20.11}$$

and

$$S^{\,(0)\,m} = \mathfrak{Re}\left\{\tfrac{1}{2}\bar{\psi}^{\dot{K}}(\beta\,\gamma^m)_{\dot{K}M}\left[\frac{\rho}{8i}J^{1/2}F_{rl}\,\gamma^{rlM}{}_N\,\psi^N - \frac{\rho}{4i}J^{-1}J_{|l}\,(\gamma^{(0)}\gamma^l)^M{}_N\,\psi^N - 2\rho l\,J^{-1/2}\psi^M\right] - \frac{\rho}{2i}\bar{\psi}^{\dot{K}}(\beta\,\gamma^{(0)})_{\dot{K}M}\,\psi^{M\|\_m}\right\}.$$

If one multiplies (20.10) on the left by $\dfrac{\rho}{i}\bar{\psi}^K(\beta\,\gamma^{(0)}\gamma^s)_{\dot{K}L}$ and takes the real part then it will follow (since $\beta\,\gamma^{(0)}\gamma^s$ is a Hermitian matrix) that:

$$\begin{aligned}&-\mathfrak{Re}\left[\frac{\rho}{2i}\bar{\psi}^{\dot{K}}(\beta\,\gamma^{(0)}\gamma^s\,\gamma^m)_{\dot{K}M}\,\psi^M{}_{\|\_m}\right]\\ &\quad= \mathfrak{Re}\left[\frac{1}{2i}\left(\rho' + \frac{\rho}{2}J^{-1}\right)J_{|m}\,\bar{\psi}^{\dot{K}}(\beta\,\gamma^{(0)}\gamma^s\gamma^m)_{\dot{K}M}\,\psi^M - \frac{\rho}{8i}J^{1/2}\,\bar{\psi}^{\dot{K}}(\beta\,\gamma^s\gamma^{rl})_{\dot{K}M}\,F_{rl}\,\psi^M\right]\end{aligned}$$

Now:

$$\mathfrak{Re}\left[\frac{\rho}{i}\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})_{\dot{K}M}\psi^{M}{}_{\|\_m}\right]$$
$$= \frac{\rho}{2i}\left[\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})_{\dot{K}M}\psi^{M}{}_{\|\_m} - \bar{\psi}^{\dot{M}}{}_{\|\_m}(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})^{*}_{\dot{M}K}\psi^{K}\right].$$

Since (7.15) implies that:

$$(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})^{*} = \gamma^{m*}\gamma^{s*}\beta\gamma^{(0)} = \beta\gamma^{m}\beta^{*-1}\beta\gamma^{s}\beta^{*-1}\beta\gamma^{(0)} = \beta\gamma^{m}\gamma^{s}\gamma^{(0)} = \beta\gamma^{(0)}\gamma^{m}\gamma^{s},$$

it will follow that:

$$\mathfrak{Re}\left[\frac{\rho}{i}\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})_{\dot{K}M}\psi^{M}{}_{\|\_m}\right]$$
$$= \frac{\rho}{2i}\left[\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})_{\dot{K}M}\psi^{M}{}_{\|\_m} - \bar{\psi}^{\dot{K}}{}_{\|\_m}(\beta\gamma^{(0)}\gamma^{s}\gamma^{m})_{\dot{K}M}\psi^{M}\right].$$

$$= \frac{\rho}{2i}\left[\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{sm})_{\dot{K}M}\psi\right]_{\|m} - \mathfrak{Re}\left[\frac{\rho}{i}\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)})_{\dot{K}M}\psi^{M\|\_s}\right],$$

such that ultimately:

$$-\mathfrak{Re}\left[\frac{\rho}{i}\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)})_{\dot{K}M}\psi^{M\|\_m}\right] = \left[\frac{\rho}{4i}\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{ml})_{\dot{K}M}\psi^{M}\right]_{\|_4 l}$$
$$+\mathfrak{Re}\left[\frac{\rho}{8i}J^{-1}J_{|l}\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{m}\gamma^{l})_{\dot{K}M}\psi^{M} - \frac{\rho}{16i}J^{1/2}F_{rl}\,\bar{\psi}^{\dot{K}}(\beta\gamma^{m}\gamma^{rl})_{\dot{K}M}\psi^{M}\right].$$

If one substitutes this into the expression for $S^{(0)\,m}$ then it will follow that:

$$S^{(0)\,m} = J^{-1}\left(\frac{\rho J}{4i}\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{mr})_{\dot{K}M}\psi^{M}\right)_{\|_4 r} - l\rho J^{-1/2}\bar{\psi}^{\dot{K}}(\beta\gamma^{m})_{\dot{K}M}\psi^{M}\,. \tag{20.12}$$

Next, it still remains to calculate $S^{(0)(0)}$ from (20.7):

$$S^{(0)(0)} = \mathfrak{Re}\left\{\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)})_{\dot{K}M}\left[\frac{\rho}{8i}J^{1/2}F_{rt}\gamma^{rlM}{}_{N}\psi^{N} - \frac{\rho}{4i}J^{-1}J_{|l}(\gamma^{(0)}\gamma^{l})^{M}{}_{N}\psi^{N} - 2\rho l\,J^{-1/2}\psi^{M}\right]\right.$$
$$\left.+\frac{2}{i}\rho\,\mu' J\,\bar{\psi}^{\dot{K}}\beta_{\dot{K}L}\psi^{L}\right\},$$

or, since $\beta\gamma^{(0)}\gamma^{(0)}\gamma^{l} = \beta\gamma^{l}$ is Hermitian:

$$\left.\begin{aligned}S^{(0)(0)} = \rho\Big[\frac{1}{8i}J^{1/2}F_{rl}\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{rl})_{\dot{K}M}\psi^M - 2l\,J^{-1/2}\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)})_{\dot{K}M}\psi^M \\ +\frac{2}{i}\mu' J\,\bar{\psi}^{\dot{K}}\beta_{\dot{K}M}\psi^M\Big].\end{aligned}\right\} \tag{20.13}$$

