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Optics

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The rays that emanate from a luminous point in a medium of uniform density may be regarded as a system of straight lines passing through this point. When these rays encounter the surface of a body that reflects or refracts, their mutual disposition experiences various modifications that give rise to all of the phenomena of optics.

Before passing to the analysis of these phenomena, we present some properties that are common to all of the sheaves of reflected or refracted rays, and in general to all of the systems of continuous straight lines that are not parallel.

[**1**]. Let:

$$m(z - z') = o(x - x'), \qquad n(z - z') = o(y - y') \tag{A}$$

be equations of a straight line that belong to a system of rays that are arranged in space according to an arbitrary analytical law, m, n, o being arbitrary functions of x', y', z'. At each point of space – i.e., for each particular value of x', y', z' – there correspond new lines that belong to the same system.

First consider the lines that belong to the points that are contiguous to the ones whose coordinates are x', y', z'; among all of the lines there is only a certain sequence of them that meets the line (A). In order to determine the locus of points that they belong to, one must express the idea that the line (A) and a contiguous line have a common point x, y, z, and then differentiate equation (A) with respect to x', y', z', by regarding x, y, z as constants, which gives:

$$dm(z - z') - m\,dz' = do(x - x') - o\,dx', \qquad dn(z - z') - n\,dz' = do(y - y') - o\,dy'. \tag{B}$$

If one eliminates $(x - x')$, $(y - y')$, $(z - z')$ then one has the result:

$$m\,do\,dy' - m\,dn\,dz' + n\,dm\,dz' - n\,do\,dx' + o\,dn\,dx' - o\,dm\,dy' = 0$$

or:

$$\left.\begin{array}{l}\left[\left(\frac{dm}{dx'}\right)dx'+\left(\frac{dm}{dy'}\right)dy'+\left(\frac{dm}{dz'}\right)dz'\right](n\,dz'-o\,dy')\\+\left[\left(\frac{dn}{dx'}\right)dx'+\left(\frac{dn}{dy'}\right)dy'+\left(\frac{dn}{dz'}\right)dz'\right](o\,dx'-m\,dz')\\+\left[\left(\frac{do}{dx'}\right)dx'+\left(\frac{do}{dy'}\right)dy'+\left(\frac{do}{dz'}\right)dz'\right](m\,dy'-n\,dx')\end{array}\right\}=0. \qquad (C)$$

This equation expresses the idea that the two lines (*A*), (*B*), and a third one:

$$dx'(z-z') = dz'(x-x'), \qquad dy'(z-z') = dz'(y-y') \qquad (D)$$

are found in the same plane. The line (*D*), which is the tangent to a curve of double curvature, indicates the direction, along which one must pass from the point x', y', z' to the contiguous points in order for the two consecutive rays to meet. The two tangents of inclination dx' / dz', dy' / dz' are coupled to each other by equation (*C*), in such a way that there is only one of them that is indeterminate, and that there is only one certain family of curves with double curvature that must satisfy equation (*C*). These curves are coupled to each other by the common property that if one considers one of them in particular then the sequence of rays (*A*) that belong to its different points meet it consecutively.

If one eliminates dx' / dz', dy' / dz' from equations (*C*), (*D*) then one will obtain the locus of all the tangents (*D*) that pass through the x', y', z'. The result of that elimination is:

$$\left.\begin{array}{l}\left[\left(\frac{dm}{dx'}\right)(x-x')+\left(\frac{dm}{dy'}\right)(y-y')+\left(\frac{dm}{dz'}\right)(z-z')\right][n(z-z')-o(y-y')]\\+\left[\left(\frac{dn}{dx'}\right)(x-x')+\left(\frac{dn}{dy'}\right)(y-y')+\left(\frac{dn}{dz'}\right)(z-z')\right][o(x-x')-m(z-z')]\\+\left[\left(\frac{do}{dx'}\right)(x-x')+\left(\frac{do}{dy'}\right)(y-y')+\left(\frac{do}{dz'}\right)(z-z')\right][m(y-y')-n(x-x')]\end{array}\right\}=0. \qquad (E)$$

This equation belongs to a conical surface of second degree that has its center at the point x', y', z', and whose nappes indicate the direction along which one must pass from that point to the contiguous points in order for the line (*A*) to meet the consecutive figure; the ray (A) itself is one of the edges of that surface.

One may already remark that any plane that passes through the center of that conical surface will be cut along two straight lines, in such a way that there are, in general, two directions at each point of a plane along which one may travel in order for the rays (*A*) to meet, no matter what the functions *m*, *n*, *o* are.

[**2**]. Now, let a curved surface be:

$$F(x, y, z) = 0, \qquad p\,dx' + q\,dy' + p\,dz' = 0. \qquad (F)$$

One of the rays (A) belongs to each point of that surface, but if one determines dx' / dz', dy' / dz' by means of equations (C), (F) then one will have, since equation (C) is of second degree, two results of the form:

$$M\,dz' = O\,dx', \qquad N\,dz' = O\,dy', \ldots \tag{s}$$
$$M'\,dz' = O'\,dx', \qquad N'\,dz' = O'\,dy', \ldots \tag{s'}$$

upon setting:

$$n = o\left(\frac{dn}{dx'}\right) - n\left(\frac{do}{dx'}\right)\beta = m\left(\frac{do}{dy'}\right) - o\left(\frac{dm}{dy'}\right)\gamma = n\left(\frac{dm}{dz'}\right) - m\left(\frac{dn}{dz'}\right),$$
$$\delta = m\left(\frac{do}{dx'}\right) - o\left(\frac{dm}{dx'}\right) + o\left(\frac{dn}{dy'}\right) - n\left(\frac{do}{dy'}\right),$$
$$\varepsilon = n\left(\frac{dm}{dx'}\right) - m\left(\frac{dn}{dx'}\right) + o\left(\frac{dn}{dz'}\right) - n\left(\frac{do}{dz'}\right),$$
$$\zeta = n\left(\frac{dm}{dy'}\right) - m\left(\frac{dn}{dy'}\right) + m\left(\frac{do}{dz'}\right) - o\left(\frac{dm}{dz'}\right),$$

$$L = [(\delta r + \varepsilon q + \zeta p)^2 + 4pq(\gamma\delta - \varepsilon\zeta) + 4pr(\beta z - \delta\zeta) + 4qr(\alpha\zeta - \delta\varepsilon) - 4\beta\gamma p^2 - 4\alpha\gamma q^2 - 4\alpha\beta r^2]$$

$$M = \zeta pq + \delta qr - \varepsilon q^2 - 2\beta pr + q\,L, \qquad N = \varepsilon pq + \delta pr - \zeta p^2 - 2\alpha qr - pL,$$
$$M' = \zeta pq + \delta qr - \varepsilon q^2 - 2\beta pr - q\,L, \qquad N' = \varepsilon pq + \delta pr - \zeta p^2 - 2\alpha qr + pL,$$
$$O = O' = -\frac{p}{r}M - \frac{q}{r}N = -\frac{p}{r}M' - \frac{q}{r}N' = 2(\alpha q^2 + \beta p^2 - \delta pq).$$

Substituting these values for dx' / dz', dy' / dz' in equation (D), one will have:

$$M(z - z') = O(x - x'), \qquad N(z - z') = O(y - y'), \ldots \tag{D}$$
$$M'(z - z') = O'(x - x'), \qquad N'(z - z') = O'(y - y'), \ldots \tag{D'}$$

equations that refer to two tangent lines to the surface (F), and whose direction indicates in which sense one must pass from the point x', y', z' to the consecutive points on that surface in order for the rays (A) to meet. One will have likewise obtained equations (D), (D') by combining the equation for the tangent plane to the surface (F) with that of the conical surface (E).

Since the equations (s), (s') are valid for each of the points of the surface (F), their integrals express two systems of curves traced on that surface, and each of them enjoys the special property that all of the rays (A) that belong to it meet consecutively; indeed, since the curve (s) or (s') satisfies equation (C) at each of these points, it is always found in the direction that the points of the consecutive rays that meet belong to.

The sequence of lines (A) that pass through the curve (s) and meet consecutively form a developable surface (S) whose edge of regression (σ) is the locus of points that meet s a second time. Since the curve (s) is cut at each of its points by one of the curves (s'), one has, in turn, that the developable surface (S) is cut along each of its generatrices by a developable surface (S') that is composed of the rays that belong to one of the curves (s').

The sequence of edges of regression (σ) of the first sequence of developable surfaces (S) form a curved surface (Σ) to which each of the rays (A) is tangent. The sequence of edges of regression (σ') of the second sequence of developable surfaces (S') form a second surface (Σ') to which all of the rays (A) are again tangent. Therefore, *any time one considers a system of straight lines that emanate from all of the points of a curved surface according to an arbitrary analytical law*, this system of lines may be regarded as the locus of intersection of two systems of developable surfaces, and since the sequence of edges of regression of these two systems of surfaces comprise the intersection of all the generatrices, one concludes that *the locus of points at which the proposed lines meet is comprised of two curved surfaces.*

[**3**]. Imagine the plane that passes through the line (A) and the tangent (D); its equation will be:

$$(nO - oN)(x - x') + (oM - mO)(y - y') + (mN - nM)(z - z') = 0. \qquad (AD)$$

Likewise, the plane that passes through the line (A) and the tangent (D') will have the equation:

$$(nO' - oN')(x - x') + (oM' - mO')(y - y') + (mN' - nM')(z - z') = 0. \qquad (AD')$$

These two planes obviously contain the consecutive lines to the line (A) by which, it is met.

In order to determine the coordinates x, y, z of the point of encounter of the lines that are found in the plane (AD), one must substitute in the equations (B) for dx' / dz, dy' / dz, their values M / O, N / O, which are found from equations (s), and after that substitution, combine the equations (A), (B), which gives, by means of equation (C), and upon setting:

$$\Lambda = \frac{nM - mN}{\left\{ m\left[\left(\frac{dn}{dx'}\right)M + \left(\frac{dn}{dy'}\right)N + \left(\frac{dn}{dz'}\right)O\right] - n\left[\left(\frac{dm}{dx'}\right)M + \left(\frac{dm}{dy'}\right)N + \left(\frac{dm}{dz'}\right)O\right]\right\}}$$

the equations:

$$x - x' = m\Lambda, \quad y - y' = n\Lambda, \quad z - z' = o\Lambda. \qquad (G)$$

The distance from the point x', y', z' to the point x, y, z, which we call R, is, as a consequence:

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 = (m^2 + n^2 + o^2)\,\Lambda^2.$$

One will obtain a similar result for the line that is found in the plane (AD') upon substituting M', N', O' for M, N, O in Λ, and one will have, upon calling the resulting value R':

$$x - x' = m\Lambda', \quad y - y' = n\Lambda', \quad z - z' = o\Lambda'. \qquad (G')$$

The distance R' from the point x', y', z' to the second point of encounter will be:

$$R'^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 = (m^2 + n^2 + o^2)\,\Lambda'^2.$$

One could collectively obtain the double values (G), (G') of $x - x'$, $y - y'$, $z - z'$ upon eliminating dx' / dz', dy' / dz' from equations (B), (F), which will produce an equation of second degree, which, when combined with equation (A), will determine the proposed values.

Let:

$$V = 0, \qquad W = 0, \ldots (ss) \qquad V' = 0, \qquad W' = 0 \qquad (s's')$$

be the integrals of equations (s), (s') that must be completed by the condition that these curves must pass through a particular point of the surface (F).

The equation of a developable surface (S) will be the result of eliminating x', y', z' from the four equations (A), (ss).

The equations of its edge of regression (σ) will be the result of the elimination of these quantities from the five equations (G), (ss). Finally, the equation of the surface (Σ), which is the locus of all the edges of regression (σ) of the sequence of surfaces (S), will be the result of eliminating x', y', z' from the three equations (G) and the equation $F(x', y', z') = 0$.

Everything that refers to the sequence of developable surfaces (S) is obtained by way of considerations that are similar to the preceding ones upon substituting the curves ($s's'$) for the curves (ss).

