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On relativistic spinning fluids

by

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Abstract. – A variational principle is formulated that yields the field equations and constitutive equations of a non-conducting charge-free perfect fluid that interacts with electromagnetism and is endowed with spins of magnetic origin. The interactions between neighboring spins are accounted for and give rise to Heisenberg exchange forces in the form of contact actions. The forms of the asymmetric energy-momentum and total spin tensors are thus constructed, as well as the expression for the spin precession velocity as a function of the different interactions. The consequences of Lorentz invariance and isotropy of the fluid are studied, and exact and approximate forms for the constitutive equations are deduced. The physical model thus-constructed is complete and may be referred to as that of a *perfect relativistic ferrofluid*. It is shown that the perfect magnetohydrodynamics model corresponds to that of a paramagnetic ferrofluid. The geometrization of the model is discussed briefly.

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1. – Introduction.

By appealing to a recent phenomenological formulation of the theory of ferromagnetism (cf., [**8**], [**12**], [**31**]) and our theory of magneto-elastic interactions (cf., [**3**], [**8**], [**24**]) in special relativity, we shall establish a model for a relativistic spinning fluid in which one takes into account not only the gyromagnetic effect, but also the interaction between the spins. According to one interpretation of the physics of continua that was given by Brown [**20**], the fluid medium is endowed with a continuous distribution of spins, and the interaction between neighboring magnetic spins (the exchange forces in Heisenberg's ferromagnetism) manifests itself by the presence of a spatial spin gradient in the thermodynamic potential that is used in the variational formulation. The latter follows the method that was introduced by Taub [**21**] and was used recently by the author [**1**] ([1]). The electromagnetic effect considered is obviously non-dissipative. The variational

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([1]) Another variational formulation of the same type as the one that was given in [**2**] for MHD is given, moreover. ["Un principe variationnel pour le schéma fluide relativiste à spin" (preprint, 1973)].

principle utilized permits us to obtain (upon taking into account the constraints imposed upon entropy, the 4-velocity, and the spin density) all of the "mechanical" conservation equations (viz., conservation of energy-momentum, conservation of the moment of energy-momentum, which is equivalent to the equation of spin evolution along a streamline), but also the detailed expressions for the total energy-momentum tensor, the total spin density, and the velocity of spin precession as a function of the different interactions: viz., matter-matter, matter-electromagnetic field, spin-spin. The latter interaction gives rise (in a "continuous" manner) to *contact* actions (viz., surface couples) that, in part, responsible for the asymmetry in the energy-momentum tensor. One likewise shows (§ **4.5**) that this asymmetry results from the fact that, locally, matter and spin have different rotation rates, which agrees with the interpretations that are given in the theories of continuous media with microstructure (cf., [**25**]). The exact form of the laws of behavior in our model (i.e., of the thermodynamic pressure and the spin interaction tensor) results from the Lorentz invariance condition when it is applied to the thermodynamic potential considered, as well as the necessary isotropy in the fluid. Some approximate laws of behavior that are called "quasi-linear" are likewise obtained. If one neglects the interaction between spins then the present formulation will revert to the simplified theory that was presented by Halbwachs [**6**]. As in that theory, the 4-velocity and the 4-momentum are not collinear. We show that the "perfect magnetohydrodynamical" model corresponds to the case in which our fluid is paramagnetic (§ **5**). Finally, we note that in some situations that are encountered in astrophysics and cosmology, the geometrization of a model for a spinning fluid is called for (§ **6**), and we briefly examine two possible geometric models. The first one (§ **6.1**) preserves the Riemannian geometric structure of classical general relativity on the condition that we must introduce a curvature term into the "conservation equation" for energy-momentum and redefine the source term in the Einstein equations, which agrees with a model that was proposed recently by Israel [**19**]. The second one (§ **6.2**) appeals to the structure of an Einstein-Cartan space U^4 with non-zero torsion that is based upon an asymmetric affine connection. In that case, the space-time torsion results, in part, by the action of Heisenberg exchange forces.

2. – Preliminaries.

The notations are those of the preceding articles [**1**], [**2**]. Let x^α, α = 1, 2, 3, 4 be a local curvilinear chart (x^4 is time-like) on M^4, which is *flat* Minkowski space-time. The curvature $R^\alpha{}_{,\beta\gamma\delta}$ of M^4 that is associated with the normal hyperbolic metric $g_{\alpha\beta}$ (of signature +, +, +, −) is zero. The symbol ∇ will indicate a covariant derivative. A fluid "particle" whose Lagrange coordinates are X^K, K = 1, 2, 3 describes a world-line C in M^4 that is time-oriented and whose equations (in local coordinates) are:

$$x^\alpha = \mathcal{X}^\alpha (X^K, s) , \tag{2.1}$$

in which s is the proper time of the "particle." (1.1) is considered to be C^m, with m greater than two. The 4-velocity u^α is defined by (c = speed of light *in vacuo*):

(2.2) $$u^\alpha = \left.\frac{\partial \mathcal{X}^\alpha}{\partial s}\right|_{X^K}, \quad g_{\alpha\beta}\, u^\alpha u^\beta + c^2 = 0.$$

The operator $\frac{D}{Ds} = \left.\frac{\partial}{\partial s}\right|_{X^K}$ is the *invariant derivative* in the direction u^α, which is defined at each event-point of $\mathcal{C}$ by:

(2.3) $$\frac{D\mathbf{A}}{Ds} = u^\alpha \nabla_\alpha \mathbf{A} = \dot{\mathbf{A}}, \qquad \forall\ \mathbf{A}.$$

It constitutes the relativistic generalization of the particule derivative. Let $M_\perp^3(x)$ be the local 3-hypersurface that is orthogonal to u^α at x along $\mathcal{C}$. The projection operator $P_{\alpha\beta}$ is defined by:

(2.4) $$\begin{cases} P_{\alpha\beta} = g_{\alpha\beta}(x) + \frac{1}{c^2} u_\alpha(x) u_\beta(x), \\ P_{\alpha\beta}(x) u^\alpha(x) = 0, \quad P_{\alpha\beta} P^{\beta\gamma} = P_{\alpha}^{\cdot\gamma}, \quad P_{\alpha}^{\cdot\alpha} = 3. \end{cases}$$

$P_{\alpha\beta}$ is useful in effecting the decomposition of a tensor into its purely-spatial, mixed, and temporal components (see, [**3**], [**4**]). In particular, one says that a tensor of arbitrary order $A^{\alpha\beta\ldots\mu}$ is completely P. U. (†) in M^4 at the event-point x on $\mathcal{C}$ if and only if u^α is a null vector of $A^{\alpha\beta\ldots\mu}$, so:

(2.5) $$A^{\alpha\beta\ldots\mu}\, u_\alpha = A^{\alpha\beta\ldots\mu}\, u_\beta = \ldots = A^{\alpha\beta\ldots\mu}\, u_\mu = 0\ .$$

Hence:

(2.6) $$(A^{\alpha\beta\ldots\mu})_\perp \equiv P_{\cdot\sigma}^{\alpha} P_{\cdot\rho}^{\beta} \cdots P_{\cdot\nu}^{\mu}\, A^{\sigma\rho\cdots\nu} \equiv A^{\alpha\beta\ldots\mu},$$

in which the symbolic notation $(\ldots)_\perp$ indicates the projection operation. A P. U. tensor has values that are essentially spatial or tri-dimensional. In a rest frame, it will reduce to the equivalent tri-dimensional concept in classical physics.

One defines the relativistic velocity gradient tensor $e_{\alpha\beta}$ and its symmetric and antisymmetric parts, which are called the *relativistic rate of deformation* and *relativistic vorticity* (or rate of rotation) tensor, by:

(2.7) $$e_{\alpha\beta} \equiv (\nabla_\beta\, u_\alpha)_\perp\ ,$$

(2.8) $$\sigma_{\alpha\beta} \equiv e_{(\alpha\beta)} \equiv \tfrac{1}{2}(e_{\alpha\beta} + e_{\beta\alpha}) = \tfrac{1}{2} \underset{u}{\mathcal{L}}\, P_{\alpha\beta}\ ,$$

(†) Translator: It is not clear to me what the abbreviation "P.U." stands for, but it seems to represent "purely spatial," based upon (2.5) and its general usage.

(2.9) $$\omega_{\alpha\beta} \equiv e_{[\alpha\beta]} \equiv \tfrac{1}{2}(e_{\alpha\beta} - e_{\beta\alpha}) ,$$

in which $\underset{u}{\mathcal{L}}$ denotes the Lie derivative with respect to the field u^α. $\sigma_{\alpha\beta}$ is an *objective* tensor, in the author's sense of the word (cf., [**5**]); $\omega_{\alpha\beta}$ is not.

ρ denotes the *invariant relativistic density* of matter. It is a *proper* invariant, i.e., when measured by an observer that is comoving with the fluid "particle." In a continuous region $\mathcal{B}$ of M^4, ρ will satisfy the equation that one calls the "continuity" equation" in one of the three forms:

(2.10) $$\frac{D\rho}{Ds} + \rho\Theta = 0, \qquad \frac{D\ln\rho}{Ds} + \Theta = 0, \qquad \nabla_\alpha(\rho u^\alpha) = 0,$$

in which $\Theta \equiv e^{\alpha}_{\cdot\alpha}$ is the dilatation.

3. – Gyromagnetic phenomena.

(*a*) The theory that we shall present here is conceived in such a manner that it takes into account the spin (kinetic spin) that is associated with the magnetic moment. That spin is of electronic origin. According to Uhlenbeck and Goudsmit, the electron spin is purely magnetic in the rest frame that is attached to the electron (cf., [**6**]). That is, the polarization 3-vector **P** is such that (2):

(3.1) $$\mathbf{P} = 0$$

in such a frame. The gyromagnetic relation (for a gyromagnetic effect that is *isotropic*, which we shall suppose) is then written:

(3.2) $$\tilde{\pi}^{\alpha\beta} = \gamma\,\tilde{S}^{\alpha\beta}, \quad \gamma \equiv -\frac{e_0}{m_0\,c}$$

(e_0 : electron charge, m_0 : rest mass of the electron, γ : gyromagnetic ratio). $\tilde{S}^{\alpha\beta}$ is the antisymmetric tensor of intrinsic spin per unit proper volume. $\tilde{\pi}^{\alpha\beta}$ is the antisymmetric tensor of magnetization-polarization (per unit proper volume), which admits the decomposition (cf., [**2**]):

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(2) Note that the hypothesis (3.1) does not imply that the polarization is zero for a body in motion. Indeed, in an instantaneous inertial frame, $\mathcal{P}^\alpha$ admits the well-known decomposition:

$$\mathcal{P}^\alpha = \left[\frac{\mathbf{P} - \mathbf{v}\times\mathbf{M}/c}{\sqrt{(1-\beta^2)}}, \frac{i\,\mathbf{v}\cdot\mathbf{P}/c}{\sqrt{(1-\beta^2)}}\right], \quad \beta = \left|\frac{\mathbf{v}}{c}\right|,$$

in which **M** is the 3-magnetization and **v** is the velocity. In such a frame (which is not at rest), the covariant condition $(3.5)_1$ leads to a polarization that is induced by the magnetization of motion and is such that $\mathbf{P} \approx \mathbf{v}\times\mathbf{M}/c$.

