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On the deformation of curved elastic plates

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Introduction. – The following investigation has the goal of developing a theory of the so-called **Bourdon** tubes. Those thin-walled metal tubes are used as aneroid barometers when the air is pumped out of them and as thermometers when they are filled with ether. They have the remarkable property that very slight pressures that act upon their external or internal surfaces can produce measurable deformations. An aneroid barometer allows fluctuations of air pressure to be measured that amount to only a fraction of a millimeter of mercury in pressure. A millimeter of mercury in pressure is a pressure of 13.6 milligrams per square millimeter. The elastic modulus of most metals corresponds to a pressure of 8000 to 20000 kilograms per square millimeter. The quotient of both pressures then lies between, say, $1 / 6 \cdot 10^{-5}$ and $1 / 15 \cdot 10^{-5}$, so it is exceptionally small. A cylindrical tube will no longer react to such a slight pressure measurably; a measurable deformation will occur for only bent tubes.

In order to lay the foundations for the theory that I speak of, it is necessary to dig somewhat deeper. I shall then begin by transforming the basic equations of the theory of elasticity into general curvilinear coordinates. In the second part, those equations will then be applied to the case in which the body considered is a thin, bent plate. The third part addresses tubes that have the form of surfaces of revolution.

Part I: Transforming the equations of elasticity into general coordinates.

1. The basic equations of elasticity theory in Cartesian coordinates. – The transformation of the equations of elasticity into general orthogonal coordinates was first carried out by **Lamé**. Simpler methods were given later by **Carl Neumann**, **Borchardt**, and **Beltrami**. For the following, it is desirable to employ, not orthogonal, but entirely arbitrary coordinates. The theory of invariants serves as means of achieving that transformation with very few calculations.

T means the kinetic energy of the body considered, U means the potential of the elastic forces, dM is the work that is done by internally-acting external forces under a virtual displacement, and $d\,(M)$ is the work done by tractions that act upon the outer surface. As is known, one will get the equations of elasticity on the basis of **Hamilton**'s principle from the equation ([1]):

$$\int_{t_0}^{t_1} \partial t\,[\delta T + \delta U + \delta M + \delta(M)] = 0 \,. \tag{1}$$

Here, t means time. The virtual displacements are chosen such that they will all vanish for the time points $t = t_0$ and $t = t_1$.

We next employ a system of Cartesian coordinates x, y, z. We denote the volume of the body by V, its surface area by S, and its density by k. The components of the displacement of a point of the body are denoted by u, v, w, and their variations by δu, δv, δw. The components of the external forces that act on a point in the interior are denoted by A, B, C, and the components of the pressure on an element of the surface by (A), (B), (C).

([1]) Cf., **Kirchhoff**, "Über das Gleichgewicht und die Bewegung einer elastischen Scheibe," Crelle's Journal, Bd. 40, *Gesammelte Abhandlungen*, pp. 237.

When expressed in terms of those quantities, the quantities that enter into (1) will have the values:

(2) $$\delta M = \int_{(V)} [A\,\delta u + B\,\delta v + C\,\delta w]\,\partial V\,,$$

(3) $$\delta(M) = \int_{(S)} [(A)\,\delta u + (B)\,\delta v + (C)\,\delta w]\,\partial S\,,$$

(4) $$T = \tfrac{1}{2}\int_{(V)} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] k\,\partial V\,.$$

Finally, under the assumption that the body is isotropic, one has:

(5) $$U = -K \int_{(V)} [\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \vartheta(\lambda_1 + \lambda_2 + \lambda_3)^2]\,\partial V\,,$$

in which λ_1, λ_2, λ_3 denote the principal dilatations.

The mechanical meaning of **Kirchhoff**'s elasticity constants K and ϑ is inferred from the following remark: If one end of a cylindrical rod of length l is fixed, while a tension of magnitude P acts upon the other one, then the rod will experience a lengthening of $l \cdot \frac{P}{2K}\frac{1+2\vartheta}{1+3\vartheta}$. A line-element that is perpendicular to the axis of the rod experiences a shortening that is equal to $\frac{\vartheta}{1+2\vartheta}$ when expressed as a fraction of the length. The elastic modulus is therefore $E = 2K\,\frac{1+2\vartheta}{1+3\vartheta}$, and the quotient of the length dilatation and the lateral contraction is $\mu = \frac{\vartheta}{1+2\vartheta}$.

2. Introduction of general coordinates. – Instead of the Cartesian coordinates $x, y, z,$ we shall now introduce general coordinates p_1, p_2, p_3 . We assume that every point is determined uniquely by those coordinates. That demands that the functional determinant $\frac{\partial(x, y, z)}{\partial(p_1, p_2, p_3)}$ must not vanish for any point in the body. We assume that the notation is chosen such that this determinant is positive.

The square of the line element:

$$ds^2 = dx^2 + dy^2 + dz^2$$

is expressed in the new coordinates in the form:

$$ds^2 = \sum a_{\lambda\mu}\, dp_\lambda\, dp_\mu \qquad (\lambda, \mu = 1, 2, 3)\,.$$

We denote that differential form by F, its determinant by A, and the subdeterminant that is adjoint to the element $a_{\lambda\mu}$ by $A_{\lambda\mu}$. The form F has the character of a positive-definite form everywhere inside of the region in question. None of the quantities $a_{\lambda\lambda}$, $A_{\lambda\lambda}$, A can vanish at any point in the region. Since:

$$a_{\lambda\mu} = \frac{\partial x}{\partial p_\lambda}\frac{\partial x}{\partial p_\mu} + \frac{\partial y}{\partial p_\lambda}\frac{\partial y}{\partial p_\mu} + \frac{\partial z}{\partial p_\lambda}\frac{\partial z}{\partial p_\mu}\;,$$

the functional determinant $\dfrac{\partial(x, y, z)}{\partial(p_1, p_2, p_3)} = \sqrt{A}$, where the symbol $\sqrt{A}$ means the positive square root, as in all of what follows.

The cosine of the angle that the direction of increasing p_λ makes with the direction of increasing p_μ is $a_{\lambda\mu} / \sqrt{a_{\lambda\lambda}\, a_{\mu\mu}}$. We distinguish an inner side and an outer side for the coordinate surface p_λ = const. We refer to the outer side as the one that lies on the side of increasing p_λ . The cosine of the angle that the outward-pointing normals to the surfaces p_λ = const., p_μ = const. make with each other is $A_{\lambda\mu} / \sqrt{A_{\lambda\lambda}\, A_{\mu\mu}}$.

Since the functional determinant $\dfrac{\partial(x, y, z)}{\partial(p_1, p_2, p_3)}$ does not vanish, by assumption, one can pose the equations:

$$\frac{\partial^2 x}{\partial p_\lambda\, \partial p_\mu} = \sum_{\nu=1}^{3} \begin{Bmatrix} \lambda\ \mu \\ \nu \end{Bmatrix} \frac{\partial x}{\partial p_\nu}, \qquad \frac{\partial^2 y}{\partial p_\lambda\, \partial p_\mu} = \sum_{\nu=1}^{3} \begin{Bmatrix} \lambda\ \mu \\ \nu \end{Bmatrix} \frac{\partial y}{\partial p_\nu}, \qquad \frac{\partial^2 z}{\partial p_\lambda\, \partial p_\mu} = \sum_{\nu=1}^{3} \begin{Bmatrix} \lambda\ \mu \\ \nu \end{Bmatrix} \frac{\partial z}{\partial p_\nu},$$

$$(\lambda, \mu = 1, 2, 3).$$

Obviously, one has $\begin{Bmatrix} \lambda\ \mu \\ \nu \end{Bmatrix} = \begin{Bmatrix} \mu\ \lambda \\ \nu \end{Bmatrix}$. The quantities $\begin{Bmatrix} \lambda\ \mu \\ \nu \end{Bmatrix}$ can be expressed in terms of the quantities $a_{\lambda\mu}$ and their first derivatives. Namely, if one multiplies the first of the foregoing equations by $\partial x / \partial p_\kappa$, the second by $\partial y / \partial p_\kappa$, and the third by $\partial x / \partial p_\kappa$, and adds them then that will give:

$$\text{(1)} \qquad \frac{\partial^2 x}{\partial p_\lambda\, \partial p_\mu}\frac{\partial x}{\partial p_\kappa} + \frac{\partial^2 y}{\partial p_\lambda\, \partial p_\mu}\frac{\partial y}{\partial p_\kappa} + \frac{\partial^2 z}{\partial p_\lambda\, \partial p_\mu}\frac{\partial z}{\partial p_\kappa} = \sum_{\nu=1}^{3} \begin{Bmatrix} \lambda\ \mu \\ \nu \end{Bmatrix} a_{\nu\kappa} = \tfrac{1}{2}\left[-\frac{\partial a_{\lambda\mu}}{\partial p_\kappa} + \frac{\partial a_{\lambda\kappa}}{\partial p_\mu} + \frac{\partial a_{\mu\kappa}}{\partial p_\lambda} \right]$$

$$(\lambda, \mu, \kappa = 1, 2, 3).$$

In place of the displacement components u, v, w, we introduce the quantities:

(2) $$\xi_\lambda = \frac{\partial x}{\partial p_\lambda} u + \frac{\partial y}{\partial p_\lambda} v + \frac{\partial z}{\partial p_\lambda} w \qquad (\lambda = 1, 2, 3)$$

and simultaneously replace the force components A, B, C and the traction components (A), (B), (C) with the components P_λ and (P_λ), which are defined by the equations:

(3) $$\delta M = \sum_{\lambda=1}^{3} P_\lambda \, \delta\xi_\lambda , \qquad \delta(M) = \sum_{\lambda=1}^{3} (P_\lambda) \, \delta\xi_\lambda .$$

Equation (2) shows that $\xi_\lambda / \sqrt{a_{\lambda\lambda}}$ is the projection of the displacement of a point in the direction of increasing p_λ .

We set $\delta\xi_2 = 0$, $\delta\xi_3 = 0$, and take $\delta\xi_1$ to be positive. Thus, the direction of the virtual displacement whose components are the quantities δu, δv, δw will coincide with the direction of the outward-pointing normal to the surface p_1 = const., and the absolute value of the displacement is $\delta\xi_1 / \sqrt{a_{11}}$. Since the work done by that displacement is equal to $P_1 \, \delta\xi_1$, $P_1 / \sqrt{a_{11}}$ will be the projection of the force whose components are the quantities A, B, C onto the outward-pointing normal to the surface p_1 = const. The mechanical meaning of the remaining quantities P_λ and (P_λ) is explained analogously.

It follows from (2) that:

$$\xi_1 \, dp_1 + \xi_2 \, dp_2 + \xi_3 \, dp_3 = u \, dx + v \, dy + w \, dz .$$

That equation shows that the quantities ξ_1, ξ_2, ξ_3 and dp_1, dp_2, dp_3 are considered to be contragredient variables.

3. The form *du dx + dv dy + dw dz*. – Along with the differential form that represents the square of the line element, yet another quadratic form is also meaningful for the theory of elasticity that determines the dilatation of the line element.

It is the form:

$$du \, dx + dv \, dy + dw \, dz$$

$$= \frac{\partial u}{\partial x} dx^2 + \frac{\partial v}{\partial y} dy^2 + \frac{\partial w}{\partial y} dz^2 + 2 \cdot \tfrac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) dy \, dz + 2 \cdot \tfrac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) dz \, dx + 2 \cdot \tfrac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx \, dy .$$

Let the expression for that form in the new coordinates be:

$$\Phi = \sum \alpha_{\lambda\mu} \, dp_\lambda \, dp_\mu \qquad (\lambda, \mu = 1, 2, 3)$$

Here, one has:

(1) $$\alpha_{\lambda\mu} = \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial p_\lambda}\frac{\partial x}{\partial p_\mu}+\frac{\partial v}{\partial p_\lambda}\frac{\partial y}{\partial p_\mu}+\frac{\partial w}{\partial p_\lambda}\frac{\partial z}{\partial p_\mu}\right)+\left(\frac{\partial u}{\partial p_\mu}\frac{\partial x}{\partial p_\lambda}+\frac{\partial v}{\partial p_\mu}\frac{\partial y}{\partial p_\lambda}+\frac{\partial w}{\partial p_\mu}\frac{\partial z}{\partial p_\lambda}\right)\right].$$

From equations [§ **2**, (2)], one has:

$$\frac{\partial \xi}{\partial p_\mu} = \frac{\partial u}{\partial p_\mu}\frac{\partial x}{\partial p_\lambda}+\frac{\partial v}{\partial p_\mu}\frac{\partial y}{\partial p_\lambda}+\frac{\partial w}{\partial p_\mu}\frac{\partial z}{\partial p_\lambda}+u\frac{\partial^2 x}{\partial p_\lambda\,\partial p_\mu}+v\frac{\partial^2 y}{\partial p_\lambda\,\partial p_\mu}+w\frac{\partial^2 z}{\partial p_\lambda\,\partial p_\mu} \qquad (\lambda, \mu = 1, 2, 3),$$

and on considering [§ **2**, (2)], it will follow that:

(2) $$\alpha_{\lambda\mu} = \tfrac{1}{2}\left(\frac{\partial \xi_\lambda}{\partial p_\mu}+\frac{\partial \xi_\mu}{\partial p_\lambda}\right)-\sum_{\nu=1}^{3}\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}\xi_\nu \qquad (\lambda, \mu = 1, 2, 3).$$

We now summarize those forms of the system that is determined by the quadratic forms F and Φ that will be used in what follows. We let $\Delta(\lambda) = \Delta_3 - \Delta_2\,\lambda + \Delta_1\lambda^2 - \Delta_0\lambda^4$ denote the determinant of the quadratic form $\Phi - \lambda F$. Here, we obviously have $\Delta_0 = A$. We let G denote the contravariant of the form F and let Ψ denote the simultaneous contravariant of the forms F and Φ, which is linear in the coefficients of those two forms. We then have:

$$G = \sum A_{\lambda\mu}\,\xi_\lambda\,\xi_\mu \qquad (\lambda, \mu = 1, 2, 3).$$

We set:

$$\Psi = \sum \beta_{\lambda\mu}\,\xi_\lambda\,\xi_\mu .$$

Here, we have:

(3) $$\beta_{\lambda\mu} = \beta_{\mu\lambda} = \sum_{\nu=1}^{3}\sum_{\kappa=1}^{3}\alpha_{\nu\kappa}\frac{\partial A_{\lambda\mu}}{\partial a_{\nu\kappa}} \qquad (\lambda, \mu = 1, 2, 3).$$

The contravariant Ψ can also be defined as the coefficient of ρ in the development of the determinant:

$$\begin{vmatrix} \alpha_{11}-\rho\,a_{11} & \alpha_{12}-\rho\,a_{12} & \alpha_{13}-\rho\,a_{13} & \xi_1 \\ \alpha_{12}-\rho\,a_{12} & \alpha_{22}-\rho\,a_{22} & \alpha_{23}-\rho\,a_{23} & \xi_2 \\ \alpha_{13}-\rho\,a_{13} & \alpha_{23}-\rho\,a_{23} & \alpha_{33}-\rho\,a_{33} & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & \end{vmatrix}$$

in powers of ρ.