[**4**]. The two developable surface (S), (S'), which meet along the line (A), are cut by each of these lines at a particular angle whose expression we shall determine. In order to do this, we observe that the plane (AD) that passes through the ray (A) and the tangent (D) contains two consecutive generatrices of the developable surface (S), so it is consequently tangent to surface, and by the same reasoning, the plane ($A'D'$) is tangent to the surface (S'), in such a way that the angle between these two planes is the angle at which the proposed surfaces meet. Now, if one refers to the angle between the two planes (AD), (AD') by τ then one will have:

$$\tau = \frac{(m^2+n^2+o^2)(MM'+NN'+OO') - (mM+nN+oO) + (mM+nN+oO)(mM'+nN'+oO')}{[(m^2+n^2+o^2)(M^2+N^2+O^2) - (mM+nN+oO)^2]^{1/2}[(m^2+n^2+o^2)(M'^2+N'^2+O'^2) - (mM'+nN'+oO')^2]^{1/2}}$$

an expression that will always be a function of x', y', z', at least when one has:

$$(m^2 + n^2 + o^2)(MM' + NN' + OO') - (mM + nN + oO')(mM' + nN' + oO') = 0,$$

independently of the values of x', y', z'. In this particular case, one has $\cos\tau = 0$, and the developable surfaces (S), (S') all cut at a right angle. Upon replacing MNO, $M'N'O'$ in that equation with their values, it becomes:

$$\left.\begin{array}{l}
\left[\varepsilon\left(\frac{dn}{dx'}\right)-n\left(\frac{do}{dx'}\right)\right][m^2(q^2+r^2)+(nq+or^2)]-\left[m\left(\frac{do}{dx'}\right)-o\left(\frac{dm}{dx'}\right)+o\left(\frac{dn}{dx'}\right)-n\left(\frac{do}{dy'}\right)\right] \\
\qquad [(m^2+n^2)\,pq+or(mq+np)-mnr^2] \\
\left[m\left(\frac{do}{dy'}\right)-o\left(\frac{dm}{dy'}\right)\right][n^2(p^2+r^2)+(mp+or)]-\left[n\left(\frac{dm}{dx'}\right)-m\left(\frac{dn}{dx'}\right)+o\left(\frac{dn}{dz'}\right)-n\left(\frac{do}{dz'}\right)\right] \\
\qquad [(m^2+o^2)\,pr+nq(mr+op)-moq^2] \\
\left[n\left(\frac{dm}{dz'}\right)-m\left(\frac{dn}{dz'}\right)\right][o^2(p^2+q^2)+(mp+nq)]-\left[n\left(\frac{dm}{dy'}\right)-m\left(\frac{dn}{dy'}\right)+m\left(\frac{do}{dz'}\right)-o\left(\frac{dm}{dz'}\right)\right] \\
\qquad [(n^2+o^2)qr+mq(nr+oq)-nop^2]
\end{array}\right\} = 0 \qquad (\mathrm{H})$$

in such a way that whenever *the functions m, n, o satisfy that equation of condition one may regard the rays* (A) *as the locus of the intersection of two systems of developable surfaces that intersect at a right angle.*

For the moment, consider the lines (A) to be the system of normals of a curved surface $F(x', y', z') = 0$, $p'\,dx' + q\,dy' + r\,dz' = 0$, so one will have $m = p$, $n = q$, $o = r$. Substituting these values in the numerator of the expression for $\cos\tau$, or in the left-hand side of equation (H), it reduces to:

$$\left[r\left(\frac{dq}{dx'}\right)-q\left(\frac{dr}{dx'}\right)+p\left(\frac{dr}{dx'}\right)-r\left(\frac{dp}{dy'}\right)+q\left(\frac{dp}{dz'}\right)-p\left(\frac{dq}{dz'}\right)\right](p^2+q^2+r^2)^2,$$

a quantity that is always null, since $p'\,dx' + q\,dy' + r\,dz'$ is the differential of the function $F(x', y', z')$, in such a way that the normals to a curved surface are always the locus of intersection of two systems of developable surfaces that intersect at a right angle. One thus concludes, conversely, that whenever the functions m, n, o satisfy the equation of condition (H) in a system of rays, there exists a series of surfaces that are normal to these rays.

Since the quantities m, n, o are given functions that do not generally satisfy equation (H), one may propose to determine the surfaces $p'\,dx' + q\,dy' + r\,dz' = 0$ by the condition that the rays (A) must be the locus of intersection of two systems of rectangular developable surfaces. Equation (H) is then the partial differential equation for the desired surface.

[**5**]. The phenomena of optics depend principally upon the locus of points at which consecutive rays meet – i.e., on the form and the position of the surfaces (Σ), (Σ') – and we remark that everything that relates to these surfaces is independent of the solution and integration of the equations, and is obtained by simple eliminations.

These surfaces might have different nappes that are situated on the same side of the surface (F) or situated with one on one side of that surface and the other, on the opposite one. Indeed, the angle between the rays (*A*) that belong to the same curve (*ss*), after having been positive, may become negative, and conversely, and it is obvious that before the change of sign, that angle will have to be zero, and the point of intersection of the two consecutive rays will go to infinity.

The point of intersection of the two consecutive rays (*A*) is, as we have seen, the same as the point of intersection of the two lines (*A*), (*B*), in such a way that when the two consecutive rays are parallel, the two lines *A*, *B* are, as well. Now, the angle that they form between them has the cosine:

$$\frac{m\,dm + n\,dn + o\,do}{(m^2+n^2+o^2)^{1/2}(dm^2+dn^2+do^2)^{1/2}},$$

and when they are parallel, that expression must be equal to unity, which gives $(m\,do - o\,dm)^2 + (n\,do - o\,dn)^2 + (m\,dn - n\,dm)^2 = 0$, or:

$$m\,do - o\,dm = 0, \qquad n\,do - o\,dn = 0. \tag{I}$$

If one replaces dx' / dz', dy' / dz' in these equations with their values, as deduced from equation (*s*), then they become:

$$\left.\begin{aligned} m\left[\left(\frac{do}{dx'}\right)M + \left(\frac{do}{dy'}\right)N + \left(\frac{do}{dz'}\right)O\right] - o\left[\left(\frac{dm}{dx'}\right)M + \left(\frac{dm}{dy'}\right)N + \left(\frac{dm}{dz'}\right)O\right] = 0, \\ n\left[\left(\frac{do}{dx'}\right)M + \left(\frac{do}{dy'}\right)N + \left(\frac{do}{dz'}\right)O\right] - o\left[\left(\frac{dn}{dx'}\right)M + \left(\frac{dn}{dy'}\right)N + \left(\frac{dn}{dz'}\right)O\right] = 0, \end{aligned}\right\} \tag{S}$$

and are the loci of the particular points of the curve (*ss*) for which the two consecutive rays cease to meet; consequently, they define a curve that cuts all of the series (*ss*).

Moreover, if one replaces dx' / dz', dy' / dz' in equation (I) with their values that are deduced from equations (*s′*) then one will have:

$$\left.\begin{aligned} m\left[\left(\frac{do}{dx'}\right)M' + \left(\frac{do}{dy'}\right)N' + \left(\frac{do}{dz'}\right)O'\right] - o\left[\left(\frac{dm}{dx'}\right)M' + \left(\frac{dm}{dy'}\right)N' + \left(\frac{dm}{dz'}\right)O'\right] = 0, \\ n\left[\left(\frac{do}{dx'}\right)M' + \left(\frac{do}{dy'}\right)N' + \left(\frac{do}{dz'}\right)O'\right] - o\left[\left(\frac{dn}{dx'}\right)M' + \left(\frac{dn}{dy'}\right)N' + \left(\frac{dn}{dz'}\right)O'\right] = 0, \end{aligned}\right\} \tag{S′}$$

which are the equations of a particular curve that meets all of the curves (*s′s′*) along the points where the rays (*A*) cease to intersect in the same sense.

The equations (*S*), (*S′*) are particular solutions of the differential equation (*C*).

If one eliminates *M*, *N*, *O* or *M′*, *N′*, *O′* from the equations (*S*) or (*S′*) and the equation $pM + qN + rO = 0$ or $pM' + qN' + rO' = 0$ [**2**] then one obtains the equation of condition:

$$\left\{p\left[m\left(\frac{do}{dz'}\right)-o\left(\frac{dm}{dz'}\right)\right]-r\left[m\left(\frac{do}{dx'}\right)-o\left(\frac{dm}{dx'}\right)\right]\right\}\left\{q\left[n\left(\frac{do}{dz'}\right)-o\left(\frac{dn}{dz'}\right)\right]-r\left[n\left(\frac{do}{dy'}\right)-o\left(\frac{dn}{dy'}\right)\right]\right\}\cdots$$
$$-\left\{p\left[n\left(\frac{do}{dz'}\right)-o\left(\frac{dn}{dz'}\right)\right]-r\left[n\left(\frac{do}{dx'}\right)-o\left(\frac{dn}{dx'}\right)\right]\right\}\left\{q\left[m\left(\frac{do}{dz'}\right)-o\left(\frac{dm}{dz'}\right)\right]-r\left[m\left(\frac{do}{dy'}\right)-o\left(\frac{dm}{dy'}\right)\right]\right\}$$
$$= 0,$$

which expresses the condition that m, n, o must satisfy in order for two consecutive rays to exist at each of the surface $p\,dx' + q\,dy' + r\,dz' = 0$; i.e., in order for one of the two systems of developable surfaces (S), (S') to be a system of cylindrical or planar surfaces. If m, n, o are given and p, q, r are unknown then the preceding condition may be regarded as the partial differential equation for the surface $p\,dx' + q\,dy' + r\,dz' = 0$ that satisfies the proposed condition.

Each of the lines (A) that belong to the curve (s) is asymptotic to two branches of one edge of regression (σ) that situated on opposite sides, and consequently the series of lines (A) that belong to the curve (s) form a surface (Σ) that is asymptotic to two nappes of the surface (Σ) that are situated on opposite sides. Similarly, the series of lines (A) that belongs to the curve (s') form a surface (Σ') that is asymptotic to two nappes of the surface (Σ').

The line of intersection of two surfaces (Σ), (Σ') – i.e., the ray (A) that corresponds to the point of intersection of the two curves (s), (s') is asymptotic to two branches of one edge of regression (σ), and to two branches of one edge of regression (σ').

Other than the two curves (s), (s'), equation (C) again has another particular solution that corresponds to the point where the curves (ss), ($s's'$) agree – i.e., the case in which one has $M = M'$, $N = N'$, $L = 0$, and consequently:

$$(\delta r + \varepsilon q + \zeta p)^2 + 4pq(\gamma\delta - \sigma\zeta) + 4pr(\varepsilon\beta + \delta\zeta) + 4qr(\alpha\zeta - \delta\varepsilon) - 4\beta\gamma p^2 - 4\alpha\gamma q^2 - 4\alpha\beta r^2 = 0.$$

The combination of that equation with the equation $F(x', y', z') = 0$ determines the curve on that surface along which (ss), ($s's'$) coincide. One may make the same observations about that equation of condition that we made about the one that we treated previously.

CATOPTRICS

[**6**]. We shall pass on to the application of that theory to optical phenomena by commencing with those products of reflected light that form the object of catoptrics.

One calls the angle between the incident ray and the normal to the surface that is reflecting or refracting the *angle of incidence* and the angle between the reflected or refracted ray and that same normal the *angle of reflection or refraction,* respectively.

The angle of incidence is always equal to the angle of reflection, and the reflected ray is always found in the plane that passes through the incident ray and the normal at the point of incidence. It is upon these principles, which are given by experiments, that we establish the analytical theory of catoptrics.

