"Über die Behandlung von Stabilitätsproblemen mit Hilfe der energetischen Methode," Zeit. angew. Math. Mech. **18** (1938), 57-73.

On the treatment of stability problems with the help of energetic methods (*).

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Abstract. – In the example of the (elastically-supported) compressed rod, it will be shown that the difference between stability problems and the problems of the "linearized" theory of elasticity are based on the fact that for stability problems, the Ansatz is that the deformation energy must include terms of order higher than two in the displacements. That conception of things allows one exhibit the connection between the energy method in the special form that is used most often for stability investigations and the principle of virtual displacements in its *general* elasticity-theoretic version. Moreover, that clarifies the behavior of the rod when one *exceeds* the critical compression.

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1. Introduction. – **Kirchhoff**'s uniqueness theorem says: An elastic body can assume one and only one equilibrium configuration under one and the same external loading. In the formulation of the energetic approach to that: The potential Π of the internal and external forces has one and only one extremal location:

$$\delta\Pi = 0\,, \tag{1.1}$$

and the extremal value is a minimum ([1]).

The uniqueness theorem is true without restriction in the realm of the linearized theory of elasticity, i.e., as long as the stresses σ, τ can be expressed linearly in terms of the distortions γ_{ik}, and the distortions can, in turn, be expressed linearly in terms of the displacements u, v, w. The function Π will then, in fact, be of degree at most two in the displacements, and the geometric picture will then immediately confirm that a "parabola" of degree two (i.e., a positive-definite quadratic form) can possess one and only one minimal location (mechanically speaking: an equilibrium configuration). Things will be different when one proceeds to examine structures whose behavior can no longer be grasped in terms of the linearized distortion-displacement

(*) The investigations in this article are a continuation of the ones that the author was urged to do by **Prof. Trefftz** (during his employment by the Deutschen Versuchsanstalt für Luftfahrt). The author is very thankful for his close collaboration with his colleague **R. Kappus** for a large part of the work (in particular, Sections 5, 6, 9).

([1]) The derivation of the principle of virtual displacements (1.1) for elastic equilibrium is in, e.g., in *Handbuch der Physik* VI, pp. 70, *et seq.* (Berlin 1928). – A careful founding of the general theory of the behavior at the limits of stability is in **E. Trefftz**: "Zur Theorie der Stabilität…" this journal **13** (1933), pp. 160. (For further literature, see *ibidem* in *Handbuch* VI, pp. 277, *et seq.*). The investigations in Section 5 make use of the **Trefftz** approach, in particular.

equations with sufficient accuracy. They are mainly bodies for which one of the dimensions is small compared to the other ones, so shells, plates, or rodlike structures. A *rod*, e.g., can suffer deflections that surpass the thickness by a multiple without the stresses needing to exceed a proportionality limit in the process, and under those circumstances, the square of the (lateral) displacements is definitely no longer small compared to the linear component. However, in that way, the deformation energy (the potential Π, resp.) will have degree higher than two in the displacements, and a higher-order "parabola" can naturally possess several extrema (mechanically: equilibrium configurations).

The problem of the (conventional) theory of stability is to determine the system of external loading under which several *neighboring* equilibrium configurations are possible ("branch" points of the elastic equilibrium). The basis for restricting the investigation to those "critical" points is found in the fact that the differential equations that characterize the elastic behavior in the super-critical domain are generally no longer linear, and an analytical treatment would be only hard to approach, while the problem can still be "linearized" at the branch points ([2]).

However, this purely-*practical* viewpoint has also led to a certain (as we would like to show, unfounded) *systematic* splitting-off of the stability problem from the remaining problems of the theory of elasticity whose mathematical expression is mostly found in a formulation of the principle of virtual displacements that deviates somewhat from the usual, and also occasionally led to the formulation of a special principle ([3]).

The principle of virtual displacements (in its usual formulation in the theory of elasticity) says that the potential of the internal and external forces:

$$\Pi = A_i + V$$

remains unchanged under a virtual (i.e., geometrically-possible) displacement from the equilibrium configuration:

$$\delta\Pi = \delta(A_i + V) = 0\ . \tag{1.2}$$

The potential of the internal forces A_i is given by the deformation energy (internal work), the potential of the external forces is given by the (negative) product of the external forces and the displacements of the points of application ([4]). In the domain of validity of the law of *proportionality*, one has $V = -\,2A_a$, when A_a denotes the work done by external forces between the null state and the final state when one runs through only equilibrium states. One also occasionally writes (1.2) in the form ([5]):

([2]) Moreover, up to now, knowing the super-critical domain was of lesser significance in practice, since "bulging" construction parts would be regarded as inadmissible. It was only in recent years in the shell constructions of aircraft construction that anyone has moved beyond assuming that large overshoots of the critical loads would be constructively inconceivable.

([3]) E. g., **Th. Pöschl**, "Über die Minimalprincipe der Elastizitätstheorie," Bau-Ing. **17** (1936), pp. 160.

([4]) Eq. (1.2) is true independently of whether the field of the external forces does or does not possess a "potential" in the sense of physics. If X_i are the external forces and u_i are their displacements then that can always be written formally as $V = -\sum X_i\, u_i$, i.e., $\delta V = - -\sum X_i\, \delta u_i$. (The forces are *not* varied.)

([5]) See, e.g., **Th. Pöschl**, *loc. cit.*

$$\delta(A_i - 2\,A_a) = 0\,. \tag{1.3}$$

As opposed to that, in stability theory, one often employs ([5]):

$$\delta(A_i - A_a) = 0 \tag{1.4}$$

as the "principle of virtual displacements."

The present article would like to show (in the classical example of the compressed rod) that stability investigations are also most conveniently carried out in connection with the *one* main equation (1.2), in which one consequently must preserve the higher-order terms in the displacements in the expression for the deformation energy. That procedure will become *necessary* from the *practical* standpoint when one wishes to investigate the relationships in the super-critical domain, and will become *desirable* from the *systematic* standpoint, because in that way, it will become clear that one can get along without any additional principle. In particular, the argument will also clarify the apparent contradiction between eqs. (1.3) and (1.4) in a simple fashion.

We shall perform the calculation itself in Sections **2** to **7** in the following way: We first derive the nonlinear distortion-displacement equations of a beam while the keeping the basic assumptions of the elementary theory of beams (cross-sections remain planar, etc.). We then exhibit the expression for the deformation energy A_i and use the demand that:

$$\delta\Pi = 0 \tag{1.5}$$

to arrive at the differential equations for the two components of the displacement, and use the sharper demand that:

$$\Pi = \text{min.} \tag{1.6}$$

to answer the question of the stability of the equilibrium configuration. In conclusion, we will treat the same problem with the help of a Ritz Ansatz for the displacements. In that way, the result of the stability analysis can be made especially beautiful to the *imagination*.

2. Connection between the distortions γ_{ik} and the displacements u, w of the rod centerline ([6]). – If $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ are the unit vectors in the directions of the axes then, with the notations of sec. **1**, the position vector of a point on the centerline of the beam before deformation will be:

$$\mathfrak{r}_0 = x\,\mathfrak{E}_1\,,$$

and

$$\mathfrak{r} = (x + u)\,\mathfrak{E}_1 + w\,\mathfrak{E}_3 \tag{2.1}$$

([6]) Cf., the corresponding calculation for the *plates* in **Marguerre-Trefftz**, "Über die Tragfähigkeit eines Plattenstreifens…," ZAMM **17** (1937), pp. 85.

after the deformation. If $\mathfrak{N}$ is the unit vector of the normal to the deformed centerline then an arbitrary point:

$$\mathfrak{R}_0 = \mathfrak{r}_0 + z\,\mathfrak{E}_3 = x\,\mathfrak{E}_1 + z\,\mathfrak{E}_3$$

will go to:

$$\mathfrak{R} = \mathfrak{r} + z\,\mathfrak{N} = (x+u)\,\mathfrak{E}_1 + w\,\mathfrak{E}_3 + z\,\mathfrak{N}\,, \tag{2.2}$$

from the basic assumption of the theory of beam bending.

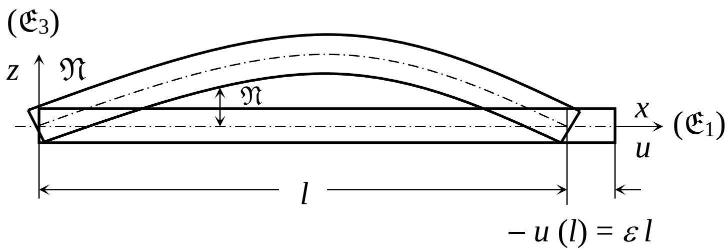

Figure 1. Compressed beam.