The energy-impulse tensor, charge-current vector, and matter invariant can now be written down immediately:

$$T^{ik} = J^{-1}U^{-1}\rho\left\{\Re\mathfrak{e}\left[\frac{1}{2i}\bar{\psi}^{\dot{K}}(\beta\gamma^i)_{\dot{K}M}\psi^{M\|_{-}k}\right] + \frac{1}{8i}J^{1/2}F_l{}^i\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{kl})_{\dot{K}M}\psi^M + (i\leftrightarrow k)\right\}. \tag{20.14}$$

If we now fix ρ [as we did with $\alpha(J)$ in no. **19**] by:

$$\rho = J\,U \tag{20.15}$$

then T^{ik} will take the usual form, up to the additional term in $F_l{}^i$ … For the charge-current vector, it will then follow that:

$$s^m = -\,U^{-1}\,J^{-1/2}\,J\left[-l\,\rho\,J^{-1/2}\,\bar{\psi}^{\dot{K}}(\beta\gamma^m)_{\dot{K}M}\psi^M + J^{-1}\left(\frac{1}{4i}\rho\,J\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{mr})_{\dot{K}M}\psi^M\right)_{\|_4 r}\right],$$

so

$$s^m = l\,\bar{\psi}^{\dot{K}}(\beta\gamma^m)_{\dot{K}M}\psi^M - \frac{1}{4i}U^{-1}J^{-3/2}\left[U\,J^2\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{mr})_{\dot{K}M}\psi^M\right]_{\|_4 r}. \tag{20.16}$$

Up to the second supplementary term, this is the known form again. Furthermore, one has:

$$\left.\begin{aligned}b = \tfrac{1}{2}U\,J\Big[\frac{1}{8i}F_{rl}\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)}\gamma^{rl})_{\dot{K}M}\psi^M - 2l\,J^{-1}\,\bar{\psi}^{\dot{K}}(\beta\gamma^{(0)})_{\dot{K}M}\psi^M \\ +\frac{1}{2i}\mu' J^{1/2}\bar{\psi}^{\dot{K}}\beta_{\,\dot{K}M}\psi^M\Big].\end{aligned}\right\} \tag{20.17}$$

One next recognizes that the supplementary terms in the equations of motion, the energy-impulse tensor, and charge-current vector cannot produce any noticeable effect in any normal experiment, since they contain either the factor $J_{|\,m}$ or $J^{-1/2}$, and are thus either based upon the variability of J or they will contain the factor 10^{-20} for our current age of the universe, as in no. **17**. Due to that factor, the influence of the extra terms seems to be lost to any observer. Speculations that the extra terms are coupled to the magnetism of rotating stellar bodies (which P. JORDAN has discussed) have not been confirmed up to now, although the extra terms that contain the field strengths yield an extra magnetic moment of the particle that is (in the first approximation) $\sqrt{\kappa/2}$ times the mechanical angular impulse (in natural units), which is a ratio that is precisely what seems to have been observed for the Earth and the Sun.

21. Vector field. Meson field. – As a further example, we consider a vector field U. Since we wish to combine charged and uncharged mesons into a symmetric theory, we assume that the U_r are not real numbers, but elements of a three-dimensional K-module (cf., no. **2**), namely, the so-called "isotopic spin space." Each U_ν (for a fixed ν!) then consists of three real components. Furthermore, a positive-definite metric is given in "isotopic spin space," which might be briefly denoted by $\mathfrak{R}_i$, such that we can imagine choosing the basis in $\mathfrak{R}_i$ in such a way that the metric (i.e., "inner") product of two quantities V and W in $\mathfrak{R}_i$ will be given by:

$$V \cdot W = V_{\mathrm{I}}\, W_{\mathrm{I}} + V_{\mathrm{II}}\, W_{\mathrm{II}} + V_{\mathrm{III}}\, W_{\mathrm{III}}\,,$$

in which V_{I}, V_{II}, … are components in $\mathfrak{R}_i$. The group of rotations in $\mathfrak{R}_i$ is defined by the metric. One axis (e.g., the III-axis) is distinguished as the "charge axis," in such a way that one has:

$$\mathfrak{T}_\rho\, U_\nu = e^{il\ln\rho\,\tau_{\mathrm{III}}} U_\nu \tag{21.1}$$

for U_ν under the transformations of $\mathfrak{P}$, in which $i\ \tau_{\mathrm{III}}$ is an infinitesimal rotation around the III-axis, so $e^{i\varphi\tau_{\mathrm{III}}}$ is a rotation around the charge axis through an angle of φ. As in the previous section, l is a real number, and as we will find once more later on, identical with the charge of the meson. In that case, the infinitesimal transformation for U_ν is then $\Pi = i\ l\,\tau_{\mathrm{III}}$.