If we let X, Y, Z be the coordinates of a luminous point then the equations of the projections of a light ray will be:

$$a(z - Z) = c(x - X), \qquad b(z - Z) = c(y - Y) \qquad \ldots\ldots\ldots \tag{a}$$

If that ray strikes a surface:

$$F(x', y', z') = 0, \qquad p\,dx' + q\,dy' + r\,dz' = 0 \tag{F}$$

at a point x', y', z' then it will be reflected along a line that we represent by the equations:

$$m(z - Z) = o(x - X), \quad n(z - Z) = o(y - Y). \tag{A}$$

Since the two lines (a), (A) form two equal angles with the normal to the surface one will express that condition by the equation:

$$\frac{ap+bq+cr}{(a^2+b^2+c^2)^{1/2}} = \frac{mp+nq+or}{(m^2+n^2+o^2)^{1/2}}\,. \tag{b}$$

In the second place, the incident ray, the reflected ray, and the normal are found in the same plane, so one will have:

$$m(cq - br) + n(ar - cp) + o(bp - aq) = 0. \tag{c}$$

The two equations (b), (c) define the conditions that must determine m / o, n / o, and one infers from them that:

$$\frac{m}{o} = \frac{pr(a^2+b^2+c^2)+(cq-br)(bp-aq)\pm(ar-cp)(ap+bq+cr)}{r^2(a^2+b^2+c^2)-(cq-br)^2-(ar-cp)^2},$$

$$\frac{n}{o} = \frac{qr(a^2+b^2+c^2)+(ar-cp)(bp-aq)\pm(br-cq)(ap+bq+cr)}{r^2(a^2+b^2+c^2)-(cq-br)^2-(ar-cp)^2}.$$

With the upper sign, one infers that $\frac{m}{o} = \frac{a}{c}$, $\frac{n}{o} = \frac{b}{c}$, which refers to the incident ray, and with the lower sign, one infers that:

$$\frac{m}{o} = \frac{2(ap+bq+cr)p - a(p^2+q^2+r^2)}{2(ap+bq+cr)r - c(p^2+q^2+r^2)},$$

$$\frac{n}{o} = \frac{2(ap+bq+cr)q - b(p^2+q^2+r^2)}{2(ap+bq+cr)r - c(p^2+q^2+r^2)},$$

which refers to a line that forms an equal angle with the normal, and which is situated in the same plane on the opposite side – i.e., on the reflected ray – and one concludes:

$$m = \lambda[2\,(ap + bq + cr)\,p - a\,(p^2 + q^2 + r^2)],$$

$$n = \lambda[2\,(ap + bq + cr)\,q - b\,(p^2 + q^2 + r^2)],$$

$$o = \lambda[2\,(ap + bq + cr)\,r - c\,(p^2 + q^2 + r^2)],$$

in which λ is an undetermined coefficient that disappears under calculation. Finally, since the point x', y', z' is common to the surface (F) and to the incident ray, one has:

$$a\,(z' - Z) = c\,(x' - X), \qquad b\,(z' - Z) = c\,(y' - Y),$$

and consequently:

$$a = \lambda'\,(x' - X'), \qquad b' = \lambda'\,(y' - Y), \qquad c = \lambda'\,(z' - Z),$$

in which λ' is a new undetermined coefficient that vanishes during the calculation. If we substitute the values of a, b, c in the expressions for m, n, o then one will have these latter quantities as functions of the coordinates X, Y, Z of the luminous point, and the coordinates x', y', z' of the point of the surface that is encountered by the incident ray.

[**7**]. From the results of paragraph [**2**], the system of reflected rays (A) may be considered as the locus of intersection of the two systems of developable surfaces (S), (S') that cut the surface (F) of the mirror along two sequences of curves (ss), ($s's'$) [**3**], and the intersection points of all these rays are found on two curved surfaces (Σ), (Σ') that we call *caustic surfaces.*

In the second place, if one substitutes the values of m, n, o in the expression for cos τ [**4**] then one finds that they satisfy the equation of condition (H), in such a way that the developable surfaces (S), (S') all intersect at right angles.

One may simplify the calculations of that substitution by observing that the values of m, n, o are independent of the position of the coordinates, so one may suppose that the reflected ray that one considers is one of the coordinate axes – for example, the z axis. One then has $m = 0$, $n = 0$, and equation (H) reduces to:

$$r\left(\frac{dn}{dx'}\right) - p\left(\frac{dn}{dz'}\right) + q\left(\frac{dm}{dz'}\right) - r\left(\frac{dm}{dy'}\right) = 0,$$

or:

$$\left.\begin{aligned}
&qr\left[p + a\left(\frac{dp}{dx'}\right) + b\left(\frac{dq}{dx'}\right) + c\left(\frac{dr}{dx'}\right)\right] + r\left(\frac{dq}{dx'}\right)(ap + bq + cr) - br\left[p\left(\frac{dp}{dx'}\right) + q\left(\frac{dq}{dx'}\right) + r\left(\frac{dr}{dx'}\right)\right] \\
&-pq\left[r + a\left(\frac{dp}{dz'}\right) + b\left(\frac{dq}{dz'}\right) + c\left(\frac{dr}{dz'}\right)\right] - p\left(\frac{dq}{dz'}\right)(ap + bq + cr) + bp\left[p\left(\frac{dp}{dz'}\right) + q\left(\frac{dq}{dz'}\right) + r\left(\frac{dr}{dz'}\right)\right] \\
&+pq\left[r + a\left(\frac{dp}{dz'}\right) + b\left(\frac{dq}{dz'}\right) + c\left(\frac{dr}{dz'}\right)\right] + q\left(\frac{dq}{dz'}\right)(ap + bq + cr) - aq\left[p\left(\frac{dp}{dz'}\right) + q\left(\frac{dq}{dz'}\right) + r\left(\frac{dr}{dz'}\right)\right] \\
&-pr\left[q + a\left(\frac{dp}{dy'}\right) + b\left(\frac{dq}{dy'}\right) + c\left(\frac{dr}{dy'}\right)\right] - r\left(\frac{dq}{dy'}\right)(ap + bq + cr) + ar\left[p\left(\frac{dp}{dy'}\right) + q\left(\frac{dq}{dy'}\right) + r\left(\frac{dr}{dy'}\right)\right]
\end{aligned}\right\} = 0. \quad (H')$$

Finally, since $m = 0$, $n = 0$ gives:

$$ar^2 - cpr = p\,(ap + bq + cr), \qquad br^2 - cqr = q\,(ap + bq + cr), \qquad bp = aq,$$

all of the terms of the preceding equation are multiplied by $ap + bq + cr$, and it reduces to:

$$r\left(\frac{dq}{dx'}\right) - q\left(\frac{dq}{dx'}\right) + p\left(\frac{dq}{dy'}\right) - r\left(\frac{dq}{dy'}\right) + q\left(\frac{dq}{dz'}\right) - p\left(\frac{dq}{dz'}\right) = 0,$$

an equation that is always true, since $p'\,dx + q'\,dy + r'\,dz = 0$ is the differential of the equation $F(x', y', z') = 0$.

If one now imagines a conical surface (C) on one of the curves (ss) and the luminous point is such that all of the rays are found in that surface then after being reflected it consecutively encounters and forms one of the developable surfaces (S). Likewise, all of the rays that are found in the conical surface (C') that passes through one of the curves ($s's'$) and through the luminous point will, after being reflected, form one of the developable surfaces (S). Therefore, we henceforth consider the incident ray (a) to be the locus of the intersection of two conical surfaces (C), (C'), and the reflected ray (A) to be the locus of intersection of two developable surfaces (S), (S').

The two conical surfaces (C), (C') intersect in the same angle as the corresponding two surfaces (S), (S'), respectively. Indeed, the two tangent planes to the surfaces (C), (C') along the line (a) form, along with the tangent plane to the mirror, a triangular pyramid (a, D, D') whose three edges are the lines (a), (D), (D'), since the lines (D), (D') are [**2**] tangents to the curves (ss), ($s's'$). Likewise, the two tangent planes to the developable surfaces (S), (S') along the line (A) form, along with the tangent plane to the mirror (F), a triangular pyramid (A, D, D') whose three lines (A), (D), (D') are the edges.

Now, the angle that is formed between the incident ray (a) and the line (D), whose cosine is:

$$\frac{aM + bN + cO}{(a^2 + b^2 + c^2)^{1/2}(M^2 + N^2 + O^2)^{1/2}},$$

and the angle that is formed between the reflected ray (A) and the same line, whose cosine is:

$$\frac{mM + nN + oO}{(m^2 + n^2 + o^2)^{1/2}(M^2 + N^2 + O^2)^{1/2}}$$

are equal, and of opposite sign, due to the equation:

$$(aM + bN + cO)(m^2 + n^2 + o^2)^{1/2} + [(mr - op)M + (nr - oq)\,N]\,(a^2 + b^2 + c^2)^{1/2} = 0,$$

which is true independently of the values of M and N.

The angle formed between the lines (a) (D'), (A) (D') are also equal to each other, in such a way that the two pyramids (aDD'), (ADD') are perfectly similar. One thus concludes that the conical surfaces (C), (C') all cut at a right angle.

[**8**]. Consider a light sheaf that is formed between four infinitely close conical surfaces (C), (C), (C'), (C'). This sheaf will have the form of a rectangular quadrangular pyramid whose edges form two small angles i, i', and the rays that it contains after being reflected will be found inside of a rectangular quadrangular surface whose edges form two new angles k, k'.

Let Δ be the distance $[(x' - X')^2 + (y' - Y')^2 + (z' - Z')^2]^{1/2}$ from the luminous point to the point x', y', z' of the surface (F), and let D be the distance from the point x', y', z' to the eye that is situated in the prolongation of the reflected ray. The light that enters the eye after being reflected has traversed a distance since it left the luminous point that is expressed by $D + \Delta$, and at that distance, the ray sheaf that starts from the luminous point will have a perpendicular section to its axis that is the rectangular quadrilateral $(D \pm R)\,k$ $(D \pm R')\,k'$. Now, since the intensity of the light is inverse to the area of the surface on which the same quantity of rays is dispersed, it is found at the place where the eye is situated after reflection, relative to what it was at the distance $D + \Delta$, by starting directly at the luminous point in the ratio of $(D + \Delta)^2\,ii'$ to $(D \pm R)\,(D \pm R')\,kk'$. One may thus express the brightness of the image at the point $x = X$, $y = Y$, $z = Z$, as seen from the point x', y', z' of the mirror, by:

$$T = \varepsilon\,\frac{(D + \Delta)^2\,ii'}{(D \pm R)(D \pm R')kk'},$$

ε being a constant coefficient that expresses the real brightness of the luminous point. One must observe that the $\pm$ signs that R and R' are affected with is independent of the positive or negative value that may result for R and R' due to their positions relative to the plane of the coordinates. These values must always be taken to be positive when they are situated in front, since D is always taken to be positive in front of the mirror.

All that remains is to determine the expression for the ratio $\frac{ii'}{kk'}$. Now, two consecutive rays (a) of the conical surface (C) form the small angle i between them, which, being equal to its sine, has the expression:

$$i^2 = \frac{(a^2+b^2+c^2)(da^2+db^2+dc^2)-(a\,da+b\,db+c\,dc)^2}{(a^2+b^2+c^2)^2},$$

and these two rays, after being reflected, form an angle k between them, whose expression is:

$$k^2 = \frac{(m^2+n^2+o^2)(dm^2+dn^2+do^2)-(m\,dm+n\,dn+o\,do)^2}{(m^2+n^2+o^2)^2}.$$

However, due to [**1**]:

$$\frac{n\,dx-m\,dy}{m\,dn-n\,dm} = \frac{o\,dx-m\,dz}{m\,do-o\,dm} = \frac{o\,dy-n\,dz}{n\,do-o\,dn},$$

one will have:

$$(m^2 + n^2 + o^2)(dm^2 + dn^2 + do^2) - (m\ dx + n\ dy + o\ dz)$$

$$= \frac{(m\,dn-n\,dm)^2}{(m\,dx-n\,dy)^2}[(m^2+n^2+o^2)(dx^2+dy^2+dz^2)-(m\,dx+n\,dy+o\,dz)^2]$$

$$\frac{i^2}{k^2}$$

$$= \frac{(m^2+n^2+o^2)^2}{(a^2+b^2+c^2)^2}\frac{(n\,dx-m\,dy)^2}{(m\,dn-n\,dm)^2}\frac{[(a^2+b^2+c^2)(da^2+db^2+dc^2)-(a\,da+b\,db+c\,dc)^2]}{[(m^2+n^2+o^2)(dx^2+dy^2+dz^2)-(m\,dx+n\,dy+o\,dz)^2]},$$

and due to the equation:

$$(m^2 + n^2 + o^2)(a\ dx' + b\ dy' + c\ dz')^2 = (a^2 + b^2 + c^2)(m\ dx' + n\ dy' + o\ dz')^2,$$

and finally:

$$\frac{i}{k} = \frac{R}{\Delta}, \qquad \frac{i'}{k'} = \frac{R'}{\Delta'}, \qquad T = \varepsilon\,\frac{(D+\Delta)^2 RR'}{(D\pm R)(D\pm R')\Delta^2},$$

when the distance Δ to the luminous point is infinite, one has:

$$T = \varepsilon\,\frac{RR'}{(D\pm R)(D\pm R')}.$$

One may already observe that when one abstracts from the term $\varepsilon\,\dfrac{(D+\Delta)^2 RR'}{\Delta^2}$, the intensity T of the reflected light will increase with $(D \pm \Delta)$, while $(D \pm R')$ will be smaller, in such a way that the *maximum* intensity will be obtained for the points where $D \pm R = 0$, $D \pm R' = 0$. Now, $D \pm R = 0$ is the equation of the caustic surface (Σ) and $D \pm R' = 0$ is that of the caustic surface (Σ'). It is therefore the intersection of these two surfaces on which one finds the *maximum* of the reflected light. When the locus of that intersection reduces to several particular points, one gives them the name of the *focus of the reflected light.*