We get the unit vector $\mathfrak{N}$ and its derivative $\mathfrak{N}_x$ from:

$$\mathfrak{N} = \frac{\mathfrak{r}_x \times \mathfrak{E}_2}{|\,\mathfrak{r}_x\,|} \tag{2.3}$$

and

$$\mathfrak{N}_x = a\,\mathfrak{r}_x\,.$$

The proportionality factor a is implied by the fact that:

$$\mathfrak{N}\,\mathfrak{r}_x = 0\,, \tag{2.4}$$

i.e.:

$$-\,\mathfrak{N}\,\mathfrak{r}_{xx} = \mathfrak{N}_x\,\mathfrak{r}_x = a\,\mathfrak{r}_x^2\,,$$

so:

$$a = -\,\frac{\mathfrak{N}\,\mathfrak{r}_{xx}}{\mathfrak{r}_x^2} = -\,\frac{(\mathfrak{r}_x \times \mathfrak{E}_2)\,\mathfrak{r}_{xx}}{|\,\mathfrak{r}_x\,|^3} = \frac{(\mathfrak{r}_x\,\mathfrak{r}_{xx}\,\mathfrak{E}_2)}{|\,\mathfrak{r}_x\,|^3}\,.$$

One will then have:

$$\mathfrak{N}_x = \frac{(1+u_x)\,w_{xx} - w_x\,u_{xx}}{[(1+u_x)^2 + w_x^2]^{3/2}}\,\mathfrak{r}_x\,, \tag{2.5}$$

and therefore:

$$\mathfrak{R}_x = \mathfrak{r}_x + z\,\mathfrak{N}_x = \mathfrak{r}_x\left(1 + z\,\frac{(1+u_x)\,w_{xx} - w_x\,u_{xx}}{[(1+u_x)^2 + w_x^2]^{3/2}}\right), \tag{2.6}$$

$$[\mathfrak{r}_x = (1+u_x)\,\mathfrak{E}_1 + w\,\mathfrak{E}_3].$$

We obtain the component γ_{11} of the distortion tensor from a consideration of the line element. The square of the distance between two points with the coordinate differences dx, dz is:

$$ds_0^2 = dx^2 + dz^2$$

before the deformation and:

$$ds^2 = d\,\mathfrak{R}^2 = (\mathfrak{R}_x\,dx + \mathfrak{N}\,dz)^2 = \mathfrak{R}_x^2\,dx^2 + dz^2 \tag{2.7}$$

after it. The change in the coefficient of dx^2:

$$\gamma_{xx} = \mathfrak{R}_x^2 - 1$$

implies the (nonlinear) component of the elastic distortion (viz., the "elongation") by means of ([7]):

$$\tfrac{1}{2}\gamma_{11} = \sqrt{1+\gamma_{xx}} - 1 = \tfrac{1}{2}\,\gamma_{xx} - \tfrac{1}{8}\,\gamma_{xx}^2 \pm \cdots, \tag{2.8}$$

so from (2.6), we have:

$$\gamma_{xx} = (2u_x + u_x^2 + w_x^2) + 2\,z\,\frac{(1+u_x)\,w_{xx} - w_x\,u_{xx}}{[(1+u_x)^2 + w_x^2]^{1/2}} + z^2\left[\frac{(1+u_x)\,w_{xx} - w_x\,u_{xx}}{(1+u_x)^2 + w_x^2}\right]^2, \tag{2.9}$$

and γ_{11} can be calculated from (2.8) and (2.9) up to the desired degree of precision.

3. Deformation energy. Potential of the internal and external forces. – As in the "ordinary" theory of beams, we start from the assumption that the part of the deformation energy that originates in the lateral forces is small compared to the part that goes back to the longitudinal forces, and with:

$$\sigma_x = \tfrac{1}{2}\,E\,\gamma_{11}, \tag{3.0}$$

we set:

$$A_i = \tfrac{1}{2}\,E\iiint\left(\frac{\gamma_{11}}{2}\right)^2 dx\,dy\,dz = \tfrac{1}{2}\,E\iiint\left[\left(\frac{\gamma_{xx}}{2}\right)^2 - \left(\frac{\gamma_{xx}}{2}\right)^2 + \cdots\right] dx\,dy\,dz\ . \tag{3.1}$$

The internal work in the elastic domain is then given completely with (3.1), together with (2.9). We will obtain a form that is useful in calculation when we develop it in powers of the differential quotients of u and w. In order to be able to perform the order of magnitude consideration that is necessary for us to do that comfortably, we write γ_{xx} in the form:

$$\tfrac{1}{2}\gamma_{xx} = \psi_0 + z\,\psi_1 + z^2\,\psi_2$$

([7]) *Handbuch der Physik*, VI, pp. 57.

and perform the integrations over y and z in (3.1). If F then means the cross-sectional area, i is its radius of inertia, l is the length of the beam, and λ_4, λ_6, … are certain numbers that depend upon the form of the cross-section and have order of magnitude unity ([8]) then the yz-integrals can be written in the following forms:

$$\iint dy\,dz = F\,, \qquad \iint z^2\,dy\,dz = i^2 F\,, \qquad \iint z^4\,dy\,dz = \lambda_4\,i^4 F\,, \qquad \iint z^6\,dy\,dz = \lambda_6\,i^6 F\,,$$

and for A_i, one has:

$$\left.\begin{aligned} A_i = \;& \tfrac{1}{2} EF \int_0^l \{[\psi_0^2 + i^2(\psi_1^2 + 2\psi_0\,\psi_2) + \lambda_4\,i^4\,\psi_2^2] \\ & -[\psi_0^2 + 3i^2(\psi_0\,\psi_1^2 + \psi_0^2\,\psi_2) + 3\lambda_4\,i^4(\psi_0\,\psi_2^2 + \psi_1^2\,\psi_2) + \lambda_6\,i^6\,\psi_2^3] + \cdots\}\,dx. \end{aligned}\right\} \tag{3.2}$$

Now, one knows that the compression u_x has order of magnitude $\left(\dfrac{i^2}{l^2}\right)$ at critical points. If we assume that the greatest deflection (the "sag" = *Biegepfeil*) f has the same order of magnitude as i then $w_x^2 \left(\sim\left[\dfrac{f^2}{l^2}\right]\right)$ will have order $\left[\dfrac{i^2}{l^2}\right]$. The increase that u_x experiences as a result of the bending is (since it must be independent of the sign of f) proportional to f^2, so it must have the same order $\left[\dfrac{f^2}{l^2}\right] = \left[\dfrac{i^2}{l^2}\right]$ as the critical value of u_x. The total compression per unit length will then have the same order of magnitude as w_x^2, while u_x^2 is small compared to that magnitude.

If we assume that viewpoint and break off after the terms of order $\left[\left(\dfrac{i^2}{l^2}\right)^3\right]$ then that will give:

$$\left.\begin{aligned} A_i &= \tfrac{1}{2} E\,F \int_0^l \Big\{\Big[\left(u_x + \tfrac{1}{2} w_x^2\right)^2 + u_x^2\left(u_x + \tfrac{1}{2} w_x^2\right) - \left(u_x + \tfrac{1}{2} w_x^2\right)^3\Big] \\ &\quad + i^2\Big[w_{xx}^2\left(1 + u_x - \tfrac{1}{2} w_x^2\right)^2 - 2 w_x\, w_{xx}\, u_{xx} - 3 w_{xx}^2\left(u_x + \tfrac{1}{2} w_x^2\right)\Big]\Big\}\,dx \\ &= \tfrac{1}{2} E\,F \int_0^l \Big\{\Big[\left(u_x + \tfrac{1}{2} w_x^2\right)^2 (1 - w_x^2) + \left(u_x + \tfrac{1}{2} w_x^2\right)\left(\tfrac{1}{2} w_x^2\right)^2\Big] \\ &\quad + i^2\,[w_{xx}^2\,(1 - 2u_x - 2 w_x^2) - 2\, w_x\, w_{xx}\, u_{xx}]\}\,dx. \end{aligned}\right\} \tag{3.3}$$

That expression for A_i is associated with a mean value for the stress:

([8]) E.g., for a rectangular cross-section $\lambda_4 = 3 / 20$, $\lambda_6 = 9 / 28$, …

$$\bar{\sigma}_x = \frac{1}{F}\int \sigma_x \, dF = E\left[\left(u_x + \tfrac{1}{2}w_x^2\right)\left(1 - \tfrac{1}{2}w_x^2\right) + \tfrac{1}{8}w_x^4\right] . \tag{3.4}$$

One remarks that the higher powers of i^4, i^6, etc., drop out in (3.3), so the cross-section functions λ_4, λ_6, … that were introduced *do not* enter into the calculations to that degree of approximation. The expression (3.3) is then true for arbitrary cross-section forms.