The following easily-proved relations exist between the forms Δ_1, Δ_2, G, and Ψ:

$$(4)\qquad \begin{cases} \Delta_1 = \sum A_{\lambda\mu}\,\alpha_{\lambda\mu} & (\lambda, \mu = 1, 2, 3), \\ \dfrac{\partial \Delta_1}{\partial \alpha_{\lambda\lambda}} = A_{\lambda\lambda}, \quad \dfrac{\partial \Delta_1}{\partial \alpha_{\lambda\mu}} = 2A_{\lambda\mu}, \quad \dfrac{\partial \Delta_2}{\partial \alpha_{\lambda\lambda}} = \beta_{\lambda\lambda}, \quad \dfrac{\partial \Delta_2}{\partial \alpha_{\lambda\mu}} = 2\beta_{\lambda\mu} & (\lambda \neq \mu). \end{cases}$$

The values that the forms F, Φ, $\Delta(\lambda)$, etc., assume when the Cartesian coordinates enter in place of the general coordinates p_1, p_2, p_3 might be denoted by F', Φ', $\Delta'(\lambda)$, etc. One then has, e.g.:

$$F' = dx^2 + dy^2 + dz^2, \qquad \Phi' = du\,dx + dv\,dy + dw\,dz\,.$$

4. Expressing the potential of the elastic forces and the kinetic energy in the new coordinates. – The principal dilatations λ_1, λ_2, λ_3 are given by the values of λ for which the determinant of the quadratic form $\Phi' - \lambda F'$ vanishes, so the roots of the equation $\Delta'(\lambda) = 0$. $\Delta(\lambda) = A\,\Delta'(\lambda)$, one then has:

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{\Delta_1}{A}, \qquad \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \frac{\Delta_2}{A}.$$

As a result:

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \frac{\Delta_1^2}{A^2} - 2\frac{\Delta_2}{A}\,.$$

One further has:

$$G(\xi_1, \xi_2, \xi_3) = A\,G'(u, v, w)\,,$$

so one will also have:

$$G\left(\frac{\partial \xi_1}{\partial t}, \frac{\partial \xi_2}{\partial t}, \frac{\partial \xi_3}{\partial t}\right) = A G'\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right) = A\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right].$$

Finally, one has:

$$\partial V = \sqrt{A}\,\partial p_1\,\partial p_2\,\partial p_3\,.$$

That then gives the following expression for the potential for the elastic forces [§ **1**, (5)]:

$$U = -\,K \int_{(V)} \left[(1+\vartheta)\frac{\Delta_1^2}{A^2} - 2\frac{\Delta_2}{A}\right]\sqrt{A}\,\partial p_1\,\partial p_2\,\partial p_3\,,$$

and the kinetic energy is [§ **1**, (4)]:

$$T = \tfrac{1}{2} \int_{(V)} \left[\sum_{\lambda,\mu} A_{\lambda\mu}\,\frac{\partial \xi_\lambda}{\partial t}\,\frac{\partial \xi_\mu}{\partial t}\right] k\sqrt{A}\,\partial p_1\,\partial p_2\,\partial p_3\,.$$

5. The basic equations of elasticity theory in general coordinates. – From the foregoing [§ **3**, (4)], one has:

$$\delta U = -\,2K\int_{(V)}\sum_{\lambda=1}^{3}\sum_{\mu=1}^{3}\left\{\left[(1+\vartheta)\frac{\Delta_1}{A}-\beta_{\lambda\mu}\right]\delta\alpha_{\lambda\mu}\right\}\frac{1}{\sqrt{A}}\partial p_1\,\partial p_2\,\partial p_3\;.$$

Here, one has [§ **3**, (2)]:

$$\delta\alpha_{\lambda\mu} = \tfrac{1}{2}\left(\delta\frac{\partial\xi_\lambda}{\partial p_\mu}+\delta\frac{\partial\xi_\mu}{\partial p_\lambda}\right)-\sum_{\nu=1}^{3}\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}\delta\xi_\nu\;.$$

We set:

(1) $$2K\left[(1+\vartheta)\frac{\Delta_1}{A}A_{\lambda\mu}-\beta_{\lambda\mu}\right] = -\,N_{\lambda\mu}\,,\quad N_{\mu\lambda} = N_{\lambda\mu}\qquad(\lambda,\mu = 1, 2, 3),$$

to abbreviate, and get:

(2) $$\delta U = \int_{(V)}\left[\sum_{\lambda=1}^{3}\sum_{\mu=1}^{3}N_{\lambda\mu}\,\delta\frac{\partial\xi_\lambda}{\partial p_\mu}-\sum_{\lambda=1}^{3}\sum_{\mu=1}^{3}\sum_{\nu=1}^{3}\begin{Bmatrix}\lambda\ \mu\\ \nu\end{Bmatrix}N_{\lambda\mu}\,\delta\xi_\nu\right]\frac{\partial p_1\,\partial p_2\,\partial p_3}{\sqrt{A}}\;.$$

We convert the integral on the right by partial integration in the known way.

The angle that the outward-pointing normals to the surface p_μ = const. make with those of the outer surface S to the body will be denoted by ω_μ . That angle is obtuse at a location where the direction of increasing p_1 enters the body, and one will accordingly have:

$$\sqrt{A_{11}}\,\partial p_2\,\partial p_3 = -\cos\omega_1\,\partial S\;.$$

The angle ω_1 is acute at an exit location, and one will accordingly have:

$$\sqrt{A_{11}}\,\partial p_2\,\partial p_3 = +\cos\omega_1\,\partial S$$

there. One then has:

$$\int_{(V)}N_{\lambda1}\delta\frac{\partial\xi_\lambda}{\partial p_1}\frac{\partial p_1\,\partial p_2\,\partial p_3}{\sqrt{A}} = \int_{(S)}\frac{N_{\lambda1}\cos\omega_1}{\sqrt{A_{11}A}}\delta\xi_\lambda\;\partial S-\int_{(V)}\frac{\partial\dfrac{N_{\lambda1}}{\sqrt{A}}}{\partial p_1}\delta\xi_\lambda\,\partial p_1\,\partial p_2\,\partial p_3\;.$$

Two analogous formulas are obtained by cyclically permuting the indices 1, 2, 3. We then get from (2) that:

(3) $$\delta U = \int\sum_{\lambda=1}^{3}\left[\frac{N_{\lambda1}}{\sqrt{A_{11}}}\cos\omega_1+\frac{N_{\lambda2}}{\sqrt{A_{22}}}\cos\omega_2+\frac{N_{\lambda3}}{\sqrt{A_{33}}}\cos\omega_3\right]\delta\xi_\lambda\cdot\frac{\partial S}{\sqrt{A}}$$

$$- \int_{(V)} \sum_{\lambda=1}^{3} \left[\frac{\partial \frac{N_{\lambda 1}}{\sqrt{A}}}{\partial p_1} + \frac{\partial \frac{N_{\lambda 2}}{\sqrt{A}}}{\partial p_2} + \frac{\partial \frac{N_{\lambda 3}}{\sqrt{A}}}{\partial p_3} + \frac{1}{\sqrt{A}} \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} \begin{Bmatrix} \mu\,\nu \\ \lambda \end{Bmatrix} N_{\mu\nu} \right] \delta\xi_\nu \, \partial p_2 \, \partial p_2 \, \partial p_3 \ .$$

Furthermore, one has (cf., § **4**):

$$\int_{t_0}^{t_1} \delta T \, \partial t \ = \int_{t_0}^{t_1} \partial t \int_{(V)} \left[\sum_{\lambda=1}^{3} \sum_{\mu=1}^{3} A_{\lambda\mu} \frac{\partial \xi_\mu}{\partial t} \delta \frac{\partial \xi_\mu}{\partial t} \right] \frac{k}{\sqrt{A}} \partial p_1 \, \partial p_2 \, \partial p_3 \ .$$

Since the variations $\delta\xi_\lambda$ vanish for $t = t_0$ and $t = t_1$, it will then follow from this by partial integration that:

$$\text{(4)} \qquad \int_{t_0}^{t_1} \delta T \, \partial t \ = \int_{t_0}^{t_1} \partial t \int_{(V)} \sum_{\lambda=1}^{3} \left[\sum_{\mu=1}^{3} A_{\lambda\mu} \frac{\partial^2 \xi_\mu}{\partial t^2} \delta\xi_\mu \right] \frac{k}{\sqrt{A}} \partial p_1 \, \partial p_2 \, \partial p_3 \ .$$

With the use of equations (3) and (4) of this article and equations (3) of article **2**, we will now get from the basic equation of the theory of elasticity [§ **1**, (1)], the differential equations for the interior of the body:

$$\text{(5)} \qquad \frac{k}{\sqrt{A}} \left[A_{\lambda 1} \frac{\partial^2 \xi_1}{\partial t^2} + A_{\lambda 2} \frac{\partial^2 \xi_2}{\partial t^2} + A_{\lambda 3} \frac{\partial^2 \xi_3}{\partial t^2} \right] = P_\lambda - \frac{\partial \frac{N_{\lambda 1}}{\sqrt{A}}}{\partial p_1} - \frac{\partial \frac{N_{\lambda 2}}{\sqrt{A}}}{\partial p_2} - \frac{\partial \frac{N_{\lambda 3}}{\sqrt{A}}}{\partial p_3} - \frac{1}{\sqrt{A}} \sum_{\mu,\nu} \begin{Bmatrix} \mu\,\nu \\ \lambda \end{Bmatrix} N_{\mu\nu}$$

$$(\lambda = 1, 2, 3),$$

and the condition on the outer surface:

$$\text{(6)} \qquad \frac{1}{\sqrt{A}} \left[\frac{N_{\lambda 1}}{\sqrt{A_{11}}} \cos\omega_1 + \frac{N_{\lambda 2}}{\sqrt{A_{22}}} \cos\omega_2 + \frac{N_{\lambda 3}}{\sqrt{A_{33}}} \cos\omega_3 \right] = -\,(P)_\lambda \qquad\qquad (\lambda = 1, 2, 3) \ .$$

Here, $\sqrt{a_{11}}\,(P_1)$, $\sqrt{a_{22}}\,(P_2)$, $\sqrt{a_{33}}\,(P_3)$ mean the projections of the pressure that acts upon the outer surface onto the outward-pointing normals to the surfaces p_1 = const., p_2 = const., p_3 = const., resp. (cf., the remark in regard to § **2**, (3)]

6. Mechanical meaning of the quantities $N_{\lambda\mu}$. – The equations that were derived in the foregoing are also valid for an arbitrary subset V_0 of the total volume V of the body. Instead of the components (P_1), (P_2), (P_3) of the pressure on the outer surface, the components of the pressure that V_0 experiences from the remaining part of the body will enter in this case. We assume that a part of the outer surface of the volume V_0 coincides with one of the surfaces p_μ = const. in such a way that the outer side of the former surface covers the inner side of the latter, and we let $\Pi_{\mu 1}$,

$\Pi_{\mu 2}$, $\Pi_{\mu 3}$ denote the values that the quantities (P_1), (P_2), (P_3), resp., assume in this case. $\sqrt{a_{\lambda\lambda}}\,\Pi_{\mu\lambda}$ then means the projection of the pressure that the inner side of the surface p_μ = const. experiences onto the outward-pointing normal to the surface p_λ = const. The cosines cos ω_1, cos ω_2, cos ω_3 take on the values:

$$-\frac{A_{\mu 1}}{\sqrt{A_{11}\,A_{\mu\mu}}}\,,\quad -\frac{A_{\mu 2}}{\sqrt{A_{22}\,A_{\mu\mu}}}\,,\quad -\frac{A_{\mu 3}}{\sqrt{A_{33}\,A_{\mu\mu}}}\,,$$

resp., in which case (cf., the previous article and article **2**), and from equation (6) of the previous article, that will give:

$$\frac{1}{\sqrt{A_{\mu\mu}\,A}}\left[\frac{A_{\mu 1}}{A_{11}}N_{\lambda 1}+\frac{A_{\mu 2}}{A_{22}}N_{\lambda 2}+\frac{A_{\mu 3}}{A_{33}}N_{\lambda 3}\right]=\Pi_{\lambda\mu} \qquad (\lambda,\ \mu = 1, 2, 3).$$

Part II: Application to the case of curved, thin plates.

7. Geometric definition of the body. – We assume that the body in question is a thin plate that has the form of a curved, but not developable surface.

In order to define the form of the body geometrically, we start from a "middle surface" S_0, which can be closed or have a boundary curve L.

As for all surfaces under consideration here, we will also distinguish an outer side and an inner side for S_0 .

We then assume that there are two parallel surfaces S_+ and S_- to the surface S_0, the former of which runs across the outer side of S_0 and the latter of which runs over the inner one, and both of them possess a distance from S_0 of ε. As long as the middle surface is closed, the two surfaces S_+ and S_- define the complete boundary of the body. If that is not the case then we imagine that the normals to the surface S_0 are erected along the boundary curve. The surface strip R that defines the normal segments that lie between S_+ and S_- complete the boundary of the body.

We assume that the middle surface is continuously curved, so in particular, it is free from edges and vertices.

8. Specializing the coordinate system. – In order to fix a point on the middle surface, we employ parameters p_1, p_2 in the usual way. It will be assumed that every point of the middle surface is determined *uniquely* by the associated parameter values. In order to determine an arbitrary point of the body, we introduce the distance p_3 of the point from the middle surface. It will be considered to be positive or negative according to whether the point lies on the outer or inner side of the middle surface.

We denote the Cartesian coordinates of an arbitrary point of the body by x, y, z, the coordinates of the base-point of the normal that goes through the middle surface by x_0, y_0, z_0, and the direction cosines of the outward-pointing normal to the middle surface by X, Y, Z.

For the square of the line element, since:

$$X\,dx_0 + Y\,dy_0 + Z\,z_0 = 0 \qquad \text{and} \qquad X\,dX + Y\,dY + Z\,dZ = 0\,,$$

there will be an expression of the form:

$$(1) \qquad [a_{11}^{(0)}\,dp_1^2 + 2a_{12}^{(0)}\,dp_1\,dp_2 + a_{22}^{(0)}\,dp_2^2] + 2[c_{11}\,dp_1^2 + 2c_{12}\,dp_1\,dp_2 + c_{22}\,dp_2^2]\,p_3 + [b_{11}\,dp_1^2 + 2b_{12}\,dp_1\,dp_2 + b_{22}\,dp_2^2]\,p_3^2 + dp_3^2\,.$$

The first quadratic form on the left represents the square of the line element of the middle surface, and the form that is multiplied by p_3^2 represents the square of the line element of the GAUSSIAN sphere. The quantities $c_{\lambda\mu}$ are the so-called "second-order fundamental quantities" of the middle surface ([1]).

In the general formulas of arts. **1-4**, one must then set $a_{13} = 0$, $a_{23} = 0$, $a_{33} = 1$ in the present case. It then follows that:

$$(2) \qquad A_{11} = a_{22}\,, \quad A_{22} = a_{11}\,, \quad A_{12} = -a_{12}\,, \quad A_{13} = 0\,, \quad A_{23} = 0\,, \quad A_{33} = A = a_{11}\,a_{22} - a_{12}^2\,.$$

In regard to the quantities $\left\{\begin{matrix}\lambda\,\mu \\ \nu\end{matrix}\right\}$ that were introduced in art. **2**, for the present purposes, it would suffice to remark that since (see art. **8**):

$$\frac{\partial x}{\partial p_3} = X\,, \qquad \frac{\partial y}{\partial p_3} = Y\,, \qquad \frac{\partial z}{\partial p_3} = Z\,, \qquad X^2 + Y^2 + Z^2 = 1\,,$$

one has:

$$X\,\frac{\partial^2 x}{\partial p_\mu\,\partial p_3} + Y\,\frac{\partial^2 y}{\partial p_\mu\,\partial p_3} + Z\,\frac{\partial^2 z}{\partial p_\mu\,\partial p_3} = 0\,,$$

and as a result:

$$\left\{\begin{matrix}\mu\,3 \\ 3\end{matrix}\right\} = 0 \qquad \text{for} \qquad \mu = 1, 2, 3.$$

([1]) Cf., **Knoblauch**, *Einleitung in die Theorie der krummen Flächen*, pp. 24.

The quantities $\begin{Bmatrix}12\\ \nu\end{Bmatrix}$, $\begin{Bmatrix}23\\ \nu\end{Bmatrix}$ ($\nu = 1, 2$) will not come under consideration in what follows. Finally, as far as the quantities $\begin{Bmatrix}\lambda\,\mu\\ 1\end{Bmatrix}$, $\begin{Bmatrix}\lambda\,\mu\\ 2\end{Bmatrix}$, $\begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix}$ ($\lambda, \mu = 1, 2$) are concerned, only the values that exist on the middle surface will appear in the following, so only the ones that correspond to the value $p_3 = 0$. Those values are defined by the equations:

$$
\begin{aligned}
\frac{\partial^2 x_0}{\partial p_\lambda\,\partial p_\mu} &= \begin{Bmatrix}\lambda\,\mu\\ 1\end{Bmatrix}\frac{\partial x_0}{\partial p_1} + \begin{Bmatrix}\lambda\,\mu\\ 2\end{Bmatrix}\frac{\partial x_0}{\partial p_2} + \begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix} X\,, \\
\frac{\partial^2 y_0}{\partial p_\lambda\,\partial p_\mu} &= \begin{Bmatrix}\lambda\,\mu\\ 1\end{Bmatrix}\frac{\partial y_0}{\partial p_1} + \begin{Bmatrix}\lambda\,\mu\\ 2\end{Bmatrix}\frac{\partial y_0}{\partial p_2} + \begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix} Y \qquad (\lambda, \mu = 1, 2)\,, \\
\frac{\partial^2 z_0}{\partial p_\lambda\,\partial p_\mu} &= \begin{Bmatrix}\lambda\,\mu\\ 1\end{Bmatrix}\frac{\partial z_0}{\partial p_1} + \begin{Bmatrix}\lambda\,\mu\\ 2\end{Bmatrix}\frac{\partial z_0}{\partial p_2} + \begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix} Z\,.
\end{aligned} \tag{3}
$$

Those equations show that for $p_3 = 0$, the quantities $\begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix}$ are the negatively-taken second-order fundamental quantities of the middle surface, so $\begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix} = -\,c_{\lambda\mu}$ for $p_3 = 0$. The values of the quantities $\begin{Bmatrix}\lambda\,\mu\\ 3\end{Bmatrix}$, ($\lambda, \mu = 1, 2$), that correspond to $p_3 = 0$ are the known **Christoffel** constraints ([1]).

The angles ω_1, ω_2, ω_3 that the outward-pointing normals to the outer surface of the body define with the outward-pointing normals to the surfaces p_1 = const., p_2 = const., p_3 = const., take the following values in the present case:

along the outer surface S_+ :	$\frac{\pi}{2}$,	$\frac{\pi}{2}$,	0,
″ inner ″ S_- :	$\frac{\pi}{2}$,	$\frac{\pi}{2}$,	π,
″ boundary surface R :	ω_1 ,	ω_2 ,	$\frac{\pi}{2}$.

The conditions for the outer surface (arts. **5**, **6**) then read as follows in the present case:

([1]) Cf., e.g., **Knoblauch**, *loc. cit.*, pp. 172.

$$
(4)\qquad
\begin{array}{lll}
\text{along} & S_+ : & N_{\lambda 3} = -A(P_\lambda^+), \\
\quad'' & S_- : & N_{\lambda 3} = +A(P_\lambda^+), \\
\quad'' & R : & \dfrac{N_{\lambda 1}}{\sqrt{a_{22}}}\cos\omega_1 + \dfrac{N_{\lambda 2}}{\sqrt{a_{11}}}\cos\omega_2 = -\sqrt{A}\,(P_\lambda^{(R)}) \qquad (\lambda = 1, 2, 3)\,.
\end{array}
$$

Here, $\sqrt{a}\,(P_1^+)$, $\sqrt{a}\,(P_2^+)$, (P_3^+) mean the projections of the external pressure that acts upon the surface S_+ onto the outward-directed normals to the surface p_1 = const., p_2 = const., and p_3 = const. or S_0 . The meanings of the quantities (P_λ^-) and $(P_\lambda^{(R)})$ are then clear with no further discussion.