[**9**]. We shall apply the preceding analysis to the surfaces of revolution. We suppose that the luminous point is on the x-axis, and we choose that axis to be the axis of the surface. One will then have $Y = 0$, $Z = 0$, $a = x' - X$, $b = y'$, $c = z'$ and the equations $F(x', y', z') = 0$, $p\,dx' + q\,dy' + r\,dz' = 0$ that represent the surface of the mirror will be:

$$y'^2 + z'^2 = \Phi(x'), \qquad 2y'\,dy' + 2z'\,dz' = \Phi'(x')\,dx'.$$

Since $\Phi(x')$ is an arbitrary function of x', one deduces from this that $p = \Phi'(x')$, $q = -2y'$, $r = -2z'$. Substituting this in the general expressions for m, n, o [**6**] and setting:

$$\lambda = \frac{1}{4\varphi(x') - \varphi'^2(x') - 4(x'-X)\varphi'(x')},$$

one will have:

$$m = \frac{[4\varphi\cdot(x') - \varphi'^2\cdot(x')](x'-X) + 4\varphi\cdot(x')\varphi'\cdot(x')}{4\varphi(x') - \varphi'^2(x') - 4(x'-X)\varphi'(x')} = \Psi(x'), \quad n = y', \ o = z',$$

and upon introducing these expressions into equations (s), (s') [**2**], one has:

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = z'[1 - \Psi'(x')], \quad \varepsilon = -y'[1 - \Psi'(x')], \quad \zeta = 0,$$

$$L = 2(y'^2 + z'^2)\,[1-\Psi'(x')],\ M = 0,\ N = -4z'^2\varphi'(x')[1-\Psi'(x')],\ O = 4y'\,z'\,\varphi'(x')[1-\Psi'(x')],$$

$$M' = 8y'\varphi'(x')[1-\Psi'(x')],\ N' = 4y'^2\,\varphi'(x')[1-\Psi'(x')],\ O' = 4y'\,z'\,\varphi'(x')[1-\Psi'(x')],$$

or simply:

$$M = 0,\ N = -z'^2\varphi'(x'),\ O = y'\,z'\,\varphi'(x'),\ M' = y'\varphi'(x'),\ N' = y'^2\,\varphi'(x'),\ O' = y'\,z'\,\varphi'(x'),$$

and finally:

$$dx' = 0, \qquad y'\,dy' + z'\,dz' = 0, \qquad (s)$$

$$2\,\Phi(x')\,dz' - z'\,\Phi'(x')\,dx' = 0, \qquad y'\,dy' - z'\,dz' = 0. \qquad (s')$$

The integrals in these equations are, upon calling the arbitrary constants that complete them A and B:

$$x' = A, \qquad y'^2 + z'^2 = \Phi(A), \tag{ss}$$

$$z'^2\,(1 + B^2) = \Phi(x'), \qquad y' = B\,z'. \tag{$s's'$}$$

The first series of curves (ss) belongs to the sequence of circles that lie in a plane perpendicular to the axis of revolution, and the second series ($s's'$), to the sequence of meridians of the surface.

If one eliminates x', y', z' from the equations of the incident ray:

$$(x' - X')\,z = z'\,(z - X), \qquad y'z = z'y, \qquad (x' - X)\,y = y'(x - X), \tag{a}$$

and those of the curves (ss) then the result will be the equation of the sequence of conic surfaces (C):

$$(A - X)^2\,(y^2 + z^2) = (x - X)^2\,\Phi \cdot (A). \tag{CC}$$

These surfaces are a sequence of cones with circular bases whose axis is the axis of revolution, and whose centers are all placed at the point $x = X$, $y = 0$, $z = 0$.

If one eliminates x', y', z' from these same equations (a) and those of the curve ($s's'$) then the result will be the equation of the series of conic surfaces (C):

$$y = Bz, \tag{$C'C'$}$$

in such a way that these surfaces reduce to a series of planes that pass through the axis of revolution.

If one eliminates x', y', z' from the equations of the reflected ray:

$$\Psi(x')(z - z') = y'(x - x'), \qquad y'\,z = z'\,y, \qquad \Psi(x')(y - y') = y'(x - x'), \tag{A}$$

and those of the curve (ss) then one will have the series of developable surfaces (S):

$$\Psi^2(A)(y^2 + z^2) = \Phi(A)[x - A + \Psi(A)]^2. \tag{SS}$$

These surfaces are a sequence of cones with circular base that have the axis of revolution for their axis, and whose centers vary for each particular value of the arbitrary constant A.

If one eliminates x', y', z' from the same equation (A) and those of the curve ($s's'$) then one will have the series of developable surfaces (S):

$$y = Bz; \tag{$S'S'$}$$

these surfaces of revolution reduce to a series of planes that pass through the axis of revolution.

In order to determine the edges of regression of these developable surfaces and the caustic surfaces (Σ), (Σ'), which are the loci of the points of intersection of the all rays, one must commence by substituting the values of M, N, O, M', N', O' in the equations (G), (G') [**3**], which gives:

$$\Lambda = -1, \qquad \Lambda' = -\frac{\Psi(x')\varphi(x') - 2\varphi(x')}{\Psi(x')\varphi'(x') - 2\varphi(x')\Psi'(x')},$$

$$x - x' = -\Psi(x'), \qquad y - y' = -y', \qquad z - z' = -z', \tag{G}$$

$$R = (m^2 + n^2 + o^2)^{1/2}\,\Lambda = -[\Psi^2(x') + \Phi(x')]^{1/2}, \tag{R}$$

$$x - x' = -\Psi(x')\,\frac{\Psi(x')\varphi(x') - 2\varphi(x')}{\Psi(x')\varphi'(x') - 2\varphi(x')\Psi'(x')}, \quad y - y' = -y'\,\frac{\Psi(x')\varphi(x') - 2\varphi(x')}{\Psi(x')\varphi'(x') - 2\varphi(x')\Psi'(x')},$$

$$z - z' = -z'\,\frac{\Psi(x')\varphi(x') - 2\varphi(x')}{\Psi(x')\varphi'(x') - 2\varphi(x')\Psi'(x')}, \tag{G'}$$

$$R' = (m^2 + n^2 + o^2)^{1/2}\,\Lambda = -[\Psi^2(x') + \Phi(x')]^{1/2}\,\frac{\Psi(x')\varphi(x') - 2\varphi(x')}{\Psi(x')\varphi'(x') - 2\varphi(x')\Psi'(x')}. \tag{R'}$$

If one eliminates x', y', z' from equations (ss) and (G) then one will obtain those of the edges of regression (s) of developable surfaces (S):

$$x = A - \Psi(A), \qquad y = 0, \qquad z = 0, \tag{$\sigma\sigma$}$$

from which, one sees that each of these edges of regression reduce to a point that is situated on the x-axis.

If one eliminates the same quantities from the same equations (G) and that of the surface $y'^2 + z'^2 = \Phi(x')$ then one will have the equation of the caustic surface (Σ) which is the locus of all these edges of regression (σ), and which reduces, in this case, to the axis of revolution:

$$y = 0, \qquad z = 0. \tag{Σ}$$

If one eliminates x', y', z' from equations ($s's'$) and (G') then one will obtain the equations of the edges of regression (σ') of the developable surfaces (S), which, when one considers them in the present context, nevertheless each have an edge of regression formed by the series of consecutive rays that contain them. Indeed, one may eliminate x', y', z' directly from the last of the equations ($s's'$), and the last two of equations (G), which gives $y = Bz$. As for the other projection, if one eliminates z' from the first of equations ($s's'$) and the last of equations (G') then one will have:

$$z^2\,(1 + B^2) = 4\,\Phi^3(x')\left[\frac{1 - \Psi'(x')}{\Psi(x')\varphi'(x') - 2\varphi(x')\Psi'(x')}\right]^2.$$

However, by means of the first of equations (G'), one may determine x as a function of x' in such a way that one will have a result of the form:

$$z^2 (1 + B^2) = \chi \cdot (x), \qquad y = Bz. \qquad (\sigma'\sigma')$$

In order to determine the caustic surface (Σ), which is the locus of all these edges of regression, one must eliminate x', y', z' from equations (G) and that of the surface $y'^2 + z'^2 = \Phi(x')$, or eliminate the arbitrary constant B from the two equations ($\sigma'\sigma'$), which gives:

$$y^2 + z^2 = \chi \cdot (x), \qquad (\Sigma')$$

an equation that also belongs to a surface of revolution. The points where that surface (Σ') cuts the axis (Σ), and which are given by the equation $\chi(x) = 0$, are the ones where the maximum of the light is found [**8**], and are called the *foci* of the reflected rays.

If one would like to determine the curves (s), (s) [**5**] on the surface $y'^2 + z'^2 = \Phi(x')$, which are the loci of the particular points of the curve (ss), ($s's'$) for which the consecutive rays cease to meet in the same sense, then one must substitute the values of M, N, O, M', N', O' in the equation (s), (s'), so one will have:

$$y' z' = 0, \qquad y'^2 + z'^2 = 0, \quad \text{or} \qquad y' = 0, \quad z' = 0. \qquad (s)$$

These equations refer to the point of the surface that is met by the axis of revolution. One must meanwhile observe that since the preceding equation gives $n / m = 0$, $o / m = 0$, one may again satisfy it when one makes $m = \pm \infty$, where:

$$4\,\Phi(x') - \Phi'^2(x') - 4(x' - X)\,\Phi(x') = 0, \qquad y'^2 + z'^2 = \Phi(x'), \qquad (s)$$

equations that pertain to a circle whose reflected rays are all parallel to the axis of revolution and form a cylindrical surface (Σ) that is asymptotic to the axis (Σ).

Upon substituting the values of M, N, O, the curve (s') becomes:

$$\Psi(x')\,\Phi'(x') - 2\Phi(x')\,\Psi'(x') = 0, \qquad y'^2 + z'^2 = \Phi(x'), \qquad (s')$$

which is the equation of a circle whose reflected rays all form a conical surface (Σ) that is asymptotic to two nappes of the surface (Σ').

As for the number $T = \varepsilon \dfrac{(D+\Delta)^2 RR'}{(D \pm R)(D \pm R')\Delta^2}$ [**8**], which expresses the brightness of the image of the luminous point, one has:

$$T = \varepsilon \frac{(D+\Delta)^2[\Psi'(x')+\Phi(x')][\Psi(x')\Phi'(x')-2\Phi(x')]}{[D\mp[\Psi'(x')+\Phi(x')]^{1/2}]\{D[\Psi(x')\Phi'(x')-2\Phi(x')\Psi'(x')]\mp[\Psi'(x')+\Phi(x')]^{1/2}[\Psi(x')\Phi'(x')-2\Phi(x')]\}\Delta^2}.$$

If one proposes to determine a surface of revolution such that all of the reflected rays are parallel to its axis then one will have to satisfy the condition $n / m = 0$, $o / m = 0$, or:

$$4\,\Phi(x') - \Phi'^2(x') - 4(x' - X)\,\Phi'(x') = 0.$$

Now, upon replacing $\Phi(x')$ and $\Phi'(x')$ with their values $y'^2 + z'^2 \dfrac{2y'dy' + 2z'dz'}{dx'}$ that equation becomes:

$$(y'^2 + z'^2)\,dx'^2 - (y'\,dy' + z'\,dz')^2 - 2(x' - X)\,(y'\,dy' + z'\,dz')\,dx' = 0,$$

and its integral:

$$4(x + \eta)(X + \eta) = y'^2 + z'^2$$

represents the paraboloid of revolution that has its focus at the point $x= 0$, $y = 0$, $z = 0$, where η is the arbitrary constant that specifies each of these surfaces.

One concludes, conversely, that if a paraboloid of revolution is struck by rays parallel to its axis then all of the reflected rays will converge to the focus.

Now, determine a surface of revolution such that all of the rays that start from the point $x = X$, $y = 0$, $z = 0$ converge to a second point $x = X'$, $y = 0$, $z = 0$. Since the equations for the reflected ray are:

$$m(z - z') = o(x - x'), \quad n(z - z') = o(y - y'), \tag{A}$$

the condition that expresses the idea that all of these rays pass through the point $x = X'$, $y = 0$, $z = 0$ is:

$$-\,mz' = o(X' - x'), \qquad nz' = oy'.$$

Replacing m, n, o with their values, the second equation becomes an identity, and the first one gives:

$$[4\varphi(x') - \varphi'^2(x')]\,(x' - X) + 4\varphi(x)\,\varphi'(x) = (x' - X)\,[4\varphi(x') - \varphi'^2(x') - 4\,(x' - X)\,\varphi'(x')].$$

Now, if, as in the preceding case, one replaces $\Phi(x')$ and $\Phi'(x')$ with their values then one has:

$$(y'^2 + z'^2)\,(2x - X - X')\,dx'^2 + 2(y'^2 + z'^2 - x'^2 + xX + x\,X' - XX')\,dx'\,(y'\,dy' + z'\,dz')$$
$$+\,(2x' - X - X')\,(y'\,dy' + z'\,dz') = 0,$$

an equation whose integral is:

$$[(X - X')^2 + 4\theta^2]\,(y'^2 + z'^2 - \theta^2) + \theta^2\,(2x' - X - X)^2 = 0,$$

which represents the ellipsoids of revolution that have their foci at the points $x = X$, $y = 0$, z, $x = X'$, $y = 0$, z, θ^2 being the arbitrary constant that specifies each of these surfaces. One sees that the minor axis of that ellipsoid is expressed by 2θ and the major axis, by $[(X - X')^2 + 4\theta^2]^{1/2}$.