The study of the elastic behavior on the basis of the Ansatz (3.3) shall be carried out in Section **9**. It will be shown there that keeping the sixth-order terms, along with the fourth-order ones, means a refinement that is generally irrelevant in *practice*. We shall then initially restrict ourselves to the fourth-order terms (the second for $\bar{\sigma}_x$) and set:

$$A_i = \tfrac{1}{2} E F \left[\int_0^l \left(u_x + \tfrac{1}{2}w_x^2\right)^2 dx + i^2 \int_0^l w_{xx}^2 \, dx\right] , \tag{3.5}$$

$$\bar{\sigma}_x = E\left(u_x + \tfrac{1}{2}w_x^2\right) . \tag{3.6}$$

If ε means the compression per unit length, and $P = p\,F = -\,\bar{\sigma}_x\,F$ is the compressive force then $-(p\,F)(\varepsilon\,l)$ will be the potential of the external forces, when referred to the unstressed initial state, and the total potential will be Π (when we likewise write it for the somewhat-more-general case of aa rod with an elastic support $K\,w$):

$$\Pi = \tfrac{1}{2} E F \int_0^l \left(u_x + \tfrac{1}{2}w_x^2\right)^2 dx + \tfrac{1}{2} E I \int_0^l w_{xx}^2 \, dx + \tfrac{1}{2} K F \int_0^l w^2 \, dx - p\,F\,\varepsilon\,l . \tag{3.7}$$

The mean potential energy per unit length (divided by $E\,F$) can then be represented (with $K / E = k$) in the form:

$$\hat{\Pi} = \frac{\Pi}{E\,F\,l} = \frac{1}{2l}\int_0^l \left(u_x + \tfrac{1}{2}w_x^2\right)^2 dx + \frac{i^2}{2l}\int_0^l w_{xx}^2 \, dx + \frac{k}{2l}\int_0^l w^2 \, dx - \frac{p}{E}\varepsilon . \tag{3.8}$$

From (3.8), together with the demand that:

$$\hat{\Pi} = \text{min.}, \tag{3.9}$$

we will then arrive at all statements about the behavior of the rod at and *beyond* the stability limit.

4. Differential equations for the displacements *u*, *w*. – We imagine a beam that is rigidly articulated on the left ($x = 0$) and free to move horizontally on the left under the influence of a central compressive force P (see Fig. 1). We consider the givens (i.e., the independent variables in our problem) to be:

either the horizontal displacement of the right-hand endpoint or the compressive stress

$$u(l) = -\varepsilon\, l \qquad\qquad P = \frac{P}{F}\,.$$

From the demands of the principle of virtual displacements, the displacements at constant load shall be varied in such a way that the varied position is compatible with the geometric conditions. If one boundary point of the displacement (so, e.g., ε in our problem) is prescribed then one fixes the point under the variation in order that the work done by the (unknown!) external forces should remain in the calculation. By contrast, if the force is given then the (geometrically unconstrained) endpoint will vary with it. The work done by the external load ($P\ \delta u$ in our problem) will then enter into the calculation.

If we vary the displacement u in (3.8) [i.e., we perform a virtual displacement in the direction of the rod-axis on the beam elements, while fixing the boundary point (the load, resp.)] then from the rules of the calculus of variations, we will get:

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(u_x + \tfrac{1}{2} w_x^2\right) = 0\,, \\ &\text{with the boundary conditions:} \\ &u(0) = 0; \quad u(l) = -\varepsilon\, l\,; \;\Big|\; u(0) = 0\,, \quad \left[u_x + \tfrac{1}{2} w_x^2 + \frac{p}{E}\right]_{x=l} = 0\,. \end{aligned}\right\} \tag{4.1}$$

Varying w will give (independently of whether ε or p is thought to have been given):

$$\left.\begin{aligned} &-w_x \frac{\partial}{\partial x}\left(u_x + \tfrac{1}{2} w_x^2\right) - \left(u_x + \tfrac{1}{2} w_x^2\right) w_{xx} + i^2 w_{xxxx} + k\, w = 0\,, \\ &\text{with the boundary conditions:} \\ &w(0) = w(l) = w_{xx}(0) = w_{xx}(l) = 0\,. \end{aligned}\right\} \tag{4.2}$$

The exact integration of the system (4.1), (4.2) of simultaneous nonlinear differential equations poses no difficulty here. It follows from (4.1) that:

$$u_x + \tfrac{1}{2} w_x^2 = \text{const.} = -\varepsilon_0\,, \tag{4.3}$$

$$u = u(0) - \varepsilon_0\, x - \int_0 \tfrac{1}{2} w_x^2\, dx\,,$$

and with the use of the relation (4.3), (4.2) will assume the form:

$$w_{xxxx} + \frac{\varepsilon_0}{i^2} w_{xx} + \frac{k}{i^2} w = 0 . \tag{4.4}$$

The solution to that linear equation with constant coefficients reads:

$$w = f \sin \kappa_0 \, x + g \cos \kappa_0 \, x + f_1 \sin \kappa_1 \, x + g_1 \cos \kappa_1 \, x ,$$

in which κ_0, κ_1 are the positive roots of the biquadratic equation:

$$\kappa^4 - \frac{\varepsilon_0}{i^2} \kappa^2 + \frac{k}{i^2} = 0 , \tag{4.5}$$

which are:

$$\kappa_{0,1}^2 = \frac{1}{2} \frac{\varepsilon_0}{i^2} \pm \sqrt{\frac{1}{4} \frac{\varepsilon_0^2}{i^4} - \frac{k}{i^2}} .$$

In order to determine the six integration constants f, g, f_1, g_1, u (0), ε_0, we have the six boundary conditions:

$$g + g_1 = 0, \qquad \kappa_0^2 \, g + \kappa_1^2 \, g_1 = 0,$$
$$f \sin \kappa_0 \, l + f_1 \sin_1 \kappa_1 \, l = 0, \quad f \kappa_0^2 \sin \kappa_0 \, l + f_1 \kappa_1^2 \sin_1 \kappa_1 \, l = 0,$$

$$u\,(0) = 0 ,$$

and

$$-\varepsilon \, l = u\,(0) - \varepsilon_0 \, l - \int_0^l \tfrac{1}{2} w_x^2 \, dx , \qquad \Big| \qquad -\varepsilon_0 + \frac{p}{E} = 0. \tag{4.6}$$

That yields ([9]):

$$f_1 = g_1 = g = u\,(0) = 0 ,$$

and *either:*

$$f = 0 , \qquad w = 0 ,$$

$$\left. \begin{array}{ll|ll} \varepsilon_0 = \varepsilon, & & \varepsilon_0 = \dfrac{p}{E}, \\ u = -\varepsilon \, x, & & u = -\dfrac{p}{E} x \end{array} \right\} \tag{4.7}$$

or ([9]):

([9]) We exclude the possibility that $f = g = g_1 = 0$, $f_1 \neq 0$, since the solution $w = f_1 \sin \kappa_1 x$, with:

$$\kappa_1^2 = \frac{1}{2} \cdot \frac{\varepsilon_0}{i^0} - \sqrt{\frac{1}{4} \frac{\varepsilon_0^2}{i^4} - \frac{k}{i^2}} ,$$

will become meaningless in the special case where $k \to 0$.

$$f \neq 0\,, \qquad w = f \sin \kappa_0 x\,, \qquad \kappa_0 = \frac{\pi}{l}\,,$$

$$\varepsilon_0 = \frac{i^2 \pi^2}{l^2} + k\frac{l^2}{\pi^2} \equiv \varepsilon^4\,, \tag{4.8}$$

$$u = -\,\varepsilon^* x + \frac{\pi}{8}\frac{f^2}{l}\sin\frac{2\pi x}{l}\,. \tag{4.9}$$

As one sees, that will then give two types of equilibrium configurations: The straight one ($f = 0$) and the curved one ($f \neq 0$). The *straight* configuration will be characterized uniquely by *p or e*, while the curved one will probably be characterized uniquely by ε [because the amplitude f of the deflection can be determined by the first eq. (4.6)]:

$$\frac{\pi^2 f^2}{4l^2} = \varepsilon - \varepsilon^4\,, \tag{4.9′}$$

but not by p. Rather, it follows from the second of eqs. (4.6) that p itself cannot exceed a certain "critical" value:

$$p = E\,\varepsilon^* \equiv p^* \tag{4.9″}$$

when $f \neq 0$ ([10]). p will then be unsuitable as an independent parameter [as opposed to the situation in the corresponding plate problem ([11])]. One sees from (4.9) that f will assume non-zero *real* values only when $\varepsilon > \varepsilon^*$.