If tractions, in the narrow sense (i.e., not *tensions*) act upon the surface S_+ and S_- then (P_λ^-) will be negative and $(P_\lambda^{(R)})$ will be positive.

9. Introducing restricted assumptions. – We now make use of the assumption that the thickness 2ε of the plate considered is very small. In that way, we consistently maintain the assumption that the components of the displacement ξ_1, ξ_2, ξ_3 and their derivatives of first and second order (and correspondingly, the stress components and their derivatives) must also be considered to be continuous functions when we consider ε to be a quantity that is infinitely-small to first order. In other words: We exclude the case in which the displacement components or their derivatives of first and second order go up to an order of magnitude of $1 / \varepsilon$.

In regard to that, it should be remarked: The assumption that the components of the stresses that originate inside the body are very small and vary continuously everywhere generally demands that the components of the displacements and their first derivatives with respect to the coordinates must be very small, but it does not exclude the possibility that in some parts of the body, the quotient of the directional derivatives of a stress component with the component itself (i.e., the logarithmic derivative of the stress component) will attain the order of magnitude $1 / \varepsilon$.

Rather, it is not at all established *a priori* whether the assumption that this does not occur is even admissible.

In the following, it will be shown that this assumption is admissible for certain forms of plates, but not for others.

With our assumptions, the quantities $\alpha_{\mu\nu}$ (and naturally the corresponding principal dilatations λ_1, λ_2, λ_3) are at least infinitely-small quantities of first order. However, the case can occur in which those quantities $\alpha_{\mu\nu}$ are all infinitely-small of second order, while the derivatives in the direction of the normals to the middle surface go up to the first order of magnitude. If one imagines developing the quantities $\alpha_{\mu\nu}$ in a series of increasing powers of p_3 :

$$\alpha_{\mu\nu} = \alpha_{\mu\nu}^{(0)} + \alpha'_{\mu\nu}\, p_3 + \tfrac{1}{2}\alpha''_{\mu\nu}\, p_3^2 + \cdots$$

then if $\alpha_{\mu\nu}^{(0)}$ is a first-order quantity, one can neglect the terms $\alpha'_{\mu\nu}\, p_3$, $\tfrac{1}{2}\alpha''_{\mu\nu}\, p_3^2$, and likewise all of the ones that follow them, as infinitely-small quantities of higher order. By contrast, when $\alpha_{\mu\nu}^{(0)}$

is infinitely-small of second order, the second term $\alpha'_{\mu\nu}\, p_3$ must be preserved. It is then clear that the two cases require an entirely-different treatment.

In what follows, I shall restrict myself to the case in which the quantities $\alpha_{\mu\nu}$ are small quantities of first order and reserve the second case as something that I will return to on a later occasion.

We make two further simplifying assumptions that are admissible in the cases that occur in practice: We overlook external forces that act upon the interior of the body, so we neglect gravity, and we assume, secondly, that only a constant normal pressure acts upon each of the two outer surfaces of the plate S_+ and S_- . Along the boundary to the body, which belongs to a boundary surface R, we assume that the pressure that acts upon an element of R is perpendicular to the normal to the middle surface S_0 that goes through the surface element.

10. Consequences of the assumptions that were introduced. – By assumption, the fluctuations that the values of the functions ξ_λ , $\alpha_{\lambda\mu}$, $N_{\lambda\mu}$ experience when one advances along a normal to the middle surface S_0 are considered to be infinitely-small quantities of higher order. It then follows that: One can replace the value of one of those functions at an arbitrary point on the normal with the value that exists at the corresponding point on the middle surface. It should be remarked in regard to the functions N_{13}, N_{23}, N_{33} that since, by assumption, a normal constant pressure acts upon the surfaces S_+ and S_- , one must set (P_1^+) , (P_2^+) , (P_1^-) , (P_2^-) equal to zero in the formulas [§ **8**, (4)]. The quantities – (P_3^+) and (P_3^-) can differ by only a negligible quantity. To simplify the notation, we set:

$$-(P_3^+) = (P_3^-) = (P) .$$

(P) will be positive or negative according to whether a pressure or a tension acts upon the surfaces S_+ and S_- .

Even though the quantity $(P_3^+) + (P_3^-)$ can generally be neglected, nonetheless, (due to the smallness of ε) the quotient – $[(P_3^+) + (P_3^-)] / 2\varepsilon$ can possess a value that *cannot* be neglected. That value might be denoted by Q. The constant Q will be positive or negative according to whether a greater pressure acts upon the outer or inner surface, resp. (cf., the remark at the conclusion of art. **8**).

On the grounds of the assumed continuity of the quantities $N_{\lambda\mu}$ and their derivatives, the following equations [see § **8**, (4) and (1)] will be true for every point of the body approximately:

$$(1) \qquad N_{13} = 0 , \qquad N_{23} = 0 , \qquad N_{33} = \left(a_{11}^{(0)} a_{22}^{(0)} - a_{12}^{(0)2}\right)(P) .$$

To the same degree of approximation, one can replace the differential quotients $\partial N_{3\lambda} / \partial p_3$ with the quotients:

$$\frac{N_{3\lambda}^+ - N_{3\lambda}^-}{2\varepsilon} = \frac{-A^+(P_\lambda^+) - A^-(P_\lambda^-)}{2\varepsilon} = \frac{A^+ - A^-}{2\varepsilon}(P^-) - A^+\frac{(P_\lambda^+) + (P_\lambda^-)}{2\varepsilon} .$$

Therefore, $\partial N_{13} / \partial p_3 = 0$, $\partial N_{23} / \partial p_3 = 0$. To the approximation that is valid for $\partial N_{33} / \partial p_3$, the quotient $(A^+ - A^-)/2\varepsilon$ can be replaced with the value of the differential quotient $\partial A / \partial p$ that is calculated for the middle surface, and that value is equal to [§ **8**, (1)]:

$$a_{11}^{(0)} c_{22} + a_{22}^{(0)} c_{11} - 2 a_{12}^{(0)} c_{12} .$$

We then get the approximate value:

$$\frac{\partial N_{33}}{\partial p_3} = (a_{11}^{(0)} c_{22} + a_{22}^{(0)} c_{11} - 2 a_{12}^{(0)} c_{12})(P) + (a_{11}^{(0)} a_{22}^{(0)} - a_{12}^{(0)2}) Q ,$$

and that will give the approximate value:

$$\frac{\partial \dfrac{N_{33}}{\sqrt{A}}}{\partial p_3} = \frac{1}{\sqrt{a_{11}^{(0)} a_{22}^{(0)} - a_{12}^{(0)2}}} [\tfrac{1}{2}(a_{11}^{(0)} c_{22} + a_{22}^{(0)} c_{11} - 2 a_{12}^{(0)} c_{12})(P) + (a_{11}^{(0)} a_{22}^{(0)} - a_{12}^{(0)2}) Q] .$$

The simplifications that proved to be admissible in the foregoing are not introduced into equations (5) and (6) of art. **5**.

It is convenient to simultaneously introduce a simplification into the notations. In what follows, the values that the aforementioned functions assume for $p_3 = 0$ shall be denoted by $x, y, z, u, v, w, a_{\lambda\mu}, \alpha_{\lambda\mu}, N_{\lambda\mu}, \xi_\lambda$. Those quantities will then be considered to be functions of position on the middle surface from now on. We denote the determinants of the quadratic binary forms:

$$a_{11}\, dp_1^2 + 2 a_{12}\, dp_1\, dp_2 + a_{22}\, dp_2^2 \qquad \text{and} \qquad c_{11}\, dp_1^2 + 2 c_{12}\, dp_1\, dp_2 + c_{22}\, dp_2^2$$

by a and c, resp., and their simultaneous invariant by (a, c). Analogously, we let (a, α) denote the simultaneous invariant of the binary forms $a_{11}\, dp_1^2 + \cdots$ and $\alpha_{11}\, dp_1^2 + \cdots$ In place of the quantities $N_{11}\ N_{12}\ N_{22}$ that were originally defined as the coefficients of a simultaneous contravariant of two ternary forms (arts. **3** and **5**), we introduce the quantities $\gamma_{11} = N_{22}$, $\gamma_{22} = N_{11}$, $\gamma_{12} = - N_{12}$, which are considered to be the coefficients of a simultaneous covariant of the two binary forms $a_{11}\, dp_1^2 + \cdots$ and $\alpha_{11}\, dp_1^2 + \cdots$

It now follows from equation [§ **5**, (1)]:

$$2K\left[(1+\vartheta)\frac{\Delta_1}{A} A_{\lambda\mu} - \beta_{\lambda\mu}\right] = - N_{\lambda\mu} ,$$

that since:

$$A_{13} = 0, \qquad A_{23} = 0, \qquad A = A_{33} = a, \quad N_{13} = 0, \qquad N_{23} = 0, \qquad N_{33} = a\,(P)$$

[cf., § **8**, (2) and equation (1) of this article]:

$$\beta_{13} = 0\,, \qquad \beta_{23} = 0\,,$$

$$2\,K\,[(1 + \vartheta)\,\Delta_1 - \beta_{33}] = -\,a\,(P)\,.$$

Now:

$$\Delta_1 = \sum_{\lambda,\mu} A_{\lambda\mu}\,\alpha_{\lambda\mu} = (a,\,\alpha) + a\,\alpha_{33}\,.$$

As a result:

(3) $$\beta_{33} = +\,\frac{(P)}{2\,K}\,(1 + \vartheta)\,[(a,\,\alpha) + a\,\alpha_{33}]\,.$$

On the other hand, from their definitions [§ **3**, (3)], the quantities $\beta_{\lambda\mu}$ have the expressions:

$$\begin{aligned} &\beta_{11} = a_{22}\,\alpha_{33} + \alpha_{22}\,, && \beta_{22} = a_{11}\,\alpha_{33} + \alpha_{11}\,, && \beta_{12} = -\,a_{12}\,\alpha_{33} - \alpha_{12}\,,\\ &\beta_{13} = a_{12}\,\alpha_{23} - a_{22}\,\alpha_{13}\,, && \beta_{23} = a_{12}\,\alpha_{13} - a_{11}\,\alpha_{23}\,, && \beta_{33} = (a\,,\,\alpha)\,. \end{aligned}$$

Since:

$$\beta_{13} = 0\,, \qquad \beta_{23} = 0\,, \qquad \text{one also has} \quad \alpha_{13} = 0\,, \qquad \alpha_{23} = 0\,.$$

If one substitutes the value of β_{33} that was just found in equation (3) then it will follow that:

$$\alpha_{33} = -\,\frac{1}{1+\vartheta}\frac{(P)}{2K} - \frac{\vartheta}{1+\vartheta}\frac{(a,\alpha)}{a}\,,$$

and from that:

$$\frac{\Delta_1}{A} = \frac{(a,\alpha)}{a} + \alpha_{33} = -\,\frac{1}{1+\vartheta}\frac{(P)}{2K} - \frac{1}{1+\vartheta}\frac{(a,\alpha)}{a}\,.$$

It now follows from equations [§ **5**, (1)] by a simple calculation that:

(4) $$\gamma_{\lambda\mu} = 2K\left[\alpha_{\lambda\mu} + \left(\frac{\vartheta}{1+\vartheta}\frac{(P)}{2K} - \frac{1+2\vartheta}{1+\vartheta}\frac{(a,\alpha)}{a}\right)a_{\lambda\mu}\right] \qquad (\lambda,\,\mu = 1,\,2)\,.$$

Art. **6** will imply the equations for the stresses that prevail along the middle surface:

$$(5)\quad \begin{cases} \sqrt{a_{11}}\,\Pi_{11} = \dfrac{1}{\sqrt{a_{11}\,a_{22}\,a}}[a_{11}\,\gamma_{22} + a_{12}\,\gamma_{12}], \\ \sqrt{a_{22}}\,\Pi_{2} = \dfrac{1}{\sqrt{a_{11}\,a_{22}\,a}}[a_{22}\,\gamma_{11} + a_{12}\,\gamma_{12}], \\ \sqrt{a_{11}}\,\Pi_{12} = \dfrac{1}{\sqrt{a_{11}\,a_{22}\,a}}[a_{11}\,\gamma_{12} + a_{12}\,\gamma_{11}], \\ \sqrt{a_{22}}\,\Pi_{21} = \dfrac{1}{\sqrt{a_{11}\,a_{22}\,a}}[a_{22}\,\gamma_{12} + a_{12}\,\gamma_{22}]\,. \end{cases}$$

The mechanical meaning of the quantities $\Pi_{\lambda\mu}$ can now be expressed in the following way:

$\sqrt{a_{\lambda\lambda}}\,\Pi_{\mu\lambda}$ means the projection of the pressure that the surface that lies on the inner side of the curve p_μ = const. experiences from the outside onto the outward-directed normal to the curve p_λ = const. that contacts the middle surface. The outer side of the curve p_μ = const. is considered to be the side along which the coordinate p_μ increases.

11. The basic equations of the problem. – We must now carry out the following substitution in the equations of the theory of elasticity [§ **5**, (5) and (6)]:

$$A_{11} = a_{22}\,,\quad A_{12} = -\,a_{12}\,,\quad A_{22} = a_{11}\,,\quad A_{13} = 0\,,\quad A_{23} = 0\,,\quad A_{33} = A = a\,,$$

$$\begin{Bmatrix} 11 \\ 3 \end{Bmatrix} = -\,c_{11}\,,\quad \begin{Bmatrix} 12 \\ 3 \end{Bmatrix} = -\,c_{12}\,,\quad \begin{Bmatrix} 22 \\ 3 \end{Bmatrix} = -\,c_{22}\,,\quad \begin{Bmatrix} 33 \\ 1 \end{Bmatrix} = 0\,,\quad \begin{Bmatrix} 33 \\ 2 \end{Bmatrix} = 0\,,\quad \begin{Bmatrix} 33 \\ 3 \end{Bmatrix} = 0\,,$$

$$P_1 = 0\,,\qquad P_2 = 0\,,\qquad P_3 = 0\,,$$

$$N_{11} = \gamma_{22}\,,\quad N_{12} = -\,\gamma_{12}\,,\quad N_{22} = \gamma_{11}\,,\quad N_{13} = 0\,,\quad N_{23} = 0\,,\quad \frac{\partial N_{13}}{\partial p_3} = 0\,,\quad \frac{\partial N_{23}}{\partial p_3} = 0\,,$$

$$N_{33} = a\,(P)\,,\qquad \frac{\partial \dfrac{N_{33}}{\sqrt{a}}}{\partial p_3} = \tfrac{1}{2}\frac{(a,c)}{\sqrt{a}}(P) + \sqrt{a}\,Q\,.$$

That gives:

$$(1)\quad \frac{k}{a}\left[a_{22}\frac{\partial^2 \xi_1}{\partial t^2} - a_{12}\frac{\partial^2 \xi_2}{\partial t^2}\right] = -\,\frac{1}{\sqrt{a}}\left[\frac{\partial \dfrac{\gamma_{22}}{\sqrt{a}}}{\partial p_1} - \frac{\partial \dfrac{\gamma_{12}}{\sqrt{a}}}{\partial p_2} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}\frac{\gamma_{22}}{\sqrt{a}} - 2\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}\frac{\gamma_{12}}{\sqrt{a}} + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{\gamma_{11}}{\sqrt{a}}\right],$$

(2) $$\frac{k}{a}\left[a_{11}\frac{\partial^2\xi_2}{\partial t^2}-a_{12}\frac{\partial^2\xi_1}{\partial t^2}\right]=-\frac{1}{\sqrt{a}}\left[\frac{\partial\frac{\gamma_{11}}{\sqrt{a}}}{\partial p_2}-\frac{\partial\frac{\gamma_{12}}{\sqrt{a}}}{\partial p_1}+\begin{Bmatrix}11\\2\end{Bmatrix}\frac{\gamma_{22}}{\sqrt{a}}-2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{\gamma_{12}}{\sqrt{a}}+\begin{Bmatrix}2\,2\\2\end{Bmatrix}\frac{\gamma_{11}}{\sqrt{a}}\right],$$

(3) $$k\frac{\partial^2\xi_2}{\partial t^2}=-\tfrac{1}{2}\frac{(a,c)}{a}(P)-Q+\frac{(c,\gamma)}{a}.$$

As long as the middle surface is not closed, the following boundary conditions [cf., § **8** (4)] will have to be considered:

(4) $$\begin{cases}\dfrac{1}{\sqrt{a}}\left[\dfrac{\cos\omega_1}{\sqrt{a_{22}}}\gamma_{11}-\dfrac{\cos\omega_2}{\sqrt{a_{11}}}\gamma_{12}\right]=-(P_1^{(R)}),\\[2ex]\dfrac{1}{\sqrt{a}}\left[-\dfrac{\cos\omega_1}{\sqrt{a_{22}}}\gamma_{12}+\dfrac{\cos\omega_2}{\sqrt{a_{11}}}\gamma_{22}\right]=-(P_2^{(R)}).\end{cases}$$

Here, ω_1 means the angle that the outward-directed normal to the boundary curve that contacts the middle surface makes with the normal to the curve p_1 = const. that contacts the middle surface and points in the direction of increasing p_1 . ω_2 has an analogous meaning.

