"Di alcune nouve forme delle equazioni della dinamica, applicabili ai sistemi anolonomi," Atti della Reale Accademia Naz. dei Lincei (5) **10** (1910), 287-292.

On some new forms of the equations of dynamics that are applicable to anholonomic systems

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The idea of forming the differential equations of motion for a constrained system of points that makes use of expressions for velocity as linear functions of the independent parameters can be found in Kirchhoff's *Mechanik*. There, that principle was used to deduce the final equations of **§ 4** of Lesson 3.a from Hamilton's theorem, which were then applied, in **§ 2** of Lesson 4, to the formation of the more general differential equations of motion of a free solid body or one that has a fixed point.

Volterra has applied the same principle in his article "Sopra una classe di equazioni dinamiche," which was published in volume XXXIII (1898) of the Atti della R. Accademia delle Scienze di Torino in order to deduce a form of the equations of motion of a system of points for which the constraints are independent of time from the equations of d'Alembert and Lagrange, when reduced to an expression by Beltrami ([1]), and expressions for the total differential equations between the coordinates that are just as valid whether or not they form an integrable system; that is to say, whether the moving system is holonomic or anholonomic. The differential equations (C) that are established between time, the coordinates, and the *characteristics of motion* are equal in number to the latter: There are thus as many parameters as there are degrees of freedom in the system, by means of which, by virtue of the constraints, one expresses the components of the velocity of any point as homogeneous linear functions. Moreover, Volterra proposed, in particular, to indicate the case in which those equations are sufficient to determine the characteristics as functions of time.

Finally, Appell used the same principle of the characteristics to deduce an elegant form for the differential equations of motion from the d'Alembert and Lagrange equations that was, like the preceding ones, applicable to holonomic or non-holonomic systems, as well as the case of constraints that depend upon time in his article "Sur les mouvements de roulement – Équations analogues à celles de Lagrange" that was included in volume CXXIX of the Comptes Rendus des Séances de l'Académie des Sciences in Paris (1899), as well as in "Sur une forme Générale des équations de la dynamique," which published in volume CXXI of Crelle's Journal (1900) ([2]).

([1]) Beltrami, "Sulle equazioni dinamiche di Lagrange," Rend. R. Ist. Lombardo **28** (1895).
([2]) See also, Appell, "Les mouvements de roulement en Dynamique," § **24** (Scientia 4º), Paris, 1899.

In this brief note, permit me to show how Appell's equations and Volterra's can be deduced from a form of the equations of dynamics that is found in § **493** of my *Meccanica* ([1]), which is, in turn, deduced quite directly from Hamilton's theorem. It would seem that it has remained unnoticed, although in the following §, it will be applied to the construction of the equations of motion for a solid by a method that seems to present the advantage of greater expediency when compared to that of Kirchhoff.

We begin by recalling the very simple deduction of the equations in question. If we start from Hamilton's theorem:

$$\int_{t'}^{t''} (\delta T + \Pi)\, dt = 0, \tag{1}$$

in which if $q_1, q_2, \ldots, q_n$ denote any type of coordinates – free or not – for the moving system then we will have:

$$\delta T = \sum_{i=1}^{n} \left(\frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i \right), \qquad \dot{q}_i = \frac{dq_i}{dt}, \qquad \Pi = \sum_{i=1}^{n} Q_i\, \delta q_i\, . \tag{2}$$

Let the constraints be represented by:

$$\sum_{i=1}^{n} \Phi_{i,j}\, dq_i = T_j\, dt \qquad (j = 1, 2, \ldots, m), \tag{3}$$

by means of which the $\delta q_1, \delta q_2, \ldots, \delta q_n$ are defined by:

$$\sum_{i=1}^{n} \Phi_{ij}\, \delta q_i = 0 \qquad (j = 1, 2, \ldots, m)\, . \tag{4}$$

These last equations can always be supposed to be solved for $n - m$ of the $\delta q_1, \delta q_2, \ldots, \delta q_n$, which are, if needed, opportunely chosen to make:

$$\delta q_i = \sum_{i=1}^{n} E_{ir}\, \varepsilon_r \qquad (i = 1, 2, \ldots, n)\, , \tag{5}$$

in which the $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-m}$ are other arbitrary parameters.

Now (1), conforming with (2), can be written in the form:

$$\int_{t'}^{t''} dt \sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i \right) \delta q_i = 0,$$

which immediately yields:

([1]) *Principii della teoria matematica del movimento dei corpi*, Milan, 1896.