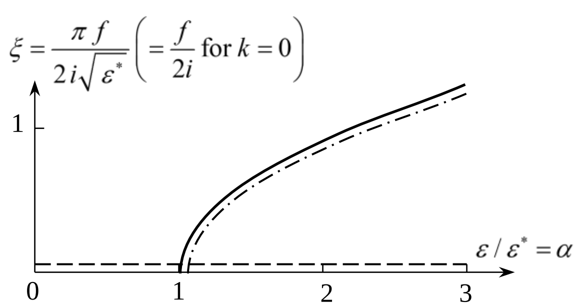

Figure 2. Sag f above ε.

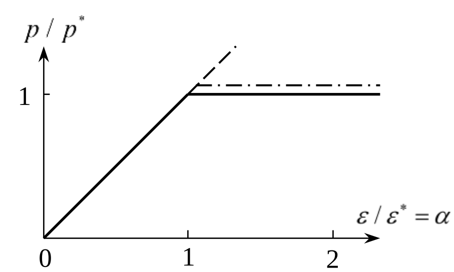

Figure 3. Load P above ε.

The dependency of the quantities f and p on ε is illustrated in Figs. 2 and 3. The solid lines are true for the solution (4.9), while the dashed lines are true for (4.7). (The dotted and dashed line suggests the result of the refined calculation in Section **9** in a grossly-magnified unit of measurement.)

([10]) In Section 9, we will see that this statement will be applicable only as long as the Ansatz (3.5) exists. On the basis of the refined Ansatz (3.3), that would imply a (generally very slight) raising of the load beyond (4.9″).

([11]) **K. Marguerre**, Luft.-Forsch. (1937), pp. 124, above.

5. Stable and labile equilibrium. – In the "ordinary" theory of elasticity, when one determines the equilibrium states, one can restrict oneself to the demand that:

$$\delta\Pi = 0\,, \qquad \text{i.e.,} \qquad \Pi = \text{extrem.,} \tag{5.1}$$

since the supplementary condition that $\delta^2\Pi > 0$ (mechanically: the stability of the equilibrium problem) is assured once and for all in that case as a result of linearization ([12]). In the present problem, however, we must expressly impose the minimum requirement, since it is only when:

$$\delta\Pi = 0\,, \qquad \delta^2\Pi > 0\,, \qquad \text{i.e.,} \qquad \Pi = \text{min.} \tag{5.2}$$

that the stable equilibrium configurations will be distinguished from the other possible (viz., labile). We specify the concept of stability by the following conventions ([13]):

1. We call an equilibrium state *stable* when the potential energy assumes a greater value for *all* neighboring states.

2. We an equilibrium state *labile* (unstable) when there exists *at least one* neighboring state that belongs to a smaller value of potential energy.

3. We speak of the *limits of stability* (i.e., an *indifferent* equilibrium state) when there is *at least one* neighboring state, in addition to the given equilibrium state, whose potential energy is just as large, but *none* for which it is smaller.

We recall (3.8) [(3.9), resp.]:

$$\hat{\Pi} = \frac{1}{2l}\int_0^l \left(u_x + \tfrac{1}{2}w_x^2\right)^2 dx + \frac{i^2}{2l}\int_0^l w_{xx}^2\, dx + \frac{k}{2l}\int_0^l w^2\, dx - \frac{p}{E}\varepsilon = \text{min.}$$

and "vary" that, i.e., we replace u with $u + \delta u$ and w with $w + \delta w$. When we order the result in powers of δu, δw and truncate after the second-order terms, that will then give:

$$\begin{aligned}\Delta\hat{\Pi} &= \hat{\Pi}(u+\delta u, w+\delta w) - \hat{\Pi}(u,w) = \delta\hat{\Pi} + \tfrac{1}{2}\delta^2\hat{\Pi} + \cdots = -\frac{1}{l}\frac{p}{E}\,\delta u\Big|_{x=l} \\ &+ \frac{1}{l}\int_0^l \left(u_x + \tfrac{1}{2}w_x^2\right)^2 \delta u_x\, dx + \frac{1}{l}\int_0^l \left[\left(u_x + \tfrac{1}{2}w_x^2\right) w_x\,\delta w_x + i^2\, w_{xx}\,\delta w_{xx} + k\, w\,\delta w\right] dx \\ &+ \frac{1}{2l}\int_0^l (\delta u_x)^2\, dx + \frac{1}{2l}\int_0^l \left[\left(u_x + \tfrac{1}{2}w_x^2 + w_x^2\right)(\delta w_x)^2 + i^2\,(\delta w_{xx})^2 + k\,(\delta w)^2\right] dx + \cdots\end{aligned}$$

([12]) *Handbuch der Physik*, VI, pp. 71 and 72.
([13]) The following definition was given (in spirit) by **E. Trefftz** occasionally in his lectures at DVL.

The demand that the *first-order terms* should vanish leads to eqs. (4.1) and (4.2). The question of stability will be answered by the *second-order* terms ([14]). When we replace u, w with the ones that result from $\delta\Pi = 0$, that, along with (4.7) (viz., the straight configuration), will yield:

$$(\delta^2 \hat{\Pi})_1 \equiv (\delta^2 \hat{A}_i)_1 = \frac{1}{l}\int_0^l [-\varepsilon(\delta w_x)^2 + i^2(\delta w_{xx})^2 + k(\delta w)^2]\,dx + \frac{1}{l}\int_0^l (\delta u_x)^2\,dx \;, \tag{5.3}$$

and with (4.9) (bent configuration):

$$(\delta^2 \hat{\Pi})_2 \equiv (\delta^2 \hat{A}_i)_2 = \frac{1}{l}\int_0^l \left[\left(-\varepsilon^* + \frac{\pi^2}{l^2} f^2 \cos^2\frac{\pi x}{l}\right)(\delta w_x)^2 + i^2(\delta w_{xx})^2 + k(\delta w)^2\right]dx + \frac{1}{l}\int_0^l (\delta u_x)^2\,dx \,. \tag{5.4}$$

We next determine the stability *limits*. From the definition that was stated above, there shall be *one* state at the stability limit for which the second variation vanishes, but *no state* for which it is negative: The value zero is then the *smallest* value that $\delta^2\Pi$ can assume at all at the limit points. Therefore, if $\delta^2\Pi$ has certain continuity properties (whose existence is explained on mechanical grounds) then the "distinguished" value $\delta^2\Pi = 0$ is, at the same time, analytically a *minimum* compared to the neighboring values, and the associated ("distinguished") system of displacements δu, δw will be determined from the condition that:

$$\delta(\delta^2\Pi) = 0 \,. \tag{5.5}$$

In our case, that demand leads to the two differential equations:

$$(\delta u)_{xx} = 0 \,, \tag{5.6}$$

$$(\delta w)_{xxxx} + \frac{\varepsilon}{i^2}(\delta w)_{xx} + \frac{k}{i^2}(\delta w) = 0 \,, \tag{5.7}$$

with the boundary conditions:

$$\left.\begin{array}{ll} & \delta w(0) = \delta w(l) = \delta w_{xx}(0) = \delta w_{xx}(l) = \delta u(0) = 0, \\ \text{and} & \delta u(l) = l \,, \quad | \quad \delta u(l) = 0. \end{array}\right\} \tag{5.8}$$

The system of solutions that satisfies the boundary conditions reads:

([14]) In the case before us (as is often the case in applications), the fact that the displacement of the point of application of the force is an *independent* variable does not enter into the second variation of the *external* force potential:

$$\Delta V = \Delta(P\,u) = P\,\Delta u = P\,\delta u\,, \qquad \text{so} \qquad \delta^2 V \equiv 0\,.$$

$$\delta u = 0 \,, \qquad \delta w = \delta f \sin \frac{\pi x}{l} \,. \tag{5.9}$$

The amplitude $\delta f \neq 0$ remains undetermined (viz., an eigenvalue problem!), and (5.7) will imply that the critical value of ε is:

$$\varepsilon_k = \frac{\pi^2 \cdot i^2}{l^2} + k \frac{l^2}{\pi^2} \,,$$

i.e., precisely the expression $\varepsilon_k = \varepsilon^*$ that is characterized by the branch point for equilibrium: The stability limit and the branch point coincide.

The questions of stability of the equilibrium configuration below and above the limit can now be answered immediately:

1. Since the stretched configuration [see (5.3)] is stable for sufficiently-small ε, it is also stable for $\varepsilon \leq \varepsilon^*$ on the grounds of continuity.

2. For $\varepsilon > \varepsilon^*$, the *stretched* configuration is *unstable*, since we can give *one* δw, namely:

$$\delta w = \delta f \sin \frac{\pi x}{l} \,,$$

for which $\delta^2 \Pi < 0$.