We define the antisymmetric tensor:

$$\Phi_{\mu\nu} = U_{\nu\,\|\,\mu} - U_{\mu\,\|\,\nu}\,, \tag{21.2}$$

such that:

$$\Phi_{\mu\nu} = U_{\nu\,|\,\mu} - U_{\mu\,|\,\nu} + i\ l\,\tau_{\mathrm{III}}\,(Y_\nu\, U_\mu - Y_\mu\, U_\nu). \tag{21.3}$$

We assume that our LAGRANGE function is:

$$L = \frac{1}{2}\left[\frac{1}{2}\alpha(J)\,\Phi_{\mu\nu}\cdot\Phi^{\mu\nu} + \beta(J)\,U_\mu\cdot U^\mu\right], \tag{21.4}$$

in which the dot · means the inner product in $\mathfrak{R}_i$. L is then invariant under arbitrary rotations in $\mathfrak{R}_i$, and above all, under $\mathfrak{P}$, and is therefore an invariant, as one would demand. The field equations read:

$$(\alpha\,\Phi^{\mu\nu})_{\|\,\mu} + \beta\, U^\nu = 0. \tag{21.5}$$

In order to calculate the affine form of the equations, in this section, we would like to deviate from the path that we took for the affine variational principle. The vector field U_μ splits affinely (after normalization) into an affine vector field u_k and a scalar field $u_{(0)}$. We then get a field theory that is intrinsically "mixed" here. From (10.23) to (10.26), one has:

$$\left.\begin{aligned}\Phi_{mn} &= e^{il\ln\eta\,\tau_{\mathrm{III}}}[\varphi_{mn} + J^{1/2}F_{mn}u_{(0)}],\\ \Phi_{n(0)} &= e^{il\ln\eta\,\tau_{\mathrm{III}}}J^{-1/2}(J^{1/2}u_{(0)})_{|\_n},\end{aligned}\right\} \tag{21.6}$$

with:

$$\varphi_{mn} = u_{n\,\|\_m} - u_{m\,\|\_n}\,, \tag{21.7}$$

and

$$u_{n\,\|\_m} = u_{n\|_4 m} - i\,l\,\varphi_m\,\tau_{\mathrm{III}}\,u_n\,.$$

Therefore, L will assume the form:

$$\left.\begin{aligned}\overset{4}{L} = \tfrac{1}{2}J^{1/2}\Big[&\frac{\alpha}{2}(\varphi_{mn} + J^{1/2}F_{mn}\,u_{(0)})\cdot(\varphi^{mn} + J^{1/2}F^{mn}u_{(0)})\\ &+\beta u_n\cdot u^n + \alpha J^{-1}(J^{1/2}u_{(0)})_{|\_n}(J^{1/2}u_{(0)})^{|\_n} + \beta\,u_{(0)}\cdot u_{(0)}\Big].\end{aligned}\right\} \tag{21.8}$$

The affine splitting of (21.5) follows from this with $\beta = \mu^2\,\alpha$:

$$\left.\begin{aligned}&\alpha^{-1}J^{-1/2}[J^{1/2}\alpha(\varphi^{mn} + J^{1/2}F^{mn}u_{(0)})]_{\|\_n} + \mu^2u^m = 0,\\ &\alpha^{-1}[J^{1/2}\alpha(J^{1/2}u_{(0)})^{\|\_n}]_{\|\_n} - \tfrac{1}{2}(\varphi^{mn} + J^{1/2}F_{mn}\,u_{(0)})F^{mn} - \mu^2u_{(0)} = 0.\end{aligned}\right\} \tag{21.9}$$

Here again, we then get the known form of the meson equations, up to supplementary terms, with μ as the rest mass of the meson.

For the calculation of the four-matter tensor, we start from formula (13.55), which we carry over to the affine case, in which we must consider $\overset{4}{\mathfrak{L}}$ to depend upon $\psi_{(M_4)}$, φ_r, and J, such that if the matter field equations $\overset{4}{\mathfrak{L}}{}^{(M_4)} = 0$ are fulfilled then we will have:

$$\overset{4}{\mathfrak{S}}{}_n{}^m = \overset{4}{\mathfrak{L}}\,\delta_n^m - \frac{\delta\mathfrak{L}}{\delta\varphi_m}\varphi_n - \left(\frac{\partial\overset{4}{\mathfrak{L}}}{\partial\varphi_{r|m}}\varphi_{r\|_4 n} + \frac{\partial\overset{4}{\mathfrak{L}}}{\partial\psi_{(M_4)|m}}\psi_{(M_4)|n}\right) + \overset{4}{\mathfrak{Z}}{}^{lm}{}_{n\|_4 l}\,, \tag{21.10}$$

with

$$\overset{4}{\mathfrak{Z}}{}^{lmn} = \tfrac{1}{2}(\overset{4}{\mathfrak{W}}{}^{lmn} + \overset{4}{\mathfrak{W}}{}^{mnl} - \overset{4}{\mathfrak{W}}{}^{nlm}) \tag{21.11}$$

and

$$\overset{4}{\mathfrak{W}}{}^{lmn} = \frac{\partial\overset{4}{\mathfrak{L}}}{\partial\varphi_{r|l}}\mathrm{P}^{nm}\,\varphi_r + \frac{\partial\overset{4}{\mathfrak{L}}}{\partial\psi_{(M_4)|l}}\mathrm{P}^{nm}\,\psi_{(M_4)}\,. \tag{21.12}$$

From (13.82), one can set:

$$\overset{4}{\mathfrak{S}}{}_n{}^m = \overset{4}{\mathfrak{L}}\,\delta_n^m - \left(\frac{\partial\overset{4}{\mathfrak{L}}}{\partial\varphi_{r|m}}\varphi_{r\|_4 n} + \frac{\partial\overset{4}{\mathfrak{L}}}{\partial\psi_{(M_4)|m}}\psi_{(M_4)\|\_n} - \left(\frac{\partial\overset{4}{\mathfrak{L}}}{\partial\varphi_{r|m}}\right)_{|r}\varphi_n\right) + \overset{4}{\mathfrak{Z}}{}^{lm}{}_{n\|_4 l}\,.$$