The connection between the quantities $\gamma_{\mu\nu}$ and the displacements is mediated by equations [§ **10**, (4) and § **3**, (1) and (2)]:

(5) $$\gamma_{\mu\nu}=2K\left[\alpha_{\lambda\mu}+\left(\frac{(P)}{2K}\frac{\vartheta}{1+\vartheta}-\frac{1+2\vartheta}{1+\vartheta}\frac{(a,\alpha)}{a}\right)a_{\lambda\mu}\right]\qquad(\lambda,\ \mu=1,\ 2)$$

and

(6) $$\begin{cases}\alpha_{\lambda\mu}&=\dfrac{1}{2}\left(\dfrac{\partial u}{\partial p_\lambda}\dfrac{\partial x}{\partial p_\mu}+\dfrac{\partial v}{\partial p_\lambda}\dfrac{\partial y}{\partial p_\mu}+\dfrac{\partial w}{\partial p_\lambda}\dfrac{\partial z}{\partial p_\mu}\right)+\dfrac{1}{2}\left(\dfrac{\partial u}{\partial p_\mu}\dfrac{\partial x}{\partial p_\lambda}+\dfrac{\partial v}{\partial p_\mu}\dfrac{\partial y}{\partial p_\lambda}+\dfrac{\partial w}{\partial p_\mu}\dfrac{\partial z}{\partial p_\lambda}\right),\\[2ex]&=\dfrac{1}{2}\left(\dfrac{\partial\xi_\mu}{\partial p_\lambda}+\dfrac{\partial\xi_\lambda}{\partial p_\mu}\right)-\begin{Bmatrix}\lambda\,\mu\\1\end{Bmatrix}\xi_1-\begin{Bmatrix}\lambda\,\mu\\2\end{Bmatrix}\xi_2+c_{\lambda\mu}\,\xi_3.\end{cases}$$

It follows from (5) that:

$$\frac{(a,\gamma)}{a}=\frac{2\vartheta}{1+\vartheta}(P)-2K\frac{1+3\vartheta}{1+\vartheta}\frac{(a,\alpha)}{a},$$

and from that:

(7) $$\alpha_{\lambda\mu}=\frac{1}{2K}\left[\gamma_{\lambda\mu}-\frac{1+2\vartheta}{1+3\vartheta}\frac{(a,\alpha)}{a}a_{\lambda\mu}+\frac{\vartheta}{1+3\vartheta}(P)\,a_{\lambda\mu}\right].$$

Weingarten denoted the expressions that define the right-hand sides of equations (1) and (2), when taken negatively, by $\gamma_a\ (p_1)$ [$\gamma_\alpha\ (p_2)$, respectively]. He proved that those expressions transform like the differentials dp_1, dp_2, which will also emerge from the development that was carried out here. He further proved that the two expressions will vanish when one replaces $\gamma_{\lambda\mu}$ with $a_{\lambda\mu}$ or $c_{\lambda\mu}$.

One can give the expressions $\gamma_a\ (p_1)$ and $\gamma_a\ (p_2)$ a form that is more convenient for many applications.

Namely, if one observes that ([1]):

$$a_{11}\left\{\begin{matrix}1\ 1\\ 1\end{matrix}\right\}+a_{12}\left\{\begin{matrix}1\ 1\\ 2\end{matrix}\right\} = \frac{1}{2}\frac{\partial a_{11}}{\partial p_1}, \qquad a_{12}\left\{\begin{matrix}1\ 1\\ 1\end{matrix}\right\}+a_{22}\left\{\begin{matrix}1\ 1\\ 2\end{matrix}\right\} = \frac{\partial a_{12}}{\partial p_1}-\frac{1}{2}\frac{\partial a_{11}}{\partial p_2},$$

$$a_{11}\left\{\begin{matrix}1\,2\\ 1\end{matrix}\right\}+a_{12}\left\{\begin{matrix}1\,2\\ 2\end{matrix}\right\} = \frac{1}{2}\frac{\partial a_{11}}{\partial p_2}, \qquad a_{12}\left\{\begin{matrix}1\ 2\\ 1\end{matrix}\right\}+a_{22}\left\{\begin{matrix}1\ 2\\ 2\end{matrix}\right\} = \frac{1}{2}\frac{\partial a_{22}}{\partial p_1},$$

$$a_{11}\left\{\begin{matrix}2\,2\\ 1\end{matrix}\right\}+a_{12}\left\{\begin{matrix}2\,2\\ 2\end{matrix}\right\} = \frac{\partial a_{12}}{\partial p_2}-\frac{1}{2}\frac{\partial a_{22}}{\partial p_1}, \qquad a_{12}\left\{\begin{matrix}2\,2\\ 1\end{matrix}\right\}+a_{22}\left\{\begin{matrix}2\,2\\ 2\end{matrix}\right\} = \frac{1}{2}\frac{\partial a_{22}}{\partial p_2}$$

then that will give:

$$(8)\qquad a_{11}\,\gamma_a\,(p_1) + a_{12}\,\gamma_a\,(p_2) = \frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial p_1}\left(\frac{a_{11}\,\gamma_{22}-a_{12}\,\gamma_{12}}{\sqrt{a}}\right)-\frac{\partial}{\partial p_2}\left(\frac{a_{11}\,\gamma_{12}-a_{12}\,\gamma_{11}}{\sqrt{a}}\right)\right]$$
$$-\frac{1}{2a}\left[\gamma_{22}\frac{\partial a_{11}}{\partial p_1}+\gamma_{11}\frac{\partial a_{22}}{\partial p_1}-2\gamma_{12}\frac{\partial a_{12}}{\partial p_1}\right],$$

$$(9)\qquad a_{12}\,\gamma_a\,(p_1) + a_{22}\,\gamma_a\,(p_2) = \frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial p_2}\left(\frac{a_{22}\,\gamma_{11}-a_{12}\,\gamma_{12}}{\sqrt{a}}\right)-\frac{\partial}{\partial p_1}\left(\frac{a_{22}\,\gamma_{12}-a_{12}\,\gamma_{22}}{\sqrt{a}}\right)\right]$$
$$-\frac{1}{2a}\left[\gamma_{22}\frac{\partial a_{11}}{\partial p_2}+\gamma_{11}\frac{\partial a_{22}}{\partial p_2}-2\gamma_{12}\frac{\partial a_{12}}{\partial p_2}\right],$$

12. Conditions for equilibrium. – I shall not go further into the theory of oscillations, but only restrict myself to a closer examination of the conditions for equilibrium.

For the case of equilibrium, the first three equations of the previous article take the form:

$$\gamma_a\,(p_1) = 0\,, \qquad \gamma_a\,(p_2) = 0\,,$$

([1]) Cf., **Knoblauch**, *loc. cit.*, pp. 170, *et seq.*

$$\frac{(c,\gamma)}{a} = \frac{1}{2}\frac{(a,c)}{a}(P) + Q.$$

We now imagine that the deformation that actually occurs is composed of a superposition of two simple deformations. The first of those two deformations corresponds to the assumption that $Q = 0$, while the second one corresponds to the assumption that $(P) = 0$.

For the first deformations, one has the equations:

$$\gamma'_a(p_1) = 0, \quad \gamma'_a(p_2) = 0, \quad \frac{(c,\gamma')}{a} = \frac{1}{2}\frac{(a,c)}{a}(P),$$

$$\alpha'_{\lambda\mu} = \frac{1}{2K}\left[\gamma'_{\lambda\mu} - \frac{1+2\vartheta}{1+3\vartheta}\frac{(a,\gamma')}{a}a_{\lambda\mu} + \frac{\vartheta}{1+3\vartheta}(P)\,a_{\lambda\mu}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u'}{\partial p_\lambda}\frac{\partial x}{\partial p_\mu} + \frac{\partial v'}{\partial p_\lambda}\frac{\partial y}{\partial p_\mu} + \frac{\partial w'}{\partial p_\lambda}\frac{\partial z}{\partial p_\mu} + \frac{\partial u'}{\partial p_\mu}\frac{\partial x}{\partial p_\lambda} + \frac{\partial v'}{\partial p_\mu}\frac{\partial y}{\partial p_\lambda} + \frac{\partial w'}{\partial p_\mu}\frac{\partial z}{\partial p_\lambda}\right].$$

For the second deformation, one has the equations:

$$\gamma''_a(p_1) = 0, \quad \gamma''_a(p_2) = 0, \quad \frac{(c,\gamma'')}{a} = Q,$$

$$\alpha''_{\lambda\mu} = \frac{1}{2K}\left[\gamma''_{\lambda\mu} - \frac{1+2\vartheta}{1+3\vartheta}\frac{(a,\gamma'')}{a}a_{\lambda\mu}\right].$$

These must be combined with the three equations that mediate the connection between the quantities $\alpha''_{\lambda\mu}$ and the components of the displacements u'', v'', w''.

We satisfy the first system of equations by the assumption that:

$$\gamma''_a(p_1) = \tfrac{1}{2}(P)\,a_{\lambda\mu}, \qquad \alpha'_{\lambda\mu} = -\frac{(P)}{4K}\frac{1-\vartheta}{1+3\vartheta}a_{\lambda\mu},$$

$$\frac{u'}{x} = \frac{v'}{y} = \frac{w'}{z} = -\frac{(P)}{4K}\frac{1-\vartheta}{1+3\vartheta}.$$

That deformation consists of a uniform compression of the middle surface under which it remains similar to its original form.

Since there is no obstacle to reducing the case in which (P) is non-zero to the case in which $(P) = 0$, I shall assume that $(P) = 0$ in what follows.

13. On the uniqueness of the solution. – The differential equations that are true for the case of equilibrium now take the form:

$$\gamma_a (p_1) = 0 , \qquad \gamma_a (p_2) = 0 , \qquad \frac{(c, \gamma)}{a} = - Q .$$

When the middle surface is not closed, the following boundary conditions must be appended:

$$\frac{1}{\sqrt{a}}\left[\frac{\cos\omega_1}{\sqrt{a_{22}}}\gamma_{22} - \frac{\cos\omega_2}{\sqrt{a_{11}}}\gamma_{12} \right] = -\left(P_1^{(R)}\right) ,$$

$$\frac{1}{\sqrt{a}}\left[-\frac{\cos\omega_1}{\sqrt{a_{22}}}\gamma_{12} + \frac{\cos\omega_2}{\sqrt{a_{11}}}\gamma_{11} \right] = -\left(P_2^{(R)}\right) .$$

The dilatations and the displacements are defined by the equations:

$$\gamma_{\lambda\mu} = 2K\left[\alpha_{\lambda\mu} - \frac{1+2\vartheta}{1+3\vartheta}\frac{(a,\alpha)}{a} a_{\lambda\mu} \right],$$

$$\alpha_{\lambda\mu} = \frac{1}{2}\left(\frac{\partial\xi_\mu}{\partial p_\lambda} + \frac{\partial\xi_\lambda}{\partial p_\mu} \right) - \begin{Bmatrix} \lambda\ \mu \\ 1 \end{Bmatrix}\xi_1 - \begin{Bmatrix} \lambda\ \mu \\ 2 \end{Bmatrix}\xi_2 + c_{\lambda\mu}\,\xi_3 .$$

These must be combined with the continuity conditions:

The quantities $\gamma_{\lambda\mu}$ and ξ_λ are single-valued and continuous on the entire surface.

If the quantities $\gamma_{\lambda\mu}$ and ξ_λ are not determined uniquely by the given conditions then there must be a system of quantities $\gamma_{\lambda\mu}$, ξ_λ that satisfy the equations that the foregoing ones go to when one sets Q equal to zero, and in the event that the surface is not closed, $\left(P_1^{(R)}\right)$ and $\left(P_2^{(R)}\right)$, as well.

The integral over the entire middle surface is:

$$J = \int\limits_{(S)} [\gamma_a(p_1)\,\xi_1 + \gamma_a(p_2)\,\xi_2]\,\partial S$$

$$= \int\limits_{(S)} \left[\frac{1}{\sqrt{a}}\left(\frac{\partial \frac{\gamma_{22}}{\sqrt{a}}}{\partial p_1} - \frac{\partial \frac{\gamma_{12}}{\sqrt{a}}}{\partial p_2} + \begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix}\frac{\gamma_{22}}{\sqrt{a}} - 2\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}\frac{\gamma_{12}}{\sqrt{a}} + \begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix}\frac{\gamma_{11}}{\sqrt{a}} \right)\xi_1 \right.$$

$$\left. + \frac{1}{\sqrt{a}}\left(\frac{\partial \frac{\gamma_{11}}{\sqrt{a}}}{\partial p_2} - \frac{\partial \frac{\gamma_{12}}{\sqrt{a}}}{\partial p_1} + \begin{Bmatrix} 1\,1 \\ 2 \end{Bmatrix}\frac{\gamma_{22}}{\sqrt{a}} - 2\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\frac{\gamma_{12}}{\sqrt{a}} + \begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix}\frac{\gamma_{11}}{\sqrt{a}} \right)\xi_2 \right] \frac{\partial p_1\,\partial p_2}{\sqrt{a}} ,$$

$$= \int_{(S)} \left[\gamma_{22} \left(-\frac{\partial \xi_1}{\partial p_1} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \xi_1 + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \xi_2 \right) + \gamma_{11} \left(-\frac{\partial \xi_2}{\partial p_2} + \begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix} \xi_1 + \begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix} \xi_2 \right) \right.$$
$$\left. - \; 2\gamma_{12} \left(-\frac{1}{2} \frac{\partial \xi_1}{\partial p_2} - \frac{1}{2} \frac{\partial \xi_2}{\partial p_1} + \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} \xi_1 + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} \xi_2 \right) \right] \frac{\partial p_1 \, \partial p_2}{\sqrt{a}} + R \,.$$

When the middle surface is closed, one must set $R = 0$. If it is bounded by a boundary curve L then R will be the integral along that curve:

$$\int_{(L)} \left[\left(\frac{\cos \omega_1}{\sqrt{a_{22}}} \gamma_{22} - \frac{\cos \omega_2}{\sqrt{a_{11}}} \gamma_{12} \right) \xi_1 + \left(\frac{\cos \omega_2}{\sqrt{a_{11}}} \gamma_{11} - \frac{\cos \omega_1}{\sqrt{a_{22}}} \gamma_{12} \right) \xi_2 \right] \frac{\partial L}{\sqrt{a}} \;\; .$$

The integral is equal to zero, due to the initial conditions.

Based upon the equations that are valid for the displacements, one now gets:

$$J = \int_{(S)} [-(\gamma, \alpha) + (c, \gamma)\, \xi_3] \, \frac{\partial p_1 \, \partial p_2}{\sqrt{a}} \,.$$

Now:

$$(c, \gamma) = 0$$

and

$$(\gamma, \alpha) = 2K \left[2\alpha - \frac{1 + 2\vartheta}{1 + \vartheta} \frac{(a, \alpha)^2}{a} \right] .$$

As a result:

$$J = 2K \int_{(S)} \left[\frac{1 + 2\vartheta}{1 + \vartheta} \frac{(a, \alpha)^2}{a} - 2 \frac{\alpha}{a} \right] \partial S \,.$$

Since the form $a_{11}\, dp_1^2 + 2a_{12}\, dp_1\, dp_2 + a_{22}\, dp_2^2$ is definite, the expression under the integral sign is non-zero and positive as long as not all three quantities $\alpha_{\lambda\mu}$ vanish.

Now since $J = 0$, it follows that the quantities $\alpha_{\lambda\mu}$ are all equal to zero. The line element of the middle surface then experiences no dilatation. It follows from this that by our differential equations and the associated continuity and initial conditions, the deformation of the middle surface is determined up to an inextensible bending.

If the middle surface is a closed surface with everywhere-positive curvature then, as **Liebmann** has proved, an inextensible bending is impossible. The question of whether (to what extent, respectively) such a bending of a closed surface is possible when its curvature is partially positive and partially negative has not been answered yet.

If the middle surface is not closed then in order for the deformation to be determined completely, further conditions must be added to the boundary conditions that were given above, e.g., the condition that the boundary curve is fixed.

However, the question of whether that condition is also sufficient to determine the deformation must remain open for now.

14. Determining the components of the displacement. – When the quantities $\gamma_{\lambda\mu}$ are known, and therefore the quantities $\alpha_{\lambda\mu}$, as well, the determination of the components of the displacement will require the integration of a linear, but not homogeneous, second-order partial differential equation, which will require only quadratures, in addition.

In order to exhibit that partial differential equation, we employ the procedure that **Weingarten** gave ([1]).

It is preferable determine the components of the displacement u, v, w, as evaluated in the direction of the Cartesian coordinates, not the quantities ξ_λ .

We introduce three new unknowns, namely:

$$\eta = \frac{1}{2\sqrt{a}}\left[\sum \frac{\partial x}{\partial p_1}\frac{\partial u}{\partial p_2} - \sum \frac{\partial x}{\partial p_2}\frac{\partial u}{\partial p_1}\right], \tag{1}$$

$$\eta_1 = \sum X \frac{\partial u}{\partial p_1}, \qquad \eta_2 = \sum X \frac{\partial u}{\partial p_2}. \tag{2}$$

Here, we wrote:

$$\sum \frac{\partial x}{\partial p_1}\frac{\partial u}{\partial p_2}$$

as an abbreviation for:

$$\frac{\partial x}{\partial p_1}\frac{\partial u}{\partial p_2} + \frac{\partial y}{\partial p_1}\frac{\partial v}{\partial p_2} + \frac{\partial z}{\partial p_1}\frac{\partial w}{\partial p_2},$$

etc.