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}-Q_i\right)\delta q_i = 0.$$

One substitutes (5) in that relation, selects one of the ε_r, equates its coefficient to 0, and sets:

$$\sum_{i=1}^{n} Q_i E_{ir} = E_r ,$$

so from (2):

$$\Pi = \sum_{r=1}^{n-m} E_r \varepsilon_r ,$$

and one will get:

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}\right)E_{ir} = E_r \qquad (r = 1, 2, \ldots, n-m) . \tag{7}$$

Those are the desired equations, and along with (3), they define a system of n differential equations, where t serves as the independent variable and the $q_1, q_2, \ldots, q_n$ are unknowns. They are valid for both holonomic and non-holonomic systems, and for constraints that are or are not independent of time.

In order to put those equations into Appell form, it is enough to observe that (3), conforming to (5), will imply that:

$$\dot{q}_i = E_i + \sum_{r=1}^{n-m} E_{ir} e_r \qquad (i = 1, 2, \ldots, n),$$

in which $e_1, e_2, \ldots, e_{n-m}$ represent the characteristics of the motion of the system considered, in such a way that:

$$E_{ir} = \frac{\partial \dot{q}_i}{\partial e_r} = \frac{\partial \ddot{q}_i}{\partial \dot{e}_r} . \tag{8}$$

Meanwhile, on the other hand, with Appell, set:

$$S = \tfrac{1}{2}\sum m(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2),$$

in which m and x, y, z represent the mass and coordinates of the generic point of the system, and the sum extends over all points, so one will have:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i} = \frac{\partial S}{\partial \ddot{q}_i} . \tag{9}$$

In fact:

$$\frac{\partial S}{\partial \ddot{q}_i} = \sum m\left(\ddot{x}\frac{\partial \ddot{x}}{\partial \ddot{q}_i}+\ddot{y}\frac{\partial \ddot{y}}{\partial \ddot{q}_i}+\ddot{z}\frac{\partial \ddot{z}}{\partial \ddot{q}_i}\right).$$

However, one has:

$$\dot{x} = \frac{\partial x}{\partial t} + \sum_{i=1}^{n} \frac{\partial x}{\partial q_i} \dot{q}_i ,$$

so:

$$\frac{\partial x}{\partial q_i} = \frac{\partial \dot{x}}{\partial \dot{q}_i} = \frac{\partial \ddot{x}}{\partial \ddot{q}_i},$$

and analogous expressions. Hence:

$$\frac{\partial S}{\partial \ddot{q}_i} = \sum m \left(\ddot{x} \frac{\partial \dot{x}}{\partial \dot{q}_i} + \ddot{y} \frac{\partial \dot{y}}{\partial \dot{q}_i} + \ddot{z} \frac{\partial \dot{z}}{\partial \dot{q}_i} \right),$$

and what will remain is:

(10) $$T = \tfrac{1}{2} \sum m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

as well as:

$$\frac{\partial S}{\partial \ddot{q}_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \sum m \left(\dot{x} \frac{d}{dt}\frac{\partial x}{\partial \dot{q}_i} + \dot{y} \frac{\partial y}{\partial \dot{q}_i} + \dot{z} \frac{\partial z}{\partial \dot{q}_i} \right) = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i};$$

therefore:

$$\frac{d}{dt}\frac{\partial x}{\partial q_i} = \frac{\partial^2 x}{\partial q_i\,\partial t} + \sum_{j=1}^{n} \frac{\partial^2 x}{\partial q_i\,\partial q_j} \dot{q}_j = \frac{\partial^2 x}{\partial t\,\partial q_i} + \sum_{j=1}^{n} \frac{\partial^2 x}{\partial q_j\,\partial q_i} \dot{q}_j = \frac{\partial}{\partial q_i}\frac{dx}{dt} = \frac{\partial \dot{x}}{\partial q_i}.$$

Having done that, from (8) and (9), (7) will become:

$$\sum_{i=1}^{n} \frac{\partial S}{\partial \ddot{q}_i} \frac{\partial \ddot{q}_i}{\partial \dot{e}_r} = E_r \qquad (r = 1, 2, \ldots, n - m)$$

or

(7′) $$\frac{\partial S}{\partial \dot{e}_i} = E_r \qquad (r = 1, 2, \ldots, n - m),$$

which are Appell's equations, when they are free of any special hypothesis about the choice of the characteristics.