[**10**] Before pressing on to the application of the preceding analysis, we present several axioms of optics that are necessary for the development of that theory.

1. The judgments that we make in the phenomena of optics on the distance and magnitude of objects are determined from the comparison of the sensations that we experience in the course of simple vision.
2. We judge the distance to an object when we know the true size, from its apparent distance, its apparent size, and its apparent brightness.
3. We relate the distance to a luminous point from the place where the rays diverge. Thus, when we perceive the image of a luminous point on a curved mirror, we relate its distance to the points of the reflected ray that are met by the consecutive rays, and consequently, to the points where that ray is tangent to the two surfaces (Σ), (Σ'). If the eye is placed between two caustics then it relates the distance from the luminous point to the only caustic that is in the direction of the visual ray; however, if the caustics are found in the direction of the visual ray then the apparent distance is a combination of the two distances from the eye to the points where the reflected ray touches it.
4. The apparent size of an object is a function of the angle between the two visual rays that subtend its extremities.
5. Finally, since the brightness of a bright object diminishes by reason of its distance, if, by a new disposition of its rays, that brightness is changed then this change will influence our judgment relating to the dimensions or the distance to the object.

[**11**]. We begin by considering a plane mirror, and we suppose that the surface is the yz-plane, in such a way that equations [**9**] $y'^2 + z'^2 = \Phi(x')$ and $2y'\,dy' + 2\,z'\,dz' = \Phi'(x')\,dx'$ become $x' = 0$, $dx' = 0$. One concludes from this that Φ is an arbitrary variable and $\Phi' = \infty$.

$$m = \Psi(x') = X, \qquad n = y', \qquad o = z',$$

$$dx' = 0, \qquad y'\,dy' + z'\,dz' = 0, \tag{s}$$

$$dx' = 0, \qquad y'\,dy' + z'\,dz' = 0, \tag{s'}$$

$$x' = 0, \qquad y'^2 + z'^2 = d^2, \tag{ss}$$

$$x' = 0, \qquad y' = Bz', \tag{$s's'$}$$

$$X^2\,(y'^2 + z'^2) = (x - X)^2\,d^2, \tag{CC}$$

$$y = Bz, \tag{$C'C'$}$$

$$X^2\,(y^2 + Bz) = (x + X)^2\,d^2, \tag{SS}$$

$$y = Bz, \tag{$S'S'$}$$

$$\Lambda = \Lambda = -1, \quad R = R' = -(X^2 + y'^2 + z'^2)^{1/2} = -\Delta,$$

$$x = -X, \qquad y = 0, \qquad z = 0, \tag{$\sigma\sigma$) ($\sigma'\sigma'$) (Σ) (Σ'}$$

$$T = e\,\frac{(D+\Delta)^2 RR'}{(D-R)(D-R')\Delta^2} = e.$$

One sees from these results that the developable surfaces (SS) that are formed by the reflected rays are a sequence of cones that are similar to the cones (CC), and which have their centers at the point $x = -X$, $y = 0$, $z = 0$.

It is to this unique point, which is the constant locus of the image, that the edges of regression ($\sigma\sigma$), ($\sigma'\sigma'$), and the caustic surfaces (Σ), (Σ') reduce.

That point is situated behind the mirror at a distance that is equal to the distance at which one finds the luminous point in front of it.

The value of T indicates that the brightness of the image is the same as that of the object.

[**12**]. Now, suppose that the proposed surface of revolution [**9**] is a sphere of radius ρ that has its center at the origin of the coordinates, so the equation of the mirror will be:

$$x'^2 + y'^2 + z'^2 = \rho'^2,$$

from which, one infers:

$$\Phi(x') = \rho^2 - x'^2, \qquad \Phi'(x') = -2x',$$

and upon substituting in m, n, o:

$$m = \Psi(x') = \frac{(\rho^2 - 2x'X)x' + \rho^2 X}{\rho^2 - 2x'X}, \qquad n = y', \qquad o = z',$$

the two sequences of curves of reflection will be:

$$x' = A, \qquad y'^2 + z'^2 = \rho'^2 - A^2, \tag{ss}$$

$$z'^2\,(1 + B^2) = \rho'^2 - x'^2, \qquad y' = Bz'. \tag{s's'}$$

Equations (ss) pertain to a series of circles that are parallel to the plane of yz and equations ($s's'$), to a series of great circles whose planes pass through the x-axis, and which project onto the xz and xy planes along a series of ellipses that all have the same major axis ρ.

The two sequences of concentric conic surfaces are:

$$(A - X)^2\,(y^2 + z^2) = (x - X)^2\,(\rho^2 - A^2), \tag{CC}$$

$$y = Bz. \tag{C'C'}$$

Equation (CC) expresses a sequence of cones with circular base, and equation ($C'C'$), a sequence of planes that pass through the x-axes.

The developable surfaces formed by the series of reflected rays that meet consecutively are:

$$[(\rho^2 - 2AX)\,A + \rho^2 X]^2\,(y^2 + z^2) = (\rho^2 - A^2)\,[(\rho^2 - 2AX)\,x + \rho^2 X]^2, \tag{SS}$$

$$y = Bz. \tag{S'S'}$$

The surfaces (SS) are a sequence of cones whose centers are found along the x-axis.

The values of Λ and Λ' are:

$$\Lambda = -1, \qquad \Lambda' = -\frac{(\rho^2 - 2x'X)(\rho^2 - x'X)}{\rho^2(\rho^2 + 2X^2 - 3x'X)},$$

in such a way that one has:

$$x = -\frac{\rho^2 X}{\rho^2 - 2x'X}, \qquad y = 0, \qquad z = 0, \tag{G}$$

$$R = (m^2 + n^2 + o^2)^{1/2}\,\Lambda = -\frac{\rho^2(\rho^2 - 2x'X + X^2)^{1/2}}{(\rho^2 - 2x'X)} = \frac{\rho^2\Delta}{(\rho^2 - 2x'X)},$$

$$\left.\begin{aligned} x &= -X\,\frac{\rho^2 - x'^2}{\rho^2(\rho^2 + 2X^2 - 3x'X)}, \\ y &= 2y'X^2\,\frac{\rho^4 + 2x'^3 X - 3\rho^2 x'X}{\rho^2(\rho^2 + 2X^2 - 3x'X)}, \\ z &= 2z'X^2\,\frac{\rho^2 - x'^2}{\rho^2(\rho^2 + 2X^2 - 3x'X)}, \end{aligned}\right\} \tag{G'}$$

$$R' = (m^2 + n^2 + o^2)^{1/2}\,\Lambda' = -\frac{(\rho^2 - 2x'X + X^2)^{1/2}(\rho^2 - x'X)}{\rho^2 + 2X^2 - 3x'X} = -\frac{\Delta(\rho^2 - x'X)}{\rho^2 + 2X^2 - 3x'X}.$$

If one eliminates x', y', z' from equations (ss) and (G) then one will have:

$$x = -\frac{\rho^2 X}{\rho^2 - 2AX}, \qquad y = 0, \qquad z = 0 \tag{$\sigma\sigma$}$$

for the equations of the edges of regression of the developable surfaces (SS). These curves reduce to a sequence of points situated on the x-axis.

The equation of the surface (Σ), which is the locus of all these edges of regression, will be:

$$y = 0, \qquad z = 0. \tag{Σ}$$

If one eliminates x', y', z' from equations ($s's'$) and (G') then one will have the equations of the edge of regression ($\sigma'\sigma'$), and if one eliminates x', y', z' from the same equation (G') and the equation $x'^2 + y'^2 + z'^2 = \rho^2$ then one will have the equation of the surface (Σ'), which is, as we have seen, a surface of revolution. However, without having recourse to that elimination, we may consider simply the generating circles of that surface, which are obviously composed of the consecutives intersections of the surfaces (SS). One will thus obtain the equation of these circles upon eliminating x', y', z' from the equations (ss) and (G), which gives:

$$x = -X\,\frac{\rho^4 + 2A^3 X - 3\rho^2 AX}{\rho^2(\rho^2 + 2X^2 - 3AX)}, \qquad y^2 + z^2 = \frac{4X^4(\rho^2 - A^2)^3}{\rho^4(\rho^2 + 2X^2 - 3AX)^2}. \tag{$\Sigma' ss$}$$

If one eliminates the arbitrary constant A from these two equations then the result will be the caustic surface (Σ').

The equations (s) of the circle on which the rays reflect parallel to the x-axis become:

$$x' = \frac{\rho^2}{2X}, \qquad y'^2 + z'^2 = (4X^2 - \rho^2)\,\frac{\rho^2}{4X^2}, \tag{s}$$

and the equation:

$$y^2 + z^2 = (4X^2 - \rho^2)\,\frac{\rho^2}{4X^2} \tag{Σ}$$

is that of the cylindrical surface that is formed by the reflected rays onto that curve.

The equations (s') of the circle whose reflected rays all form a conical surface that is asymptotic to the two nappes of the surface (Σ') are:

$$x' = \frac{\rho^2 + 2X^2}{3X}, \qquad y'^2 + z'^2 = \frac{5X^2\rho^2 - \rho^4 - 4X^4}{9X^2}, \tag{s'}$$

and the proposed conical surface is:

$$(\rho^4 + 7X^2\rho^2 - 8X^4)^2\,(y^2 + z^2) = (5X^2\rho^2 - \rho^4 - 4X^4)\,[(\rho^2 - 4X^4)\,x + 3\rho^2 X]^2. \tag{Σ'}$$

One will have likewise obtained equations (s') upon making $x = \infty$ in equations (G').

As for the number T that expresses the brightness of the image of a luminous point, one has:

$$T = e\,\frac{(D+\Delta)^2 RR'}{(D+R)(D+R')\Delta^2} = \frac{(D+\Delta)^2\rho^2(\rho^2 - x'X)}{[D(\rho^2 - 2x'X) - \Delta\rho^2][D(\rho^2 + 2X^2 - 3x'X) - \Delta(\rho^2 - x'X)]}.$$

We let R and R' have the positive sign, because in the case that we are considering these quantities will always be positive when they are behind the tangent plane, and always negative when they are in front of it.

[**13**]. We shall expose the various phenomena that are presented by convex or concave spherical mirrors by means of the preceding equations.

In order to fix ideas, we take the positive x axis to be in front of the yz-plane and negative x, on the opposite side.

(Fig. 1) If one first considers the case of parallel luminous rays then one will have $X = \infty$, and the equation of the conical surfaces (SS) will become, upon replacing the arbitrary constant A with the arbitrary constant x' that it represents:

$$(\rho^2 - 2x'^2)^2\,(y^2 + z^2) = (\rho^2 - x'^2)(\rho^2 - 2x'x)^2, \tag{SS}$$

where the positive values of x' pertain to a convex mirror and the negative values, to a concave one.

The center of the surfaces (*SS*) will have the equation:

$$x = \frac{\rho^2}{2x'}, \qquad y = 0, \qquad z = 0, \qquad (\sigma\sigma)$$

so, since x will always have the same sign as x', all of the centers will have their centers in front of the yz-plane for the convex mirror and behind that plane for the concave mirror.

The consecutive surfaces (*SS*) intersect along the circles:

$$x = x' \frac{3\rho^2 - 2x'^2}{2\rho^2}, \qquad y^2 + z^2 = \frac{(\rho^2 - x'^2)^3}{\rho^4}. \qquad (\Sigma' ss).$$

The sequence of circles ($\Sigma' ss$) forms the caustic surface (Σ'), and since all of the values of x are found between ρ and $-\rho$, one sees that x will always have the same sign as x', and that the part of the surface that pertains to the convex mirror will always be in front of the yz-plane, and the part that relates to the concave mirror will be behind it.

If one eliminates the arbitrary constant x' from the two equations ($\Sigma' ss$) then one will have the equation of the caustic surface (Σ'):

$$2x = [\rho^2 - \rho^{4/3}(y^2 + z^2)^{1/3}]^{1/2}\, [1 + 2\rho^{-2/3}(y^2 + z^2)^{1/3}]. \qquad (\Sigma')$$

That surface cuts the (Σ) axis at the points $y = 0$, $z = 0$ or $x = \pm \rho / 2$, which are the ones where the *maximum* light is found, in such a way that the focus of the reflected parallel rays is $x = \rho/2$ for the convex mirror and $x = -\rho/2$ for the concave mirror.