η is a differential invariant. Namely, the numerator of the expression η is the linear invariant of the bilinear differential form:

$$\sum_{\lambda=1}^{2}\sum_{\mu=1}^{2}\left[\sum \frac{\partial x}{\partial p_\lambda}\frac{\partial u}{\partial p_\mu}\right] dp_\lambda \, dp_\mu .$$

η_1, η_2 are the coefficients of a linear differential covariant, which will emerge immediately from the equation:

$$\eta_1 \, dp_1 + \eta_2 \, dp_2 = X \, du + Y \, dv + Z \, dw .$$

In order to see the mechanical meaning of the quantities η, η_1, η_2, we imagine, for the moment, that the origin of the Cartesian coordinates x, y, z is placed at a point of the middle surface, and the positive z-axis coincides with the outward-pointing normal. We then have:

([1]) "Über die Deformation einer biegsamen unausdehnbaren Fläche," Crelle's Journal, Bd. 100, pp. 296.

$$\eta_1\, dp_1 + \eta_2\, dp_2 = dw\,.$$

Therefore, $\eta_1 / \sqrt{a_{11}}$ is the cosine of the angle that a line element that has the direction of increasing p_1 after the deformation makes with the original direction of the surface normal. The meaning of η_2 is explained analogously.

In the case where the coordinate lines p_1 = const., p_2 = const. are perpendicular to each other, $\eta_1 / \sqrt{a_{11}}$, $\eta_2 / \sqrt{a_{22}}$ mean the components of the rotation around the line p_1 = const. (p_2 = const., resp.).

If one chooses the Cartesian coordinates x, y to be independent variables, for the moment, then one will have:

$$\eta = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial w}{\partial x}\right),$$

where η means the component of the rotation around the normal of the middle surface ([1]).

The equations:

$$\alpha_{\lambda\mu} = \tfrac{1}{2}\left[\sum \frac{\partial x}{\partial p_\lambda}\frac{\partial u}{\partial p_\mu} + \frac{\partial x}{\partial p_\mu}\frac{\partial u}{\partial p_\lambda}\right] \qquad (\lambda, \mu = 1, 2)$$

are now replaced by the equations:

$$(3)\qquad \begin{cases} \displaystyle\sum \frac{\partial x}{\partial p_1}\frac{\partial u}{\partial p_1} = \alpha_{11}, & \displaystyle\sum \frac{\partial x}{\partial p_2}\frac{\partial u}{\partial p_2} = \alpha_{22}, \\ \displaystyle\sum \frac{\partial x}{\partial p_1}\frac{\partial u}{\partial p_1} = \alpha_{12} + \eta\sqrt{a}, & \displaystyle\sum \frac{\partial x}{\partial p_2}\frac{\partial u}{\partial p_1} = \alpha_{12} - \eta\sqrt{a}\,. \end{cases}$$

They must be combined with the equation [see § **11**, (7)]:

$$(4)\qquad \alpha_{\lambda\mu} = \frac{1}{2K}\left[\gamma_{\lambda\mu} - \frac{1+2\vartheta}{1+3\vartheta} H\, a_{\lambda\mu}\right],$$

in which one has set $(a, \gamma) / a = H$, to abbreviate.

It now follows from (3) that:

$$(5)\qquad \sum \frac{\partial^2 x}{\partial p_1^2}\frac{\partial u}{\partial p_2} - \sum \frac{\partial^2 x}{\partial p_1\, \partial p_2}\frac{\partial u}{\partial p_1} = \frac{\partial \alpha_{12}}{\partial p_1} + \frac{\partial \eta\sqrt{a}}{\partial p_1} - \frac{\partial \alpha_{11}}{\partial p_2}\,.$$

In order to convert the right-hand side of that equation, we remark that:

([1]) **Weingarten**, *loc. cit.*, used φ to denote the quantity that is denoted by η here and called it the "displacement function."

$$\frac{\partial \sqrt{a}}{\partial p_1} = \sqrt{a}\left[\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\right]\frac{\partial \sqrt{a}}{\partial p_2} = \sqrt{a}\left[\begin{Bmatrix}2\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,2\\1\end{Bmatrix}\right].$$

As a result:

$$(6)\qquad \left\{\begin{aligned} &\frac{\partial \alpha_{12}}{\partial p_1}-\frac{\partial \alpha_{11}}{\partial p_2}+\frac{\partial \eta\sqrt{a}}{\partial p_1}=\sqrt{a}\left[\frac{\partial \frac{\alpha_{12}}{\sqrt{a}}}{\partial p_1}-\frac{\partial \frac{\alpha_{11}}{\sqrt{a}}}{\partial p_2}\right]\\ &\quad+\left[\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\right]\alpha_{12}-\left[\begin{Bmatrix}2\,2\\2\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\right]\alpha_{11}+\sqrt{a}\,\frac{\partial \eta}{\partial p_1}+\sqrt{a}\left[\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\right]\eta. \end{aligned}\right.$$

In order to convert the left-hand side of equation (5), we make use of the equations:

$$\frac{\partial^2 x}{\partial p_\lambda\,\partial p_\mu} = \begin{Bmatrix}\lambda\;\mu\\1\end{Bmatrix}\frac{\partial x}{\partial p_1}+\begin{Bmatrix}\lambda\;\mu\\2\end{Bmatrix}\frac{\partial x}{\partial p_2}-c_{\lambda\mu}\,X \qquad (\lambda,\ \mu = 1, 2),$$

and the equations that are obtained from them by cyclically permuting x, y, z. When we recall (3), we will get:

$$\sum \frac{\partial^2 x}{\partial p_1^2}\frac{\partial u}{\partial p_2}-\sum \frac{\partial^2 x}{\partial p_1\,\partial p_2}\frac{\partial u}{\partial p_1}$$

$$= \begin{Bmatrix}11\\1\end{Bmatrix}[\alpha_{12}+\eta\sqrt{a}]+\begin{Bmatrix}11\\2\end{Bmatrix}\alpha_{22}-c_{11}\,\eta_2-\begin{Bmatrix}1\,2\\1\end{Bmatrix}\alpha_{11}-\begin{Bmatrix}1\,2\\2\end{Bmatrix}[\alpha_{12}-\eta\sqrt{a}]+c_{12}\,\eta_1 .$$

We substitute that value in equation (5) and simultaneously employ (6). That will give:

$$\sqrt{a}\left[\frac{\partial \frac{\alpha_{12}}{\sqrt{a}}}{\partial p_1}-\frac{\partial \frac{\alpha_{11}}{\sqrt{a}}}{\partial p_2}\right]-\begin{Bmatrix}2\,2\\2\end{Bmatrix}\alpha_{11}+2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\alpha_{12}-\begin{Bmatrix}11\\2\end{Bmatrix}\alpha_{22} = -\sqrt{a}\,\frac{\partial \eta}{\partial p_1}-c_{11}\,\eta_2+c_{12}\,\eta_1 .$$

With the use of **Weingarten**'s notation, one can write:

$$\alpha_a\,(p_2) = \frac{1}{\sqrt{a}}\frac{\partial \eta}{\partial p_1}+\frac{c_{11}\,\eta_2-c_{12}\,\eta_1}{a} .$$

Now, as a result of (4):

$$\alpha_a(p_2) = \frac{1}{2K}\left[\gamma_a(p_2) - \frac{1+2\vartheta}{1+3\vartheta} H\, a_a(p_2) - \frac{1+2\vartheta}{1+3\vartheta}\left(a_{11}\frac{\partial H}{\partial p_2} - a_{12}\frac{\partial H}{\partial p_1}\right)\right],$$

so since $\gamma_a(p_2) = 0$ (art. **12**) and $a_a(p_2) = 0$ (art. **11**):

(7) $$c_{11}\,\eta_2 - c_{12}\,\eta_1 = -\sqrt{a}\,\frac{\partial\eta}{\partial p_1} - \frac{1}{2K}\frac{1+2\vartheta}{1+3\vartheta}\left(a_{11}\frac{\partial H}{\partial p_2} - a_{12}\frac{\partial H}{\partial p_1}\right).$$

When one switches p_1 and p_2, so $-\eta$ will enter in place of η, that will give:

(8) $$c_{22}\,\eta_2 - c_{12}\,\eta_2 = \sqrt{a}\,\frac{\partial\eta}{\partial p_2} - \frac{1}{2K}\frac{1+2\vartheta}{1+3\vartheta}\left(a_{22}\frac{\partial H}{\partial p_1} - a_{12}\frac{\partial H}{\partial p_2}\right).$$

We solve equations (7) and (8) for η_1 and η_2. We obtain:

(9) $$\left\{\begin{aligned} c\,\eta_1 &= \sqrt{a}\left(\quad c_{11}\frac{\partial\eta}{\partial p_2} - c_{12}\frac{\partial\eta}{\partial p_1}\right) - \frac{1}{2K}\frac{1+2\vartheta}{1+3\vartheta}\left(A_{11}\frac{\partial H}{\partial p_1} + A_{12}\frac{\partial H}{\partial p_2}\right),\\ c\,\eta_2 &= \sqrt{a}\left(-c_{22}\frac{\partial\eta}{\partial p_1} + c_{12}\frac{\partial\eta}{\partial p_2}\right) - \frac{1}{2K}\frac{1+2\vartheta}{1+3\vartheta}\,a\left(A_{21}\frac{\partial H}{\partial p_1} + A_{22}\frac{\partial H}{\partial p_2}\right).\end{aligned}\right.$$

Here, one sets:

(10) $$\left\{\begin{aligned} a\,A_{11} &= a_{22}\,c_{11} - a_{12}\,c_{12}, \quad & a\,A_{12} &= a_{11}\,c_{12} - a_{12}\,c_{11},\\ a\,A_{21} &= a_{22}\,c_{12} - a_{12}\,c_{22}, \quad & a\,A_{22} &= a_{11}\,c_{22} - a_{12}\,c_{12},\end{aligned}\right.$$

to abbreviate. The quantities $A_{\lambda\mu}$ have a simple meaning. Namely, one has the equations:

$$\frac{\partial X}{\partial p_\lambda} = A_{\lambda 1}\frac{\partial x}{\partial p_1} + A_{\lambda 2}\frac{\partial x}{\partial p_2} \qquad (\lambda = 1, 2)$$

and four more equations that are obtained from them by cyclically permuting x, y, z.

Now, it follows from (2) and (3) that:

$$\frac{\partial\eta_1}{\partial p_2} - \frac{\partial\eta_2}{\partial p_1} = \sum \frac{\partial X}{\partial p_2}\frac{\partial u}{\partial p_1} - \sum \frac{\partial X}{\partial p_1}\frac{\partial u}{\partial p_2}$$

$$= [A_{21}\,\alpha_{11} + A_{22}\,(\alpha_{12} - \eta\sqrt{a})] - [A_{11}\,(\alpha_{12} + \eta\sqrt{a}) + A_{12}\,\alpha_{22}]$$
$$= [A_{21}\,\alpha_{11} - A_{12}\,\alpha_{22} - (A_{11} - A_{22})\,\alpha_{12}] - \eta\sqrt{a}\,(A_{11} + A_{22}).$$

One can write the expression in the first bracket on the right-hand side in the form:

$$-\frac{1}{a}\begin{vmatrix} a_{11} & a_{12} & a_{22} \\ c_{11} & c_{12} & c_{22} \\ \alpha_{11} & \alpha_{12} & \alpha_{22} \end{vmatrix} = -\frac{1}{2K\,a}\begin{vmatrix} a_{11} & a_{12} & a_{22} \\ c_{11} & c_{12} & c_{22} \\ \gamma_{11} & \gamma_{12} & \gamma_{22} \end{vmatrix} .$$

We denote the determinant on the right (which is the simultaneously invariant of the three quadratic forms $a_{11}\,dp_1^2 + \cdots$, $c_{11}\,dp_1^2 + \cdots$, $\gamma_{11}\,dp_1^2 + \cdots$) by (a, c, γ), to abbreviate.

One further has $A_{11} + A_{22} = (a, c) / a$. We then get:

$$\frac{1}{\sqrt{a}}\left[\frac{\partial\eta_2}{\partial p_1} - \frac{\partial\eta_1}{\partial p_2}\right] = \frac{1}{2K}\frac{(a,c,\gamma)}{a^{3/2}} + \frac{(a,c)}{a}\eta \ .$$

If we introduce the values (9) for η_1 and η_2 then we will get the partial differential equations for η:

$$(11)\left\{\begin{aligned} &\frac{1}{\sqrt{a}}\frac{\partial}{\partial p_1}\left[\frac{\sqrt{a}}{c}\left(c_{22}\frac{\partial\eta}{\partial p_1} - c_{12}\frac{\partial\eta}{\partial p_2}\right)\right] + \frac{1}{\sqrt{a}}\frac{\partial}{\partial p_2}\left[\frac{\sqrt{a}}{c}\left(c_{11}\frac{\partial\eta}{\partial p_2} - c_{12}\frac{\partial\eta}{\partial p_1}\right)\right] + \frac{(a,c)}{a}\eta \\ &= \frac{1}{2K}\frac{1+2\vartheta}{1+3\vartheta}\frac{1}{\sqrt{a}}\left\{\frac{\partial}{\partial p_1}\left[\frac{a}{c}\left(A_{21}\frac{\partial H}{\partial p_1} + A_{22}\frac{\partial H}{\partial p_2}\right)\right] + \frac{\partial}{\partial p_2}\left[\frac{a}{c}\left(A_{11}\frac{\partial H}{\partial p_1} + A_{12}\frac{\partial H}{\partial p_2}\right)\right]\right\} - \frac{1}{2K}\frac{(a,c,\gamma)}{a^{3/2}} . \end{aligned}\right.$$

If η is determined then equations (9) will yield η_1 and η_2 , and the derivatives of the components u, v, w can then be calculated. The determination of the components themselves then requires nothing more than quadratures.

(to be continued)

"Über die Deformation gekrümmter elastischer Platten," Archiv. d. Math. u. Phys. (3) **6** (1904), 260-283.

(continuation)

Part III: A more precise examination of the special case in which the middle surface is a surface of revolution.

15. Definition of the surface. Choice of coordinate system. – We apply the general theory to the case in which the middle surface is a surface of revolution. In so doing, we restrict ourselves to the case in which the curve whose rotation generates the surface is an oval whose symmetry axis is perpendicular to the axis of rotation. We choose the axis of rotation to be the z-axis and imagine that it is vertical. We choose the plane $z = 0$ to be the symmetry plane of the surface – i.e., the *equatorial plane*. We denote the distance from a point on the surface to the axis of rotation by r. The quantities r and z are considered to be functions of a parameter p. We denote the derivatives of r and z with respect to p by r', z', etc., and denote the derivative of the line element on the meridian curve by s'. The quantities r and s' are everywhere positive. The parameter p is chosen in such a way that r' is positive at the highest point of the meridian curve and negative at the lowest one. The reciprocal value of the positively-taken radius of curvature of the meridian curve is then $R = (z'\,r'' - r'\,z'')\,/\,s'^2$.

For the second parameter, we employ the angle that an arbitrary meridian defines with the initial meridian that goes through the x-axis. We denote that second parameter by q (which deviates from the notation that we have used up to now).

We now get the following representation for the coordinates of a point in the middle surface and the direction cosines of the outward-pointing normal:

$$(1) \qquad \left\{ \begin{array}{l} x = r\cos q, \qquad y = r\sin q, \\ X = -\dfrac{z'}{s'}\cos\, q, \quad X = -\dfrac{z'}{s'}\sin\, q, \quad Z = \dfrac{r'}{s'}. \end{array} \right.$$

One easily verifies that the determinant:

$$\begin{vmatrix} \dfrac{\partial x}{\partial p_1} & \dfrac{\partial y}{\partial p_1} & \dfrac{\partial z}{\partial p_1} \\ \dfrac{\partial x}{\partial p_2} & \dfrac{\partial y}{\partial p_2} & \dfrac{\partial z}{\partial p_2} \\ X & Y & Z \end{vmatrix} = r\,s'$$

is then positive, which would correspond to the convention that was made before (cf., the beginning of art. **2**).

We get the following values for the first and second order fundamental quantities:

$$
(2)\qquad \begin{cases} a_{11} = s'^2, & a_{12} = 0, \quad a_{22} = r^2, \qquad a = r^2 s'^2, \\ c_{11} = \dfrac{z' r'' - r' z''}{s'} = s'^2 R, & c_{12} = 0, \quad c_{22} = -\dfrac{r z'}{s'}, \quad c = -r z' s' R. \end{cases}
$$

It is preferable to introduce the displacements in the directions of increasing r and q in place of the displacement components u, v. We set:

$$
u = \rho \cos q - \sigma \sin q, \qquad v = \rho \sin q + \sigma \cos q .
$$

We then have [see § **11**, (6) and § **14**, (2)]:

$$
(3)\qquad \begin{cases} \alpha_{11} = \sum \dfrac{\partial u}{\partial p}\dfrac{\partial x}{\partial p} = r' \dfrac{\partial \rho}{\partial p} + z' \dfrac{\partial w}{\partial p}, \\ \alpha_{22} = \sum \dfrac{\partial u}{\partial q}\dfrac{\partial x}{\partial q} = r \dfrac{\partial \sigma}{\partial q} + r\sigma, \\ \alpha_{12} - r s' \eta = \sum \dfrac{\partial u}{\partial p}\dfrac{\partial x}{\partial q} = r \dfrac{\partial \sigma}{\partial q}, \\ \alpha_{12} + r s' \eta = \sum \dfrac{\partial u}{\partial q}\dfrac{\partial x}{\partial p} = r' \dfrac{\partial \sigma}{\partial q} - r'\sigma + z' \dfrac{\partial w}{\partial q}, \\ \eta_1 = \sum X \dfrac{\partial u}{\partial p} = -\dfrac{z'}{s'}\dfrac{\partial \rho}{\partial p} + \dfrac{r'}{s'}\dfrac{\partial w}{\partial p}, \\ \eta_2 = \sum X \dfrac{\partial u}{\partial q} = -\dfrac{z'}{s'}\left(\dfrac{\partial \rho}{\partial q} - \sigma\right) + \dfrac{r'}{s'}\dfrac{\partial w}{\partial q}. \end{cases}
$$

Equations [§ **14**, (9)] yield:

$$
(4)\qquad \begin{cases} \eta_1 = -\dfrac{s'^2}{r'}\dfrac{\partial \mu}{\partial q} + \dfrac{1}{E} r \dfrac{s'}{z'}\dfrac{\partial H}{\partial p}, \\ \eta_2 = -\dfrac{r}{s' R}\dfrac{\partial \eta}{\partial p} - \dfrac{1}{E}\dfrac{1}{R}\dfrac{\partial H}{\partial p}. \end{cases}
$$

Here, the elastic modulus $E = 2K\dfrac{1+3\vartheta}{1+2\vartheta}$ (art. **1**, conclusion) was introduced in place of the **Kirchhoff** constant on the right-hand sides.