3. For $\varepsilon > \varepsilon^*$, the *curved* configuration is *stable*, because one has:

$$\frac{1}{l} \int_0^l [-\varepsilon (\delta w_x)^2 + i^2 (\delta w_{xx})^2 + k (\delta w)^2] \, dx \geq 0$$

for the one component in (5.4), as one might also assume of δw, and that is added to a positive term:

$$\frac{\pi^2}{l^3} \int_0^l f^2 \cos^2 \frac{\pi x}{l} (\delta w)^2 \, dx \,.$$

6. Deriving the usual Ansätze by the procedure that was given here. – The investigation of the behavior at the stability limit that was carried out in Section **5** allows one to exhibit the connection between the usual stability considerations and the argument that was presented here systematically.

Ordinarily, in stability analysis, one restricts oneself to examining the behavior at the stability *limit*. One assumes that an equilibrium figure (the line, in our case) is *known* and examines only the "transition" to the curved ones. In that way, one imagines that *either* the ends of the rod are absolutely fixed, so one interprets the transition as a purely-internal rearrangement of the energy, *or* that one imparts an infinitesimal displacement to the rod-end, i.e., one imagines that external work has been performed on it that the system will absorb as internal energy. In formulas:

$$A_i = 0 \qquad\qquad | \qquad\qquad A_i - A_a = 0 \, . \tag{6.1}$$

In that way, one understands A_a and A_i in both cases to mean the total work *minus* the amount of work that was expended (stored up, resp.) before attaining the stability limit. If one denotes the (desired!) critical compression $-u_x$ by ε_0 and the critical stress by p_0 then:

$$A_i = \tfrac{1}{2} EF \left[i^2 \int_0^l w_{xx}^2 \, dx + k \int_0^l w^2 \, dx - \varepsilon_0 \int_0^l w_x^2 \, dx \right], \qquad \left| \qquad \left. \begin{aligned} A_i &= \tfrac{1}{2} EF \left[i^2 \int_0^l w_{xx}^2 \, dx + k \int_0^l w^2 \, dx \right], \\ A_0 &= \tfrac{1}{2} F \, p_0 \int_0^l w_x^2 \, dx . \end{aligned} \right\} \right. \tag{6.2}$$

One obtains the first formula in a way that is similar to the way that one obtained (3.7) above when one drops the terms in w_x^4 (restriction to "small" deflections) and the second one from the demand that no new extension energy should be performed during the transition from the straight to the curved configuration, i.e., that the bent beam should have the same length as the stretched one. It will then follow that the ends must be closer to each other as a result of the bending by a displacement:

$$u_1 = - \tfrac{1}{2} \int_0^l w_x^2 \, dx ,$$

such that the external force will do an amount of work $- p_0 \, F \, u_1 = \tfrac{1}{2} \, p_0 \, F \int_0^l w_x^2 \, dx$.

If one now defines the stability limit to be the place at which the condition (6.1) is fulfilled for *one* well-defined displacement $w(x)$, while the following inequality is true for the other one:

$$A_i > 0 \qquad\qquad | \qquad\qquad A_i - A_a > 0 \, ,$$

then A_i ($A_i - A_a$, resp.) will assume its smallest value for that $w(x)$, and one must then have:

$$\delta A_i = 0 \qquad\qquad | \qquad\qquad \delta A_i - \delta A_a = 0 \, . \tag{6.3}$$

The double statement:

$$\begin{aligned} A_i &= \tfrac{1}{2} EF \left[i^2 \int_0^l w_{xx}^2 \, dx + k \int_0^l w^2 \, dx - \varepsilon_0 \int_0^l w_x^2 \, dx \right] \\ &= \min. = 0, \end{aligned} \qquad \left| \qquad \left. \begin{aligned} A_i - A_0 = \tfrac{1}{2} EF \Bigg[& i^2 \int_0^l w_{xx}^2 \, dx + k \int_0^l w^2 \, dx \\ & - \frac{p_0}{E} \int_0^l w_x^2 \, dx \Bigg] = \min. = 0 \end{aligned} \right\} \right. \tag{6.4}$$

is free of contradictions, since the expressions (6.4) are *homogeneous* quadratic functions of w, from **Euler**'s theorem on homogeneous functions ([15]).

The connection between the *first* of eqs. (6.4) and our previous argument [eq. (5.3)] is obvious. Eq. (6.4) says nothing but the fact that the second variation should have a minimum of value 0, except that simply w is written in place of δw in (6.4). That notation also has a factual basis, in addition to its convenience: The distinguished displacement δw (which will make $\delta^2\Pi = 0$) is not only geometrically possible (i.e., virtual) at the stability *limit*, but also mechanically possible, i.e., the displacement that actually *occurs without* varying the *external* load. Nonetheless, the notation δw unconditionally offers some advantages:

1. Because the δ-symbol expresses the connection with the second variation (as we could show above, it is not f itself that remains "undetermined" when one prescribes ε, but only δf).

2. Because the more convenient notation (formally) complicates the transition to the super-critical region, since the infinitely-small displacements *at* the limit and the finite displacements *beyond* the limit must be expressed by the same symbol.

The connection between the *second* of eqs. (6.4) and our previous arguments can be exhibited in the following way: We inferred from the demand that the *second*-order terms in δu, δw should produce a minimum value of zero that $\delta u = 0$ [eq. (5.9)]. However, the condition $\delta^2\Pi = \delta(\delta^2\Pi) = 0$ will also be satisfied when δu is small of *higher* order than δw, so e.g., $\sim (\delta w)^2$. The non-zero term $(\delta u)^2$ no longer belongs to the *second*-order terms at all and remains unaffected by the calculation that was performed above. On the other hand, the boundary term:

$$\left(u_x + \tfrac{1}{2} w_x^2\right)\delta u\,\Big|_{x=l} + \frac{p}{E}\,\delta u\,\Big|_{x=l}$$

must be counted among the second-order terms [cf., (4.1)] such that with the special (basically arbitrary) Ansatz:

$$\delta u_x = -\tfrac{1}{2}(\delta w_x)^2\,, \qquad \text{i.e.,} \qquad \delta u\,|_{x=l} = -\tfrac{1}{2}\int_0^l (\delta w_x)^2\,dx\,, \tag{6.5}$$

one will get:

$$(\delta^2\Pi)_1 = \frac{1}{l}\int_0^l [-\varepsilon(\delta w_x)^2 + i^2(\delta w_{xx})^2 + k(\delta w)^2]\,dx - \frac{1}{l}\left(u_x + \tfrac{1}{2}w_x^2\right)\int_0^l (\delta w_x)^2\,dx - \frac{p}{E\,l}\int_0^l (\delta w_x)^2\,dx\ ,$$

in place of (5.3). At the stability limit, one has $-\varepsilon = u_x + \tfrac{1}{2}w_x^2 = -\varepsilon^*$, since the first and fourth summands cancel. What then remains (with $p = p_0$) is:

([15]) In approximate calculations for the determination of the critical load (viz., the Ritz Ansatz), one often follows the **Timoshenko** procedure and appeals to only the condition that $A_i - A_a = 0$ *explicitly*.

$$(\delta^2 \hat{\Pi})_1 = \frac{i^2}{l}\int_0^l (\delta w_{xx})^2\,dx + \frac{k}{l}\int_0^l (\delta w)^2\,dx - \frac{p_0}{E\,l}\int_0^l (\delta w_x)^2\,dx\ , \tag{6.6}$$

i.e., precisely the expression (6.4), except that it is written in terms of δw, instead of w.

With that (undoubtedly somewhat artificial) interpretation, the second Ansatz in (6.4):

$$\boldsymbol{A}_i - \boldsymbol{A}_a = \delta(\boldsymbol{A}_i - \boldsymbol{A}_a) = 0 \tag{6.7}$$

can be justified by the "extended" principle of virtual displacements. Nonetheless, and despite the "intuitiveness" of the usual derivation, the notation (6.7) is not advisable, on the following grounds:

1. The expression $\boldsymbol{A}_a = -\,p\,F\,u\,|_{x=l}$ is only a function of the displacement w directly, and the use of the δ-symbol for a 'variation," by which δu must be first expressed in terms of w by means of:

$$u = -\tfrac{1}{2}\int_0 w_x^2\,dx\ ,$$

contradicts the notation of the calculus of variations, in which δu and δw must be treated as being mutually independent.

2. In complicated problems, it is often very difficult to write down the expression for the external work with the help of "intuition" directly, and since the splitting of the extension work into an external and an internal part is still arbitrary by an Ansatz of the type of (6.5), one will often prefer the equations:

$$\boldsymbol{A}_i = 0\ , \qquad \delta\boldsymbol{A}_i = 0 \qquad \text{[i.e., properly-speaking, } \delta^2 A_i = 0,\ \delta(\delta^2 A_i) = 0]$$

in the spirit of the *first* Ansatz, even on *practical* grounds.

