"Ueber die Differentialgleichungen der Mechanik," Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse **2** (1905), 91-116.

On the differential equations of mechanics

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Presented by **D. Hilbert** on 25 February 1905

Translated by D. H. Delphenich

The present investigation was inspired by the discussion of the question of whether Hamilton's principle did or did not retain its validity in the case of non-holonomic condition equations. In order to briefly clarify the gist of the question, I shall restrict myself to the simple case in which the motion of a single material point is to be determined. For the sake of simplicity, I shall assume that no forces act on the point, while its degrees of freedom shall be restricted by one constraint equation.

We initially assume that the constraint equation includes only the coordinates of the point, but not the velocities, so it has the form:

$$f(x, y, z) = 0 . \tag{1}$$

When Hamilton's principle is applied to that case, it will demand that the variation of the integral:

$$\Omega = \int_{t_1}^{t_2} \tfrac{1}{2} m (x'^2 + y'^2 + z'^2)\, dt$$

must vanish for all variations, δx, δy, δz that are compatible with the constraint equation (1), and assume the value zero for $t = t_1$ and $t = t_2$. That will imply the following differential equations for the coordinates:

$$m \frac{dx'}{dt} = \rho \frac{\partial f}{\partial x}, \qquad m \frac{dy'}{dt} = \rho \frac{\partial f}{\partial y}, \qquad m \frac{dz'}{dt} = \rho \frac{\partial f}{\partial z} .$$

We now let the "non-holonomic" constraint equation appear in place of the "holonomic" constraint equation:

$$\varphi x' + \psi y' + \chi z' = 0 . \tag{2}$$

The φ, ψ, χ in it mean functions of x, y, z that do not satisfy any equation of the form:

$$\varphi : \psi : \chi = \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} ,$$

such that equation (2) will not be integrable with no further restrictions.

One also often writes the constraint equation (2) in the form:

$$\varphi\, dx + \psi\, dy + \chi\, dz = 0 . \tag{2.a}$$

The two forms (2) and (2.a) of the constraint equation correspond to different ways of conceptualizing the situation: When one uses the form (2.a), one considers x, y, z to be independent variables, whereas when one uses equation (2), one considers those quantities to be functions of t.

If one demands that the variations δx, δy, δz satisfy equation (2) then one will get the equation:

$$\begin{aligned} &\varphi\,\delta x' + \psi\,\delta y' + \chi\,\delta z' + x'\,\delta\varphi + y'\,\delta\psi + z'\,\delta\chi \\ &= \delta x' + \psi\,\delta y' + \chi\,\delta z' \\ &+ x'\left(\frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z\right) + y'\left(\frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y + \frac{\partial\psi}{\partial z}\delta z\right) + z'\left(\frac{\partial\chi}{\partial x}\delta x + \frac{\partial\chi}{\partial y}\delta y + \frac{\partial\chi}{\partial z}\delta z\right) = 0 . \end{aligned} \tag{8}$$

In order for the variation $\delta\Omega$ for to vanish for all variations that satisfy that equation, as is known, the differential equations:

$$\begin{aligned} m\frac{dx'}{dt} &= \frac{d\rho\varphi}{dt} - \rho\left(\frac{\partial\varphi}{\partial x}x' + \frac{\partial\psi}{\partial y}y' + \frac{\partial\chi}{\partial z}z'\right), \\ m\frac{dy'}{dt} &= \frac{d\rho\psi}{dt} - \rho\left(\frac{\partial\varphi}{\partial x}x' + \frac{\partial\psi}{\partial y}y' + \frac{\partial\chi}{\partial z}z'\right), \\ m\frac{dz'}{dt} &= \frac{d\rho\chi}{dt} - \rho\left(\frac{\partial\varphi}{\partial x}x' + \frac{\partial\psi}{\partial y}y' + \frac{\partial\chi}{\partial z}z'\right) \end{aligned} \tag{4}$$

are necessary and sufficient.

For the motion that actually takes place, it is not *those* differential equations that are valid, but the following ones:

$$\frac{dx'}{dt} = \rho\,\varphi , \qquad \frac{dy'}{dt} = \rho\,\psi , \qquad \frac{dz'}{dt} = \rho\,\chi .$$

One will arrive at those differential equations when one demands that the variation $\delta\Omega$ must vanish for all of the variations δx, δy, δz that satisfy the condition that ([1]):

$$\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z = 0 . \tag{6}$$

([1]) See Hölder, "Über die Prinzipien von Hamilton und Maupertuis," Göttinger Nachrichten (1896), pp. 122.

In his *Mechanik*, Hertz referred to the path that is determined by equations (4) as the "geodetic" path, while he referred to the one that is determined by equations (5) as the "straightest" path.