(3.3)
$$\begin{cases} \pi^{\alpha\beta} = \dfrac{1}{c}(\mathcal{P}^{\alpha}u^{\beta} - \mathcal{P}^{\beta}u^{\alpha}) + \dfrac{1}{ic}\eta^{\alpha\beta\gamma\delta}M_{\gamma}\,u_{\delta}, \\ \mathcal{P}^{\alpha} \equiv \dfrac{1}{c}\pi^{\alpha\beta}u_{\beta}, \quad M_{\alpha} \equiv \dfrac{1}{2ic}\eta_{\alpha\beta\gamma\delta}\,\pi^{\beta\gamma}u^{\delta}, \\ \mathcal{P}^{\alpha}u_{\alpha} = M^{\beta}u_{\beta} = 0, \end{cases}$$

in which $\mathcal{P}^{\alpha}$ and M_{β} are P.U. 4-vectors of polarization and volumetric magnetization, resp. ([3]). Outside of matter:

(3.4)
$$\tilde{\pi}^{\alpha\beta} = 0 \Leftrightarrow \mathcal{P}^{\alpha} = 0, \quad M_{\beta} = 0 .$$

The covariant transcription of the hypothesis (3.1) is the *Frenkel condition* [**7**]:

(3.5)
$$\mathcal{P}^{\alpha} \equiv 0, \qquad \tilde{\pi}^{\alpha\beta}u_{\beta} = \gamma\,\tilde{S}^{\alpha\beta}u_{\alpha} = 0,$$

which is a condition that is weaker than (3.4). Instead of the volumetric quantities $\tilde{\pi}^{\alpha\beta}$, M_{α}, and $\tilde{S}^{\alpha\beta}$, one prefers to employ the same quantities when they are defined per unit proper mass: $\pi_{\alpha\beta}$, $\mathcal{M}_{\alpha}$, and $S^{\alpha\beta}$. One will then have:

(3.6)
$$S^{\alpha\beta} = \gamma^{-1}\pi_{\alpha\beta} = \frac{\gamma^{-1}}{i\,c}\eta^{\alpha\beta\gamma\delta}\mathcal{M}_{\sigma}u_{\rho}, \qquad \mathcal{M}_{\sigma} \equiv M_{\sigma}/\rho .$$

It then follows that when one defines the axial P. U. spin 4-vector s_{α}, (3.6) can be written in the 4-vectorial form:

(3.7)
$$s_{\alpha} = \gamma^{-1}\mathcal{M}_{\alpha}, \quad \text{with} \quad s_{\alpha} \equiv \frac{1}{2i\,c}\eta_{\alpha\beta\gamma\delta}\,S^{\beta\gamma}u^{\delta}.$$

(*b*) Imagine that the magnetic moment at a point **X** of the usual physical space $\mathbb{E}^3$ can only turn. Its norm is then fixed in $\mathbb{E}^3$. The covariant transcription of that hypothesis in $M_{\perp}^{3}(x)$ is written:

(3.8)
$$\boxed{P_{\alpha\beta}(x)\mathcal{M}^{\alpha}(x)\mathcal{M}^{\beta}(x) = \text{const.} \quad \text{along } \mathcal{C}.}$$

It then follows that:

([3]) One recalls that:

$$\eta_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}\sqrt{|\,g\,|}, \qquad \eta^{\alpha\beta\gamma\delta} = \frac{\varepsilon_{\alpha\beta\gamma\delta}}{\sqrt{|\,g\,|}}, \qquad g = \det(g_{\alpha\beta}),$$

in which $\varepsilon_{\alpha\beta\gamma\delta}$ is the completely-antisymmetric permutation symbol.

(3.8)
$$\mathcal{M}_\alpha \left(\frac{D\mathcal{M}_\alpha}{Ds} \right)_\perp = 0 \qquad \text{along } \mathcal{C}\,.$$

A solution to that equation is given in [**8**]. The temporal evolution of $\mathcal{M}^\alpha$ (or of the spin s^α) along $\mathcal{C}$ is described by the following *kinematical* equations:

(3.9)
$$\boxed{\begin{gathered} \frac{D\mathcal{M}^\alpha}{Ds} = \left(\Omega^{\alpha}_{\cdot\,\beta} + \frac{1}{c^2} u^\alpha \frac{Du_\beta}{Ds} \right) \mathcal{M}^\beta, \\ \left(\frac{D\mathcal{M}^\alpha}{Ds} \right)_\perp = \Omega^{\alpha}_{\cdot\,\beta} \mathcal{M}^\beta, \quad \Omega_{\alpha\beta} = -\Omega_{\beta\alpha}, \quad \Omega_{\alpha\beta}\, u^\beta = 0, \end{gathered}}$$

in which the antisymmetric P.U. tensor $\Omega_{\alpha\beta}$ represents the *angular velocity* of the spin that one agrees to determine. As $(3.9)_2$ shows, that P. U. tensor measures the velocity of spin precession in the inertial frame, while the term $\frac{1}{c^2} u^\alpha \frac{Du_\beta}{Ds}$ in $(3.9)_1$ represents Fermi-Walker transport along $\mathcal{C}$. Since $\Omega^{\alpha\beta}$ is P.U, one associates it with its dual π_α such that:

(3.10)
$$\pi_\alpha = \frac{1}{i\,c} \eta_{\alpha\beta\gamma\delta}\, \Omega^{\beta\gamma} u^\delta\,, \qquad\qquad \pi_\alpha\, u^\alpha = 0\,.$$

Consider the *real* work W done by the spin during a finite rotation in an inertial frame. The couple that is associated with the spin is $-\left(\frac{Ds^\alpha}{Ds} \right)_\perp$, so one will have:

(3.11)
$$W = -\frac{Ds^\alpha}{Ds} \pi_\alpha = -\frac{1}{2} \frac{DS^{\alpha\beta}}{Ds} \Omega_{\alpha\beta}\,.$$

With the aid of $(3.9)_1$, one shows that (cf., [**8**]):

(3.12)
$$\boxed{W = 0.}$$

The couple that is produced by the spin is then of d'Alembert type. Spin obviously represents an effect of a gyroscopic nature. It is not possible to construct a kinetic energy of rotation in integrated form for spin. One can consider only an "already-varied" form ([4]).

([4]) See the commentaries on this subject in the classical theory of micro-magnetism [**9**], [**10**], [**11**].

(*c*) *Spatial variation of spin*. – If $\mathcal{M}^\alpha$ keeps a constant norm [cf., (3.8)] along a world-line $\mathcal{C}$ that is described by (2.1) then its norm can vary when one passes to a neighboring world-line to $\mathcal{C}$ ([5]); i.e., if one considers a spatial variation:

$$\mathcal{C}:\ x^\alpha = \mathcal{X}^\alpha(X^K, s) \mapsto \tilde{\mathcal{C}} : \tilde{x}^\alpha = \tilde{\mathcal{X}}^\alpha(\tilde{X}^K = X^K + \delta X^K, s).$$

That can be written locally in $M_\perp^3(x)$ in the obvious form (cf., [**5**]):

$$\tilde{\mathcal{X}}^\alpha(\tilde{X}^K, s) - x^\alpha(X^K, s) = (\delta x^\alpha)_\perp + O(|\,\delta\mathbf{x}\,|^2)\,. \tag{3.13}$$

Hence:

$$\tilde{\mathcal{M}}_\alpha(\tilde{\mathbf{x}}) - \mathcal{M}_\alpha(\mathbf{x}) \equiv \mathfrak{M}_{\alpha\beta}(\mathbf{x})\,(\delta x^\beta)_\perp + O(|\,\delta\mathbf{x}\,|^2)\,, \tag{3.14}$$

in which the projection operator takes it value at **x**, and one defines the *relativistic magnetization gradient tensor* $\mathfrak{M}_{\alpha\beta}$ by:

$$\boxed{\mathfrak{M}_{\alpha\beta}(\mathbf{x}) = (\nabla_\beta \mathcal{M}_\alpha)_\perp, \quad \mathfrak{M}_{\alpha\beta}\,u^\alpha = 0, \quad \mathfrak{M}_{\alpha\beta}\,u^\beta = 0.} \tag{3.15}$$

Since that tensor is asymmetric and essentially spatial, that P. U. tensor will have nine independent components, in general.

4. – "Perfect ferrofluid" model.

4.1 – Introduction.

In this paragraph, we propose to start from a variational principle and establish the field equations and laws of behavior that correspond to the *relativistic perfect ferrofluid* model. We intend that to mean a model that corresponds to an electromagnetic fluid that presents a continuous distribution of spins of magnetic origin and is *non-dissipative*. In particular, the fluid considered will have *an infinite electrical conductivity* σ. That is, when the conduction 4-current j^α, which one can show to have the form:

$$j^\alpha = \sigma\,\mathcal{E}^\alpha, \tag{4.1}$$

([5]) Three-dimensionally, that signifies that the norm of the magnetization varies from point to point in $\mathbb{E}^3$.

under acceptable simplifying hypotheses (linear, isotropic law, uncoupled with any other transport phenomena), in which $\mathcal{E}^\alpha$ is the P.U. *electric current* 4-vector, is added to the convection current $q\,u^\alpha$ (q : volumetric electric charge), that can produce a finite total current only if ([6]):

(4.2)
$$\boxed{\mathcal{E}^\alpha = 0.}$$

It then follows that the action associated with an electromagnetic field in an open subset $\overset{\circ}{\mathcal{B}}$ of M^4 (viz., a tube generated by a material body B) is generally written in the form (dv_4 : Riemannian volume element on M^4):

(4.3)
$$\mathcal{A}_{(em)} = \int_{\overset{\circ}{\mathcal{B}}} \left(\tfrac{1}{4} F^{\alpha\beta} F_{\beta\alpha} - \tfrac{1}{2} \tilde{\pi}^{\alpha\beta} F_{\beta\alpha} \right) dv_4 .$$

When one takes (3.5), (4.2), $(3.6)_2$, and the following decomposition of the magnetic flux tensor $F_{\alpha\beta}$ (cf., [**12**], [**3**]) into account:

(4.4)
$$\begin{cases} F_{\alpha\beta} = \dfrac{1}{c}(\mathcal{E}_\beta u_\alpha - \mathcal{E}_\alpha u_\beta) + \dfrac{1}{ic}\eta_{\alpha\beta\gamma\delta}\mathcal{B}^\gamma u^\delta \\ \mathcal{E}^\alpha \equiv \dfrac{1}{c} F^{\alpha\beta} u_\beta, \quad \mathcal{B}^\alpha \equiv \dfrac{1}{2ic}\eta^{\alpha\beta\gamma\delta} F_{\beta\gamma} u_\beta, \quad \mathcal{E}^\alpha u_\alpha = 0, \quad \mathcal{B}^\alpha u_\alpha = 0, \end{cases}$$

($\mathcal{B}^\alpha$: P.U. magnetic induction4-vector), the two contributions to the integrand, which represent the energy of the free electromagnetic field and the energy of the magnetic doublet, respectively, (4.3) will reduce to:

(4.5)
$$\boxed{\tilde{\mathcal{A}}_{(em)} = -\int_{\overset{\circ}{\mathcal{B}}} \rho \left(\frac{1}{2\rho} \mathcal{B}^\alpha \mathcal{B}_\alpha - \mathcal{M}^\alpha \mathcal{B}_\alpha \right) dv_4 .}$$

For the sake of simplicity, we set $q \equiv 0$. The fact that the model considered is non-dissipative implies that we have conservation of *specific entropy* η along a streamline, namely:

(4.6)
$$\boxed{\frac{D\eta}{Ds} = 0 \quad \text{along } \mathcal{C}.}$$

([6]) See the remark concerning that equation in the note ([9]) on pp. 160 of reference [**2**].

4.2 – Conditions on the variations.

4.2.1. – **Definition**: Here, we shall link up with the determination of the conservation equations (viz., the laws of motion). We then agree to vary the world-line of the fluid "particles." That variation can be defined by:

$$\delta : x^\alpha(\mathcal{C}) \mapsto x^\alpha(\hat{\mathcal{C}}) = x^\alpha + \varepsilon\, \xi^\alpha, \qquad \delta x^\alpha = \varepsilon\, \xi^\alpha, \tag{4.7}$$

in which ε is infinitely small and ξ^α is a 4-vector field that is not necessarily P.U. The resultant variation of a tensor **A** is written:

$$\delta \mathbf{A} = \varepsilon \underset{\xi}{\mathcal{L}} \mathbf{A}, \tag{4.8}$$

in which $\underset{\xi}{\mathcal{L}}$ denotes the Lie derivative with respect to the field ξ^α. In particular, for $\mathcal{B}^\lambda$ (cf., [**13**], pp. 86):

$$\delta\mathcal{B}^\lambda = (\nabla_\rho \mathcal{B}^\lambda)\, \delta x^\rho - \mathcal{B}^\rho \nabla_\rho (\delta x^\lambda) + \mathcal{B}^\lambda \nabla_\rho (\delta x^\rho) . \tag{4.9}$$

Hence:

$$\delta\left(\tfrac{1}{2}\mathcal{B}^\lambda \mathcal{B}_\lambda\right) = (\mathcal{B}^\lambda \mathcal{B}_\lambda g^{\alpha\beta} - \mathcal{B}^\alpha \mathcal{B}^\beta)\nabla_\beta (\delta x_\alpha) + \nabla_\alpha\left(\tfrac{1}{2}\mathcal{B}^\lambda \mathcal{B}_\lambda\right)\delta x^\alpha , \tag{4.10}$$

$$\delta(\mathcal{M}^\lambda \mathcal{B}_\lambda) = \mathcal{B}_\lambda\, \delta\mathcal{M}^\lambda + \mathcal{M}_\lambda (\nabla_\rho \mathcal{B}^\lambda)\, \delta x^\rho - (\mathcal{M}^\alpha \mathcal{B}^\beta - \mathcal{B}^\lambda \mathcal{B}_\lambda g^{\alpha\beta})\nabla_\beta (\delta x_\alpha) . \tag{4.11}$$

The variation $\delta\mathcal{M}^\lambda$ has a special character because $\mathcal{M}^\lambda$ must satisfy the constraint (3.8) along a world-line $\mathcal{C}$. That variation is given in the following paragraph.