The further investigation takes an essentially different form according to whether the middle surface is or is not intersected by the axis of rotation. In the first case, the surface will be everywhere convex-outward, and in the second case, we will be dealing with a tubular surface.

16. Middle surface with everywhere-positive curvature. – We first consider the case in which the middle surface is a simply-connected, everywhere convex-outward surface. In order to split off the elastic deformation from the motion of the body, when considered to be rigid, we assume (and this is obviously permissible) that the connecting line between the two poles of the surface experiences no elastic deformation and that its bisecting point is fixed. Furthermore, the surface element that goes through the upper pole might experience no rotation around the vertical. Obviously, every point of the surface will be displaced only in its meridian plane, and all meridian curves will be deformed in the same way.

It follows from this that the quantities γ_{11}, γ_{12}, γ_{22}, ρ, and w depend upon only the parameter p, and the quantities σ, η, and η_2 vanish [cf., the remark in § **14**, (2)]. The third of equations (3) in the previous article shows that α_{12}, and therefore γ_{12} will vanish, as well, as a result.

In order to determine the quantities γ_{11}, γ_{12}, γ_{22}, we employ equations (8) and (9) of art. **11**. They give:

(1) $$\frac{d}{dp}\left(\frac{s'\gamma_{22}}{r}\right)-\frac{s''}{r}\gamma_{22}-\frac{r'}{s'}\gamma_{11}=0\,,$$

(2) $$\frac{d}{dp}\left(\frac{r\,\gamma_{12}}{s'}\right)=0\,,$$

to which we add the equation:

(3) $$\frac{(c,\gamma)}{a}=\frac{R}{r^2}\gamma_{22}-\frac{z'}{r\,s'^3}\gamma_{11}=Q\,.$$

Since γ_{12} cannot become discontinuous at the poles of the surface of rotation, equation (2) demands that $\gamma_{12}=0$, which agrees with remark that was made above.

It follows from (1) that:

$$s'\frac{d\dfrac{\gamma_{22}}{r}}{dp}-\frac{r'}{s'}\gamma_{11}=0\,.$$

If we introduce the value of γ_{11} that follows from (3) into that equation then that will give:

$$s'\frac{d\dfrac{\gamma_{22}}{r}}{dp}-\frac{s'^2\,r'\,R}{z'}\frac{\gamma_{22}}{r}+\frac{r\,r'\,s'^2}{z'}Q=0\,.$$

We multiply that by z'/s'^2. The coefficient of γ_{22}/r will be:

$$-\frac{r'(z'r''-r'z'')}{s'^3}=-\frac{z'(r'r''+z'z'')-s'^2\,z''}{s'^3}=\frac{z''}{s'}-\frac{z'\,s''}{s'^2}=\frac{d\dfrac{z'}{s'}}{dp}\,.$$

We then get $\frac{d}{dp}\left(\frac{z'\gamma_{22}}{s' r}\right) = - r r' Q$, and as a result:

$$\frac{z'\gamma_{22}}{s' r} = -\frac{Q}{2} r^2 + \text{const.} \tag{4}$$

In order to determine the integration constant, it should be pointed out that our system of surface coordinates p, q violates the previous assumption that the determinant of the quadratic form that represents the square of the line element should not vanish anywhere on the surface. As a result, yet another condition must be added to the continuity conditions that the quantities γ_{11}, γ_{22} must satisfy:

The differential invariant $H = \frac{(a, \gamma)}{a} = \frac{\gamma_{11}}{s'^2} + \frac{\gamma_{22}}{r^2}$ must also remain continuous at the poles. It then follows that the integration constant that we speak of is set equal to zero.

We then get:

$$\gamma_{22} = -\frac{Q}{2}\frac{r^3 s'}{z'},$$

$$\gamma_{11} = -\frac{Q}{2} s'^2 \left[2r \frac{s'}{z'} + r^2 \frac{s'^2}{z'^2} R\right],$$

$$H = -\frac{Q}{2}\left[3r \frac{s'}{z'} + r^2 \frac{s'^2}{z'^2} R\right].$$

For $z' = 0$, one also has $r = 0$, so the quotient r / z' will remain continuous, and as a result, γ_{11}, γ_{22} and H will be continuous on the entire middle surface.

When we introduce the constants $E = 2K \frac{1+3\vartheta}{1+2\vartheta}$ and $\mu = \frac{\vartheta}{1+2\vartheta}$ (cf., art. **1**, conclusion), equations [§ **14**, (4)] will give:

$$\alpha_{11} = \frac{1}{E}\left[\mu \gamma_{11} - \gamma_{22}\frac{s'^2}{r^2}\right], \qquad \alpha_{22} = \frac{1}{E}\left[\mu \gamma_{22} - \gamma_{11}\frac{r^2}{s'^2}\right]. \tag{5}$$

The first two of equations (3) in art. **15** imply that $r'\rho' + z' w' = \alpha_{11}$, $r\ \rho = \alpha_{22}$. The determination of w will be completed by the remark that that quantity must vanish on the grounds of the symmetry in the equatorial plane.

One easily verifies that the values of ρ and w thus found will be single-valued and continuous on the entire middle surface.

For the sake of brevity, I shall pass over the application of the fully-developed general formulas to examples.

17. Deformation of a tube. – We shall now assume that the middle surface has the form of a bent circular tube. However, it does not define a complete closed ring, but is bounded by two meridian planes. The angle through which the given oval must rotate in order to generate the middle surface will be denoted by $2\pi - 2\beta$.

We imagine that the open ends of the tube are closed by caps. These caps (as well as the tube itself) are acted upon by the normal pressure (P_3^+) from the outside and the normal pressure (P_3^-) from the inside. That pressure carries over to the boundaries of the tube: On each of those boundaries, a force acts perpendicularly to the boundary plane whose total magnitude is the product of the pressure difference $(P_3^+) + (P_3^-)$ and the area of the oval whose rotation generals the middle surface. How that force is distributed over the boundary depends upon the form of the endcaps, and likewise depends upon the component of the pressure that acts upon the boundary whose direction falls in the plane of the boundary. When the distance from the generating oval to the axis of rotation is somewhat considerable with respect to its dimensions, one can assume that the influence of the form of the endcaps is noticeable only in the close vicinity of the ends of the tube, but that at some distance from the boundaries, the elastic stresses that originate in the tube are independent of it. To that extent that this oversight is permissible, the remark that the stresses (so the quantities γ_{11}, γ_{12}, γ_{22}) depend upon the parameter *p*, but not the parameter *q*, will also be true here. We choose the middle meridian plane, relative to which the surface of the tube is symmetric, to be the initial meridian (as the plane $y = 0$). We think of the point at which the surface intersects the *x*-axis, which lies next to the axis of rotation, as fixed, and we further assume that under elastic deformation, that element of the meridian curve that goes through that point does not change in direction and the surface element that goes through it does not change its location, which are assumptions that are obviously permissible.

On the grounds of symmetry, one finds that the displacement component ρ is an even function of the parameter *q*, and the component σ is an odd function of *q*. The component *w* must be independent of *q*, since we have indeed neglected all forces that can bring about a dependency upon that the parameter *q*. Of the components of the rotation η_1 / s', η_2 / r, η [cf., the remark in § **14**, (2)], the first one (viz., the component of the rotation around the parallel circle) must be independent of *q*, while the other two (viz., the components of the rotation around the meridian and the surface normal) are odd functions of *q*.

The first of equations [§ **15**, (4)] shows that $\partial\eta / \partial q$ is independent of *q*. As a result, η is the product of *q* with a function of *p*. The third of equations [§ **15**, (3)] shows that the quantity α_{12} (just like σ and η) is an odd function of *q*. Now, since, by assumption, the quantities γ_{11}, γ_{12}, γ_{22}, and as a result, the quantities α_{11}, α_{12}, α_{22} [see § **14**, (4)], as well, are independent of *q*, one must have $\alpha_{12} = 0$, and as a result, $\gamma_{12} = 0$, as well.

Up to now, no closer determinations of the parameter *p* have been made. We now assume that the parameter *p* is chosen such that the points of the middle surface that lie symmetrically to the equatorial plane correspond to opposite values of the parameter *p*. Let $p = 0$ at the point on the meridian curve that lies next to the axis of rotation. Let $p = \pi / 2$ at the highest point of the meridian curve and let $p = \pi$ at the point that is furthest from the axis of rotation. We index the aforementioned three points by the numbers 0, 1, 2, respectively. We denote the values that any function f of p assumes that those points by f_0, f_1, f_2, resp.

The points 0 and 2 are the endpoints of the symmetry axis of the oval, and the tangent to the point 1 is parallel to the symmetry axis. r and z are periodic functions of p with period 2π, and indeed r is an even function, while z is odd.

The displacement components ρ and σ are even functions of p, and w is an odd function of p.

From the convention that was made above, $\rho = 0$ for $p = 0$, $q = 0$.

18. Integrating the differential equations of the problem. – The integration of the differential equations of elasticity theory proceeds in the present case just as it did in the case of art. **16**. We first get [equation (4) of the cited art.]:

$$\frac{z'}{s'}\frac{\gamma_{22}}{r} = -\frac{Q}{2}r^2 + \text{const.}$$

In order for γ_{22} to remain continuous at the point 1, where $z' = 0$, the expression on the right-hand side must vanish at that point. One then has:

(1) $$\gamma_{22} = -\frac{Q}{2}\frac{s'}{z'}\,r\,(r^2 - r_1^2)\,.$$

From the equation [see § **16**, (3)]:

$$\frac{(c,\gamma)}{a} = \frac{R}{r^2}\gamma_{22} - \frac{z'}{r\,s'^2}\gamma_{11} = Q\,,$$

one will get:

(2) $$\gamma_{11} = -\frac{Q}{2}s'^2\left[2\frac{s'}{z'}r + \frac{s'^2}{z'^2}R\,(r^2 - r_1^2)\right].$$

If one writes γ_{11} in the form:

(2.a) $$\gamma_{11} = -\frac{Q}{2}s'\left[2\,r\,z' + \frac{d}{dp}\left(\frac{r'}{z'}(r^2 - r_1^2)\right)\right]$$

then one will see that γ_{11} is continuous along the entire oval.

It follows from (1) and (2.a) that:

(3) $$H = \frac{\gamma_{11}}{s'^2} + \frac{\gamma_{22}}{r^2} = -\frac{Q}{2}\left[2\,r\frac{z'}{s'} + \frac{1}{s'}\frac{d}{dp}\left(\frac{r'}{z'}(r^2 - r_1^2)\right) + \frac{s'}{z'}\frac{r^2 - r_1^2}{r}\right].$$

The function η must next be determined.

The expression on the right-hand side of the partial differential equation that this function satisfies [§ **14**, (11)] vanishes, since $A_{21} = 0$, $\gamma_{12} = 0$, and H is independent of q. The second term on the left vanishes because $c_{12} = 0$ and η is a linear function of q (cf., the previous art.). We then get [see § **15**, (2)]:

$$\frac{1}{r\,s'}\frac{\partial}{\partial p}\left(\frac{r}{R\,s'}\frac{\partial\eta}{\partial p}\right)+\left(R-\frac{z'}{s'}\frac{1}{r}\right)\eta = 0\,. \tag{4}$$

A further condition must be added: The left-hand side of equation [§ **14**, (8)] vanishes for $p = \pi / 2$ and arbitrary values of q. Therefore, for $p = \pi / 2$ and arbitrary values of q, one has:

$$\frac{\partial\eta}{\partial q} = \frac{1}{2K}\frac{1+2\vartheta}{1+3\vartheta}\frac{r}{s'}H' = \frac{1}{E}\frac{r}{s'}H'\,.$$

We denote the value that the derivative $\frac{dH}{ds} = \frac{H'}{s'}$ assumes for $p = \pi / 2$ by $Q\,h$. The constant h is characteristic of the deformation of the tube.

We will satisfy the differential equation (4) and the auxiliary conditions when we set:

$$\eta = \frac{Q}{E}h\,r_1\frac{r'}{s'}q\,. \tag{5}$$

It now follows from [§ **14**, (8)], when one recalls [§ **14**, (8)], that:

$$\eta_1 = -\frac{1}{E}\left[Q\,h\,r_1\frac{r'\,s'}{z'}-\frac{r'\,s'}{z'}H'\right]. \tag{6}$$

One further has:

$$\frac{d\dfrac{r}{s'}}{dp} = \frac{s'\,r''-r'\,s''}{s'^2} = \frac{(r'^2+z'^2)\,r''-r'\,(r'\,r''+z'\,z'')}{s'^3} = z'\,R\,,$$

so

$$\frac{d\eta}{dp} = \frac{Q\,h\,r_1}{E}z'\,R\,q\,.$$

If we introduce that value into [§ **14**, (7)] then it will follow that:

$$\eta_2 = -\frac{Q\,h\,r_1}{E}r\frac{z'}{s'}q\,. \tag{7}$$

The quantities α_{11}, α_{22} are determined from equations [see art. **16**, (5)]:

$$\alpha_{11} = \frac{1}{E}\left[\mu\,\gamma_{11}-\frac{s'^2}{r^2}\gamma_{22}\right],\qquad \alpha_{22} = \frac{1}{E}\left[\mu\,\gamma_{22}-\frac{r^2}{s'^2}\gamma_{11}\right]. \tag{8}$$

Equations (3) of art. **15**, which determine the components of the displacement, take the form:

$$(9) \qquad r'\frac{\partial \rho}{\partial p} + z'\,w' = \alpha_{11}\,,$$

$$(10) \qquad r\frac{\partial \sigma}{\partial p} + r\,\rho = \alpha_{22}\,,$$

$$(11) \qquad \frac{\partial \sigma}{\partial p} = -\frac{Q\,h\,r_1}{E}\,r'\,q\,,$$

$$(12) \qquad \frac{\partial \sigma}{\partial p} - \sigma = \frac{Q\,h\,r_1}{E}\,r\,q\,,$$

$$(13) \qquad -\frac{z'}{s'}\frac{\partial \rho}{\partial p} + \frac{r'}{s'}\,w' = \eta_1\,.$$

The last equation in that system can be dropped since it is identical to (12).

Equation (9) shows that $\partial\rho / \partial p$ is independent of q. We set $\rho = \rho_0 + \bar{\rho}$. Here, ρ_0 means the value that ρ assumes for $p = 0$, and $\bar{\rho}$ is a function of p that vanishes for $p = 0$.

It follows from (11) that:

$$(14) \qquad \sigma = -\frac{Q\,h\,r_1}{E}\,r\,q + \tau\,,$$

where τ means a function of q.

If we differentiate equation (10) with respect to q and eliminate ρ and σ by using equations (12) and (14) then that will give:

$$\frac{\partial^2 \tau}{\partial q^2} + \tau = 0\,.$$

σ is an odd function of q, and the same thing is true for τ. Therefore, one has:

$$\tau = \text{const.}\,\sin q\,.$$

If we denote the integration constant by $\kappa\,r_0^2\,Q / E$ then we will have:

$$(15) \qquad \sigma = -\frac{Q}{E}\,[h\,r_1\,r\,q - \kappa\,r_0^2 \sin q]\,.$$

In order to determine the constant κ, we set $p = 0$ and $q = 0$ in equation (10). For that system of values, the parameter is $\rho = 0$, so:

$$\frac{\gamma_{22}}{r^2} = -\frac{Q}{2}\frac{r_0^2 - r_1^2}{r_0}, \qquad \frac{\gamma_{11}}{s'^2} = -\frac{Q}{2}[2r_0 + R_0\,(r_0^2 - r_1^2)], \qquad \text{[(1) and (2)]}$$

and as a result (8):

$$\frac{\alpha_{22}}{r^2} = -\frac{Q}{2E}\left[\mu\frac{r_0^2 - r_1^2}{r_0} - 2r_0 - R_0\,(r_0^2 - r_1^2)\right].$$

A simple calculation gives:

(16)
$$\kappa = h\frac{r_1}{r_0} + 1 + \frac{r_0^2 - r_1^2}{2r_0^2}(\mu - r_0\,R_0).$$

It follows from (10), when one recalls (15), that:

(16)
$$\rho = \frac{\alpha_{22}}{r} + \frac{Q}{E}[h\,r_1\,r - \kappa\,r_0^2\cos q].$$

In particular, for $p = 0$:

(18)
$$\rho_0 = \frac{Q}{E}\kappa\,r_0^2(1 - \cos q).$$

If one multiplies equation (9) by z'/s'^2 and (13) by r'/s' and adds them then it will follow that:

$$w' = \frac{z'}{s'^2}\alpha_{11} + \frac{r'}{s'}\eta_1,$$

so since $w = 0$ for $p = 0$:

$$w = \int_0^p \left[z'\frac{\alpha_{11}}{s'^2} + \frac{r'}{s'}\eta_1\right]dp.$$

Now, one has (6):

$$\int_0^p \frac{r'}{s'}\eta_1\,dp = -\frac{1}{E}\int_0^p \left[Q\,h\,r_1\frac{r'^2}{z'} - r\frac{r'}{s'}H'\right]dp.$$

Upon partial differentiation, the expression on the right-hand side will go to:

$$\frac{1}{E}\frac{r\,r'}{z'}(H - H_1) - \frac{1}{E}\int_0^p \left[Q\,h\,r_1\frac{r'^2}{z'} + \frac{d\,\frac{r\,r'}{z'}}{dp}(H - H_1)\right]dp,$$

in which one must consider that $r' = 0$ at the point 0.