With (13.81), one has:

$$\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} \varphi_{r\|_4 n} - \left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} \right)_{|r} \varphi_n = \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} F_{nr} - \left(\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} \varphi_n \right)_{\|_4 r} .$$

It likewise follows from (21.11) and (21.12) that:

$$\overset{4}{\mathfrak{Z}}{}^{lm}{}_n = \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} \varphi_n + \overset{4}{\mathfrak{Z}}{}^{lm}{}_n \qquad \text{with} \qquad \underline{\overset{4}{\mathfrak{Z}}}{}^{lmn} = \tfrac{1}{2}(\underline{\overset{4}{\mathfrak{W}}}{}^{lmn} + \underline{\overset{4}{\mathfrak{W}}}{}^{mnl} - \underline{\overset{4}{\mathfrak{W}}}{}^{nlm})$$

and

$$\underline{\overset{4}{\mathfrak{W}}}{}^{lmn} = \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \psi_{(M_4)|l}} \mathrm{P}^{nm} \psi_{(M_4)} .$$

It will then ultimately follow from (21.10) that:

$$\boxed{\overset{4}{\mathfrak{S}}{}_n{}^m = \overset{4}{\mathfrak{L}} \delta_n^m - \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{(M_4)|m}} \psi_{(M_4)\|\_n} - \frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} F_{nr} + \underline{\overset{4}{\mathfrak{Z}}}{}^{lm}{}_{n\|_4 l} .} \tag{21.13}$$

If $\varphi_{r\,|\,m}$ enters into only in the form F_{mr} then:

$$\frac{\partial \overset{4}{\mathfrak{L}}}{\partial \varphi_{r|m}} = 2 \frac{\partial \overset{4}{\mathfrak{L}}}{\partial F_{mr}} .$$

Since $\overset{4}{\mathfrak{S}}{}^{nm}$ is symmetric in n, m, one can also write:

$$\left.\begin{aligned} \overset{4}{S}{}^{nm} &= \overset{4}{L} g^{nm} - \frac{1}{2}\left(\frac{\partial \overset{4}{L}}{\partial \psi_{(M_4)|m}} \psi_{(M_4)}{}^{\|\_n} + (m \leftrightarrow n) \right) \\ &\quad - \frac{1}{2}\left(\frac{\partial \overset{4}{L}}{\partial F_{mr}} F^n{}_r + (m \leftrightarrow n) \right) + \frac{1}{2}\left(\underline{\overset{4}{W}}{}^{mnl} + (m \leftrightarrow n) \right)_{\|_4 l} . \end{aligned}\right\} \tag{21.14}$$

If we apply this to (21.8) then we will get:

$$\frac{\partial \overset{4}{L}}{\partial u_{r|m}} = \alpha J^{1/2} (\varphi^{mr} + J^{1/2} F^{mr} u_{(0)}),$$

$$\frac{\partial \overset{4}{L}}{\partial u_{r|m}} = \alpha\,(J^{1/2}\,u_{(0)})^{\lfloor m},$$

$$\frac{\partial \overset{4}{L}}{\partial F_{mr}} = \frac{\alpha}{2}\;J\,u_{(0)}\cdot(\varphi^{mr} + J^{1/2}\,F^{mr}\,u_{(0)}),$$

and therefore:

$$\underline{\overset{4}{W}}{}^{mnl} = \alpha\,J^{1/2}\,[(\varphi^{ml} + J^{1/2}\,F^{ml}\,u_{(0)})\cdot u^n - (\varphi^{mn} + J^{1/2}\,F^{mn}\,u_{(0)})\cdot u^l\,],$$

from which it will follow further that:

$$\underline{\overset{4}{W}}{}^{mnl} + \underline{\overset{4}{W}}{}^{nml} = \alpha\,J^{1/2}\,[(\varphi^{ml} + J^{1/2}\,F^{ml}\,u_{(0)})\cdot u^n + (m \leftrightarrow n\,).$$

With (21.9), one will get:

$$\left(\underline{\overset{4}{W}}{}^{mnl} + (m\leftrightarrow n)\right)_{\|_4 l} = -\,2\mu^2\,\alpha\,J^{1/2}\,u^m\cdot u^n + [\alpha\,J^{1/2}\,(\varphi^{ml} + J^{1/2}\,F^{ml}\,u_{(0)})\cdot u^n{}_{\lfloor l} + (m\leftrightarrow)].$$

(21.14) can therefore be written:

$$\left.\begin{aligned}\overset{4}{S}{}^{nm} = \overset{4}{L}\,g^{nm} - \mu^2\alpha J^{1/2}u^m\cdot u^n - \alpha J^{1/2}(\varphi^{mr} + J^{1/2}F^{mr}u_{(0)})\\ \cdot(\varphi^n_r + J^{1/2}F^n{}_r\,u_{(0)}) - \frac{\alpha}{2}[(J^{1/2}u_{(0)})^{\lfloor n} + (m\leftrightarrow n)].\end{aligned}\right\}\tag{21.15}$$