Taking (4.8) into account, one easily calculates the following expressions:

$$\delta u^\alpha = u^\beta \nabla_\beta (\delta x^\alpha) , \tag{4.12}$$

$$\delta\rho = -\rho P^{\alpha\beta} \nabla_{(\alpha} (\delta x_{\beta)}) , \tag{4.13}$$

$$\delta(dv_4) = g^{\alpha\beta} \nabla_{(\alpha} (\delta x_{\beta)})\, dv_4 , \tag{4.14}$$

$$\delta(\rho\, dv_4) = -\rho \frac{u^\alpha u^\beta}{c^2} \nabla_{(\beta}(\delta x_{\alpha)})\, dv_4 , \tag{4.15}$$

$$\delta P_{\alpha\mu} = \frac{2}{c^2} u^\lambda \nabla_\lambda (\delta x^\gamma)\, P_{\gamma(\alpha} u_{\mu)} , \tag{4.16}$$

and one notes that for any $\boldsymbol{\phi}$:

(4.17) $$\delta(\nabla_\alpha \boldsymbol{\phi}) = \nabla_\alpha(\delta\boldsymbol{\phi}) - (\nabla_\beta \boldsymbol{\phi})\, \nabla_\alpha(\delta x^\beta)\ .$$

4.2.2. – **Variation of the magnetization**: Imagine that $\mathcal{M}^k$ (or the 4-vector s^α) is *rigidly linked* (because its normal is constant along $\mathcal{C}$; cf., 3.8) with an orthonormal triad of spacelike vectors $\{\mathbf{a}_{(K)}, K = 1, 2, 3\}$that is defined at the point $\mathbf{x}$ of $\mathcal{C}$. The vectors $\mathbf{a}_{(K)}$ and their reciprocals $\mathbf{a}^{(K)}$ are contained in $M_\perp^3(\mathbf{x})$, so they are P.U., and their quadri-dimensional components satisfy the relations (cf., [**14**]):

(4.18) $$a_{(K)}^{\cdot\ \alpha} u_\alpha = 0, \quad a_{(K)}^{\cdot\ \alpha} a^{(K)}_{\ \cdot\beta} = P^\alpha_{\ \cdot\beta}, \quad a_{(K)}^{\cdot\ \alpha} a^{(L)}_{\ \cdot\alpha} = \delta^L_K\ .$$

In the course of the variation (4.7), the $\mathbf{a}_{(K)}$ remain orthonormal and P.U., in the sense of (4.18). The resultant variation of $\mathbf{a}_{(K)}$ that is defined at $\mathbf{x}$ along $\mathcal{C}$ admits a decomposition into an essentially spatial (P.U.) part that is denoted by $(\delta\mathbf{a}_{(K)})_\perp$ and a time-like part that is denoted by $(\delta\mathbf{a}_{(K)})_\parallel$ and parallel to $u^\alpha(\mathbf{x})$. One then has:

(4.19) $$\begin{cases} \delta a_{(K)}^{\cdot\ \alpha} = (\delta a_{(K)}^{\cdot\ \alpha})_\perp + (\delta a_{(K)}^{\cdot\ \alpha})_\parallel, \\ (\delta a_{(K)}^{\cdot\ \alpha})_\perp u_\alpha = 0, \quad (\delta a_{(K)}^{\cdot\ \alpha})_\parallel \equiv A_K\, u^\alpha. \end{cases}$$

When one varies $(4.18)_3$, multiplies the result by $a_{(L)}^{\cdot\ \beta}$, sums over L, and uses $(4.18)_2$, one will get:

(4.20) $$\begin{cases} (\delta a_{(K)}^{\cdot\ \alpha})_\perp = \delta\omega^\beta_{\ \cdot\alpha}\, a_{(K)}^{\cdot\ \alpha}, \\ \delta\omega^\beta_{\ \cdot\alpha} \equiv -a_{(L)}^{\cdot\ \beta}(\delta a^{(L)}_{\ \cdot\ \alpha}), \quad u_\beta\, \delta\omega^\beta_{\ \cdot\alpha} = 0, \quad \delta\omega^\beta_{\ \cdot\alpha}\, u^\alpha = 0. \end{cases}$$

One shows that the tensor $\delta\omega_{\alpha\beta}$ is *antisymmetric* by varying $(4.18)_2$. Finally, if one multiplies $(4.19)_1$ by u_α and takes into account $(4.19)_3$ then one will get:

(4.21) $$A_K = -\frac{1}{c^2} u_\alpha \delta a_{(K)}^{\cdot\ \alpha} = \frac{1}{c^2} a_{(K)}^{\cdot\ \alpha}\, \delta u_\alpha\ .$$

The latter equality comes from $(4.18)_1$. If one takes (4.19), $(4.20)_1$, (4.21), and (4.12) into account then $(4.19)_1$ can be written:

(4.22) $$\delta a_{(K)}^{\cdot\ \alpha} = \left[\delta\omega^\alpha_{\ \cdot\beta} + \frac{1}{c^2} u^\alpha \frac{D}{Ds}(\delta x_\beta)\right] a_{(K)}^{\cdot\ \beta}\ .$$

The two contributions in brackets provide an infinitesimal rotation of $\mathbf{a}_{(K)}$ in $M_\perp^3(\mathbf{x})$ (or of an inertial frame) and the varied form of the Fermi-Walker transport of constant norm $\mathbf{a}_{(K)}$ along $\mathcal{C}$, respectively. Let $\mathcal{M}^K$ (K = 1, 2, 3) be the *constant* non-holonomic components of $\mathcal{M}^\alpha$ in $\{\mathbf{a}_{(K)}\}$. One then has:

$$\mathcal{M}^\alpha = \mathcal{M}^K a_{(K)}^{\cdot\;\alpha}, \quad \mathcal{M}^K = \mathcal{M}^\alpha a^{(K)}_{\cdot\;\alpha}, \quad \delta\mathcal{M}^K = 0\,.$$

Hence, with (4.22):

$$\text{(4.23)} \qquad \boxed{\delta\mathcal{M}^K = \left[\delta\omega^\alpha_{\cdot\beta} + \frac{1}{c^2} u^\alpha \frac{D}{Ds}(\delta x_\beta)\right]\mathcal{M}^\alpha, \quad (\delta\mathcal{M}^\alpha)_\perp = \delta\omega^\alpha_{\cdot\beta}\,\mathcal{M}^\beta.}$$

Those varied expressions correspond to the finite expressions (3.9). When one takes $(4.23)_2$ into account, (4.11) can be written:

$$\text{(4.24)} \qquad \delta(\mathcal{M}^\lambda\,\mathcal{B}_\lambda) = \mathcal{B}^{[\alpha}\mathcal{M}^{\beta]}\,\delta\omega_{\alpha\beta} + \mathcal{M}_\lambda\,(\nabla_\rho\,\mathcal{B}^\lambda)\,\delta x^\rho - (\mathcal{M}^\alpha\,\mathcal{B}^\beta - \mathcal{B}^\lambda\,\mathcal{M}_\lambda\,g^{\alpha\beta})\,\nabla_\beta\,(\delta x_\alpha)\,.$$

4.2.3. **Kinetic energy of the spin rotation**. – We saw above that one can consider only a form of the kinetic energy of spin rotation that Hertz called "already varied." If we take the forms of the expressions (3.11) and (3.12) for W into account then we can set:

$$\text{(4.25)} \qquad \boxed{\delta W = \int_{\overset{\circ}{\mathcal{B}}} \frac{DS^{\alpha\beta}}{Ds}\left[\delta\omega^\alpha_{\cdot\beta} + \frac{1}{c^2} u^\alpha \frac{D}{Ds}(\delta x_\beta)\right] dv_4,}$$

in which $\delta\omega_{\alpha\beta}$ is arbitrary (i.e., virtual), in order to account for the spin in the open subset $\overset{\circ}{\mathcal{B}}$ of M^4. If we take into account the fact that $\delta\omega_{\alpha\beta}$ and $S^{\alpha\beta}$ are P.U., along with their symmetries, then (4.25) can also be written:

$$\boxed{\delta W = \int_{\overset{\circ}{\mathcal{B}}} \left[\tfrac{1}{2}\rho\left(\frac{DS^{\alpha\beta}}{Ds}\right)_\perp \delta\omega^\alpha_{\cdot\beta} + \frac{1}{c^2}\rho\,S^{\alpha\beta}\,\frac{Du_\alpha}{Ds}\,u_\gamma\nabla_\gamma(\delta x_\beta)\right] dv_4.}$$

That is:

$$\text{(4.26)} \qquad \delta W = \int_{\overset{\circ}{\mathcal{B}}} \left[\tfrac{1}{2}\rho\left(\frac{DS^{\alpha\beta}}{Ds}\right)_\perp \delta\omega^\alpha_{\cdot\beta} - \nabla_\beta\left(\frac{1}{c^2}\rho\,S^{\alpha\beta}\,\dot{u}_\gamma\,u^\beta\right)\delta x_\alpha\right] dv_4 + \int_{\overset{\circ}{\mathcal{B}}} \nabla_\beta\left(\frac{1}{c^2}\rho\,S^{\alpha\gamma}\,\dot{u}_\gamma\,u^\beta\,\delta x_\alpha\right) dv_4\,.$$

4.2.4. **Variation of the energy of the electromagnetic field**. – If one takes into account (4.13)-(4.15) and (4.10) and (4.24) then when one starts from (4.5) and rearranges the terms, one will get:

$$\text{(4.27)} \qquad \delta\tilde{\mathcal{A}}_{(em)} = -\int_{\overset{\circ}{\mathcal{B}}} [(\nabla_\beta\,T^{\alpha\beta}_{(em)})\delta x_\alpha - \rho\,\mathcal{B}^{[\alpha}\mathcal{M}^{\beta]}\,\delta\omega_{\alpha\beta}]\,dv_4 + \int_{\overset{\circ}{\mathcal{B}}} \nabla_\beta(T^{\alpha\beta}_{(em)}\,\delta x_\alpha)\,dv_4\,,$$

in which one has set:

(4.28)
$$T^{\alpha\beta}_{(em)} = \left(\tfrac{1}{2}\mathcal{B}^{\lambda}\mathcal{B}_{\lambda} - \rho\,\mathcal{B}^{\lambda}\mathcal{M}_{\lambda}\right)\frac{u^{\alpha}u^{\beta}}{c^2} - \mathcal{B}^{\alpha}\mathcal{B}^{\beta} + \rho\,\mathcal{M}^{\alpha}\mathcal{B}^{\beta} + \left(\tfrac{1}{2}\mathcal{B}^{\lambda}\mathcal{B}_{\lambda} - \rho\,\mathcal{B}^{\lambda}\mathcal{M}_{\lambda}\right)P^{\alpha\beta}.$$

Remark (*i*). – In the present formulation, we have not considered a proper variation of an electromagnetic parameter (viz., the electromagnetic 4-potential A_α), because we have not sought to deduce the Maxwell equations in the variational formulation (compare with [**15**], [**16**], [**1**]). We shall indicate only what must be done if we were to look for that. Classically, we have:

(4.29)
$$F_{\alpha\beta} = 2\,\nabla_{[\alpha}A_{\beta]}\ .$$

If we introduce the P.U. *magnetic 4-potential* a_α and the *scalar electric potential* φ then thanks to the decomposition of A_α on $M_\perp^3(\mathbf{x})$ and along u_α, we will have:

(4.30)
$$A_\alpha = \frac{\varphi}{c}\,u_\alpha + a_\alpha\,, \qquad \varphi \equiv -\frac{1}{c}\,A_\alpha\,u^\alpha, \qquad a_\alpha\,u^\alpha = 0\ .$$

If we substitute (4.30) in (4.29) and then substitutes the result of that in $(4.4.)_2$ and $(4.4)_3$ then we will get:

(4.31)
$$\mathcal{E}_\alpha = -(\nabla_\alpha\varphi)_\perp - \frac{1}{c}\left(\underset{u}{\mathcal{L}}\,a_\alpha\right)_\perp - \frac{1}{c^2}\,\varphi\,\frac{Du_\alpha}{Ds},$$

(4.32)
$$\mathcal{B}_\alpha = \frac{1}{i\,c}\,\eta^{\alpha\beta\gamma\delta}u_\delta\nabla_\alpha a_\lambda + \frac{1}{i\,c^2}\,\eta^{\alpha\beta\gamma\delta}\,\omega_{\gamma\beta}\,u_\delta,$$

which are representations that are valid inside of deformable matter in motion, and which reduce to the following classical tri-dimensional relations in a *rest frame:*

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{a}}{\partial t}, \qquad \mathbf{B} = \text{rot}\ \mathbf{a}\ .$$

In the present study, $\mathcal{E}_\alpha = 0$; one only needs to consider the P.U. magnetic potential a_α then. If one then introduces Weiss's "gauge-invariant" variation (cf., [**1**], [**15**]) then one will exhibit a proper variation of a_α and a variation that is due to the variation of the world-line $\mathcal{C}$. The variation of $\mathcal{B}^\alpha$ and the proper variation of a_α will then permit one to obtain one of the groups of Maxwell equations (with $\mathcal{E}^\alpha = \mathcal{P}^\alpha = 0$, $q = 0$) in 4-vectorial form, while the other group is satisfied identically by (4.32). We shall not do those calculations here.

Remark (*ii*). – The energy-momentum tensor of the electromagnetic field in matter that is defined by (4.28) is nothing but the tensor that was introduced by Grot and Eringen [**17**], namely:

(4.33) $$T^{\alpha\beta}_{(em)} = -F^{\ \alpha}_{\cdot\ \mu} G^{\mu\beta} + \tfrac{1}{4} F_{\mu\nu} F^{\nu\mu} g^{\alpha\beta}, \quad G^{\mu\beta} \equiv F^{\mu\beta} - \pi^{\mu\beta},$$

when one takes $(4.4)_1$, $(3.3)_1$, and the hypotheses (3.5) and (4.2) into account. Recall that the asymmetric tensor (4.33) that was used before in [**1**], [**3**], and [**16**] is different from the ones that were introduced by Abraham and Minkowski, but it is close to the one that Suttorp and Groot [**18**] constructed, as well as the tensor that was considered by Israel [**19**].

4.3. – Variation of the internal energy.

4.3.1. Introduction. – In continuum physics, the internal specific energy *e* serves to represent the thermodynamic state of the system and the internal interactions (or "internal efforts"). For an *electromagnetic fluid*, the thermodynamic state variables are the density ρ, the entropy η, and the magnetic moment density $\mathcal{M}^\alpha$. The dependency of *e* upon ρ for a fluid represents the matter-matter interaction, since it leads to the definition of thermodynamic pressure, which is a notion that is equivalent to that of normal constraint. In the classical theory of micro-magnetism ([**20**], [**9**]), it is shown that the functional dependency of specific internal energy on the magnetization gradient permits the phenomenological representation of the Heisenberg exchange forces – i.e., the interactions between neighboring spins. In our study, we then take:

(4.34) $$\boxed{e = e(\rho, \eta, \mathcal{M}^\alpha, \mathfrak{M}_{\alpha\beta}),}$$

in which $\mathfrak{M}_{\alpha\beta}$ is the P.U. tensor that was defined in (3.15). Since it results from an infinitesimal approximation in the neighborhood of a world-line $\mathcal{C}$ [cf., (3.14)], the definition (3.15) shows that the interaction that was represented in (4.34) by the intermediary of the dependency on $\mathfrak{M}_{\alpha\beta}$ is a short-range one, or, in the language of continuum mechanics, it is a "contact action" ([7]). One says that one has a *theory of the first gradient* because one takes into account only the first gradient of the magnetization in the expression (4.34). It will be shown later on that the dependency of *e* with respect to $\mathcal{M}^\alpha$ serves to represent the matter-spin interactions.

4.3.2. Consequence of Lorentz invariance. – The specific internal energy *e* is a Lorentz invariant. Consequently, if one considers an infinitesimal transformation in the form (in rectangular coordinates):

(4.35) $$\overset{*}{x}{}^{\alpha} = (\delta^\alpha_\beta + \varepsilon\, Q^\alpha_{\ \beta}) x^\beta, \quad Q_{\alpha\beta} = -Q_{\beta\alpha},$$

([7]) One knows from the work of Dirac, Heisenberg, and Bloch that the intensity of the interactions between spins decreases very rapidly with distance.

in which ε is an infinitesimal, then one will obtain the following transformation laws for the quantities $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$:

(4.36)
$$\overset{*}{\mathcal{M}}{}^\alpha = \mathcal{M}^\alpha + \varepsilon Q^\alpha_\beta \mathcal{M}^\beta ,$$
$$\overset{*}{\mathfrak{M}}{}^{\alpha\beta} = \mathfrak{M}^\alpha + \varepsilon (\mathfrak{M}^{\alpha\lambda} Q^\beta_{\cdot\lambda} + \mathfrak{M}^{\mu\beta} Q^\alpha_{\cdot\mu}) + O(\varepsilon^2).$$

With $\rho = \overset{*}{\rho}, \eta = \overset{*}{\eta},$ and (4.36), one will verify that the condition $e = \overset{*}{e}$ for any $Q_{\alpha\beta}$ (up to order ε^2) leads to the equation:

(4.37)
$$\mathcal{L}^{[\alpha\beta]} \equiv \frac{\partial e}{\partial \mathcal{M}_{[\alpha}} \mathcal{M}^{\beta]} + \frac{\partial e}{\partial \mathfrak{M}_{[\alpha|\mu|}} \mathfrak{M}^{\beta]}_{\cdot\mu} + \frac{\partial e}{\partial \mathfrak{M}_{\mu[\alpha}} \mathfrak{M}_\mu^{\cdot\beta]} = 0 ,$$

which is an equation that one can project onto $M^3_\perp(\mathbf{x})$ and in the direction of u^α at any point $\mathbf{x}$ on $\mathcal{C}$ and get:

(4.38)
$$(\mathcal{L}^{[\alpha\beta]})_\perp = 0 , \qquad u_\alpha \mathcal{L}^{[\alpha\beta]} = 0 .$$

The second of those equations, which is valid for all $\mathcal{M}^\alpha$ and any $\mathfrak{M}_{\alpha\beta}$ [considered as independent variables, from (4.34)], is satisfied if:

(4.39)
$$\frac{\partial e}{\partial \mathcal{M}_{[\alpha}} u_\alpha = 0, \quad \frac{\partial e}{\partial \mathfrak{M}_{[\alpha|\mu|}} u_\alpha = 0, \quad \frac{\partial e}{\partial \mathfrak{M}_{[\alpha|\mu|}} u_\beta = 0.$$

One can then define the P.U. fields ${}^l\mathcal{B}^\alpha$ and $\tau^{\alpha\beta}$ by:

(4.40)
$$\boxed{{}^l\mathcal{B}^\alpha \equiv -\left(\frac{\partial e}{\partial \mathcal{M}_\alpha}\right)_\perp = -\frac{\partial e}{\partial \mathcal{M}_\alpha}, \quad \tau^{\alpha\beta} \equiv \rho \left(\frac{\partial e}{\partial \mathfrak{M}_{\alpha\beta}}\right)_\perp = \rho \frac{\partial e}{\partial \mathfrak{M}_{\alpha\beta}}.}$$

Equation (4.38)1 then implies the condition:

(4.41)
$$\boxed{-\rho\, {}^l\mathcal{B}^{[\alpha} \mathcal{M}^{\beta]} + \tau^{[\alpha}_{\cdot\mu} \mathfrak{M}^{\beta]\mu} + \tau^{\mu[\alpha} \mathfrak{M}_\mu^{\cdot\beta]} = 0.}$$

The P.U. 4-vector field ${}^l\mathcal{B}^\alpha$ is homogeneous to a magnetic induction; one calls it the *local magnetic induction*, or *magnetic anisotropy field*, by analogy with the theory of micro-magnetism (cf., [**9**]). From what was said above, one can call the P.U. tensor $\tau^{\alpha\beta}$, which generally has nine independent components, the *spin interaction tensor*. One can now calculate δe .

4.3.3. Variation. – One immediately has:

(4.42) $$\rho\,\delta e = \frac{p}{\rho}\,\delta\rho + \rho\,\theta\,\delta\eta - {}^{l}\mathcal{B}_\alpha\,(\delta\mathcal{M}^\alpha)_\perp + \tau^{\alpha\beta}\,(\delta\mathfrak{M}_{\alpha\beta})_\perp\,,$$

in which one defines the *thermodynamic pressure p* and the *proper temperature* θ in the usual manner by:

(4.43) $$\boxed{p \equiv \rho^2 \left.\frac{\partial e}{\partial\rho}\right|_{\eta,\mathcal{M}^\alpha,\mathfrak{M}_{\alpha\beta}}\,, \quad \theta \equiv \left.\frac{\partial e}{\partial\eta}\right|_{\eta,\mathcal{M}^\alpha,\mathfrak{M}_{\alpha\beta}}\,.}$$

By using (4.8), the constraint (4.6), and (2.10), one can show that:

(4.44) $$\rho\,\theta\,\delta\eta = \rho\,\theta\,\eta\,\frac{u^\alpha u^\beta}{c^2}\nabla_{[\beta}(\delta x_{\alpha]})\,.$$

A lengthy calculation that requires the use of (3.15), (4.16), (4.17), (3.9), and (4.23), and which we shall not perform explicitly, leads to the expression for the last term in (4.42). We get:

(4.45) $$\tau^{\alpha\beta}\,\delta\mathfrak{M}_{\alpha\beta} = \nabla_\mu\,(M^{\beta\alpha\mu}\,\delta\omega_{\alpha\beta}) - (\nabla_\mu\,\tau^{[\alpha|\mu|})\,\mathcal{M}^{\beta]}\,\delta\omega_{\alpha\beta}$$
$$-\left(\frac{2}{c^2}M^{\mu\alpha\nu}e_{\mu\nu}u^\beta + \frac{2}{c^2}M^{\gamma\mu(\alpha}u^{\beta)}\Omega_{\mu\gamma} + \tau_\mu^{\cdot\beta}\mathfrak{M}^{\mu\alpha}\right)\nabla_\beta(\delta x_\alpha)\,,$$

in which we define the P.U. tensor $M^{\gamma\nu\alpha}$ by:

(4.46) $$\boxed{M^{\mu\alpha\nu} \equiv \mathcal{M}^{[\gamma}\tau^{\mu]\alpha}, \quad M^{\gamma\mu\alpha} = -M^{\mu\gamma\alpha}, \quad M^{\gamma\mu\alpha}u_\gamma = 0, \quad M^{\gamma\mu\alpha}u_\alpha = 0.}$$

If we then substitute the results (4.14), (4.44), $(4.23)_2$, and 4.45) in (4.42) then that will give:

(4.47) $$\rho\,\delta e = -\left(\frac{2}{c^2}M^{\mu\alpha\nu}e_{\mu\nu}u^\beta + \frac{2}{c^2}M^{\gamma\mu[\alpha}u^{\beta]}\Omega_{\mu\gamma} + \tau_\mu^{\cdot\beta}\mathfrak{M}^{\mu\alpha} - \rho\,\theta\,\eta\,\frac{u^\alpha u^\beta}{c^2} + p\,P^{\alpha\beta}\right)\nabla_\beta(\delta x_\alpha)$$
$$-\,[\rho\,{}^{l}\,\mathcal{B}^{[\alpha}\,\mathcal{M}^{\beta]} + (\nabla_\mu\,\tau^{[\alpha\,|\,\mu\,|})\,\mathcal{M}^{\beta]}]\,\delta\omega_{\alpha\beta} + \nabla_\mu\,(M^{\beta\alpha\mu}\delta\omega_{\alpha\beta})\,.$$

Finally, in the course of variation, one must take into account the constraint $(2.2)_2$ that is imposed upon the 4-velocity. In order to do that, one introduces a Lagrange multiplier $\mathfrak{M}$. One then writes the action that represents the matter that is associated with $\overset{\circ}{\mathcal{B}} \subset M^4$ in the form:

(4.48) $$\mathcal{A}_{(m)} = -\int_{\overset{\circ}{\mathcal{B}}} \rho\left[e - \tfrac{1}{2}\mathfrak{M}(u^\alpha u_\alpha + c^2)\right]dv_4\,.$$

The variation of $\mathfrak{M}$ will yield the constraint $(2.2)_2$. When one takes (4.15), (4.47), and (4.12) into account, one will then have:

(4.49) $$\delta\mathcal{A}_{(m)} = -\int_{\overset{\circ}{\mathcal{B}}} \left\{ (\nabla_\beta \tilde{T}^{\alpha\beta})\,\delta x_\alpha - [\rho^l\, \mathcal{B}^{[\alpha}\mathcal{M}^{\beta]} + (\nabla_\beta \tau^{[\alpha|\mu|})\mathcal{M}^{\beta]}]\,\delta\omega_{\alpha\beta} \right\} dv_4$$

$$+\int_{\overset{\circ}{\mathcal{B}}} \nabla_\beta\,(\tilde{T}^{\alpha\beta}\,\delta x_\alpha - M^{\beta\alpha\mu}\delta\omega_{\alpha\beta}\, dv_4\ ,$$

in which one sets:

(4.50) $$\tilde{T}^{\alpha\beta} \equiv \rho\left(\mathfrak{M} + \frac{\psi}{c^2}\right)u^\alpha u^\beta + p\,P^{\alpha\beta} + \mathfrak{M}^{\mu\alpha}\,\tau_\mu^{\ \cdot\beta} + \frac{2}{c^2}M^{\mu\alpha\nu}e_{\mu\nu}\,u^\beta + \frac{2}{c^2}M^{\gamma\mu[\alpha}u^{\beta]}\Omega_{\mu\nu},$$

in which ψ is the specific free energy that is defined by the Legendre transformation:

(4.51) $$\psi\,(\rho,\,\theta,\,\mathcal{M}^\alpha,\,\mathfrak{M}_{\alpha\beta}) = e\,(\rho,\,\theta,\,\mathcal{M}^\alpha,\,\mathfrak{M}_{\alpha\beta}) - \eta\,\theta\,,$$

with:

(4.52) $$\boxed{p = \rho^2\frac{\partial\psi}{\partial\rho},\quad \eta = -\frac{\partial\psi}{\partial\theta},\quad {}^l\mathcal{B}^\alpha = -\left(\frac{\partial\psi}{\partial\mathcal{M}_\alpha}\right)_\perp,\quad \tau^{\alpha\beta} = \rho\left(\frac{\partial\psi}{\partial\mathfrak{M}_{\alpha\beta}}\right)_\perp}$$

from now on.

4.4. – Variational principle for the "perfect ferrofluid" model.

4.4.1. Statement. – *The conservation laws and the laws of behavior of the "relativistic perfect ferrofluid" model follow from the variational principle:*

(4.53) $$\boxed{\delta\mathcal{A}_{(m)} + \delta\tilde{\mathcal{A}}_{(em)} + \delta W = 0}$$

for any variation δx_α *of the streamlines* $\mathcal{C}$ *and any arbitrary rotation* $\delta\omega_{\alpha\beta}$ *of the spin in an inertial frame, where the expressions* $\mathcal{A}_{(m)}$, $\tilde{\mathcal{A}}_{(em)}$ *, and* δW *are given by* (4.48), (4.5), *and* (4.25), *respectively.*

4.4.2. Proof. – Combining the results (4.49), (4.27), and (4.26), the variational principle (4.53) will lead to:

(4.54) $$-\int_{\overset{\circ}{\mathcal{B}}} \left\{ \left(\nabla_\beta T^{\alpha\beta}_{(\text{tot})}\right)\delta x_\alpha - \left[\tfrac{1}{2}\rho\left(\frac{DS^{\alpha\beta}}{Ds}\right)_\perp + \rho\mathcal{B}^{[\alpha}_{\text{eff.}}\,\mathcal{M}^{\beta]}\right]\delta\omega_{\alpha\beta}\right\} dv_4 + O(\partial\mathcal{B}) = 0,$$

in which $O\,(\partial\mathcal{B})$ means "modulo a surface term," and one defines:

(4.55) $$T^{\alpha\beta}_{(\text{tot})} \equiv \tilde{T}^{\alpha\beta} + \frac{\rho}{c^2}S^{\alpha\gamma}\,\frac{Du_\gamma u^\beta}{Ds} + \tilde{T}^{\alpha\beta}_{(em)},$$

(4.56) $$\mathcal{B}^\alpha_{\text{eff.}} \equiv \mathcal{B}^\alpha + {}^l\mathcal{B}^\alpha + \rho^{-1}\nabla_\mu\tau^{\alpha\mu},$$

(4.57) $$O(\partial B) \equiv \int_{\overset{\circ}{\mathcal{B}}} \nabla_{\mu} (T^{\alpha\mu}_{(\mathrm{tot})}\, \delta x_{\alpha} - M^{\beta\alpha\mu} \delta\omega_{\alpha\beta})\, dv_4 \,.$$

The last expression is, in fact, a surface integral over the hypersurface $\partial\mathcal{B}$, which is the boundary of the domain $\mathcal{B}$ in M^4, and $\overset{\circ}{\mathcal{B}}$ is an open subset of the latter. One supposes that either the variations δx_α and $\delta\omega_{\alpha\beta}$ are annulled on $\partial\mathcal{B}$ or that $\partial\mathcal{B}$ is pushed out to infinity. Meanwhile, note that $M^{\beta\alpha\mu}$ enters on $\partial\mathcal{B}$ in a term of the form $M^{\beta\alpha\mu} N_\mu$ [N_μ : unit (spacelike) normal oriented external to $\partial\mathcal{B}$] as a factor of the term $\delta\omega_{\alpha\beta}$. That shows that the interaction between neighboring magnetic spins manifests itself in the form of a *contact action*, which is a surface couple here. Since the expression (4.54) is valid for any continuous region of $\overset{\circ}{\mathcal{B}}$ in M^4 and any variations δx_α and $\delta\omega_{\alpha\beta}$, it leads to local conservation equations in $\overset{\circ}{\mathcal{B}}$:

(4.58) $$\nabla_{\mu} (T^{\alpha\mu}_{(\mathrm{tot})} = 0,$$

(4.59) $$\left(\frac{DS^{\alpha\beta}}{Ds}\right)_{\perp} = 2\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]}_{\mathrm{eff.}}.$$

The first of these equations obviously expresses the conservation of energy-momentum. The second one provides the evolution of the magnetic spin $S^{\alpha\beta}$ in an inertial frame (in covariant form). It remains to establish the expressions for $\Omega_{\alpha\beta}$ and $\mathfrak{M}$ that enter into $\tilde{T}^{\alpha\beta}$ and to show that (4.59) is nothing but the equation of conservation of the moment of energy-momentum.

4.4.3. Velocity of spin precession. – Express the left-hand side of (4.59) with the aid of $(3.6)_1$ and multiply the result by (1 / *i c*) $\eta_{\alpha\nu\gamma\delta}$. Upon rearranging the indices, one will get:

$$\left(\frac{D\mathcal{M}_{\gamma}}{Ds}\right)_{\perp} = \left(-\frac{\gamma}{ic}\eta_{\gamma\alpha\beta\delta}\, \mathcal{B}^{\beta}_{\mathrm{eff.}}\, u^{\delta}\right)\mathcal{M}^{\alpha}.$$

When one identifies that with $(3.9)_2$, one will see that since $\Omega_{\alpha\beta}$ is P.U., the only possibility is:

(4.60) $$\Omega_{\alpha\beta} \equiv -\frac{\gamma}{ic}\eta_{\gamma\alpha\beta\delta}\, \mathcal{B}^{\beta}_{\mathrm{eff.}}\, u^{\delta}.$$

From (4.56), the velocity of spin precession in an inertial frame is provided by the conjugate action of the Maxwell magnetic induction $\mathcal{B}^\alpha$, the magnetic anisotropy field ${}^{l}\mathcal{B}^\alpha$, and the interaction between neighboring magnetic spins by the intermediary of $\tau^{\alpha\beta}$. If one introduces the P.U. 4-

vector π^α that was defined in (3.10) and takes (3.6) into account then one can write (4.59) in the form ([8]):

(4.61)
$$\boxed{\begin{gathered}\left(\frac{D\pi^{\alpha\beta}}{Ds}\right)_\perp = 2\pi^{[\alpha}\mathcal{M}^{\beta]},\\ \pi^\alpha = -\gamma\,\mathcal{B}^\alpha_{\text{eff.}} = -\gamma(\mathcal{B}^\alpha + {}^l\mathcal{B}^\alpha + \rho^{-1}\nabla_\mu\tau^{\alpha\mu}).\end{gathered}}$$

4.4.4. Conservation of the moment of energy-momentum. – Upon developing equation (4.59), it will give:

(4.62)
$$\tfrac{1}{2}\rho\left(\frac{D\pi^{\alpha\beta}}{Ds}\right)_\perp = \rho\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]} + \rho\mathcal{M}^{[\alpha\,l}\mathcal{B}^{\beta]} + \rho^{-1}\mathcal{M}^{[\alpha}\,\nabla_\mu\tau^{\beta]\mu}.$$

However, from (4.55), (4.50), and (4.28):

(4.63)
$$T^{[\alpha\beta]}_{(\text{tot})} = \mathfrak{M}^{\mu[\alpha}\,\tau_\mu^{\cdot\,\beta]} + \frac{2}{c^2}M^{\mu[\alpha|\nu|}e_{\mu\nu}u^{\beta]} + \rho\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]} + \frac{\rho}{c^2}S^{[\alpha|\gamma|}\frac{Du_\gamma}{Ds}u^{\beta]}.$$

If one eliminates $\rho\ \mathcal{M}^{[\alpha}\ \mathcal{B}^{\beta]}$ from these two equations and takes into account the invariance condition (4.41), which is likewise valid when ψ is the thermodynamic potential, and regroups the terms then one will arrive at the equation:

(4.64)
$$\tfrac{1}{2}\rho\left[\left(\frac{DS^{\alpha\beta}}{Ds}\right)_\perp + \frac{2}{c^2}S^{[\alpha|\gamma|}u^{\beta]}\frac{Du_\gamma}{Ds}\right] - T^{[\alpha\beta]}_{(\text{tot})} = \left[(\nabla_\mu M^{\alpha\beta\mu})_\perp - \frac{2}{c^2}M^{[\alpha|\mu\nu|}u^{\beta]}e_{\mu\nu}\right].$$

As one easily verifies, the terms in […] are nothing but the terms $DS^{\alpha\beta} / Ds$ and $\nabla_\mu M^{\alpha\beta\mu}$, but not projected onto $M^3_\perp$. When one takes $(2.10)_1$ and (2.3) into account, (4.63) will then be written in the canonical form of a conservation law for the moment of energy-momentum:

(4.65)
$$\boxed{\nabla_\mu\mathcal{S}^{\alpha\beta\mu} - T^{[\alpha\beta]}_{(\text{tot})} = 0,}$$

in which one defines the total spin tensor $\mathcal{S}^{\alpha\beta\mu}$ by:

(4.66)
$$\boxed{\mathcal{S}^{\alpha\beta\mu} \equiv \frac{\rho}{2}S^{\alpha\beta}u^\mu - M^{\alpha\beta\mu},\quad \mathcal{S}^{\alpha\beta\mu} = -\mathcal{S}^{\beta\alpha\mu},\quad \mathcal{S}^{\alpha\beta\mu}u_\alpha = 0.}$$

Note that if equations (4.59) and (4.65) are equivalent, and if (4.65) lends itself to geometrization better then the form of equation (4.59) or $(4.61)_1$ is more interesting because it contains the expression for the velocity of spin precession. The kinematical equation $(3.9)_1$ is likewise equivalent to equations (4.59) and (4.65), up to γ, although its form is not a kinematical

([8]) Recall that for an isolated electron of magnetic moment **m**, one has $\partial\mathbf{m} / \partial t = \boldsymbol{\omega}_\text{B} \times \mathbf{m}$, classically, where $\boldsymbol{\omega}_\text{B} = -\gamma\,\mathbf{B}$ is the Larmor precession.

consequence of the constraint (3.8). The interest in the present variational formulation is in the fact that it provides the form of the precession tensor as a function of the various interactions, as well as the coupling that is present in the energy-momentum tensor.