Among the surfaces (*SS*), there are two that reduce to planes perpendicular to the x-axis. Now, the equation of these two surfaces is valid independently of the values of $y'^2 + z'^2$. Introducing that condition in the equation (*SS*) one has $\rho^2 - 2x'^2 = 0$, $\rho^2 - 2x'x = 0$, or:

$$x = x' = \pm \frac{\rho}{\sqrt{2}}. \qquad (SP)$$

These two planes touch the surface (Σ') along two circles whose equation comes about upon substituting that value of x' into the equations ($\Sigma' ss$), which gives:

$$x = \pm \frac{\rho}{\sqrt{2}}, \qquad y^2 + z^2 = \frac{\rho^2}{8}.$$

The equations of the circle on which the rays are reflected parallel to the x-axis are:

$$x' = 0, \qquad y^2 + z^2 = \rho^2, \qquad (s)$$

and the cylindrical surface that is formed by these rays is:

$$y^2 + z^2 = \rho^2; \tag{Σ}$$

i.e., it envelops the sphere along the great circle that separates the concave mirror and the convex mirror.

As for the circle whose reflected rays form a conical surface that is asymptotic to the two nappes of the surface (Σ), one will have:

$$x' = X, \qquad y^2 + z^2 = -\frac{4}{9}X^2, \tag{s'}$$

an imaginary result, from which one concludes that the surface (Σ′) does not have infinite nappes.

In order to obtain the values of R and R', one will observe that $\Delta = X = \infty$, and one will have:

$$R = \frac{\rho^2}{2\chi'}, \qquad R' = \frac{\chi'}{2}, \tag{R)(R'}$$

so R and R' are always of the same sign and situated on the same side of a tangent plane. In the second place, since the caustic surface (Σ′) is found between the planes $x = \pm \rho / \sqrt{2}$, (SP), and the cylindrical surface $y^2 + z^2 = \rho^2$, (Σ) will always refer to the interior of the sphere, in such a way that:

1. *Inside the convex mirror*, all of the rays meet behind the tangent plane, and *the image is situated behind the mirror.*

2. *Inside the concave mirror*, all of the rays meet in front of the tangent plane, and *the image is situated in front of the mirror.*

One may already conclude from this, by induction, that when the parallel rays strike an arbitrary convex mirror, they diverge after being reflected, and when they strike an arbitrary concave mirror they converge after being reflected.

As for the term that expresses the intensity of the light, one has:

$$T = e\,\frac{(D+\Delta)^2 RR'}{(D+R)(D+R')\Delta^2} = e\,\frac{\rho^2 x'}{(2x'D+\rho^2)(2D+x')}.$$

One will have the *maximum* intensity of the light at the points for which $2x'D + \rho^2 = 0$, $2D + x' = 0$, equations from which one deduces $x' = \pm \rho$, values which, when substituted in (Σ′ss), give $x = \pm \rho / 2$, $R = R' = \pm \rho / 2$, which are results that conform to the already-known position of the focus of the parallel reflected rays on the convex and concave spherical mirrors.

[**14**]. Now, suppose that X has a value that is finite, but larger than ρ, in such a way that one has $X = f\rho$, $f > 1$.

First of all, we determine the limits of the convex mirror and the concave mirror. If, at the luminous point $x = X$, $y = 0$, $z = 0$, one envelopes the sphere by a conical surface then the circle of contact will be the limit of the two mirrors. Now, the tangent plane to the sphere at a point x', y', z' is $x'(x - x') + y'(y - y') + z'(z - z') = 0$, and if one expresses the idea that it passes through the point $x = X$, $y = 0$, $z = 0$ then one will have the condition $x'X - x'^2 - y'^2 - z'^2 = 0$, or $x' = \rho^2 / X = \rho / f$.

The convex mirror will thus be found between the limits:

$$x' = \rho \qquad \text{and} \qquad x' = \rho / f,$$

and the concave mirror, between the limits:

$$x' = \rho / f \qquad \text{and} \qquad x' = - \rho.$$

Having said this, the surfaces (*SS*) will be:

$$[(\rho - 2x'f)\, x' + \rho^2 f]^2\, (y^2 + z^2) = (\rho^2 - x'^2)[(\rho - 2x'f)\, x' + \rho^2 f]^2. \qquad (SS)$$

The position of the center of these cones will be:

$$x' = \frac{\rho^2 f}{2x'f - \rho}, \qquad y = 0, \qquad z = 0. \qquad (\sigma\sigma)$$

This value is positive from $x' = \rho$ up to $2x'f - \rho = 0$, or $x' = \rho/2f$. Now, since this latter limit refers to the concave mirror, x will always be positive for the convex mirror. It is negative for the concave mirror from $x' = \rho/2f$ to $x' = - \rho$.

The consecutive surfaces (*SS*) intersect along the circles:

$$x = \frac{3x'f^2\rho^2 - f\rho^3 - 2f^2x'^2}{\rho^2 + 2f^2\rho^2 - 3x'f\rho}, \qquad y^2 + z^2 = \frac{4f^4(\rho^2 - x'^2)^3}{(\rho^2 + 2f^2\rho^2 - 3x'f\rho)^2} \qquad (\Sigma' ss)$$

and the locus of all of these circles, which one obtains by eliminating x' from these two equations, is the surface (Σ').

The surfaces (*SS*) that reduce to planes are:

$$x = \rho\, \frac{1 \pm (1 + 8f^2)^{1/2}}{4f}. \qquad (SP)$$

The surfaces are situated with one of them in front of the *yz*-plane and the other one, behind it. Moreover, one may observe that for any f the two values of x will always be found between the limits $x = \pm\, \rho$.

The equations of the circle on which the rays are reflected parallel to the x-axis are:

$$x' = \frac{\rho}{2f}, \qquad y'^2 + z'^2 = \rho^2\, \frac{4f^2 - 1}{4f^2}, \qquad (s)$$

and the cylindrical surface that is composed of these rays is:

$$y^2 + z^2 = \rho^2 \, \frac{4f^2 - 1}{4f^2}.$$

No matter what the values of f are the radius of that surface is always less than that of the sphere.

As for the circle (s), whose reflected rays form a conical surface that is asymptotic to the two nappes of the surface (Σ'), one will have:

$$x' = \rho \, \frac{1 + 2f^2}{3f}, \qquad y'^2 + z'^2 = \rho^2 \, \frac{5f^2 - 1 - 4f^4}{9f^2}. \tag{s'}$$

Now, since $f > 1$, these expressions are imaginary and indicate that the caustic surface (Σ') is composed of just one nappe:

$$R = \frac{\Delta\rho}{2x'f - \rho}, \qquad R' = \frac{\Delta(x'f - \rho)}{\rho + 2f^2\rho - 3x'f},$$

$$T = e \, \frac{(D+\Delta)^2 RR'}{(D+R)(D+R')\Delta^2} = e \, \frac{(D+\Delta)^2 \rho(x'f - \rho)}{[D(2x'f - \rho) + \Delta\rho][D(\rho + 2f^2\rho - 3x'f) + \Delta(x'f - \rho)]}.$$

1. First, consider these results at the limits of the *convex* mirror: $x' = \rho$, $x' = \rho / f$.

Between these limits, R and R' are always positive and situated on the same side of the tangent plane. In the second place, since the caustic surface (Σ') is found between the plane (SP) and the cylindrical plane (Σ), it will be found to fill up the entire interior of the sphere, in such a way that *the image of the luminous point will always be situated behind the tangent plane to the mirror.*

Moreover, R is found between the limits $R = \Delta \dfrac{1}{2f - 1}$ and $R = \Delta$, and R', between the limits $R' = \Delta \dfrac{1}{2f - 1}$, $R' = 0$, and one always has $R < \Delta$, $R' < \Delta$, Δ always being the distance from the luminous point $x = X$, $y = 0$, $z =$ to the point x', y', z' where one finds its reflection. Now, the distance Δ' from the luminous point to the point x', y', z' of the mirror, namely, $\dfrac{R + R'}{2}$, is found between the limits $\Delta' = \Delta$, $\Delta' = \Delta/2$, and one always has $\Delta' < \Delta$, in such a way that the point $x = X$, $y = 0$, $z = 0$ *will always be seen at a distance from the surfaces that is less than the real distance.*

One has, from paragraph [**8**], the two ratios $\dfrac{i}{k} = \dfrac{R}{\Delta}$, $\dfrac{i'}{k'} = \dfrac{R'}{\Delta}$, and consequently, $i < k$, $i' < k'$, so two consecutive rays that start from the luminous point and subtend an angle i will, after being reflected, subtend an angle $k > i$; i.e., they will diverge even more. Conversely, if two converging rays subtend an angle k then after being reflected they will subtend an angle $i < k$. If, in the medium considered, a sequence of luminous rays starts from the point $x = X$, $y = 0$, $z = 0$, one considers a sequence of visual rays that start from this point, then since one will always have $i < k$, $i' < k'$, one will also have that the sum of $i = I$ < the sum of $k = K$, and the sum of $i' = I'$ < the sum of $k' = K'$. Therefore, if two extremities of a line O subtend an angle K' then its image will be seen at an angle of $I' < K'$, from which, one concludes that *the apparent size of objects see on the convex mirror is less than their actual size.*

Since the caustics (Σ), (Σ') are situated behind the convex mirror, as they are with the plane mirror, these objects will be seen in the same sense as in that mirror; i.e., *they keep the same position as they would be perceived to have under simple viewing.*

Finally, since one always has $T < e$, *the brightness of the different points of the image is always less than their real brightness.*

2. Consider the same quantities in the concave mirror, and to begin with, in the spherical zone that is found between the limits $x' = \rho / f$, $x' = \rho / 2f$.

R is found between the limits $R = \mathrm{D}$ and $R = \infty$, and R', between the limits $R' = 0$, $R' = -\Delta\,\dfrac{1}{2f^2-1}$, and one always has $R > \Delta$, $-R' < \Delta$, so R is situated behind the tangent plane and R' is in front of that plane. However, since one always has $R > -R'$, the distance $\Delta' = \dfrac{R+R'}{2}$ will always be measured in the sense of R, and consequently, behind the tangent plane, in such a way that *the image will be situated behind the mirror.*

In the second place, since the distance Δ' is found between the limits $\Delta' = \Delta/2$, $\Delta' = \infty$, *the distance to the image, after having been less than the distance to the object, will increase to infinity.*

From the preceding, one will have $i > k$, $-i' < k'$, in such a way that *the object will appear to be increased in the sense of the angles i, k, and diminished in the sense of the angles i′, k′*, for the eye that is situated at the point $x = X$, $y = 0$, $z = 0$, if the distance D is $> \dfrac{2\Delta R'}{\Delta - R'}$.

Since the line R, and consequently, the caustic (Σ) are situated behind the tangent plane, *the image will be erect as in the plane mirror in the sense of the angles i, k;* however, since the caustic (Σ') is situated in front of the tangent plane, *the image will be reversed in the sense of the angles i′, k′.*

Thus, $x' = \rho / 2f$ up to $x' = 0$, $R = R'$ are of the same sign and situated in front of the tangent plane. They are found between the limits $R = -\infty$, $R = -\Delta$, $R' = -\Delta\,\dfrac{1}{2f^2-1}$, $R' = -\Delta\,\dfrac{1}{2f^2-1}$, and one has $-R > \Delta$, $-R' < \Delta$, $-i > k$, $-i' < k'$, in such a way that *the image is increased in the sense of the angles i, k, diminished in the sense of the angles i′, k′, and reversed in the two senses if $D > R$.*

3. Now, examine the part of the concave mirror that is formed by the hemisphere that is found between the limits $x' = 0$, $x' = -\rho$, while R and R' have the same sign and are situated in front of the tangent plane; they are found between the limits $R = -\Delta$, $R =$

$- \Delta \dfrac{1}{2f+1}$, $R' = - \Delta \dfrac{1}{2f^2-1}$, . One always has $- R < \Delta$, $- R' < \Delta$, $- i < k$, $- i' < k'$, in such a way that the image is both diminished and reversed in the sense of the angles i, k, i', k' if $D > \dfrac{2\Delta R}{\Delta - R}$ and $> \dfrac{2\Delta R'}{\Delta - R'}$.

One may remark that in general the image is deformed, and that it is all the more so that the ratios i/k, i'/k' or the quantities R and R' differ from each other.