Substituting this into (15.4), while setting $\alpha = J\;U$ (as above), will give the energy-impulse tensor:

$$\begin{aligned}T_{ik} = {} & [(\varphi_i{}^r + J^{1/2}\,F_i{}^r\,u_{(0)})\cdot(\varphi_{kr} + J^{1/2}\,F_{kr}\,u_{(0)})\\ & - \tfrac{1}{4}(\varphi_i{}^r + J^{1/2}\,F_i{}^r\,u_{(0)})\cdot(\varphi_{kr} + J^{1/2}\,F_{kr}\,u_{(0)})\\ & + \mu^2\left(u^m\cdot u^n - \frac{1}{2}u_r\cdot u^r g_{ik}\right) - \tfrac{1}{2}\mu^2\,u_{(0)}\cdot u_{(0)}\,g_{ik}\\ & - \tfrac{1}{2}J^{1/2}\,(J^{1/2}\,u_{(0)})_{\lfloor r}\cdot(J^{1/2}\,u_{(0)})_{\lfloor r}\,g_{ik} + \tfrac{1}{2}J^{-1/2}\,[(J^{1/2}\,u_{(0)})_{\lfloor r}\cdot u_{(0)}{}^{\lfloor k} + (i\leftrightarrow k)].\end{aligned}$$

We can calculate the four-matter tensor from (13.80). With the use of the result above for $\partial\overset{4}{L}/\partial\psi_{(M_4)|k}$, etc., we will then get:

$$\begin{aligned}t^m = {} & i\,l\,\alpha J^{1/2}\,(\varphi^{mr} + J^{1/2}\,F^{mr}\,u_{(0)})\;\tau_{\text{III}}\,u_r - i\,l\,\alpha\,(J^{1/2}\,u_{(0)})^{\lfloor m}\cdot\tau_{\text{III}}\,u_r\\ & - \left[\alpha J\,u_{(0)}\cdot(\varphi^{mr} + J^{1/2}F^{mr}u_{(0)})\right]_{\|_4 r},\end{aligned}$$

and therefore, from (15.7), one will have the charge-current vector (with $\alpha = J\,U$):

$$\left.\begin{aligned} s^m = i\,l\,(\varphi^{mr} + J^{1/2}F^{mr}u_{(0)})\cdot\tau_{\text{III}}\,u_r + i\,l\,J^{-1/2}(J^{1/2}u_{(0)})^{|\_m}\cdot\tau_{\text{III}}\,u_{(0)} \\ +U^{-1}J^{-1/2}[J^2U\,u_{(0)}\cdot(\varphi^{mr} + J^{1/2}F^{mr}u_{(0)})]_{\|_4 r}. \end{aligned}\right\} \tag{21.16}$$

If one drops the small supplementary terms then one will get the following expressions for the energy-impulse tensor and charge-current vector:

$$\left.\begin{aligned} T_{ik} \sim \Big[\varphi_i^{\ r}\cdot\varphi_{kr} - \tfrac{1}{4}\varphi^{lr}\cdot\varphi_{lr}\,g_{ik} + \mu^2\left(u_i\cdot u_k - \tfrac{1}{2}u^r\cdot u_r\,g_{ik}\right) \\ +u_{(0)|\_i}\cdot u_{(0)|\_k} - \tfrac{1}{2}u_{(0)|\_r}\cdot u_{(0)}^{|\_r}g_{ik} - \tfrac{1}{2}\mu^2 u_{(0)}\cdot u_{(0)}g_{ik}\Big], \\ s^m \sim i\,l\,\varphi^{mr}\cdot\tau_{\text{III}}\,u_r + i\,l\,u_{(0)}^{|\_m}\cdot\tau_{\text{III}}\,u_{(0)}. \end{aligned}\right\} \tag{21.17}$$

As for the meaning of the supplementary terms, one can make the same statements as the ones that were made at the conclusion of the previous section. The charge of the particle that corresponds to the field is given by the eigenvalues $+\,l$, 0, $-\,l$ of the operator $l\,\tau_{\text{III}}$.

22. Coupling thr matter fields with each other. – In this final section, we would like to show, in connection with the arguments of A. PAIS, which are valid for constant J, how projective geometry will suggest a somewhat large symmetry for the coupling of the matter fields with each other than the considerations of affine geometry alone.

For the coupling of mesons and nucleons, we must represent the nucleon field (proton-neutron field) by a spinor field, just like the electron wave field that was described in no. **20**. We achieve that by regarding the components Ψ^K of the nucleon spinors, not as complex numbers, but as elements of a two-dimensional $K(i)$-module $\mathfrak{R}_l$. An element of this module is given by two (complex) components. Furthermore, let $\mathfrak{R}_l$ be an (irreducible) representation module for the group of rotations in isotopic spin space. The representation is known to be the unitary group $\mathfrak{u}_2$ in two dimensions. In particular, the infinitesimal rotations $i\,\tau_{\text{I}}$, $i\,\tau_{\text{II}}$, $i\,\tau_{\text{III}}$ around the axes I, II, III, resp., in spin space are associated with three Hermitian operators then, which we will also denote by τ_{I} , τ_{II} , τ_{III} that act upon the elements of $\mathfrak{R}_l$ and can be formally represented by exactly the same matrices as the PAULI spin matrices (in which those matrices naturally act upon only $\mathfrak{R}_l$).

The transformations of the group $\mathfrak{P}$ are true for the elements Ψ^K:

$$\mathfrak{T}_\rho\,\Psi^K = e^{il\,\ln\rho(\tau_{\text{III}} - \frac{1}{2})}\,\Psi^K.$$

The charge of the elementary particle that corresponds to the Ψ^K field (under quantization) is equal to an eigenvalue of the operator:

$$\frac{1}{i}\Pi = l\left(\tau_{\text{III}} - \tfrac{1}{2}\right),$$

and is thus equal to $l\left(\tfrac{1}{2}-\tfrac{1}{2}\right) = 0$ (neutrons) or $l\left(-\tfrac{1}{2}-\tfrac{1}{2}\right) = -l$ (protons), resp. [We have denoted the charge of the proton by (– l), since we called the charge of the electron l in no. **20**.]