4.4.5. Determining the multiplier $\mathfrak{M}$. – That calculation is very tedious, and we shall give only the main steps to it. One must first consider the following intermediate results:

(*a*) Taking (3.12) into account and multiplying (4.64) by $\Omega_{\alpha\beta}$ will give:

$$\left(T^{[\alpha\beta]}_{(\text{tot})} + \nabla_\mu M^{\alpha\beta\mu}\right)_\perp \Omega_{\alpha\beta} = 0,$$

namely, along with (4.63):

$$\left(\nabla_\mu M^{\alpha\beta\mu}\right)_\perp \Omega_{\alpha\beta} = -(\mathfrak{M}^{\mu[\alpha}\,\tau_\mu^{\cdot\,\beta]} + \rho\,\mathcal{B}^{[\alpha}\mathcal{M}^{\beta]})\,\Omega_{\alpha\beta}. \tag{4.67}$$

(*b*) By virtue of the condition (4.41), one has:

$$\rho\,{}^{l}\mathcal{B}^{[\alpha}\mathcal{M}^{\beta]}\,\Omega_{\alpha\beta} = -(\tau^{[\alpha}_{\cdot\,\mu}\,\mathfrak{M}^{\beta]\mu} + \tau^{\mu[\alpha}\,\mathfrak{M}_\mu^{\cdot\,\beta]})\,\Omega_{\alpha\beta}. \tag{4.68}$$

(*c*) One calculates $D\psi / Ds$ by starting from (4.51) and using $(2.10)_1$, (4.6), (4.52), and $(3.9)_1$. The calculation, which is simple, but somewhat lengthy, and is similar to what one did for the variations that led to (4.47), leads to the result:

$$\rho\frac{D\psi}{Ds} = -p\,\Theta - \rho\frac{D(\eta\theta)}{Ds} - \rho\,{}^{l}\mathcal{B}^{[\alpha}\mathcal{M}^{\beta]}\,\Omega_{\alpha\beta} - \tau^{\mu\beta}\,\mathfrak{M}_\mu^{\cdot\,\alpha}e_{\alpha\beta} + \tau^{\mu[\alpha}\,\mathfrak{M}^{\beta]}_{\cdot\,\mu}\,\Omega_{\alpha\beta} + M^{\lambda\mu\nu}A_{\mu\lambda\nu}, \tag{4.69}$$

in which $\mathcal{A}_{\mu\lambda\nu}$ is the (P.U.) kinematical quantity that was introduced in [**8**]:

$$\begin{cases} \mathcal{A}_{\mu\lambda\nu} \equiv \left(\nabla_\nu\,\Omega_{\mu\lambda} + \dfrac{1}{c^2}\Omega_{\mu\lambda}\,\dfrac{Du_\nu}{Ds} + \dfrac{1}{c^2}(\nabla_\nu\,u_{[\mu}\,\dfrac{Du_{\lambda]}}{Ds}\right)_\perp, \\ \mathcal{A}_{\mu\lambda\nu} = -\mathcal{A}_{\lambda\mu\nu}, \quad \mathcal{A}_{\mu\lambda\nu}\,u^\mu = 0, \quad \mathcal{A}_{\mu\lambda\nu}\,u^\nu = 0, \end{cases} \tag{4.70}$$

which relativistically generalizes the notion of the gradient of the velocity of precession. In order to determine the value of $\mathfrak{M}$, we then employ the same method as in [**1**] (which was introduced by Taub [**21**]): We project (4.58) along the direction of u_α in such a manner as to obtain the equation of "conservation of energy." When we take (4.55), (4.50), (4.28), and $(2.10)_1$ into account, it will become:

$$-\left(\rho c^2\,\frac{D\mathfrak{M}}{Ds} + \rho\frac{D\psi}{Ds} + p\,\Theta + \mathfrak{M}^{\mu\alpha}\,\tau_\mu^{\cdot\,\beta}e_{\alpha\beta}\right) + A = -\rho\,\mathcal{B}^{[\alpha}\mathcal{M}^{\beta]}\,\Omega_{\alpha\beta},$$

or

$$A \equiv u_\alpha \nabla_\beta \left(\frac{\rho}{c^2} S^{\alpha\gamma} \frac{Du_\gamma}{Ds} + \frac{2}{c^2} M^{\mu\alpha\nu} e_{\mu\nu} u^\beta + \frac{2}{c^2} M^{\gamma\mu[\alpha} u^{\beta]} \Omega_{\mu\gamma} \right).$$

The calculation of A is done by successively using the results (4.67)-(4.69). The term $D\psi / Ds$ is eliminated, and the differential equation that determines $\mathfrak{M}$, which is valid along the streamline $\mathcal{C}$, is finally written:

(4.71)
$$\frac{D}{Ds} (c^2\, \mathfrak{M} - \eta\, \theta) = 0 .$$

When one integrates that along $\mathcal{C}$ and introduces the constant of integration c^2 (viz., the rest energy per unit of proper mass of the fluid), one will then have:

(4.72)
$$\boxed{\mathfrak{M} = 1 + \frac{\eta\theta}{c^2}.}$$

4.4.6. Summary of the results. – The conservation laws and the laws of behavior that correspond to the "relativistic perfect ferrofluid" model in an open subset $\overset{\circ}{\mathcal{B}}$ of M^4 are equations (4.58), (4.59) or (4.65), $(2.10)_1$, (4.6), and (4.52), to which one agrees to add the constraint (3.8) and Maxwell's equations, corresponding to the simplification hypotheses (3.5) and (4.2) ([9]). When one takes the results (4.50), (4.28), (4.55), and (4.72) into account, one can write the total energy-momentum tensor $T^{\alpha\beta}_{(\mathrm{tot})}$ in the decomposed canonical form:

(4.73)
$$T^{\alpha\beta}_{(\mathrm{tot})} \equiv \omega_{(\mathrm{tot})}\, u^\alpha u^\beta + p^\alpha u^\beta + \bar{p}^\alpha u^\beta - t^{\beta\alpha} - \bar{t}^{\beta\alpha}_{(em)} ,$$

with

(4.74)
$$\begin{cases} \omega_{(\mathrm{tot})} \equiv \rho \left[1 + c^{-2} \left(\psi + \eta\theta + \dfrac{1}{2\rho} \mathcal{B}^\lambda \mathcal{B}_\lambda - \mathcal{M}^\lambda \mathcal{B}_\lambda \right) \right], \\ p^\alpha \equiv \dfrac{1}{c^2} \left(\rho S^{\alpha\gamma} \dfrac{Du_\gamma}{Ds} + 2 M^{\mu\alpha\nu} e_{\mu\nu} + M^{\gamma\alpha\nu} \Omega_{\mu\lambda} \right), \\ \bar{p}^\alpha \equiv \dfrac{1}{c^2} M^{\gamma\mu\beta} \Omega_{\mu\lambda}, \\ t^{\beta\alpha} \equiv -p\, P^{\alpha\beta} - \tau^{\mu\beta} M_{\mu}^{\cdot\alpha}, \\ \bar{t}^{\beta\alpha}_{(em)} \equiv \mathcal{B}^\alpha \mathcal{B}^\beta - \rho \mathcal{M}^\alpha \mathcal{B}^\beta - \left(\tfrac{1}{2} \mathcal{B}^\lambda \mathcal{B}_\lambda - \rho \mathcal{B}^\lambda \mathcal{M}_\lambda \right) P^{\alpha\beta}, \end{cases}$$

in which $\Omega_{\gamma\alpha}$ and $M^{\gamma\mu\alpha}$ are given by (4.60) and $(4.46)_1$, respectively, and p, θ, ${}^l\mathcal{B}^\alpha$, and $\tau^{\alpha\beta}$ are determined from equations (4.52) when one starts from ψ.

([9]) Those Maxwell equations are given in 4-vectorial form in reference [**3**].

4.5. – Equation of conservation of energy.

One can give a remarkable form to the equation of "conservation of energy." When one transforms the third term on the right-hand side of (4.69) with the aid of (4.68), one will have:

$$\rho \frac{D(\psi + \eta\theta)}{Ds} = -p\,\Theta - \tau_{\mu\beta}\, M_{\mu}^{\cdot\alpha}(e_{\alpha\beta} - \Omega_{\alpha\beta}) + M^{\lambda\mu\nu} \mathcal{A}_{\mu\lambda\nu}. \tag{4.75}$$

However, according to $(4.74)_4$ and the definition $\Theta \equiv P^{\alpha\beta}\, \sigma_{\alpha\beta}$, one has:

$$t^{(\beta\alpha)} = -(p\, P^{\alpha\beta} + \tau^{\mu(\beta}\, \mathfrak{M}_{\mu}^{\cdot\alpha)}), \quad t^{[\beta\alpha]} = -\tau^{\mu[\beta}\, \mathfrak{M}_{\mu}^{\cdot\alpha]}. \tag{4.76}$$

With (4.51) and (4.76), one sees that (4.75) is nothing but ([10]):

$$\boxed{\rho \frac{De}{Ds} = t^{(\beta\alpha)} \sigma_{\alpha\beta} + t^{[\beta\alpha]} \nu_{\alpha\beta} + M^{\alpha\beta\gamma} \mathcal{A}_{\beta\alpha\gamma},} \tag{4.77}$$

in which the P.U. antisymmetric tensor $\nu_{\alpha\beta}$ represents the relative velocity of spin precession with respect to the rotation of the fluid:

$$\boxed{\nu_{\alpha\beta} = \omega_{\alpha\beta} - \Omega_{\alpha\beta} = -\nu_{\beta\alpha}, \quad \nu_{\alpha\beta}\, u^{\alpha} = 0.} \tag{4.78}$$

The relation (4.77) is valid only for reversible thermodynamic processes ([10]). Consistent with its interpretation in the thermodynamics of continuous media (cf., Germain [**22**] and de Groot and Mazur [**25**]), and the notion of duality that is inherent to that kind of thermodynamics, the expression (4.77) signifies that $t^{(\beta\alpha)}$, $t^{[\beta\alpha]}$ (the symmetric and antisymmetric parts of the *relativistic constraint tensor*), and $M^{\lambda\mu\nu}$ are "forces" that are derivable from the potential e, the kinematical quantities $\sigma_{\alpha\beta}$, $\nu_{\alpha\beta}$, and $\mathcal{A}_{\mu\lambda\nu}$ are the corresponding "generalized velocities." It then follows that we have an interpretation for the coupling of the spin field to the velocity field of the fluid that is analogous to the one that is described in the phenomenological or statistical theories of continuous media with micro-structure ([11]).