In order to determine the points where the *maximum* of the reflected light is found, one sets, in the expression for T:

$$D(2x'f - \rho) + \Delta\rho = 0, \quad D(\rho + 2f^2\rho - 3x'\rho) + \Delta(x'f - \rho) = 0,$$

which gives $x' = \pm \rho$, a value which, when substituted in $(\Sigma' ss)$, gives $x = \rho \dfrac{f}{\pm 2f - 1}$ for the position of the focus.

The upper sign belongs to the convex mirror and the lower sign to the concave mirror.

[**15**]. (Fig. III) Place the luminous point on the surface itself of the sphere; i.e., suppose that $X = \rho$, $f = 1$. The light strikes only the concave surface, and the equations of the preceding paragraph become:

$$[(\rho - 2x')x' + \rho^2]^2 (y^2 + z^2) = (\rho^2 - x'^2)\,[(\rho - 2x')x + \rho^2]^2 \qquad (SS)$$

$$x = \frac{\rho^2}{2x' - \rho}, \qquad y = 0, \qquad z = 0, \qquad (\sigma\sigma)$$

$$x = \frac{2x'^2 + 2x'\rho - \rho^2}{3\rho}, \qquad y^2 + z^2 = \frac{4(\rho^2 - x'^2)(\rho + x')^2}{9\rho^2}, \qquad (\Sigma' ss)$$

$$x = \rho\,\frac{1 \pm 3}{4}, \qquad \text{or} \qquad x = \rho, \qquad x = -\frac{\rho}{2}, \qquad (SP)$$

$$x' = \frac{\rho}{2}, \qquad y'^2 + z'^2 = \rho^2\,\frac{3}{4}, \qquad (s)$$

$$y'^2 + z'^2 = \rho^2\,\frac{3}{4}, \qquad (\Sigma)$$

$$x' = \rho, \qquad y'^2 + z'^2 = 0, \qquad (s')$$

$$R = \frac{\Delta\rho}{2x' - \rho}, \qquad R' = -\frac{\Delta}{3}, \qquad T = e\,\frac{(D + \Delta)^2}{[D(2x' - \rho) + \Delta\rho](3D - \Delta)}.$$

One sees that since the caustic surface (Σ') is found between the planes (SP) and the cylindrical surface (Σ), it is further contained in its entirety in the interior of the sphere, except that they touch at the point $x = \rho$.

From $x' = \rho$ to $x' = \rho / 2$, R is situated behind the mirror and R' is in front, $R > \Delta$, $- R' < \Delta$, $i > k$ $- i' < k'$, $\Delta' = \dfrac{R + R'}{2} = \dfrac{\Delta(2\rho - x')}{3(2x' - \rho)} > \Delta$. Thus, the image will always be situated behind the mirror if the eye is placed at the point $x = X$, $y = 0$, $z = 0$. It will be augmented in the sense of the angles i, k due to the fact that $\frac{R'}{\Delta} = \frac{i'}{k'} = \frac{1}{3}$; they will be diminished by a third in the sense of the angles i', k'.

When $D = \infty$, it will be erect in the sense of the angles i, k and reversed in the sense of the angles i', k'.

Beyond the limit $x' = \rho / 2$, since the lines R and R' have the same sign and are situated in front of the mirror, *the image will be reversed in the two senses* if $D > R$. It will continue to be augmented in the sense of the angles i, k up to $x' = 0$, and then diminished up to $x' = -\rho$ if $D > \frac{2\Delta R}{\Delta - R}$, and it will always be diminished by a third in the sense of the angles i', k' when $D = \infty$.

By making $D(2x' - \rho) + \Delta\rho = 0$, $3D - \Delta = 0$, one obtains $x' = -\rho$ for the position of the focus, and $x' = -\rho / 3$ upon substituting in ($\sigma\sigma$) or ($\Sigma' ss$).

[**16**]. (Fig. IV and V) We shall now suppose that the luminous point is in the interior of the sphere, in such a way that the following observations relate only to the concave spherical mirror. We thus make X positive, but less that ρ.

The position of the center of the surfaces (ss) is, as we saw in [**13**]:

$$x = \frac{\rho^2 X}{2x'X - \rho^2}, \qquad y = 0, \qquad z = 0. \tag{$\sigma\sigma$}$$

This value is positive for the concave mirror that is in front of the yz-plane, as long as $2x'X - \rho^2$. The limit of that value is thus obtained by the equation $2x'X - \rho^2 = 0$ or $x' = \frac{\rho^2}{2X}$; however, since x' may not be larger than r, the largest value of that limit will be $\rho = \frac{\rho^2}{2X}$, and consequently the smallest value of X, $X = \rho / 2$. Thus, as long as X is found between ρ and $\rho / 2$, the centers of the cones (SS) that pertain to the anterior mirror will be positive from $x' = \rho$ to $x' = \frac{\rho}{2X}$; however, if X is less than $\rho / 2$ then these centers will be negative.

1. First, suppose $X > \rho / 2$ – i.e., $X = f\rho$, f being a number that is greater than 1/2 and less than 1 – and consider what the equations of the paragraph [**15**] become under that hypothesis.

The surfaces (SS) intersect along the circles:

$$x = \frac{3x'f^2\rho^2 - f\rho^3 - 2f^2x'^3}{\rho^2 + 2f^2\rho^2 - 3x'f\rho}, \qquad y^2 + z^2 = \frac{4f^4(\rho^2 - x'^2)^3}{(\rho^2 + 2f^2\rho^2 - 3x'f\rho)^2}. \tag{$\Sigma' ss$}$$

Now, for the mirror that is anterior to the *yz*-plane, *x* is positive and equal to $\rho\,\dfrac{f}{2f-1}$ when $x' = \rho$, while it becomes infinite when:

$$\rho^2 + 2f^2\rho^2 - 3x' f\rho = 0 \qquad \text{or} \qquad x' = \rho\,\frac{1+2f^2}{3f}.$$

After this limit, the denominator becomes positive, and *x* suddenly passes from positive infinity to negative infinity. This indicates that the caustic surface (Σ') is composed of two infinite nappes.

The denominator continues to increase, the circles ($\Sigma' ss$) approach the *yz*-plane, and exceeds it again when the numerator becomes positive. It then approaches it until it terminates behind that plane with the negative value $x = -\rho\,\dfrac{f}{2f+1}$, which happens when $x' = -\rho$.

One obtains the *maxima* and *minima* of the values of *x* by differentiating the equation ($\Sigma' ss$) with respect to x', and upon setting $dx / dx' = 0$, which gives:

$$2f x'^3 - x'^2 \rho(1 + 2f^2) + f^2\rho^3 = 0,$$

an equation whose three roots are:

$$x' = f\rho, \qquad x' = \rho\,\frac{1+(1+8f^2)^{1/2}}{4f}, \qquad x' = \rho\,\frac{1-(1+f^2)}{4f}.$$

The substitution of the first one in the equation ($\Sigma' ss$) determines a circle that contains the points of the edge of regression of all the edges ($\sigma'\sigma'$), and which is an edge of regression of the caustic surface (Σ').

The second value of x' is obviously impossible, since it gives $x' > \rho$. The third root, when substituted in the equation ($\Sigma' ss$), determines the circle of the rear nappe of the surface (Σ'), which is the furthest away from the *yz*-plane. Since the equation:

$$x' = \rho\,\frac{1\pm(1+8f^2)^{1/2}}{4f}, \tag{SP}$$

which gives the equations of the surfaces (*SS*) that reduce to a plane, is identical with the last two roots of the equation of third degree that we just considered, it is susceptible to the same observations. Thus, since the first value is impossible, there is only one of the surfaces (*SS*) in this case that reduce to a plane perpendicular to the *x*-axis. The equations of the circle on which the rays reflect parallel to the *x*-axis are:

$$x' = \frac{\rho}{2f}, \qquad y'^2 + z'^2 = \rho^2\,\frac{4f^2-1}{4f^2}. \tag{s}$$

and the cylindrical surface that is formed by these rays is:

$$y^2 + z^2 = \rho\,\frac{4f^2-1}{4f^2}. \tag{Σ}$$

As for the circle (s'), whose reflected rays form the conical surface that is asymptotic to the two nappes of the surface (Σ), one will have:

$$x' = \rho\,\frac{1+2f^2}{3f}, \qquad y'^2 + z'^2 = \rho^2\,\frac{5f^2-1-4f^4}{9f^2}, \tag{s'}$$

and the conical surface proposed will have the equation [**13**].

$$(1 + 7f^2 - 8f^4)^2\,(y'^2 + z'^2) = (5f^2 - 1 - 4f^4)[(1 - 4f^2)\,x + 3f\rho]^2. \tag{Σ'}$$

The value of $y'^2 + z'^2$ in equation (s') is real only when it is positive, in such a way that one will obtain its limits from the equation $5f^2 - 1 - 4f^4 = 0$, or $f^2 = \frac{5\pm 3}{8}$, which reduces to $f = 1$, $f = 1/2$ in the case considered; however, among the values of f that are found between these limits, there is one of them that will give a *maximum* for $y'^2 + z'^2$, and one obtains it from the condition $\frac{d(y'^2+z'^2)}{df} = 0$, which gives $f = 1/\sqrt{2}$. In this case, the value of $(y'^2 + z'^2)^{1/2}$ becomes $\rho / 3$, in such a way that all of the values of f, $(y'^2 + z'^2)^{1/2}$, may not be larger than $\rho / 3$.

Now, examine the quantities:

$$R = \frac{\Delta\rho}{2x'f-\rho}, \qquad R' = \frac{\Delta(x'f-\rho)}{\rho-2f^2\rho-3x'f},$$

under the hypothesis $f > 1/2$ and $f < 1$.

R and R' are positive and situated behind the tangent plane from $x' = \rho$ to $\rho + 2f^2\rho - 3x'f = 0$ or $x' = \rho\,\frac{1+2f^2}{3f}$, in such a way that *the image of the luminous point is always situated behind the mirror.*

(Fig. IV and V). In the second place, within the same limits $R > \Delta$, $R' > \Delta'$, $i > K$, $i' > K'$, two contiguous rays that subtend and angle i will, after being reflected, subtend an angle $K < i$; i.e., they will diverge less after reflection. Conversely, if two converging rays subtend an angle k then after being reflected they will subtend an angle $i > k$; i.e., they converge even more, and by the reasoning of paragraph [**17**], one concludes that when the object is situated at a distance from the surface that is less than ρ / 2, *its* *apparent size surpasses its real size* in the part of the mirror that is found between the limits that are analogous to $x' = \rho$, $x' = \rho\,\dfrac{1+2f^2}{3f}$.

(Fig. VI and VII) This consequence is obvious, from the preceding, when the rays that start at the extremities of the object *o* converge to the mirror and are then reflected towards the eye ω; however, one imagines that there is a position for the eye at which these rays are parallel, and that ultimately, if the eye moves away from the mirror then these rays, if they are to reflect towards it, must diverge, since, for example, the ones that start at the extremities of the object o', and which reflect towards the eye ω'. In that case, one concludes, *a fortiori*, that the image O' of the object will be larger than the object o' itself. Since the lines *R* and R', as well as the caustics, are situated behind the tangent plane, as in the case of the plane mirror, the image will not be reversed. Finally:

$$T = e\,\frac{(D+\Delta)^2\rho(x'f-\rho)}{[D(2x'f-\rho)+\Delta\rho][D(\rho+2f^2\rho-3x'f)+\Delta(x'f-\rho)]},$$

which constantly gives $T > e$ within the same limits, so *the brightness* of the image will always be greater than that of the object, which contributes to diminishing the apparent distance or augmenting the apparent size.

(Fig. IV and V). At the limit $x' = \rho\,\dfrac{1+2f^2}{3f}$, R' passes from positive infinity to negative infinity; however, one again has $-i' > k'$, up to the limit $x' = f\rho$, which gives $R' = \Delta$, $-i' = k'$. Therefore, between these limits, for the eye that is situated at the point $x = X$, $y = 0$, $z = 0$, *the image of the object is augmented, but reversed in the sense of the angles* i', k' *if* $D > R'$.

As for *R*, it does not change sign between these limits, and one always has $R > D$, $i > k$, in such a way that *the image continues to be augmented and erect in the sense of the angles i, k*; i.e., in the sense that is perpendicular to the plane that passes through the *x*-axis.

(Fig. IV) If $f > 1 / 2f$ – i.e., if $f > 1/\sqrt{2}$, as one has between the limits $x' = f\rho$, $x' = \rho / 2f$ – *R* is positive and situated behind the tangent plane $R > \Delta$, $i > K$, R' negative and situated in front of the tangent plane $-R' < \Delta$, $-i' < k'$, *the object will appear to be augmented and erect in the sense of the angles i, k, and diminished and reversed in the sense of the angles* i', k', *if* $D > \dfrac{2\Delta R'}{\Delta - R'}$.