If τ is the (operator)-vector in isotropic spin space $\mathfrak{R}_l$, with the components τ_{I}, τ_{II}, τ_{III}, then the real quantities:

$$\left.\begin{aligned} M_\mu &= g_1(J)(\bar{\Psi}^{\dot{K}}, \tau(\beta\gamma_\mu)_{\dot{K}L}\psi^L, \\ N_{\mu\nu} &= \mu_2^{-1} g_2(J)(\bar{\Psi}^{\dot{K}}, \tau(\beta\gamma_\mu)_{\dot{K}L}\psi^L \end{aligned}\right\} \tag{22.1}$$

will be elements of $\mathfrak{R}_l$, in which the symbol (…, …) will suggest the Hermitian (inner, invariant under $\mathfrak{u}_2$) product of two elements in $\mathfrak{R}_l$.

The Ψ^K experience a representative transformation $D^{\supset}$ under a rotation D in isotopic spin space. If one consider the elements of the matrices to be invariant elements and subjects the Ψ^K in (22.1) to the transformation $D^{\supset}$ then, as is known, one will have:

$$D\,M_\mu = g_1\left(D^{\supset *}\,\bar{\Psi}^{\dot{K}}, \tau(\beta\gamma_\mu)_{\dot{K}L}\,D^{\supset}\Psi^L\right), \text{ etc.}$$

In particular, when one sets D equal to a rotation around the III-axis through an angle l ln ρ, one will have:

$$\mathfrak{T}_\rho\, M_\mu = g_1\left(e^{-il\ln\rho(\tau_{\text{III}}-\frac{1}{2})}\,\bar{\Psi}^{\dot{K}}, \tau(\beta\gamma_\mu)_{\dot{K}L}\,e^{il\ln\rho(\tau_{\text{III}}-\frac{1}{2})}\,\Psi^L\right) = e^{il\ln\rho\tau_{\text{III}}}\;M_\mu,$$

for a transformation $\mathfrak{T}_\rho$ in $\mathfrak{P}$, in which τ_{III} once more acts as an operator on $\mathfrak{R}_i$ itself in the last expression.

The expressions (22.1) are suitable for introducing a coupling of mesons and nucleons for the purpose of explaining the nuclear forces, when one sets, with:

$$H_{\mu\nu} = \Phi_{\mu\nu} + N_{\mu\nu},$$

the LAGRANGE function equal to:

$$\left.\begin{aligned} L = \frac{\alpha}{4} H_{\mu\nu}\cdot H^{\mu\nu} + \frac{\beta}{2} U_\mu \cdot U^\mu - \alpha U_\mu \cdot M^\mu \\ \mathfrak{Re}\,\frac{1}{i}\left[\rho\left(\bar{\Psi}^{\dot{K}}, (\beta\gamma_\mu)_{\dot{K}L}\,\Psi^L{}_{\|\mu}\right) + \sigma\left(\bar{\Psi}^{\dot{K}}, \beta_{\dot{K}L}\Psi^L\right)\right]. \end{aligned}\right\} \tag{22.2}$$

L can then be written:

$$L = L_m + L_n + \alpha\left(\tfrac{1}{2}\Phi_{\mu\nu}\cdot N^{\mu\nu} + \tfrac{1}{4}N_{\mu\nu}\cdot N^{\mu\nu} - U_\mu \cdot M^\mu\right), \tag{22.3}$$

in which L_m and L_n are the LAGRANGE functions for free mesons (free nucleons, resp.) that were discussed in nos. **21** and **20**. However, the expression (20.16) for s^m in no. **20** (and at a corresponding place), namely, $l\,\bar{\psi}^{\dot{K}}(\beta\gamma^m)_{\dot{K}M}\psi^M$, must be replaced with $l\left(\bar{\psi}^{\dot{K}}, (\tau_{\mathrm{III}} - \tfrac{1}{2})(\beta\gamma_\mu)_{\dot{K}M}\,\psi^M\right)$, and the product of two spinors must always be regarded as a Hermitian product (…, …). From the remarks above, L invariant under $\mathfrak{P}$ (as it must be) and also under the group of rotation in $\mathfrak{R}_i$.

The field equations (21.5) go to:

$$(\alpha H^{\nu\mu})_{\|\mu} + \beta U^\nu = \alpha M^\nu. \tag{22.4}$$

When we consider $\alpha = \rho$ (= $J\,U$), it will follow that the field equations (20.4) will be:

$$\gamma^{\mu L}{}_M\left(\Psi^M{}_{\|\mu} + \frac{1}{2}\frac{\rho'}{\rho}J_{|\mu}\Psi^M\right) + \mu\Psi^L - ig_1 U_\mu\cdot\tau\,\gamma^{\mu L}{}_M\Psi^M + \frac{i}{2}\frac{g_2}{\mu}H_{\mu\nu}\cdot\gamma^{\mu L}{}_M\Psi^M = 0. \tag{22.5}$$

No difficulties will arise in the calculation of the affine splitting of these two equations, so it will follow directly from (22.1), with the normalized spinor $\Psi^K = e^{il\ln\eta\,(\tau_{\mathrm{III}}-\frac{1}{2})}\psi^K$, that:

$$\left.\begin{aligned} M_m &= g_1\left(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma_m)_{\dot{K}L}\psi^L\right), & M_{(0)} &= g_1\left(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma_{(0)})_{\dot{K}L}\psi^L\right), \\ N_{nm} &= \frac{1}{\mu}g_2\left(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma_{nm})_{\dot{K}L}\psi^L\right), & N_{(0)n} &= \frac{1}{\mu}g_2\left(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma_{(0)}\gamma_n)_{\dot{K}L}\psi^L\right). \end{aligned}\right\} \tag{22.6}$$