([10]) The complete equation for the general thermodynamic process in relativistic media with spin was obtained by another method in the reference [**24**] [equation (2.4)]:

$$\rho \frac{De}{Ds} + P_{\beta}^{\cdot\gamma}\, \nabla_{\gamma}\, q^{\beta} + \rho h = t^{(\beta\alpha)} \sigma_{\alpha\beta} + M^{\alpha\beta\gamma} \mathcal{A}_{\beta\alpha\gamma} + \mathcal{E}_{\gamma}\, j^{\gamma},$$

in which $t^{\beta\alpha}$ and $M^{\alpha\beta\gamma}$ present conservative parts that are derived from a potential, but also dissipative parts, and q^{β} and h are the P.U. heat flux 4-vector and the massive heat source, respectively.

([11]) "The spin field is a kinematical macroscopic representation of the internal angular momentum of spinning molecules and is dynamically coupled to the fluid velocity by means of the collisional interactions of the translating and rotating molecules. This coupling was described somewhat earlier by Born [**26**] and later by Grad [**27**], who regarded the difference between the fluid vorticity $\frac{1}{2}\nabla \times \mathbf{U}$ and the molecular spin precession $\mathbf{W}$ as the kinematical strain field that is responsible for giving rise to an antisymmetric state of stress." ([**25**], pp. 914).

4.6. – Consequences of the isotropy of the fluid and linearization.

The model that was established in paragraph 4.4 includes laws of behavior – for ${}^l\mathcal{B}^\alpha$ and $\tau^{\alpha\beta}$ (or $M^{\alpha\beta\mu}$) – that one can qualify as *nonlinear* ones, because the potentials ψ or e are functions that are not specified by their tensorial arguments. Moreover, we have not taken into account an essential property that is coupled with the specific character of fluids: Whether relativistic or not, *fluids are necessarily isotropic* ([12]). The potential e [eq. (4.34)] must then be necessarily an *isotropic* function of its arguments. Since the tensorial arguments $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$ are P.U., so they are essentially equivalent to tri-dimensional arguments ([13]), one can employ the recent theorems of Wang [**29**] on the representations of isotropic functions. Since $\mathcal{M}^\alpha$ is P.U. and must satisfy the constraint (3.8), $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$ will have only two and nine independent scalar components, respectively, so $N = 11$. According to Wang [**29**], since the dimensionality of space is n ($n = 3$ here, because the representation space of $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$ is $M_\perp^3$), in order for the scalar e to be an isotropic function, it is necessary and sufficient that e must be a function of $N - n (n - 1) / 2 = 11 - 3 = 8$ mutually functionally-independent invariants that are constructed by starting from $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$. One will then have:

$$e = e\,(\rho,\ \eta,\ I_{(\alpha)}\,;\ \alpha = i, \dots, 8)\,. \tag{4.79}$$

The invariants can be chosen from the lists that were produced by Spencer [**30**] when one notes that $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$ behave like axial vectors. One can take:

$$\left\{\begin{aligned}
&I_{(1)} = \tfrac{1}{2}\mathcal{M}^\alpha\mathcal{M}_\alpha, \quad I_{(2)} = \tfrac{1}{2}(\mathfrak{M}^\alpha_{\cdot\alpha})^2, \quad I_{(3)} = \tfrac{1}{2}\mathfrak{M}^{\alpha\beta}\,\mathfrak{M}_{\alpha\beta}, \quad I_{(3)} = \tfrac{1}{2}\mathfrak{M}^{\alpha\beta}\,\mathfrak{M}_{\beta\alpha}, \\
&I_{(5)} = \tfrac{1}{2}\,\mathrm{sign}\,(\mathcal{M}^\gamma)\mathcal{M}^\alpha\mathcal{M}^\beta\,\mathfrak{M}_{(\alpha\beta)}, \quad I_{(6)} = \tfrac{1}{2}\,\mathcal{M}^\alpha\mathcal{M}^\beta\mathcal{M}^\gamma\,\mathfrak{M}_{(\alpha\beta)}\,\mathfrak{M}_{(\gamma\delta)}, \\
&I_{(7)} = \tfrac{1}{3}\mathfrak{M}^{\alpha\beta}\,\mathfrak{M}_{\beta\gamma}\,\mathfrak{M}^\gamma_{\cdot\alpha}, \quad I_{(8)} = \tfrac{1}{2}\mathfrak{M}^{\alpha\beta}\,\mathfrak{M}_{\beta\gamma}\,\mathcal{M}^\gamma\mathcal{M}_\alpha,
\end{aligned}\right. \tag{4.80}$$

which satisfies the criterion of functional independence. According to (4.40), one will have:

$${}^l\mathcal{B}^\alpha = -\sum_{(\beta)}\alpha_{(\beta)}\frac{\partial I_{(\beta)}}{\partial\mathcal{M}_\alpha}, \quad \tau^{\alpha\beta} = \rho\sum_{(\beta)}\alpha_{(\beta)}\frac{\partial I_{(\beta)}}{\partial\mathfrak{M}_{\alpha\beta}}, \tag{4.81}$$

([12]) One can show that this is a consequence of the fact that fluids have no "memory" of an earlier configuration. The proof in the classical mechanics of continuous media is given in [**28**].

([13]) The notion of isotropy (and the crystallographic group) is linked with the usual tri-dimensional Euclidian concept of physical space. Isotropy must then be studied in an orthonormal frame at **x** on $\mathcal{C}$ of the type that was introduced in (4.18). Meanwhile, the invariants obtained in (4.80) are identical to the ones that one constructs by starting with the non-holonomic components of $\mathcal{M}^\alpha$ and $\mathfrak{M}_{\alpha\beta}$ in such a frame.

in which the scalar coefficients $\alpha_{(\beta)}$, $\beta = 1, \dots, 8$ are again functions of ρ, η, and $I_{(\beta)}$. Some *finite* expressions for ${}^{l}\mathcal{B}^{\alpha}$ and $\tau^{\alpha\beta}$ are implied immediately from equations (4.81) and (4.80). They are the *exact* laws of behavior for the "perfect ferrofluid" model. We shall not give those expressions but will consider some *quasi-linear* laws of behavior. Let $O(\mathcal{M})$ be the order of magnitude of the components of $\mathcal{M}^{\alpha}$. Suppose that $O(\mathfrak{M}_{\alpha\beta}) = O(\mathcal{M})$. The invariants $I_{(\beta)}$, $\beta = 1, \dots, 4$ are $O(\mathcal{M}^2)$, and all of the other invariants that one can construct with the aid of $\mathcal{M}^{\alpha}$ and $\mathfrak{M}_{\alpha\beta}$ are of higher order. In order to obtain isotropic linear laws of behavior, it will then suffice to consider only invariants of order $O(\mathcal{M}^2)$. With that hypothesis, when one starts from (4.81) and (4.80), one will get:

(4.82) $$\begin{cases} {}^{l}\mathcal{B}^{\alpha} = -\alpha_{(1)}\mathcal{M}^{\alpha}, \\ \tau^{\alpha\beta} = \rho[\alpha_{(2)}\,\mathfrak{M}^{\gamma}_{\cdot\,\gamma}\,P^{\alpha\beta} + \alpha_{(3)}\,\mathfrak{M}^{\alpha\beta} + \alpha_{(4)}\,\mathfrak{M}^{\beta\alpha}], \end{cases}$$

(4.83) $$M^{\gamma\alpha\beta} \equiv \mathcal{M}^{[\gamma}\,\tau^{\alpha]\,\beta} = r\,[\alpha_{(2)}\,\mathfrak{M}^{\mu}_{\cdot\,\mu}\mathcal{M}^{[\gamma}P^{\alpha]\,\beta} + \alpha_{(3)}\,\mathcal{M}^{[\gamma}\mathfrak{M}^{\alpha]\,\beta} + \alpha_{(4)}\,\mathcal{M}^{[\gamma}\mathfrak{M}^{\beta]\,\alpha}]\,,$$

or now:

(4.84) $$\alpha_{(\beta)} = \alpha_{(\beta)}\,(\rho,\,\eta,\,\mathcal{M}^2),\ b = 1, \dots, 8, \qquad \mathcal{M}^2 \equiv P^{\alpha\beta}\,\mathcal{M}_{\alpha}\,\mathcal{M}_{\beta}\,.$$

Equations (4.82) and (4.83) are *quasi-linear* laws of behavior for a perfect ferromagnetic fluid ([14]). One remarks that:

(*a*) ${}^{l}\mathcal{B}^{\alpha}$ is collinear with $\mathcal{M}^{\alpha}$. From (4.59), the magnetic anisotropy field plays no role in that fluid. Nonetheless, it will play a role if one preserves the complete expression (4.79).

(*b*) There are obviously no magnetostriction or piezomagnetism effects in the fluid [cf., $(4.74)_4$].

(*c*) The terms that represent the interaction between neighboring magnetic spins in the expressions $(4.74)_{2-4}$ are always $O(\mathcal{M}^2)$. The same thing is true for the exchange force term that contributes to the right-hand side of equation (4.59).

Using a method that is already classical (cf., [**31**]), one can show that the approximation that was made above is equivalent to the one that one will obtain by considering a limited development of e in the form:

(4.85) $$e\,(\rho,\,\eta,\,\mathcal{M}^{\alpha},\,\mathfrak{M}_{\alpha\beta}) = \tilde{e}\,(\rho,\,\eta) + \tfrac{1}{2}\,\alpha_{(1)}\,P^{\alpha\beta}\,\mathcal{M}_{\alpha}\,\mathcal{M}_{\beta}$$

([14]) $\tau^{\alpha\beta}$ is linear in $\mathfrak{M}_{\alpha\beta}$, but $M^{\gamma\alpha\beta}$ is $O(\mathcal{M}^2)$. Equation (4.59) is intrinsically nonlinear.

$$+\frac{1}{2}(\alpha_{(2)}P^{\alpha\beta}P^{\gamma\delta}+\alpha_{(2)}P^{\alpha\gamma}P^{\beta\delta}+\alpha_{(2)}P^{\alpha\delta}P^{\gamma\beta})\,\mathfrak{M}_{\alpha\beta}\,\mathfrak{M}_{\gamma\delta}+O(\mathcal{M}^3)\,,$$

in which e is *positive definite* up to $O(\mathcal{M}^3)$ if and only if:

$$\alpha_{(1)}\geq 0\,,\qquad 3\alpha_{(2)}+\alpha_{(3)}+\alpha_{(4)}\geq 0,\qquad \alpha_{(4)}+\alpha_{(3)}\geq 0\geq \alpha_{(4)}-\alpha_{(3)}\,.$$

Meanwhile, from (3.8), the second term in the development (4.85) is constant along $\mathcal{C}$ (if $\alpha_{(1)}$ = const.); it is therefore unnecessary to consider it.

4.7. – Comments.

(*a*) To our knowledge, the model constructed in this paragraph is the only complete model of a relativistic spinning fluid in which one tries to take into account the "phenomenological" nature of the interaction between spins. One notes that there is generally something interesting about the variational formulation that permits one to obtain expressions of the type (4.76), which would have been difficult to postulate.

(*b*) In the approximation that is called "semi-classical," in which one neglects the Heisenberg exchange forces, so $2\mathcal{S}^{\alpha\beta\mu}=\rho\,S^{\alpha\beta}\,u^{\mu}$, and the equations (4.59), $(4.74)_{2\text{-}4}$, (4.60), and (4.63), they reduce to:

$$\text{(4.86)}\qquad \begin{cases} \left(\dfrac{DS^{\alpha\beta}}{Ds}\right)_{\perp}=2\mathcal{M}^{[\alpha}\mathcal{M}^{\beta]}, \\ p^{\alpha}=\dfrac{\rho}{c^2}S^{\alpha\beta}\dfrac{Du_{\gamma}}{Ds},\quad \tilde{p}^{\alpha}=0,\quad t^{\alpha\beta}=-p\,P^{\alpha\beta}, \\ \Omega_{\alpha\beta}=-\dfrac{\gamma}{i\,c}\eta_{\alpha\beta\gamma\delta}\,\mathcal{B}^{\gamma}u^{\delta}\equiv-\gamma\,F_{\alpha\beta}, \\ T^{[\alpha\beta]}_{(\text{tot})}=\rho\,\mathcal{M}^{[\alpha}\mathcal{M}^{\beta]}+p^{[\alpha}u^{\beta]}. \end{cases}$$

One then recovers the simplified theory of relativistic spinning fluids that was presented by Halbwachs [**6**].