At the limit $x' = \rho / 2f$, *K* will suddenly pass from positive infinity to negative infinity, but one again has $-R > \Delta$, $-i > k$ up to $x = 0$, which gives $R = -\Delta$. Therefore, within these limits, *the object will appear to be augmented and reversed in the sense of the angles i, k, and diminished in the sense of the angles* i', k'. On the contrary, if $f < 1 / 2f$ or $f < 1/\sqrt{2}$, as it is between the limits $x' = r / 2f$, $x' = 1 / fr$, *R* is negative $-R > D$, $-i' > k$, R' is negative, $-R' > \Delta$, $-i' > k'$, *and the image will augmented and reversed in the two senses if* $D > R$.

Beyond the limit $x' = f\,\rho$, R is negative, $-R > \Delta$, $-i > k$, up to $x' = 0$, which gives $R = -\Delta$, $-i = k$, R' is negative, $-R' < \mathrm{D}$, $-i < k'$. Within these limits, *the object will appear to be augmented and reversed in the sense of the angles* i, k *if* $D > R$ *and diminished and reversed in the sense of the angles* i', k' *if* $D > \dfrac{2\Delta R'}{\Delta - R'}$.

As for the second concave mirror, which is found between the limits $x' = 0$, $x' = -\rho$, since one constantly has R and R' are negative and situated in front of the tangent plane, and moreover, $-R < \Delta$, $-i < K$, $-R' < \Delta$, $-i' < K'$, *the image will always be reversed and diminished, as long as* $D > \dfrac{2\Delta R}{\Delta - R}$ and $> \dfrac{2\Delta R'}{\Delta - R'}$.

If, as in the preceding case, one substitutes the values $x' = \pm\rho$ into equations ($\sigma\sigma$) or ($\Sigma' ss$), one has for the position of the focus:

$$x = \rho\,\frac{f}{\pm 2f - 1}.$$

The upper sign always gives $x > \rho$ and the lower sign, $-x < \rho$.

[**17**]. (Fig. VIII) If the luminous point is at a distance of $X = \rho / 2$ then one will have, upon replacing f with the value 1/2 in the equations of paragraph [**14**]:

$$[2(\rho - x')\,x' + \rho^2]^2\,(y^2 + z^2) = (\rho^2 - x')\,[2(\rho - x')\,x + \rho^2] \qquad (SS)$$

$$x = \frac{\rho^2}{2(x' - \rho)}, \qquad y = 0, \qquad z = 0, \qquad (\sigma\sigma)$$

$$x = \frac{3x'\rho^2 - 2\rho^3 - 2x'^3}{6\rho(\rho - x')}, \qquad y^2 + z^2 = \frac{(\rho^2 - x'^2)(\rho + x')^2}{9\rho^2}, \qquad (\Sigma' ss)$$

$$x' = \rho, \qquad y'^2 + z'^2 = 0, \qquad (s), (s')$$

$$R = \frac{\Delta\rho}{x' - \rho}, \qquad R' = \frac{\Delta(x' - 2\rho)}{3(\rho - x')},$$

$$T = e\,\frac{(D + \Delta)^2\rho(x' - 2\rho)}{[D(x' - \rho) + \Delta\rho][3D(\rho - x') + \Delta(x' - 2\rho)]}.$$

One may remark, in particular, that the values of x in the equations ($\sigma\sigma$), (Σ') are all negative and found between $x = -\infty$ and $x = -\rho / 4$, and one will make observations about the other equations that analogous to those of the preceding paragraph.

(Fig. IX) Once again, let $X = \rho / 4$, $f = 1/4$, and one will have:

$$[(4\rho - 2x')\,x' + \rho^2]^2\,(y^2 + z^2) = (\rho^2 - x'^2)\,[(4\rho - 2x')\,x' + \rho^2]^2, \qquad (SS)$$

$$x = \frac{\rho^2}{2x' - 4\rho}, \qquad y = 0, \qquad z = 0, \qquad (\sigma\sigma)$$

$$x = \frac{3x'\rho^2 - 2\rho^3 - 2x'^3}{6\rho(3\rho - 2x')}, \qquad y^2 + z^2 = \frac{(\rho^2 - x'^2)^3}{9\rho^2(3\rho - 2x')^2}, \qquad (\Sigma' ss)$$

$$x = \rho\left[1 \pm \left(\frac{3}{2}\right)^{1/2}\right], \qquad (SP)$$

$$x' = 2\rho, \qquad y'^2 + z'^2 = -3\rho^2, \qquad (s)$$

$$x' = \frac{3}{2}\rho, \qquad y'^2 + z'^2 = -\frac{5}{4}\rho^2, \qquad (s')$$

$$R = \frac{2\Delta\rho}{x' - 2\rho}, \qquad R' = \frac{2\Delta(x' - 4\rho)}{9\rho - 6x'},$$

$$T = e\,\frac{(D + \Delta)^2\, 4\rho(x' - 4\rho)}{[D(x' - 2\rho) + 2\Delta\rho][D(9\rho - 6x') + 2\Delta(x' - 4\rho)]}.$$

Finally, if $X = 0$, $f = 0$:

$$x'^2\,(y^2 + z^2) = (\rho^2 - x'^2)\, x^2, \qquad (SS)$$

$$x = 0, \qquad y = 0, \qquad z = 0, \qquad (\sigma\sigma)(\mathrm{S}')$$

$$x = \rho\,\frac{1 \pm 1}{0}, \qquad (SP)$$

$$x' = \infty, \qquad y'^2 + z'^2 = -\infty, \qquad (s)(s')$$

$$R = -\Delta, \qquad R' = -\Delta, \qquad T = e.$$

[**19**] The eye that is placed at a point $x = X'$, $y = Y'$, $z = Z'$ might perceive several images of the luminous point on the mirror $F(x', y', z') = 0$. Indeed, since the equation of the reflected ray is:

$$m(z - z') = o(x - x'), \qquad n(z - z') = o(y - y'),$$

one expresses the idea that this ray passes through the eye by the condition:

$$m(Z' - z') = o(X' - x'), \qquad n(Z' - z') = o(Y' - y').$$

Now, if, by means of these two equations and that of the mirror $F(x', y', z')$, one determines the coordinates x', y', z' of the point on the surface that corresponds to the image of the luminous point then one will obtain as many results as one has real values for x', y', z', in such a way that if these equations give four real values for x', y', z' then the luminous point will be perceived by the eye at four different points, and so on, if this result has a number of real roots that is greater or less.

[**20**] If one would like to determine the figure on the mirror $F(x', y', z') = 0$ that will be seen by the eye that is situated at the point $x = X, y = Y, z = Z$ due to a curve $x = fz, y = fz$ then one will eliminate x, y, z from the equations:

$$m(z - z') = o(x - x'), \quad n(z - z') = o(y - y'), \quad x = fz, \;\; y = fz.$$

The result $R = 0$ at x', y', z' will be the equation of a surface that intersects the mirror $F(x', y', z') = 0$ along the desired curve in such a way that upon alternately eliminating x', y', z' from these two equations, one will have the projections of that curve on the three coordinate planes.

If it is a curved surface $\Pi = 0$, $d\Pi = \pi\, dx + \pi'\, dy + \pi''\, dz = 0$, where one may eliminate the apparent contour on the mirror $F(x', y', z') = 0$, then one eliminates x, y, z between the four equations:

$$m(z - z') = o(x - x'), \quad n(z - z') = o(y - y'), \quad \Pi = 0, \quad d\Pi = \pi\, dx + \pi'\, dy + \pi''\, dz = 0,$$

and the result $R = 0$ at x', y', z' will be the equation of a surface that intersects the mirror $F(x', y', z') = 0$ along the desired curve.

These results serve to determine the form of the image, modified by the curvature of the mirror. They might also serve to determine the form that the mirror must have in order for the image of a given body to present itself in a given form after reflection.

Indeed, if we suppose that the eye is placed at the point $x = X, y = Y, z = Z$, the image with which the object is to be seen being given, then one will envelop its contour with a conical surface that has its center at the point $x = X, y = Y, z = Z$, and whose equation will consequently be:

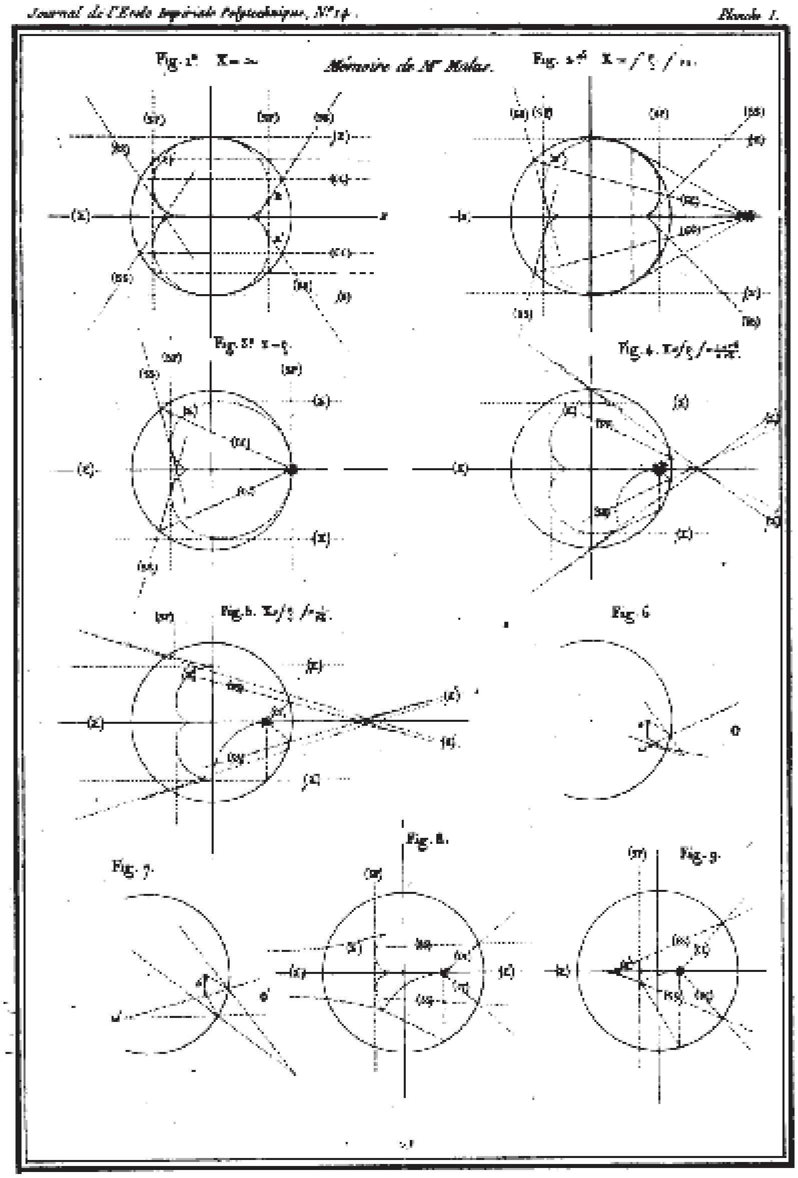
$$\Phi\left(\frac{z' - Z}{x' - X}, \frac{z' - Z}{y' - Y}\right) = 0,$$

Φ being a given function.

Now, if one combines this equation with $R = 0$ then one will obtain the value of p/r and that of q/r at x', y', z' when one substitutes them in $p\, dx' + q\, dy' + r\, dz' = 0$, and the integral of the latter will be the desired equation of the mirror $F(x', y', z') = 0$.

Up to now, we have supposed that R and R' are on the same side of the mirror, so the distance Δ' [**14**] from the image of a luminous point to the point x', y', z' is $\frac{R + R'}{2}$. We observe here that one has $\Delta' = \frac{R + R'}{2}$ only when $R - R'$ is very small with respect to $D + R'$, a condition that is, moreover, necessary in order for the image to be distinct. Indeed, we confirm, by addressing the dioptrics, that the term that expresses the sharpness of the image is of the form $\frac{[\mu(D+R')+\sqrt{(R-R')}]^2}{(R-R')^2}$, in such a way that the smaller that $R - R'$ gets, the more this term approaches infinity, and the more the vision is almost perfectly distinct, which is, in general, the most important case to consider. Meanwhile, we then determine the exact value of Δ'; however, since this calculation demands considerations that depend upon the structure of the eye and the theory of refractions, we have postponed it until the second part of this memoir.

After having presented the theory of dioptrics, we apply the results of our analysis to the construction of optical instruments.


Mémoire de M.r Malus.
Fig. 1.re
Fig. 2.e
Fig. 3.e
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.