With the help of (21.9), one will then get from (22.4) that:

$$\left.\begin{aligned} &\alpha^{-1}J^{-1/2}\left[J^{1/2}\alpha\left(\varphi^{mn} + J^{1/2}F^{mn}u_{(0)} + \frac{g_2}{\mu}\left(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma^{mn})_{\dot{K}L}\psi^L\right)\right)\right]_{\|\_n} + \mu^2 u^m \\ &\qquad = g_1(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma^m)_{\dot{K}L}\psi^L), \\ &\alpha^{-1}\left[J^{-1/2}\alpha\left(J^{1/2}u_{(0)}\right)^{|\_n}\right]_{\|\_n} - \tfrac{1}{2}J^{1/2}[\varphi_{mn} + J^{1/2}F_{mn}u_{(0)} \\ &\qquad + \frac{g_2}{\mu}\left(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma^{mn})_{\dot{K}L}\psi^L\right)\Big]F^{mn} - \mu^2 u_{(0)} = -\,g_1(\bar{\psi}^{\dot{K}}, \tau(\beta\gamma^{(0)})_{\dot{K}L}\psi^L), \end{aligned}\right\} \tag{22.7}$$

while (22.5), with (20.10), will go to:

$$\left.\begin{aligned}
&\gamma^{mL}_{\ \ M}\left[\psi^{M}{}_{\|_m}+\frac{1}{2}\left(\frac{\rho'}{\rho}+\frac{1}{2}J^{-1}\right)J_{|m}\psi^{M}\right]+\mu\,\psi^{L}+\tfrac{1}{8}J^{1/2}F_{rl}\,(\gamma^{(0)}\gamma^{rl})^{L}_{\ M}\,\psi^{M}\\
&-i\,g_2\,u_m\cdot\tau\,\gamma^{mL}_{\ \ M}\psi^{M}-i\,g_2\,u_{(0)}\cdot\tau\,\gamma^{(0)L}_{\ \ M}\psi^{M}\\
&+\frac{i}{2}\frac{g_2}{\mu}\left[\varphi_{mn}+J^{1/2}F_{mn}u_{(0)}+\frac{g_2}{\mu}(\bar{\psi}^{\dot K},\tau(\beta\,\gamma_{mn})_{\dot K L}\psi^{L})\right]\cdot\tau\,\gamma^{mnL}_{\ \ \ M}\psi^{M}\\
&+i\,\frac{g_2}{\mu}\left[J^{-1/2}(J^{1/2}u_{(0)})_{|_n}+\frac{g_2}{\mu}(\bar{\psi}^{\dot K},\tau(\beta\,\gamma_{(0)})_{\dot K L}\psi^{L})\right]\cdot\tau(\gamma^{n}\,\gamma^{(0)})\psi^{M}=0.
\end{aligned}\right\}\qquad(22.8)$$

The advantage of the projective theory of relativity emerges here (when one neglects the "small supplementary terms" for most applications): We have obtained two affine fields: one vector and one scalar. Nevertheless, we have only two coupling numbers g_1 and g_2 as parameters, although, from a purely affine standpoint, we have been able to replace the number g_1 that appears in the bottom equation of (22.7) with a third one g_3 . However, in another respects, the same difficulty will exist in the theory that is presented here that exists in the affine theory, namely, the fact that the Ansatz (22.1), (22.2) is not the only one possible.

The affine LAGRANGE function $\overset{4}{L}$, the energy-impulse tensor T_{ik} , and the charge-current vector s^m can be easily written down with the help of the derivatives of the previous section:

$$\overset{4}{L}=\overset{4}{L}_{n}+\overset{4}{L}{}^{\supset}_{m},$$

in which $\overset{4}{L}_{n}$ is the LAGRANGE function for the free nucleons:

$$L_n = J^{1/2}\,\mathrm{Re}\,\frac{1}{i}\Big[\rho(\bar{\psi}^{\dot K},(\beta\,\gamma^{m})_{\dot K M}\psi^{M}{}_{\|\_m})+\frac{\rho}{8}J^{1/2}F_{rl}\,(\bar{\psi}^{\dot K},(\beta\,\gamma^{(0)}\gamma^{rl})_{\dot K M}\psi^{M})$$
$$+\ \frac{\rho}{4}J^{-1}J_{|l}\left[(\bar{\psi}^{\dot K},(\beta\,\gamma^{l})_{\dot K M}\psi^{M})+\sigma(\bar{\psi}^{\dot K},\beta_{\dot K M}\psi^{M})\right],$$

and $\overset{4}{L}{}^{\supset}_{m}$ arises from $\overset{4}{L}$ quite simply using (21.8) when one replaces:

$$\varphi_{mn}+J^{1/2}F_{mn}\,u_{(0)}\qquad\text{with}\qquad\varphi_{mn}+J^{1/2}F_{mn}\,u_{(0)}+N_{mn}\,,\tag{22.9}$$

and

$$(J^{1/2}\,u_{(0)})_{|\_n}\qquad\text{with}\qquad(J^{1/2}\,u_{(0)})_{|\_n}+J^{1/2}\,N_{n\,(0)}\tag{22.10}$$

everywhere, and adds a term:

$$-\,J^{\,1/2}\,\alpha\,(u_n\cdot M_n+u_{(0)}\cdot M_{(0)})$$

to it. Likewise, the energy-impulse tensor will arise from the sum of the energy-impulse tensors for the nucleons according to (20.14) and that of the mesons according to (21.17) when one performs the replacements (22.9) and (22.10) everywhere.