(*c*) The reversible thermodynamic model that is obtained here can be completed by the study of irreversible phenomena (e.g., viscosity, magnetic spin relaxation, electric and heat conduction) using an analysis that is analogous to what was done in [**24**].

5. – The "perfect magnetohydrodynamical" model as a limiting case.

Consider the case in which the fluid that was previously examined is *paramagnetic*. In that case, the notions of magnetic spin and exchange forces (or interactions between neighboring spins) have no meaning. As a consequence, $S^{\alpha\beta}$ will be zero, and e cannot depend upon $\mathfrak{M}_{\alpha\beta}$; hence, $\tau^{\alpha\beta}$ and $M^{\alpha\beta\gamma}$ will be zero. Meanwhile, $\mathcal{M}^{\alpha}$ is non-zero, because the gyromagnetic relation (3.7) is not valid. The total energy-momentum tensor becomes symmetric. The hypothesis (3.8) no longer means anything, and one cannot construct a kinematical relationship such as $(3.9)_1$. One is reduced to considering $\delta\mathcal{M}^{\alpha}$ to be an independent variation in the variational formulation of paragraph 4.4. That independent variation leads to the *equilibrium equation* between the Maxwellian induction and the local magnetic induction:

$$\mathcal{B}^{\alpha} + {}^{l}\mathcal{B}^{\alpha} = 0 . \tag{5.1}$$

Hence, with the law of behavior $(4.40)_1$:

$$\mathcal{B}^{\gamma} = \left(\frac{\partial e}{\partial \mathcal{M}_{\alpha}}\right)_{\perp} . \tag{5.2}$$

In the context of the linear approximation to the isotropic fluid that was discussed in the preceding paragraph, that will become:

$$\mathcal{B}^{\alpha} = \alpha_{(1)}\,(\rho,\,\eta,\,\mathcal{M}^2)\,\mathcal{M}^{\alpha}, \qquad \alpha_{(1)} \geq 0. \tag{5.3}$$

The magnetic induction and the magnetization are collinear. One sets:

$$\alpha_{(1)} \equiv \rho\,\frac{\mu}{\mu-1}\,, \qquad \mu = \mu\,(\rho,\,\eta,\,\mathcal{M}^2) > 1 \tag{5.4}$$

(if $\mu \mapsto 1$ then one must have $|\,\mathcal{M}^{\alpha}\,| \mapsto 0$). With the relation $\mathcal{H}^{\alpha} \equiv \mathcal{B}^{\alpha} - \rho\,\mathcal{M}^{\alpha}$, one will get:

$$M^{\alpha} \equiv \rho\,\mathcal{M}^{\alpha} = (\mu - 1)\,\mathcal{H}^{\alpha}, \qquad \mathcal{B}^{\alpha} = \rho\,\mathcal{H}^{\alpha}. \tag{5.5}$$

The coefficient μ is the magnetic permeability of the fluid then. If one replaces $\mathcal{M}^{\alpha}$ and $\mathcal{B}^{\alpha}$ with their values (5.5) in (4.73), in which one sets:

$$S^{\alpha\beta} = M^{\alpha\beta\gamma} = \mathfrak{M}_{\alpha\beta} = \tau^{\alpha\beta} = 0$$

and uses the approximation (4.85), namely:

$$e\,(\rho,\,\eta,\,\mathcal{M}^{\alpha}) = \tilde{e}\,(\rho,\,\eta) + \tfrac{1}{2}\,\alpha_{(1)}\,\mathcal{M}^{\alpha}\mathcal{M}_{\beta} = \tilde{e} + \frac{1}{2}\frac{\mu\,(\mu-1)}{\rho}\,\mathcal{H}^{\lambda}\mathcal{H}_{\lambda}\,,$$

then one will get the expression for the energy-momentum tensor for the "perfect magnetohydrodynamical" model from the present model:

(5.6) $$T^{\alpha\beta}_{(\mathrm{tot})} = \left[\rho f + \frac{\mu}{2c^2}(3-\mu)\mathcal{M}^2\right]u^{\alpha}u^{\beta} - \mu\mathcal{H}^{\alpha}\mathcal{H}^{\beta} + \left[p + \mu\left(1-\frac{\mu}{2}\right)\mathcal{H}^2\right]g^{\alpha\beta}, \quad \mathcal{H}^2 = \mathcal{H}^{\lambda}\mathcal{H}_{\lambda},$$

in which one defines the *index f* of the fluid by:

$$f \equiv 1 + c^{-2}\left[\tilde{e}\,(\rho,\eta) + \frac{p}{\rho}\right].$$

In our analysis, we started from an energy-momentum tensor for the electromagnetic field that is different from the one that was considered by Lichnerowicz ([**12**], pp. 87-96). Meanwhile, under the astrophysical conditions where μ is slightly different from unity, we can write $3 - \mu \neq 2$ and $1 - \mu / 2 \neq 1 / 2$, and (5.6) will reduce to:

$$T^{\alpha\beta}_{(\mathrm{tot})} = \left(\rho\, f + \frac{\mu}{c^2}\mathcal{H}^2\right)u^{\alpha}u^{\beta} - \mu\,\mathcal{H}^{\alpha}\mathcal{H}^{\beta} + \left(p + \frac{\mu}{2}\mathcal{H}^2\right)g^{\alpha\beta}\,,$$

which is, in fact, Lichnerowicz's perfect magnetohydrodynamical tensor ([**12**], pp. 150), up to the signature of the metric $g_{\alpha\beta}$.

6. – Geometrization.

One knows that the nuclear or electronic spin has a non-negligible influence on the gravitational field of a macroscopic body only when the latter was compressed in such a way that its dimensions are of the same order as the Compton wavelength of the electron. Meanwhile, if the "particles" considered are rotating proto-galaxies with turbulent currents or "primitive black holes" then the influence of spins can be significant during the first state of the evolution of the universe or during the phenomenon of "gravitational collapse" {in particular, it is possible that any singularity of the metric can be avoided if one takes spin into account {cf., Kopczynski [**32**], Hehl and von der Heyde [**40**])}. Under those conditions, the idealization of the medium as a gaseous cloud of "particles" endowed with spin and the physical models that were proposed above will require geometrization in the context of general relativity. Meanwhile the construction of a geometric structure that would be associated with not-necessarily-symmetric systems of energy-momentum tensors remains an open problem. In conclusion, we shall briefly examine several possibilities.

6.1. – Riemannian space-time V^4.

This is the simplest-possible generalization: The geometric structure remains that of quadri-dimensional Riemannian space-time V^4 with the symmetric normal hyperbolic metric $g_{\alpha\beta}$. Following Belinfante and Rosenfeld [**34**], the conservation equation (4.58) is modified by writing ([15]):

$$\nabla_\beta T^{\alpha\beta}_{(\text{tot})} = -R^{\alpha}_{\cdot\,\beta\gamma\delta}\,\mathcal{S}^{\gamma\delta\beta}, \tag{6.1}$$

in which $R^{\alpha}_{\cdot\,\beta\gamma\delta}$ is the curvature tensor of V^4 is a "Mathisson 4-force." Equation (4.65) is not modified. If one now introduces the tensor $J^{\alpha\beta}$:

$$J^{\alpha\beta} \overset{\text{def}}{=} T^{\alpha\beta}_{(\text{tot})} + \nabla_\gamma\left(\mathcal{S}^{\alpha\gamma\beta} + \mathcal{S}^{\beta\gamma\alpha} - \mathcal{S}^{\alpha\beta\gamma}\right) \tag{6.2}$$

– and from (6.1) and (4.65), it is *symmetric* and *conserved* – then one can *postulate* the Einstein equations in the form:

$$G^{\alpha\beta} = -\kappa J^{\alpha\beta} \quad \left(\kappa \equiv \frac{8\pi k}{c^4}\right), \tag{6.3}$$

in such a way that the Bianchi identities are satisfied, and (6.3) is *the only geometry-source relation* of the theory ([16]).

6.2. – Einstein-Cartan space-time U^4.

Costa de Beauregard [**39**], Weyl [**35**], Sciama [**36**], Kibble [**37**], and Hehl [**38**], as far as they are concerned, followed the work of E. Cartan and assumed that there must exist a profound relationship between the spin density tensor and the torsion tensor of the connection of a non-Riemannian space-time U^4: The local existence of a spin density can induce torsion in the corresponding region of the universe in the same way that the presence of matter induced a local curvature in the universe in classical general relativity. Thus, those authors coupled the spin tensor ([17]) to the antisymmetric part of the affine connection (which preserves the metric) ([18]):

([15]) In V^4, one defines $R^{\alpha}_{\cdot\,\beta\gamma\delta}$, the Ricci tensor $R_{\beta\gamma}$, and the Einstein tensor $G^{\alpha\beta}$ by:

$$2\,\nabla_{[\delta}\nabla_{[\gamma}A_{\beta} = A_a\,R^{\alpha}_{\cdot\,\beta\gamma\delta}\,, \quad R_{\beta\gamma} \equiv R^{\alpha}_{\cdot\,\beta\gamma\alpha}\,, \qquad G^{\alpha\beta} \equiv R^{\alpha\beta} - \tfrac{1}{2}\,g^{\alpha\beta}\,R^{\gamma}_{\cdot\,\gamma}\,,$$

respectively.

([16]) Compare with Israel [**19**], who utilized an elementary model from kinetic theory.

([17]) The general expression for $\mathcal{S}^{\alpha\beta\mu}$ has the form:

$$\mathcal{S}^{\alpha\beta\mu} = a^\alpha\,u^\beta\,u^\gamma - b^\alpha\,u^\alpha\,u^\gamma + b_1^{\alpha\beta}u^\gamma - b_2^{\beta\gamma}u^\alpha - b_3^{\alpha\gamma}u^\beta - M^{\alpha\beta\mu}\,.$$

The expression (4.66) corresponds to the particular case of $a^\alpha = b^\alpha = b_2^{\alpha\beta} = 0$.

([18]) Hehl and von der Heyde [**40**] utilized the *modified* torsion tensor:

(6.4) $$K_{\alpha\beta}^{\cdot\cdot\gamma} = \kappa \mathcal{S}_{\alpha\beta}^{\cdot\cdot\gamma}, \quad K_{\alpha\beta}^{\cdot\cdot\gamma} \equiv \Gamma_{[\alpha\beta]}^{\gamma}.$$

The corresponding field equations are given by Hehl and von der Heyde [**40**]. Their structure is close to that of (6.1), (6.3), and 4.65). If one accepts such a description then one will get a possible interpretation of the torsion $K_{\alpha\beta}^{\cdot\cdot\gamma}$. Indeed, according to the interpretation that is given to the dependency of the thermodynamic potential with respect to $\mathfrak{M}_{\alpha\beta}$ in paragraph 4.3.1, the tensor $M^{\alpha\beta\gamma}$ of $(4.66)_1$ is supposed to represent the action of Heisenberg exchange forces in the form of contact actions (cf., § 4.4.2). It then follows that even if one does not take into account gyromagnetic phenomena in $\mathcal{S}^{\alpha\beta\gamma}$, the term $M^{\alpha\beta\gamma}$ will be coupled with the torsion on U^4 by the relation (6.4): *Torsion can geometrically represent the spin-spin "contact" action.* Our heuristic model thus comes back to Hehl's conjecture ([**38**], [**40**]).

Finally, note that a geometric structure that is substantially equivalent can be obtained by starting from a geometric Lagrangian on a pseudo-Riemannian manifold V^4 with an asymmetric connection, where V^4 is the diagonal submanifold of a certain eight-dimensional manifold V^8 (see the prolongation process of A. Crumeyrolle in Clerc [**41**]).

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$$\overline{K}_{\alpha\beta}^{\cdot\cdot\gamma} \equiv K_{\alpha\beta}^{\cdot\cdot\gamma} + \delta_{\alpha}^{\gamma} K_{\beta\mu}^{\cdot\cdot\mu} - \delta_{\beta}^{\gamma} K_{\alpha\mu}^{\cdot\cdot\mu},$$

instead of $K_{\alpha\beta}^{\cdot\cdot\gamma}$ in $(6.4)_1$. Sciama [**36**] considered an asymmetric $g_{\alpha\beta}$, in addition.

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