We then get:

$$w = \frac{1}{E}\frac{r\,r'}{z'}(H - H_1) + \int_0^p z'\frac{\alpha_{11}}{s'^2}\,dp - \frac{1}{E}\int_0^p \left[Q\,h\,r_1\frac{r'^2}{z'} + \frac{d\,\dfrac{r\,r'}{z'}}{dp}(H - H_1)\right] dp\,.$$

Before we enter into a discussion of the formulas that we developed, it is first necessary for us to calculate the constants h and κ.

19. Determining the constants h **and** κ. – For the determination of the constant h, it is preferable to choose the first parameter to be the arc-length along the meridian curve, as measured from the point 0.

h was the value that the quotient $\frac{1}{Q}\frac{dH}{ds}$ assumed at the point 1, and one has [§ **18**, (3)]:

$$-\frac{H}{Q} = r\frac{dz}{ds} + \frac{1}{2}\frac{ds}{dz}\frac{r^2 - r_0^2}{r} + \frac{1}{2}\frac{d}{ds}\left[\frac{dr}{dz}(r^2 - r_0^2)\right].$$

We set $(r^2 - r_0^2)/\frac{dr}{dz} = \psi$ and get:

$$-\frac{H}{Q} = r\frac{dz}{ds} + \frac{1}{2}\frac{\psi}{r} + \frac{1}{2}\frac{d\left(\dfrac{dr}{dz}\psi\right)}{ds}\,.$$

At the point 1, we have $\frac{dz}{ds} = 0$, $\frac{dr}{ds} = 1$, $\frac{d^2r}{ds^2} = 0$, and as a result:

(1) $$-h = r_1\left(\frac{d^2z}{ds^2}\right)_1 - \frac{1}{2}\frac{\psi_1}{r_1^2} + \frac{1}{2r_1}\left(\frac{d\psi}{ds}\right)_1 + \frac{1}{2}\left(\frac{d^2\psi}{ds^2}\right)_1 + \frac{1}{2}\left(\frac{d^3r}{ds^3}\right)_1\psi_1\,.$$

We next express the derivatives of r and z in terms of the curvature R and its derivatives. One has:

$$R = \frac{d^2r}{ds^2}\frac{dz}{ds} - \frac{d^2z}{ds^2}\frac{dr}{ds} = \sqrt{\left(\frac{d^2r}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}\,,$$

$$\frac{dR}{ds} = \frac{d^3r}{ds^3}\frac{dz}{ds} - \frac{d^3z}{ds^3}\frac{dr}{ds}\,,$$

$$\frac{d^2R}{ds^2} = \frac{d^4r}{ds^4}\frac{dz}{ds} - \frac{d^4z}{ds^4}\frac{dr}{ds} + \frac{d^3r}{ds^3}\frac{d^2z}{ds^2} - \frac{d^3z}{ds^3}\frac{d^2r}{ds^2}\,.$$

Furthermore:

$$\frac{d^2r}{ds^2}\frac{dr}{ds}+\frac{d^2r}{ds^2}\frac{dz}{ds} = 0\,,$$
$$\frac{d^2r}{ds^2}\frac{dr}{ds}+\frac{d^2r}{ds^2}\frac{dz}{ds} = -R^2\,.$$

As a result:

$$\left(\frac{d^3r}{ds^3}\right)_1 = -\ R_1^2\,,\qquad \left(\frac{d^2z}{ds^2}\right)_1 = -R_1\,,\qquad \left(\frac{d^3z}{ds^3}\right)_1 = -\left(\frac{dR}{ds}\right)_1,\qquad \left(\frac{d^4z}{ds^4}\right)_1 = -\left(\frac{d^2R}{ds^2}\right)_1 + R_1^3\,.$$

Hence, $\frac{dz}{ds}\psi = r^2 - r_1^2$, and as a result:

$$\frac{dz}{ds}\frac{d\psi}{ds}+\frac{d^2z}{ds^2}\psi = 2r\frac{dr}{ds}\,,$$

$$\frac{dz}{ds}\frac{d^2\psi}{ds^2}+2\frac{d^2z}{ds^2}\frac{d\psi}{ds}+\frac{d^2z}{ds^2}\psi = 2r\frac{d^2r}{ds^2}+2\left(\frac{dr}{ds}\right)^2,$$

$$\frac{dz}{ds}\frac{d^3\psi}{ds^3}+3\frac{d^2z}{ds^2}\frac{d^2\psi}{ds^2}+3\frac{d^3z}{ds^3}\frac{d\psi}{ds}+\frac{d^4z}{ds^4}\psi = 2r\frac{d^3r}{ds^3}+6\frac{d^2r}{ds^2}\frac{dr}{ds}\,.$$

A consideration of (2) then gives the following equations for the values of the function ψ and the derivatives at the point 1:

$$(3)\qquad \left\{\begin{aligned} &\psi_1 = -2\frac{r_1}{R_1},\quad \left(\frac{d\psi}{ds}\right)_1 = \frac{r_1}{R_1^2}\left(\frac{dR}{ds}\right)_1 - \frac{1}{R_1},\\ &\left(\frac{d^2\psi}{ds^2}\right)_1 = \frac{2}{3}\frac{r_1}{R_1^2}\left(\frac{d^2R}{ds^2}\right)_1 - \frac{r_1}{R_1^3}\left(\frac{dR}{ds}\right)_1^2 + \frac{1}{R_1^2}\left(\frac{dR}{ds}\right)_1. \end{aligned}\right.$$

If we substitute that value in (1) then we will get:

$$(4)\qquad h = -\frac{1}{3}\frac{r_1}{R_1^2}\left(\frac{d^2R}{ds^2}\right)_1 + \frac{1}{2}\frac{r_1}{R_1^3}\left(\frac{dR}{ds}\right)_1^2 - \frac{1}{R_1^2}\left(\frac{dR}{ds}\right)_1 - \frac{1}{2}\frac{1}{r_1\,R_1}\,.$$

The constant κ is then determined by that [see equation (16) of the previous article].

The differential invariant that appears in the expression for h as a factor of r_1 :

$$J = \frac{1}{2}\frac{1}{R^3}\left(\frac{dR}{ds}\right)^2 - \frac{1}{3}\frac{1}{R^2}\frac{d^2R}{ds^2}$$

depends upon the curvature and its derivatives of first and second order. Therefore, when two curves have five-point contact, that invariant will have the same value for both curves at that contact point.

In order to assess the dependency of the invariant J on the curvature of the curve precisely, we calculate J for an arbitrary point of an ellipse.

We represent the coordinates of a point of the ellipse in the form $a \sin p$, $b \cos p$. That will give:

$$J = -\frac{a^2-b^2}{ab}\frac{\cos 2p}{s'} - \frac{3}{8}\frac{(a^2-b^2)^2}{ab}\frac{\sin^2 2p}{s'^3}\,.$$

Here, one has $s' = \sqrt{a^2\cos^2 p + b^2 \sin^2 p}$. The absolute value of J attains a maximum at the endpoints of the axes, and indeed at the endpoints of the a-axis $J = \frac{a^2-b^2}{ab^2}$, while $J = \frac{b^2-a^2}{a^2 b}$ at the endpoints of the b-axis. When the distance from the surface of the tube to the axis of rotation is sufficiently large in comparison to the dimensions of the cross-section of the surface, the terms on the right-hand side of (4) that are multiplied by r_1 will be definitive in determining the order of magnitude of h. When one is then trying to achieve a greatest-possible value of h one will then choose the cross-section of the tube in such a way that it possesses curvatures of the same type at the highest and lowest point that an ellipse possesses at the endpoints of its major or minor axis, resp.

Under the assumption that J is sufficiently large, we will get the following approximate values for h and κ [see § **18**, (16)]: $h = r_1 J$, $\kappa = h + 1 = r_1 J + 1$.

20. On the continuity and boundary conditions. – The formulas that were developed in art. **18** show that the quantities γ_{11}, γ_{22}, ρ, σ, and w' are single-valued and continuous on the entire middle surface. However, since the middle surface is *doubly*-connected, the single-valuedness and continuity of the derivatives are still not sufficient to guarantee the single-valuedness of the function w. Rather, one must further demand that the integral $\int_0^{2\pi} w'\,dp$ will vanish when it is extended over the meridian curve. If one recalls that w' is an even function of p and that the derivative r' vanishes for $p = \pi$ then one can also replace that condition with the following one [§ **18**, (19)]:

(1) $$w_2 = \int_0^{\pi} z' \frac{\alpha_{11}}{s'^2}\,dp - \frac{1}{E}\int_0^{\pi}\left[Q\,h\,r_1\frac{r'^2}{s'^2} + \frac{d\,\frac{r\,r'}{z'}}{dq}(H - H_1)\right]dp = 0\,.$$

In general, the modulus of periodicity $2w_2$ is non-zero, so the assumptions that we started from in art. **9** will not be admissible in the general case. However, we will prove in what follows that the cross-section of the tube can be chosen in many ways that make the modulus of periodicity $2w_2$ vanish. There is then an extended class of cases in which the theory that was developed here finds it place. Those cases are precisely the ones that are important in practice. That is because, obviously, it is preferable to give a tube of the type that is considered here, which might serve for barometric or thermometric measurements, such a form that the stresses that occur are as continuous as possible, where the word *continuous* is used in a practical sense, not a purely-abstract one.

The irregularity in operation that many aneroid barometers exhibit is probably due to the fact that the continuity conditions that were assumed here are not fulfilled.

In the following statements, we shall assume that the condition $w_2 = 0$ is fulfilled.

The pressure that a surface element in the body that is perpendicular to a parallel circle experiences from the side of decreasing q is equal to γ_{11} / s'^2. It acts in the direction of increasing q [§ **10**, (5)].

The pressure that a surface element that is perpendicular to a meridian experiences from the side of decreasing p is γ_{22} / r^2. When the distance from the tube to the axis of rotation is large compared to its dimensions, one will have the following values, in the first approximation:

$$\frac{\gamma_{11}}{s'^2} = - Q\left[r_1 \frac{z'}{s'} + r_1 \frac{d}{dp}\left(\frac{r'(r-r_1)}{z'}\right)\right],$$

$$\frac{\gamma_{22}}{r^2} = - Q \frac{s'}{z'} (r - r_1) \qquad \text{[§ } \mathbf{18}\text{, (1) and (2.a)].}$$

The pressure that acts in the direction of the parallel circle then rises to an appreciably larger value than the one that acts in the direction of the meridian.

The force [§ **18**, (2.a)] that acts from the inside outward in the direction of increasing q upon an element of the boundary that lies on the side of positive q is:

$$\frac{\gamma_{11}}{s'^2} \cdot 2\varepsilon\, ds = - 2Q\varepsilon\left[r_1 \frac{z'}{s'} + \frac{1}{s'} \frac{d}{dp}\left(\frac{r'}{z'}(r^2 - r_1^2)\right)\right].$$

Thus, the total force that acts upon the boundary is:

$$- 2Q\varepsilon \int_0^{2\pi} r\, z' dp = 2Q\varepsilon \int_0^{2\pi} (r_1 - r)\, z' dp .$$

The integral on the right-hand side represents the area that the meridian curve encloses. The factor $2Q\varepsilon$ in front of the integral sign is equal to (art. **10**) – $[(P_3^+) + (P_3^-)]$. The force that acts upon the boundary and originates in the elasticity of the tube is then (as theory would demand) equal and opposite to the forces that originates in the pressure difference (cf., art. **17**).

One has the following values [art. **18**, (15) and (18)] for the displacement of the end of the tube that lies on the side of positive q ($q = \pi - \beta$):

$$\sigma_0 = -\frac{Q}{E} r_0 [h\, r_1 (\pi - \beta) - \kappa\, r_0 \sin \beta] , \qquad \rho_0 = \frac{Q}{E} \kappa\, r_0^2 \; (1 + \cos \beta) .$$

If r_1 is large and β is small compared to the difference $r_1 - r_0$ then we will get (see art. **19**, conclusion), $\sigma_0 = -\frac{Q}{E} J\, r_1^3$, $\rho_0 = \frac{Q}{E} J\, r_1^2$ in the first approximation. The displacement of the tube in the tangential direction is appreciably larger than the displacement in the radial direction.

The two displacement components have opposite signs.

If the tube has the same type of curvature at its highest point as an ellipse has at the endpoints of its major axis then J will be positive, so when Q is positive (so the external pressure is greater than the internal pressure), the end of the tube will shrink back in the tangential direction. If the tube has the same type of curvature at the highest point that an ellipse has at the endpoints of its minor axis, so J is negative, then the end of the tube will move forward.

21. Analytic representation of the meridian curve. – It remains to be proved that the cross-section of the tube can be chosen in such a way that the integral $w_2 = \int_0^\pi w' \, dp$ will vanish. To that end, it is necessary to make certain assumptions about the analytical representation of the oval at which the surface of the tube is intersected by a meridian plane.

It can be proved that w_2 will necessarily be non-zero when one chooses the oval to be an ellipse, and as far as I can see, w_2 cannot vanish at all when the oval possesses *two* symmetry axes. That can be proved in full rigor for the case in which r_1 is very large compared to the dimensions of the cross-section. One will arrive at ovals with *only one* symmetry axis, which are useful for our purposes, most simply in the following way: We assume an ellipse with one axis that falls along the r-axis (it can be the major or minor axis of the ellipse) and assume that this ellipse contacts the other axis of the oval that is parallel to the axis of rotation at five points. We now assign any point of the ellipse to the point of the oval at which the outward-pointing normal has the same direction. The coordinates of a point of the ellipse can be represented in the form $r_1 - b \cos p$, $a \sin p$. The following differential equations are true for the coordinates of the corresponding point of the oval:

$$r' = b \sin p\, (1 + \nu m) , \qquad z' = a \cos p\, (1 + \nu m) , \tag{1}$$

and the initial conditions $r = r_1$ for $p = \pi / 2$, $z = 0$ for $p = 0$. Here, m means an everywhere-continuous periodic function of p with a period of 2π, and ν is a constant whose absolute value is smaller than the maximum of the absolute value of $1/m$. Furthermore, the function m is considered to be given, while ν is an auxiliary parameter.

In order for the r-axis to be the symmetry axis of an oval, m must be an even function of p, so it will be a single-valued function of cos p. Moreover, we would like to assume that m is an odd function of cos p. That assumption is not essential for the following considerations, but it will lead to a significant simplification in the proof.

In order for z to be a periodic function of p, it is necessary that $\int_0^{\pi} m\cos p\,dp = 0$. Since $m\cos p$ is an even function of cos p, that condition can also be replaced with:

$$\int_0^{\pi/2} m\cos p\,dp = 0 .$$

It follows from that equation that $z = a$ for $p = \pi/2$, as required. We denote the differences between the coordinates of a point on the ellipse and the corresponding point on the oval by $\nu\, b\, u \cos p$ and $\nu\, a\, v \sin p$, resp. One then has:

$$\text{(3)} \qquad \begin{cases} r = r_1 - b\cos p\,(1+\nu u), \qquad z = a\sin p\,(1-\nu v), \\ \dfrac{d(u\cos p)}{dp} = -m\sin p, \quad \dfrac{d(v\sin p)}{dp} = -m\cos p. \end{cases}$$

It follows from the penultimate equations that since $\dfrac{du}{dp} = -\dfrac{du}{d\cos p}\sin p$:

$$u + \frac{du}{d\cos p}\cdot \cos p = m ,$$

and from that:

$$\int_0^{\pi/2} \frac{du}{d\cos p}\,dp = \int_0^{\pi/2} \frac{m}{\cos p}\,dp - \int_0^{\pi/2} \frac{u}{\cos p}\,dp .$$

Now, one has:

$$\int_0^{\pi/2} \frac{u}{\cos p}\,dp = \int_0^{\pi/2} \frac{dp}{\cos^2 p} \int_p^{\pi/2} m(p_1)\sin p_1\,dp_1 = \int_p^{\pi/2} m(p_1)\sin p_1\,dp_1 \int_0^{p_1} \frac{dp}{\cos^2 p}$$

$$= \int_p^{\pi/2} m(p_1)\frac{\sin^2 p_1}{\cos p_1}\,dp_1 = \int_0^{\pi/2} \frac{m}{\cos p}\,dp - \int_0^{\pi/2} m\cos p\,dp .$$

The second integral vanishes due to (2), so as a result:

$$\text{(4)} \qquad \int_0^{\pi/2} \frac{du}{d\cos p}\,dp = 0 .$$

It still remains for us to express the idea that the oval contacts the associated ellipse at five points that correspond to the parameter values $p = \pm\,\pi / 2$.