A third objection concerns the question of whether one should apply the variation symbol δ to an "external work" $\boldsymbol{A}_a$ at all. That is because, *in general*, the external work can be expressed in terms of the forces and displacements in *several* ways: On those grounds, the notation (1.3), in particular, for the principle of virtual displacements is definitely unsuitable ([16]). However, from that viewpoint, there are no objections to the notation (6.7), because since the load p_0 changes by only *higher*-order quantities in the lateral displacement w being varied under the transition from the straight to the curved line, p_0 will not be affected by the variation symbol. That is, we can replace $\boldsymbol{A}_a$ in with a potential difference $\boldsymbol{V}$ directly in (6.7), and if one would like to overlook the

([16]) Thus, e.g., for the compressed rod below the critical point, twice the external work can be written in three forms: $E\,\varepsilon^2$, $p\,\varepsilon$, p^2 / E, which are to be varied (relative to ε!). One really means the second form in that, but that is no longer clearly recognizable, as a result of using the notation $2A_a$ instead of $-V$ [see footnote ([4])].

objection that was formulated in 1. then the δ-symbol in $\delta\boldsymbol{A}_a = -\delta\boldsymbol{V}$ will definitely be unambiguous: That demands the variation of the lateral displacement while fixing the external load.

With the use of the symbol V, (6.7) will read:

$$\boldsymbol{A}_i + \boldsymbol{V} = \delta(\boldsymbol{A}_i + \boldsymbol{V}) = 0\,. \tag{6.8}$$

That notation has the advantage that it makes it clear to what extent the first part of that expression expresses something different from the usual law of energy $A_i - A_a = 0$. It has the disadvantage that the formal similarity between the second part of the state and the principle (1.2) makes it tempting to refer to formula (6.8) (as often happens in the applications) as a special form of the principle of virtual displacements. It emerges from our derivation that this is *not* the case: The principle (1.2) answers the question of the equilibrium configuration that results under *given* loads (or boundary conditions). By contrast, formula (6.8) produces the second equilibrium form that is possible at the branch point, and the *desired* value of the load at which the equilibrium begins to be multi-valued.

When one appeals to the Ansatz (6.7) [(6.8), resp.] for investigating the branch points of the elastic equilibrium (which is not inconvenient for practical calculations in many cases), one should avoid the terminology “principle of virtual displacements” [which will also eliminate the “contradiction” between formulas (1.3) and (1.4) that was mentioned to begin with] in all cases.

7. Interpreting the results up to now with the help of a Ritz Ansatz. – The results of Sections **4** and **5** can be illustrated very beautifully when we convert the variational problem into an ordinary minimum problem by employing the **Ritz** method. We choose the solutions of the differential equations (4.2) and (4.1):

$$w = f\sin\frac{\pi x}{l}\,, \qquad u_x + \tfrac{1}{2}w_x^2 = -\,\varepsilon + \frac{\pi^2}{4}\frac{f^2}{l^2} \tag{7.1}$$

to be a “**Ritz** Ansatz” that satisfies all boundary conditions. The minimal requirement:

$$\left.\begin{aligned}\hat{\Pi} &= \frac{1}{2l}\int_0^l \left(u_x + \tfrac{1}{2}w_x^2\right)^2 dx + \frac{i^2}{2l}\int_0^l w_{xx}^2\,dx + \frac{k}{2}\int_0^l w^2\,dx - \frac{p}{E}\varepsilon\,,\\ &= \frac{1}{2}\left(\varepsilon - \frac{\pi^2 f^2}{4l^2}\right)^2 + \frac{i^2\pi^4 f^2}{4l^4} + \frac{f^2 k}{4} - \frac{p}{E}\varepsilon = \frac{\pi^4 f^4}{32 l^4} - \frac{\pi^2 f^2}{4l^2}(\varepsilon - \varepsilon^*) - \frac{p}{E}\varepsilon \overset{!}{=} \min.\end{aligned}\right\} \tag{7.2}$$

implies two equations for determining the “free value” f and the dependency of the load p on the compression ε [which must coincide exactly with the previous results (see pp. 9)]:

$$\frac{\partial \hat{\Pi}}{\partial \varepsilon} = \varepsilon - \frac{\pi^2 f^2}{4l^2} - \frac{p}{E} = 0 , \quad \frac{\partial \hat{\Pi}}{\partial f} = \frac{f \pi^2}{2l^2}\left[\varepsilon^* - \left(\varepsilon - \frac{\pi^2 f^2}{4l^2}\right)\right] = 0 , \tag{7.3}$$

so *either:*

$$f = 0 , \quad p = E\,\varepsilon \tag{7.4}$$

or:

$$f \neq 0 , \qquad \frac{p}{E} = \varepsilon - \frac{\pi^2 f^2}{4l^2} = \varepsilon^* = i^2 \frac{\pi^2}{l^2} + k \frac{l^2}{\pi^2} . \tag{7.5}$$

The stability question is again answered by the second variation, which is the second derivative here:

$$\frac{\partial^2 \hat{\Pi}}{\partial \varepsilon^2} = 1 , \quad \frac{\partial^2 \hat{\Pi}}{\partial f^2} = \frac{\pi^2}{2l^2}\left[\varepsilon^* - \varepsilon + 3\frac{\pi^2 f^2}{4l^2}\right] . \tag{7.6}$$

One sees that:

for $\varepsilon < \varepsilon^*$, one has $\delta^2 \hat{\Pi} > 0$,

for $\varepsilon > \varepsilon^*$ and $\frac{\pi^2 f^2}{4l^2} = \varepsilon - \varepsilon^*$, one has $\frac{\partial^2 \hat{\Pi}}{\partial f^2} = \frac{\pi^2}{l^2}(\varepsilon - \varepsilon^*) > 0$,

for $\varepsilon > \varepsilon^*$ and $f = 0$, one has $\frac{\partial^2 \hat{\Pi}}{\partial f^2} = \frac{\pi^2}{2l^2}(\varepsilon^* - \varepsilon) < 0$,

i.e., the straight configuration is stable for $\varepsilon < \varepsilon^*$ and unstable for $\varepsilon > \varepsilon^*$, while the curved configuration is always stable if it is mechanically-possible to begin with (so for $\varepsilon > \varepsilon^*$).

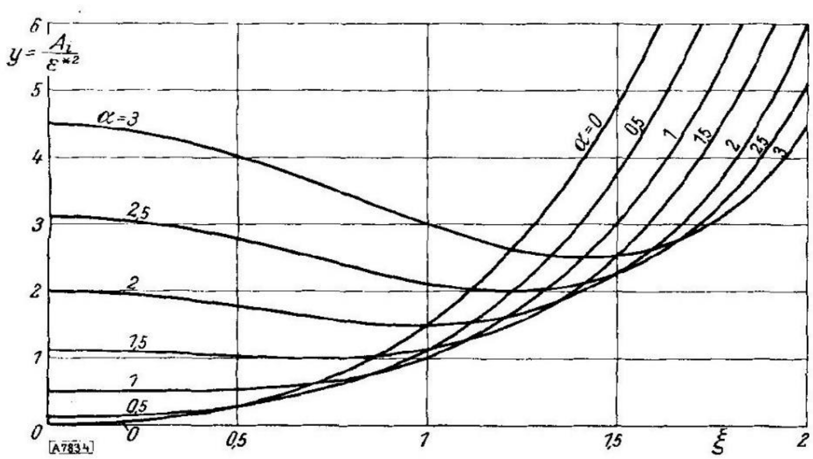

Figure 4. Dependency of the deformation energy on the sag $f = \xi \frac{2}{\pi} \cdot l \sqrt{\varepsilon^*}$

with the compression $\varepsilon = a\varepsilon^*$ as the parameter.

Fig. 4 illustrates the result of eqs. (7.3)…(7.6). The dependency of f $\left(\xi = \frac{f\pi}{2l\sqrt{\varepsilon^*}}, \text{resp.}\right)$ with ε $\left(a = \frac{\varepsilon}{\varepsilon^*}, \text{resp.}\right)$ as a parameter is applied to $\hat{A}_i$:

$$\frac{\hat{A}_i}{\varepsilon^{*2}} = \tfrac{1}{2}\,\xi^4 - \xi^2(a-1) + \tfrac{1}{2}\,a^2, \qquad \text{resp.} \tag{7.7}$$

The potential of the external forces is not added to that because that would bring with it another parameter (viz., the load) that would be harder to ignore, but would not be affected by the minimal demand relative to f. (The principle requires that the external loads must be kept *constant*!)