One will get Volterra's equations when one puts (7) into the form:

$$\sum_{i=1}^{n} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right) \frac{\partial \dot{q}_i}{\partial \dot{e}_r} = E_r ,$$

conforming to (8), or:

$$\frac{d}{dt} \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial e_r} = \sum_{i=1}^{n} \frac{dE_{ir}}{dt} \frac{\partial T}{\partial \dot{q}_i} + \sum_{i=1}^{n} E_{ir} \frac{\partial T}{\partial q_i} + E_r ,$$

or finally:

$$\frac{d}{dt}\frac{\partial T}{\partial e_r} = \sum_{i=1}^{n}\frac{dE_{ir}}{dt}\frac{\partial T}{\partial \dot{q}_i} + \sum_{i=1}^{n} E_{ir}\frac{\partial T}{\partial q_i} + E_r \qquad (r = 1, 2, \ldots, n - m)\,. \tag{7''}$$

Those equations properly apply to any type of coordinates $q_1, q_2, \ldots, q_n$ and to constraints that also depend upon time. Suppose that the $q_1, q_2, \ldots, q_n$, with $n = 3\nu$, represent the orthogonal Cartesian coordinates x, y, z of a system of ν points, and that the constraints are independent of time. Conforming to (10), one will have:

$$\frac{\partial T}{\partial q_i} = 0, \qquad \frac{\partial T}{\partial \dot{q}_i} = m_i\,\dot{q}_i = m_i\sum_{u=1}^{n-m} E_{iu}e_u \qquad (i = 1, 2, \ldots, 3\nu\,;\ m_i = m_{i+1} = m_{i+2})\,.$$

Furthermore:

$$\frac{dE_{ir}}{dt} = \sum_{j=1}^{n}\frac{\partial E_{ir}}{\partial q_j}\dot{q}_j = \sum_{v=1}^{n-m} e_v \sum_{j=1}^{n}\frac{\partial E_{ir}}{\partial q_j}E_{jv}\,.$$

Therefore, under those hypotheses, the preceding equations will become:

$$\frac{d}{dt}\frac{\partial T}{\partial e_i} = \sum_{u=1}^{n-m}\ \sum_{v=1}^{n-m} b_{rv}^{(u)}\,u_u\,e_v\,, \tag{7'''}$$

in which

$$b_{rv}^{(u)} = \sum_{j=1}^{3\nu} E_{jv}\sum_{j=1}^{3\nu} m_i\,\frac{\partial E_{ir}}{\partial q_j}E_{iu}\,.$$

The simple form to which Volterra's equations (C) reduce (except for different symbols) makes use of the assumed relations and follows from the indicated operations.

We conclude with some brief observations on the possibility and legitimacy of using Hamilton's theorem.

It seems to me that as far as the possibility is concerned, the equation that translates that theorem can always be considered to be a reduction of a more concise form of d'Alembert's and Lagrange's equations that is almost spontaneous from the way that the equations of motion in general coordinates are deduced.

As for their legitimacy, with the exception of Appell, who addressed that topic in his article "Sur les équations de Lagrange et le principe d'Hamilton" in volume XXVI of the Bulletin de la Société mathématique de France (1898), one can object that the proof of the incompatibility of:

$$d\,\delta x = \delta\,dx, \qquad d\,\delta q_i = \delta\,dq_i \tag{11}$$

in the anholonomic case is based upon the deduction of:

$$\delta\,[dx - (A_1\,dq_1 + A_2\,dq_2)] = 0$$

from the equations that translate the constraint:

$$dx - (A_1\, dq_1 + A_2\, dq_2) = 0\ .$$

Now, that signifies that the virtual motion is forced to satisfy the same constraint as the effective motion. Conforming to the usual canon, it is the variation that relates to the passage from a virtual motion that is defined by (11) to:

$$\delta x - (A_1\, \delta q_1 + A_2\, \delta q_2) = 0$$

in the case at hand. In that case, Appell's argument shows how holonomity is the necessary and sufficient condition for the virtual motion to coincide with the motion that satisfies the same constraints as the effective motion ([1]).

([1]) Cf., Hölder, "Ueber die Principien von Hamilton und Maupertuis," § **6**, Nachrichten der Gesellschaft der Wissenschaften in Göttingen (1896).