We denote the differential of arc-length of the ellipse by dt and its curvature by P. One then has:

$$t' = \sqrt{a^2 \cos^2 p + b^2 \sin^2 p}\,, \qquad P = \frac{a\,b}{t'^3}\,.$$

The following relations exist between the quantities t', P and the corresponding quantities s', R on the oval:

(5) $$s' = (1 + \nu\,m)\;t', \qquad R = \frac{P}{1+\nu\,m}\,.$$

Since m is an odd function of cos p, the function m itself and its derivatives of even order will vanish for $p = \pi / 2$. We now assume that the first derivative $m' = 0$ is also true for $p = \pi / 2$. Under that assumption, one has that:

$$R = P, \qquad \frac{dR}{ds} = \frac{dP}{dt}, \qquad \frac{d^2R}{ds^2} = \frac{d^2P}{dt^2}$$

for $p = \pi / 2$. That will then imply a five-point contact, as required. If follows from the assumption that was introduced above that $m / \cos^3 p$ will remain finite and continuous for $p = \pi / 2$. Equations (3) show that u and v are odd functions of cos p, like m, and that the quotients $u / \cos^3 p$ and $v / \cos^3 p$ remain continuous for $p = \pi / 2$.

We summarize the properties of the functions m, u, v that are obtained from the assumptions that were introduced up to now:

(A)
$$\left\{\begin{array}{ll} 1) & m, u, v \text{ are single-valued and continuous odd functions of } \cos\,p. \\ 2) & \dfrac{m}{\cos^3 p}, \dfrac{u}{\cos^3 p}, \dfrac{v}{\cos^3 p} \text{ are continuous for } p = \pi / 2. \\ & \\ & \text{In addition, we would like to assume:} \\ 3) & \text{The maximum of the absolute value of } r - r_1 \text{ is smaller than } r_1. \end{array}\right.$$

Equations (2) and (4) are added to that.

That assumption will suffice for the proof that w_2 can be made to vanish by suitable conditions on the parameter ν. Moreover, for practical applications, one will demand that the oval takes as simple a form as possible. In order to achieve that, we introduce a second group of parameters. We assume:

(B)
1) The functions m and $du / d \cos p$ change sign at *only one* location in the interval $0, \pi / 2$. [At least one change of must occur due to (2) and (4).]

2) The functions u and v are everywhere - positive in the same intervals, and indeed $u = 1$ for $p = 0$.

Since u is equal to zero for $\cos p = 0$ and positive for $\cos p > 0$, $du / d \cos p$ will be positive for small positive values of $\cos p$ and negative for $\cos p = 1$. The same thing will be true of the function m.

The simplest function m that satisfies the conditions that were posed is $m = 20 \cos^3 p - 24 \cos^5 p$. In that case, one will have:

$$u = 5 \cos^3 p - 4 \cos^5 p , \qquad v = 4 \cos^5 p .$$

22. Proof that there exist forms for the cross-section that correspond to the assumptions of the theory. – We now prove that once the function m has been combined with the constants a, b, one can choose the quantities r_1 and ν in a number of ways that will make the integral [§ **20**, (1)] $w_2 = \int_0^\pi w' \, dp$ vanish.

We introduce the values that were given in (3) of the previous article into equations (1) and (2) of art. **18** and order them in powers of r_1. γ_{11} / s'^2 is a linear function of r_1. γ_{22} / r^2 can be developed in an advancing series of decreasing powers of r_1 [since, by assumption (A.3 of the last article), the absolute value of $r - r_1$ is smaller than r_1]. That series begins with a term that is independent of r_1. The series developments of H and α_{11} / s'^2 begin with terms that includes r_1 [cf., § **18**, (3) and (8)], the series development of w' begins with a term that includes r_1^2 [see, § **18**, (19)]. We then get a series for w' that takes the form:

$$w' = A r_1^2 + B r_1 + C + \frac{D}{r_1} + \frac{E}{r_1^2} + \cdots , \tag{1}$$

and that will imply an analogous series development for w_2 :

$$w_2 = (A) r_1^2 + (B) r_1 + (C) + \frac{(D)}{r_1} + \frac{(E)}{r_1^2} + \cdots , \tag{2}$$

In that series, the coefficients of even powers of r_1 are odd functions of the parameter ν, while the coefficients of odd powers of r_1 are even functions of ν. In particular, (A) is an odd function of ν.

To prove that, we remark: If we simultaneously replace r_1 with $-r_1$, ν with $-\nu$, and p with $\pi - p$ in equations (3) of the previous article then the quantities $-m$, $-u$, $-v$ will enter in place of the quantities m, u, v, resp., and correspondingly, $-r$, $-z'$ will enter in place of r, z', resp.; the quantities z, r', s' will remain unchanged. The quantities γ_{11}/s'^2 and γ_{22}/r^2 also remain unchanged under the given exchanges, while $-w'$ will enter in place of w'.

We now imagine that the coefficients A, B, C, … of the series that represents w' are developed into series that advance in increasing powers of ν. The coefficients of the doubly-infinite series that now represents w' are single-valued functions of cos p, and indeed the coefficient of $r_1^i \nu^j$ will be an even or odd function of cos p according to whether the sum of the exponents $i + j$ is an odd or even number, resp., because the w' will also change sign under a simultaneous sign change of r_1, ν, and cos p.

The integral that extends from 0 to π will have the value zero for an odd function of cos p. Therefore, in the series development of the quantity w_2 that we obtain by term-wise integration, only those products $r_1^i \nu^j$ for which the sum of the exponents $i + j$ is an odd number will occur. Q.E.D. It is now easy to complete the proof that was mentioned to begin with.

To that end, we assign an arbitrarily-chosen positive value τ to the parameter ν that must satisfy only the one condition that $1 / \tau$ must be less than the maximum of the absolute value of the function m. We then choose the constant r_1 to be large enough that the absolute value of the first term in the series (2) is greater than the sum of the absolute values of all following terms. w_2 will then have the same sign as (A). We fix the value of r_1 and let ν decrease from τ to $-\tau$. The final value of w_2 will have the opposite sign to the initial value because (A) is an odd function of ν. As a result, since w_2 is a continuous function of ν, it must vanish for at least one value of ν that lies between τ and $-\tau$. With that, we have proved: There are forms for the cross-section for which the assumptions of our theory are fulfilled.

23. Approximate determination of the constant ν. – In the foregoing, we made use of only the assumptions (A) of art. **21**. We shall now add assumptions (B) and assume, moreover, that the dimensions of the cross-section of the tube are small compared to its distance from the axis of rotation, so the semi-axes a, b are small compared to r_1.

We will get an approximate value for the parameter ν when we preserve only the first two terms on the right-hand side of equation (2) in the previous article:

$$w_2 = (A) r_1^2 + (B) r_1 + (C) + \cdots = 0 .$$

We develop the quantities (A) and (B) in series that proceed in increasing powers of ν and keep only the first term in each series, so the term that is multiplied by ν in the odd function (A) and the absolute term in the even function (B).

We must now calculate the approximate values for the individual quantities that appear in equation [§ **20**, (1)]. One has:

$$w_2 = \int_0^\pi z' \frac{\alpha_{11}}{s'^2} dp - \frac{1}{E}\int_0^\pi \left[Q\,h\,r_1 \frac{r'^2}{z'} + \frac{d\,\frac{r\,r'}{z'}}{dp}(H - H_1) \right] dp \ .$$

From [art. **18**, (1) and (2.a)], one has:

$$\frac{\gamma_{11}}{s'^2} = -\ Q\left[r\frac{z'}{s'} + \frac{1}{2s'}\frac{d}{dp}\left(\frac{r'}{z'}(r^2 - r_1^2) \right) \right] ,$$

$$\frac{\gamma_{22}}{r^2} = -\ Q\,\frac{s'}{z'}\,\frac{r + r_1}{2r}\,(r - r_1) \ .$$

We introduce the values that were given in [art. **21**, (1), (3), and (5)] here and immediately suppress all terms that do not come under consideration in the calculation of our approximate value. We will get:

$$\frac{r'}{z'}(r^2 - r_1^2) = -\ \frac{b}{a}\tan p\ [2 r_1\, b \cos p\,(1 + \nu\, u) - b^2 \cos^2 p + \cdots]$$
$$= -\ 2\frac{b^2}{a}[r_1 \sin p - \tfrac{1}{4} b \sin 2p + \nu\, r_1\, u \sin p + \cdots] \ ,$$

$$\frac{1}{2}\frac{d}{dp}\left[\frac{r'}{z'}(r^2 - r_1^2) \right] = -\ 2\frac{b^2}{a}\left[r_1 \cos p - \tfrac{1}{2} b \cos 2p + \nu\, r_1 \left(m \cos p - \frac{du}{d\cos p} \right) + \cdots \right] ,$$

In the last term that was written out on the right-hand side, use was made of the identities:

$$m = u + \frac{du}{d\cos p}\cdot \cos p \ ,$$

$$\frac{d(u \sin p)}{dp} = u \cos p - \frac{du}{d\cos p}\sin^2 p = m \cos p - \frac{du}{d\cos p} \ .$$

We further have:

$$\frac{1}{s'} = \frac{1}{(1 + \nu\, m)\, t'} = \frac{1}{t'}\ (1 - \nu\, m + \ldots) \ .$$

As a result:

$$\frac{1}{2}\frac{1}{s'}\frac{d}{dp}\left[\frac{r'}{z'}(r^2 - r_1^2) \right] = -\ \frac{b^2}{a}\left[r_1 \cos p - \tfrac{1}{2} b \cos 2p + \nu\, r_1 \left(m \cos p - \frac{du}{d\cos p} \right) + \cdots \right] .$$

Hence:

$$r\frac{z'}{s'} = [r_1 - b\cos p + \ldots]\frac{a\cos p}{t'} = r_1\frac{a\cos p}{t'} - ab\frac{\cos^2 p}{t'},$$

so

(1)
$$\frac{\gamma_{11}}{s'^2} = -\frac{Q}{at'}\left[r_1(a^2-b^2)\cos p - b(a^2-b^2)\cos^2 p - \tfrac{1}{2}b^3 - \nu r_1 b^2\frac{du}{d\cos p} + \cdots\right].$$

Moreover:

(2)
$$\frac{\gamma_{22}}{r^2} = Q\frac{b}{a}t' + \cdots = \frac{Qb}{t'a}\left[(a^2-b^2)\cos^2 p + b^2 + \cdots\right].$$

That will give:

(3)
$$H = -\frac{Q}{at'}\left[r_1(a^2-b^2)\cos p - 2(a^2-b^2)b\cos^2 p - \tfrac{3}{2}b^3 - \nu r_1 b^2\frac{du}{d\cos p} + \cdots\right],$$

$$H_1 = \tfrac{3}{2}Q\frac{b^2}{a}.$$

Now:

$$\frac{rr'}{z'} = \frac{b}{a}\tan p[r_1 - b\cos p] = r_1\frac{b}{a}\tan p - \frac{b^2}{a}\sin p,$$

$$\frac{d\frac{rr'}{z'}}{dp} = r_1\frac{b}{a}\frac{1}{\cos^2 p} - \frac{b^2}{a}\cos p.$$

As a result:

(4)
$$\frac{d\frac{rr'}{z'}}{dp}(H - H_1)$$

$$= -\frac{Qb}{t'a^2}\left[r_1^2(a^2-b^2)\frac{1}{\cos p} - 3r_1(a^2-b^2)b + \tfrac{3}{2}r_1 b^2\frac{t'-b}{\cos^2 p} - \nu r_1 b^2\frac{1}{\cos^2 p}\frac{du}{d\cos p} + \cdots\right].$$

Furthermore, one has (art. **19**):

$$Qhr_1\frac{r'^2}{z'} = Qr_1^2\frac{a^2-b^2}{a^2}\frac{\sin^2 p}{\cos p}(1 + \nu m).$$

As a result:

(5)
$$Qhr_1\frac{r'^2}{z'} + \frac{d\frac{rr'}{z'}}{dp}(H - H_1)$$

$$= -\frac{Q}{a^2t'}\left[r_1^2(a^2-b^2)\frac{b - t'\sin^2 p}{\cos p} - 3r_1(a^2-b^2)b + \tfrac{3}{2}r_1 b^2\frac{t'-b}{\cos^2 p}\right]$$

$$+\ Q r_1^2 \nu \left[\frac{a^2-b^2}{a^2}\left(\frac{m}{\cos p} - m\cos p\right) + \frac{b^2}{a^2}\,\frac{\frac{du}{d\cos p}}{t'\cos^2 p} \right].$$

Finally, [§ **18**, (8) and eq. (1) of this article]:

$$z'\frac{\alpha_{11}}{s'^2} = a\cos p\ \frac{\mu}{E}\cdot\frac{\gamma_{11}}{s'^2}\cdots = -\ \mu\frac{Q}{E}r_1\frac{(a^2-b^2)\cos^2 p}{t'}.$$

When we recall that the integral of an odd function of cos p from 0 to π will vanish and that the integral from 0 to π of an even function will be equal to twice the integral from 0 to $\pi / 2$ [cf., also § **21**, (2)], we will have:

(6)
$$w_2 = -\ 2\frac{Q}{E}r_1^2\,\nu\int_0^{\pi/2}\left[\frac{a^2-b^2}{a^2}\,\frac{m}{\cos p} + \frac{b^3}{a^2}\,\frac{\frac{du}{d\cos p}}{t'\cos^2 p}\right]dp$$

$$-\ 2\frac{Q}{E}r_1\int_0^{\pi/2}\left[\mu\cdot\frac{(a^2-b^2)\cos^2 p}{a^2} + \frac{3(a^2-b^2)b^2}{a^2 t'} - \frac{3}{2}\frac{b^3}{a^2}\frac{t'-b}{t'\cos^2 p}\right]dp.$$

We set:

$$w_2 = -2\,\frac{Q}{E}\,r_1[r_1\,\nu L + M] + \cdots,$$

to abbreviate, where L and M are independent r_1 and ν. By assumption [art. **21**, (B)], the two functions m and $du / d\cos p$ are negative at the lower limit of the integration, positive at the upper one, and change sign only once in the integration interval. Furthermore [art. **21**, (2) and (4)]:

$$\int_0^{\pi/2} m\cos p\,dp = 0 \qquad \text{and} \qquad \int_0^{\pi/2}\frac{du}{d\cos p}dp = 0.$$

Since the function $1 / \cos^2 p$ increases monotonically in the integration interval, the integral $\int_0^{\pi/2}\frac{1}{\cos^2 p}\cdot m\cos p\,dp$ will be positive. If $a > b$ then the function $\frac{1}{t'} = \frac{1}{\sqrt{a^2\cos^2 p + b^2\sin^2 p}}$ will increase monotonically, so the integral $\int_0^{\pi/2}\frac{1}{t'\cos^2 p}\cdot\frac{du}{d\cos p}dp$ will be likewise positive. It then follows that L is positive for $a > b$. That will also still be true for $a = b$.

If $a < b$ then the sign of L cannot be determined from the outset. However, it is easy to see that L will become negative for very small values of the quotient a / b. Namely, if we denote the value of p for which $du / d \cos p = 0$ by p_1 then the first integral on the right-hand side of the equation:

$$\int_0^{\pi/2} \frac{1}{t' \cos^2 p} \cdot \frac{du}{d \cos p} dp = \int_0^{p_1} \frac{1}{t' \cos^2 p} \cdot \frac{du}{d \cos p} dp + \int_{p_1}^{\pi/2} \frac{1}{t' \cos^2 p} \cdot \frac{du}{d \cos p} dp$$

will be negative and its absolute value will grow beyond all limits when one fixes the value of b and lets a converge to zero. The second integral on the right will take on a finite positive value. As a result, the integral on the left will have a negative value for a sufficiently-small value of a, and the same thing will be true of L.

It follows from the identity:

$$\begin{aligned}\frac{d}{dp}[\tan p \cdot (t' - b)] &= \frac{t' - b}{\cos^2 p} - \frac{(a^2 - b^2)\sin^2 p}{t'} \\ &= \frac{(a^2 - b^2)\cos^2 p + b^2}{t' \cos^2 p} - \frac{b}{\cos^2 p} - \frac{a^2 - b^2}{t'} \\ &= -\frac{b^2}{t'} + t' - \frac{1}{\cos^2 p}\left(b - \frac{b^2}{t'}\right)\end{aligned}$$

that

$$\int_0^{\pi/2} \frac{1}{\cos^2 p}\left(b - \frac{b^2}{t'}\right) dp = \int_0^{\pi/2} \left(t' - \frac{b^2}{t'}\right) dp = (a^2 - b^2) \int_0^{\pi/2} \frac{\cos^2 p \, dp}{t'} .$$

We then get:

$$M = (a^2 - b^2) \int_0^{\pi/2} \left[\mu \frac{\cos^2 p}{t'} + \frac{3b^2}{a^2} \frac{1 - \frac{1}{2}\cos^2 p}{t'} \right] dp .$$

M has the same sign as $a^2 - b^2$.

The quantities L and M then have the same sign in the case where $a > b$, as well as in the case where $a < b$, and at the same time, the quotient a / b is small.

Now, the equation $w_2 = 0$ gives $\nu = -\frac{1}{r_1}\frac{M}{L}$ in the first approximation, so ν will be negative. Hence, the oval in which the meridian plane cuts the tube will be flattened on the side of the axis of rotation and sharpened on the opposite side.

Munich, September 1902