The figure makes the change in "type" of the curves under the transition from the sub-critical domain to the super-critical ones especially beautiful, namely, the coincidence of the maximum and minimum for $a = 1$. It becomes quite clear here that the opinion that is occasionally expressed that the essence of the stability problem is not the appearance of large displacements is incorrect: Although the displacements are small *at the moment* of the transition ($f_{\min} \to 0$), the behavior of the body will still be determined by the "possibility" of large deflections; mathematically speaking, by the fact that there is *just no* ordinary minimum at the critical point. [One has $\delta\Pi$ = $\delta^2\,\Pi$ = $\delta(\delta^2\,\Pi)$ = 0 there.]

It must be remarked, moreover, that the fact that $\partial^2\,\Pi / \partial f^2 > 0$ does not *prove* that the curved configuration is actually stable, because from our previous definition, we must have $\delta^2\,\Pi > 0$ for *any* ([17]) conceivable variation δw, and not just under the special assumption that $(w + \delta w) = (f + \delta f) \cdot \sin \pi x / l$ ([18]). However, the approximate calculation with the help of the **Ritz** Ansatz makes the stability intuitively *plausible*.

8. The infinitely-long elastically-supported compressed rod. – The determination of the buckling load of an infinitely-long elastically-supported beam (and similarly, the infinitely-long plate of shell) is not possible with the use of the differential equations (4.1) and (4.2) alone. The demand of (pure) periodicity of the lateral displacement w that enters in place of the "boundary" conditions in this problem does not suffice to determine all constants: The *length of the period l* remains undetermined.

We obtain the equation for determining l when we rewrite the requirement (3.9) for l:

$$\hat{\Pi} = \text{min.} \tag{8.1}$$

separately as:

([17]) See above, pp. 13.

([18]) $\Pi(f + \delta f) - \Pi(f) = \frac{\delta\Pi}{\delta f}\,\delta f + \frac{1}{2}\frac{\delta^2\,\Pi}{\delta f^2}(\delta f)^2 + \cdots$, so $= \delta^2\Pi = \frac{\delta^2\,\Pi}{\delta f^2}(\delta f)^2$.

$$\frac{\partial \hat{\Pi}}{\partial l} = 0 \,. \tag{8.2}$$

We get the desired relation from (8.2) when we apply the demand on $\hat{\Pi}$ in the form (3.8), while observing the fact that *l* enter into the integrals as an upper limit and as a parameter in *w*, or when we employ the integrated form (7.2) directly:

$$\frac{\partial \hat{\Pi}}{\partial l} = \left(\varepsilon - \frac{\pi^2 f^2}{4l^2}\right)\frac{\pi^2 f^2}{4}\cdot\frac{2}{l^2} - \frac{i^2\,\pi^4 f^2}{l^5} = -\,\frac{\pi^2 f^2}{2l^3}\left[-\left(\varepsilon - \frac{\pi^2 f^2}{4l^2}\right) + 2\,\frac{i^2\,\pi^2}{l^2}\right], \tag{8.3}$$

which is much more convenient. It then follows from (8.3) and (7.5) that:

$$\frac{i^2\,\pi^2}{l^2} = k\,\frac{l^2}{\pi^2}\;,$$

or

$$l^4 = \frac{i^2\,\pi^4}{k} = \frac{E}{K}\,i^2\,\pi^4 \,. \tag{8.4}$$

The wavelength *l* is independent of ε (*f*, resp.). Beyond the critical compression, it still keeps the value that it assumed at the critical point itself then.

Ordinarily, one gets the result (8.4) from the demand that at the critical point ($f = 0$), the ε in eq. (7.5), so:

$$\varepsilon_k = i^2\,\frac{\pi^2}{l^2} + k\,\frac{l^2}{\pi^2}\,,$$

should have a smallest value relative to *l* :

$$\frac{\partial \varepsilon_k}{\partial l} = -\,\frac{2i^2\,\pi^2}{l^2} + 2k\,\frac{l}{\pi^2} = 0 \,. \tag{8.5}$$

For the most part, the condition (8.5) cannot be justified as a demand that is otherwise intuitively obvious. From our previous argument, it is equivalent to the demand that the second variation of $\hat{\Pi}$ must have a minimum relative to *l* (at the critical point). In fact, it follows from (7.2) [(5.3), resp.]:

$$\delta^2\hat{\Pi} = -\,\varepsilon_k\,\frac{\pi^2(\delta f)^2}{2l^2} + \frac{i^2\,\pi^4(\delta f)^2}{2l^4} + \frac{k(\delta f)^2}{2} = 0 \,. \tag{8.6}$$

Now, we have:

$$\frac{\partial \varepsilon_k}{\partial l} = - \frac{\dfrac{\partial}{\partial l}(\delta^2 \hat{\Pi})}{\dfrac{\partial}{\partial \varepsilon_k}(\delta^2 \hat{\Pi})} ,$$

and therefore (8.5) says, in fact, nothing but the fact that:

$$\frac{\partial}{\partial l}(\delta^2 \hat{\Pi}) = 0 . \tag{8.7}$$

However, one observes that only the *critical* value of l can be determined from (8.7) [(8.5), resp.], while one will get its value for *arbitrarily*-given ε from (8.3).

9. Behavior of the rod under very large deflections [Ansatz (3.3)]. – The investigations up to now have led to the result, *inter alia*, that the load remains exactly constant beyond the buckling limit:

$$p = E\,\varepsilon^* = E\left(\frac{i^2\,\pi^2}{l^2} + k\,\frac{l^2}{\pi^2}\right) .$$

Naturally, that is correct only as long as the elastic behavior of the beam is characterized sufficiently precisely by the Ansatz (3.5) for the deformation energy. That is no longer the case for large deflections, since the term $(f / l)^6$ will no longer be *negligible* compared to the term $(f / l)^4$. One includes the variability of the compressive force (in the second approximation) when one appeals to the Ansätze (3.3) and (3.4).

The differential equations that arise from (3.3) on the basis of the requirement $\delta^2 \hat{\Pi} = 0$ can be given with no difficulty. One finds that the first one is (up to higher-order terms):

$$\frac{\partial}{\partial x}\left(u_x + \tfrac{1}{2}\,w_x^2 + \tfrac{1}{8}\,w_x^4\right) = 0 , \tag{9.1}$$

and when one introduces:

$$u_x + \tfrac{1}{2}\,w_x^2 + \tfrac{1}{8}\,w_x^4 = \text{const.} = -\,\hat{\varepsilon}$$

into the second one, that will give (we restrict ourselves to the case of $k = 0$):

$$w_{xxxx} + \left[\frac{\hat{\varepsilon}}{i^2}(1-\varepsilon^*) + \zeta(x)\right] w_{xx} = 0 , \quad \text{with} \quad \zeta(x) = 2\,w_x \cdot w_{xxx} + w_{xx}^2 - \frac{3}{2}\frac{\pi^2}{l^2}\,w_x^2 \tag{9.2}$$

(and again, up to higher-order terms). An *exact* integration of eq. (9.2) is probably not possible. However, since the newly-added terms are small in comparison to the previous ones [eq. (4.4)] with a ratio of $(f / l)^2$ (i.e., they represent only correction terms), it is entirely sufficient in practice to regard their influence *approximately*. The fastest way to reach that goal here is the **Ritz** procedure.

As a starting point, we choose the solution of the previous, "next lowest," differential equations:

$$\left.\begin{aligned} w &= f \sin \frac{\pi x}{l}, \\ u_x + \tfrac{1}{2} w_x^2 &= -\varepsilon_0. \end{aligned}\right\} \tag{9.3}$$

If we set:

$$\frac{\pi^2 f^2}{4 l^2} = \varphi , \tag{9.4}$$

to abbreviate, then the connection (4.9′) ([19]):

$$\varepsilon_0 = \varepsilon - \varphi \tag{9.5}$$

will once more exist between the (unknown) integration constants and the displacement of the right-hand side of the rod $u(l) = -\varepsilon\, l$, which is thought to be given.

If we substitute (9.3) in (3.3) then that will give:

$$\Pi = \tfrac{1}{2} E F \int_0^l \left[\varepsilon_0 \left\{ \varepsilon_0 (1 - w_x^2) - \tfrac{1}{4} w_x^4 \right\} + i^2 \{ w_{xx}^2 (1 + 2\varepsilon_0 - w_x^2) + 2 w_x^2 w_{xx}^2 \} \right] dx + P\, u(l) ,$$

or

$$\hat{\Pi} = \frac{\Pi}{E F l} = \tfrac{1}{2} \varepsilon_0^2 - \varepsilon_0^2 \varphi - \tfrac{3}{4} \varepsilon_0 \varphi^2 + \varepsilon^* \varphi \{ 1 + 2\varepsilon_0 + \varphi \} - \frac{p}{E} \varepsilon .$$

Upon observing (9.5), the minimal requirement:

$$\frac{\partial \hat{\Pi}}{\partial \varepsilon} = 0 , \qquad \frac{\partial \hat{\Pi}}{\partial \varphi} = 0$$

will imply the following two equations for the connection between p, ε^*, and φ:

$$\varepsilon_0 (1 - 2\varphi) - \tfrac{3}{4} \varphi^2 + 2\varepsilon^* - \frac{p}{E} = 0 ,$$

$$-\ \varepsilon_0 (1 - 2\varphi + \varepsilon_0) + \tfrac{3}{4} \varphi^2 - \frac{3}{2} \varepsilon^* \varphi + \varepsilon^* (1 - 2\varepsilon_0 + 2\varphi - 2\varphi) = 0 .$$

It follows from the second one that ([20]):

([19]) In the notations of Fig. 2: $\xi^2 = a - \varepsilon_0 / \varepsilon^*$.