What is interesting now is the dependency of the coupling numbers g_1 and g_2 on J. However, it is precisely at this point that one can make no purely deductive statements from the theory. The theory that is presented here is in adequate agreement with the experiments in nuclear forces and nuclear structure when one suitably chooses the coupling numbers to have order of magnitude unity in natural units. On that basis, one would have the right to regard g_1 and g_2 as independent of J.

Almost exactly the same coupling problem occurs with the interaction of the meson field with the electron-neutrino field. However, experiments concerned with the β-instability of mesons have shown here that the values of the coupling numbers are small ($\sim 10^{-20}$ in natural units), so they are probably proportional to $J^{1/2}$. A purely deductive basis for this fact in the projective theory, which P. JORDAN arrived at inductively, is not possible at present. Generally it allows one to represent such behavior with no further assumptions. If one sets $g_2 \sim J^{1/2}$ then it will be at least noteworthy that, e.g., φ_{mn} will enter into the field equation (22.8) in a manner that is similar to the way that F_{mn} enters into it, so a term $\frac{1}{8} J^{1/2} F_{mn} (\gamma^{(0)} \gamma^{mn})^L{}_M \psi^M$ will appear in the equation in a purely deductive way that will then correspond to a term $\frac{i}{2} g_2(J) \mu^{-1} \varphi_{mn} \tau \gamma^{mnL}{}_M \psi^M$. This analogy at least raises the suspicion that a deductive basis might be possible in the manner that will be suggested in the next chapter.

CHAPTER IV

GLIMPSE OF POSSIBLE EXTENSIONS OF THE THEORY

Whereas the combination of gravitation and electromagnetism into a unified geometric theory of the laws for those fields and their coupling to other matter fields can admit a deductive basis, to a certain extent, as well as yielding an extension of EINSTEIN's general (affine) theory of relativity that implies a foundation for JORDAN's cosmology, the last three sections have shown the precise limits of the theory, since despite an agreement with the affine theory, some unforeseeable possibilities will also exist for matter fields, and especially for their mutual couplings, as a result of the demand of greatest simplicity. If one would like to go further along the lines of geometrization then the question will arise of whether it is not possible to also incorporate the meson field into a unified geometric field theory. The spinor field seems to show no point of application for the development of such ideas, so one might possibly already consider it from the standpoint of quantization (perhaps as singularities of the geometric continuum). In contrast to the geometrization of the meson field, one has some ideas that SCHRÖDINGER sought to develop in some recent papers on the basis of the affine theory. His theory goes back to the ideas of EDDINGTON-EINSTEIN in the year 1923. The EDDINGTON-EINSTEIN attempt to introduce the electromagnetic field started with the introduction of displacement quantities $\Gamma^{\lambda}_{\mu\nu}$ (which are initially still assumed to be symmetric in μ, ν, which SCHRÖDINGER also dropped) that are more general than CHRISTOFFEL's three-index symbols, with the result that the field equations, which had precisely the form of a vector meson theory, practically coincided with MAXWELL's equations for a very small "rest mass" (which one could not, however, set equal to zero). There is a very strong temptation then to carry over the ideas of EDDINGTON, EINSTEIN, and SCHRÖDINGER to the present theory, which is a problem that P. JORDAN already came to grips with. An extension of this theory in that respect has the advantage that one can obtain not only the field laws for mesons in a deductive form, but also the coupling of other matter fields with mesons.

The starting point for the extension is to abandon the metric $g_{\mu\nu}$ such that the basis quantities then appear to be the displacement quantities $\Gamma^{\rho}_{\mu\nu}$ (which are symmetric in μ, ν) and a covariant vector Y_μ with $X^\nu\, Y_\nu = 1$. For the parallel displacement by ξ^ν, one will then have:

$$\delta_{\|}\, \alpha^{\rho} = -\, \xi^{\nu}\, \Gamma^{\rho}_{\nu\mu}\, \alpha^{\mu} + \xi^{\nu}\, Y_{\nu}\, \Pi \alpha^{\rho}\,.$$

As before, the curvature tensor $R_{\mu\nu}{}^{\lambda}{}_{\rho}$ and the field $F_{\mu\nu} = Y_{\nu\,|\,\mu} - Y_{\mu\,|\,\nu}$ can be defined by the $\Gamma^{\rho}_{\nu\mu}$. One can derive $R_{\mu\rho} = R_{\mu\nu}{}^{\nu}{}_{\rho}$ and $R_{\mu\nu}{}^{\lambda}{}_{\lambda}$ from the curvature tensor by contraction, the latter of which does not vanish, as it would it if were based upon a metric, but defines precisely an antisymmetric tensor field that essentially agrees with $R_{\mu\rho} - R_{\rho\mu}$, by which the meson field can be represented. Generally, one gets only uncharged mesons in that way, since $R_{\mu\nu}{}^{\lambda}{}_{\lambda}$ is a normal tensor.

To what extent more far-reaching generalizations (e.g., complex $\Gamma^{\rho}_{\nu\mu}$, ones that are asymmetric in ν, μ, or similar things) can product new physical viewpoints, and likewise, whether a geometric theory of mesons would in itself yield the β-decay probabilities of atomic nuclei in a purely deductive way, which P. JORDAN inferred from an order-of-magnitude analysis, and which was suggested at the end of no. **22**, all of this will first be shown only by a more precise elaboration of the theory.

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