([20]) Thus [from (9.6)], in Fig. 2, one has: $a = 1 + \varepsilon^* + \xi \left(1 + \frac{1}{2} \varepsilon^* + \frac{3}{4} \varepsilon^* \xi^2 \right)$.

$$\varepsilon_0 = \varepsilon^*(1+\varepsilon^*+\tfrac{1}{2}\varphi)+\tfrac{3}{4}\varphi^2 \ ,$$

and therefore, the first one will give:

$$\frac{p}{E} = \varepsilon^*\left(1+\varepsilon^*+\tfrac{1}{2}\varphi\right) + \text{higher-order terms.}$$

One then has ([21]):

$$p = E\,\varepsilon^*\left(1+\varepsilon^*+\frac{\pi^2 f^2}{8l^2}\right), \qquad P = \frac{E\,I\,\pi^2}{l^2}\left(1+\frac{\pi^2 i^2}{l^2}+\frac{\pi^2\ f^2}{8l^2}\right).$$

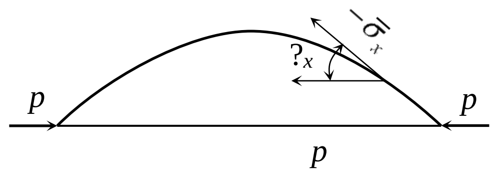

Figure 5.

We will arrive at the same result by the detour to the differential equations (9.1) and (9.2). The first equation says that (see Fig. 5) the horizontal force:

$$\left.\begin{aligned} p &= -\frac{\bar{\sigma}_x}{1-\frac{1}{2}w_x^2} = E\left(u_x+\tfrac{1}{2}w_x^2+\tfrac{1}{8}w_x^2\right) \quad \text{([see eq.(3.4)]} \\ \text{must be } \textit{constant}: \quad & \qquad p = E\,\hat{\varepsilon} = \text{const.,} \end{aligned}\right\} \tag{9.7}$$

and according to **Galerkin** ([22]), when we multiply the second equations by *w* and integrate, while employing the Ansatz (9.3), we will get:

$$\hat{\varepsilon} = \varepsilon^*(1+\varepsilon^*+\varphi/2) \ ,$$

i.e., the relation (9.6) again.

One sees from (9.7) that the critical stress:

$$p_k = E\,\varepsilon^*(1+\varepsilon^*) \tag{9.8}$$

is somewhat higher than in the elementary theory. The very slight increase goes back to the enlargement of the part due to the bend by $(1+2\varepsilon^*)$, which counteracts the enlargement of the extension part by $(1+\varepsilon^*)$. The excess in the raising of the elastic compression that enters in at the critical point:

([21]) In Fig. 2, $\frac{p}{p^*} = 1+\varepsilon^*\left(1+\frac{1}{2}\xi^2\right) = 1+\varepsilon^*\left(1+\frac{1}{2}(a-1)\right)$.

([22]) See, e.g., ZAMM **16** (1936), pp. 353/354.

$$p = p_k \left(1 + \frac{\pi^2 f^2}{8 l^2} \right) \tag{9.9}$$

is likewise very small.

If we denote the percentage increase by ψ:

$$\psi = 100 \; \frac{p - p_k}{p_k}$$

then:

$$\frac{f}{l} = \frac{1}{10\pi} \sqrt{8\psi} \; ,$$

and the bending stress that belongs to that deflection will be:

$$\bar{\sigma} = E \, w'' \frac{h}{2} \approx E \frac{h}{l} \cdot \frac{\pi^2}{2} \cdot \frac{f}{l} = 0.444 \, E \frac{h}{l} \sqrt{\psi} \qquad (h = \text{height of the beam})$$

so only a few percent of overshoot in the elastic domain will remain only for a *very small* ratio h / l .

Appendix

Behavior of the rod in the super-critical domain according to Trefftz. – Eq. (9.9) agrees with the approximate result of the so-called "exact theory" of the **Euler** rod ([1]). That theory starts from the "exact" differential equation of the buckled rod:

$$P \, w + \frac{E \, I}{\rho} = 0 \tag{1}$$

$\left(\text{with } \dfrac{1}{\rho} = \dfrac{w_{xx}}{(\sqrt{1 + w_x^2})^3} \right)$, which can be integrated with the help of elliptic functions ([2]).

Another (very simple) way of deriving eq. (9.9) was given by **E. Trefftz**. Starting from the assumption that the arclength of the beam did not change under the deformation, **Trefftz** used the principle of virtual displacements in the form:

$$\delta (\boldsymbol{A}_i + \boldsymbol{V}) = \delta \boldsymbol{A}_i - P \; \delta u = 0 \; , \tag{2}$$

([1]) *Handbuch der Physik* VI, pp. 202, eq. (10).
([2]) *Handbuch der Physik* VI, pp. 201.

in which $\boldsymbol{A}_i$ is the work done by bending, δu is the displacement of the endpoint *as a result* of the deflection w, and P is the force of compression (which remains constant under the variation). **Trefftz** introduced s as the longitudinal coordinate in place of x. One will then have:

$$\boldsymbol{A}_i = \tfrac{1}{2}\int_0^l E\,I\left(\frac{1}{\rho}\right)^2 ds\,, \qquad P\,u = P\int_0^l \left(1-\frac{dx}{ds}\right) ds\,, \tag{3}$$

and since:

$$\left.\begin{aligned} &ds^2 = dx^2 + dw^2, \qquad \text{so} \qquad \frac{dx}{ds} = \sqrt{1-w_s^2} \approx 1-\tfrac{1}{2}w_s^2 - \tfrac{1}{8}w_s^4\,, \\ \text{and}\quad &\left(\frac{1}{\rho}\right) = \frac{d\arcsin w_s}{ds} = \frac{w_{ss}}{\sqrt{1-w_s^2}} \approx w_{ss}(1+w_s^2)^{1/2}, \end{aligned}\right\} \tag{4}$$

that can be written:

$$\boldsymbol{A}_i = \tfrac{1}{2}E\,I\int_0^l w_{ss}^2(1+w_s^2)\,ds\,, \qquad P\,u = \frac{P}{2}\int_0^l \left(w_s^2 - \tfrac{1}{4}w_s^4\right) ds\,. \tag{5}$$

If one now makes the **Ritz** Ansatz for w :

$$w = f\sin\frac{\pi\,x}{l} \tag{6}$$

then the two integrals (5) can be performed:

$$\boldsymbol{A}_i = \frac{E\,I}{2}\left\{\frac{\pi^4 f^2}{2l^3} + \frac{\pi^6 f^6}{8l^5}\right\}, \qquad P\,u = P\left\{\frac{\pi^2 f^2}{4l} + \frac{3}{64}\frac{\pi^4 f^4}{l^3}\right\}, \tag{7}$$

and the demand that:

$$\delta(\boldsymbol{A}_i + \boldsymbol{V}) = \delta(\boldsymbol{A}_i + P\,u) = \frac{d}{df}(\boldsymbol{A}_i - P\,u)\,\delta f = 0$$

will imply that:

$$\frac{E\,I\,\pi^2}{l^2}\cdot\frac{\pi^2 f}{2l}\left\{1+\frac{1}{2}\frac{\pi^2 f^2}{l^2}\right\} = P\,\frac{\pi^2 f}{2l}\left\{1+\frac{3}{8}\frac{\pi^2 f^2}{l^2}\right\}, \tag{8}$$

so *either:*

$$f = 0$$

or:

$$P = \frac{E\,I\,\pi^2}{l^2}\cdot\frac{1+\dfrac{1}{2}\dfrac{\pi^2 f^2}{l^2}}{1+\dfrac{3}{8}\dfrac{\pi^2 f^2}{l^2}} = E\,F\,\varepsilon^*\left(1+\frac{\pi^2 f^2}{8l^2}\right). \tag{9}$$

Eq. (9) agrees with the result (9.9) that we obtained with out systematic method, up to the factor $(1+\varepsilon^*)$ (which is missing here).