There is no doubt that the fact that the motion that actually takes place will result along the straightest path and not along a geodetic path must be regarded as an experimental fact. By contrast, opinions differ as to whether that fact agrees with Hamilton's principle or whether the validity of that principle is restricted to the case of holonomic constraint equations. Hertz was of the latter opinion ([1]). He based that upon the fact that from the principles of the calculus of variations in the case of a non-holonomic constraint equation (2), the variations must satisfy equation (3), and not equation (6). Hölder maintained the opposite viewpoint. He distinguished two types of variations that are compatible with the constraint equation (2.a). They are characterized by equations (3) and (6). Once that distinction has been made, one can obviously say: Hamilton's principle is also true for the case of a non-holonomic constraint equation, but when one assumes that the admissible virtual displacements are defined by equation (6).

Hamel ([2]) went one step further: He considered the variations that are defined in the calculus of variations to be special cases of virtual displacements. For the former, the following well-known relations are true:

$$\delta x' = \delta \frac{dx}{dt} = \frac{d\,\delta x}{dt}\,, \text{ etc.,}$$

while they are not true for the latter, in general. Therefore, six mutually-independent defining data come under consideration for a virtual displacement:

$$\delta x,\ \delta y,\ \delta z, \qquad \delta x',\ \delta y',\ \delta z'.$$

In order to define the virtual displacements that are compatible with the non-holonomic constraint equation (2.a), Hamel proceeded as follows: He first considered the point to be free and set:

$$\varphi\, dx + \psi\, dy + \chi\, dz = d\vartheta .$$

If ϑ were a true coordinate then the condition for the virtual displacements that $\delta\vartheta = 0$ would follow from the equation $d\vartheta = 0$. Hamel also kept that constraint equation $\delta\vartheta = 0$ in the case where ϑ is a generalized (improper) coordinate. He ultimately came to the same definition of an admissible virtual displacement as Hölder then.

One is free to choose whether one prefers the viewpoint of Hertz or Hölder since what one cares to understand the term "admissible" variation to mean in a given case is a matter of definition. However, as long one decides to accept the extension of the concept of a variation that Hölder introduced, one must demand to know which types of variation should be regarded as admissible, except for a viewpoint that is established for each application to mechanics, in principle, and one

([1]) *Mechanik*, pp. 23.
([2]) "Die Lagrange-Eulerschen Gleichungen in der Mechanik," Zeit. Math. Phys.. Bd. 50, Heft 1. "Ueber die virtuellen Verschiebungen in der Mechanik," Math. Ann., Bd. 59, pp. 416.

must further demand that the relations between the differential equations to which the different types of variation will lead must be verified.

One can satisfy those demands when one starts from an obvious generalization of the concept of the singular solution for a differential equation. However, that concept suggests itself even more so for the present question: Based upon it, one can exhibit the dynamical equations for unfree systems without making use of the principle of virtual displacements. One can get around all difficulties that might give rise to disputes in that way.

I. – On the singular integrals of a system of first-order differential equations.

1. – Let a system of m first-order differential equations be given:

$$\frac{dx_\nu}{dt} = X_\nu \qquad \nu = 1, 2, \ldots, n\,. \tag{1}$$

The $X_1, X_2, \ldots, X_n$ in this mean single-valued functions of the independent variable t and the dependent variables $x_1, x_2, \ldots, x_n$.

Let the general solution of those differential equations be:

$$x_\nu = f_\nu(a_1, a_2, \ldots, a_n, t) \qquad \nu = 1, 2, \ldots, n, \tag{2}$$

in which $a_1, a_2, \ldots, a_n$ mean integration constants.

In order to arrive at the singular integrals of the differential equations (1), we now consider the available constants $a_1, a_2, \ldots, a_n$ to be functions of the independent variable t.

We denote the differential quotients that this assumption corresponds to by $\left(\frac{dx_\nu}{dt}\right)$. We shall save the notation $\frac{dx_\nu}{dt}$ for the differential quotients that are defined under the assumption that the quantities do not depend upon t.

We will then have:

$$\left(\frac{dx_\nu}{dt}\right) = \frac{dx_\nu}{dt} + \sum_{\mu=1}^{n} \frac{\partial x_\nu}{\partial a_\mu}\frac{da_\mu}{dt} \qquad \nu = 1, 2, \ldots, n, \tag{3}$$

and when we recall (1), it will then follow that:

$$\left(\frac{dx_\nu}{dt}\right) = X_\nu + \sum_{\mu=1}^{n} \frac{\partial x_\nu}{\partial a_\mu}\frac{da_\mu}{dt} \qquad \nu = 1, 2, \ldots, n. \tag{4}$$

If we would like to demand that those differential equations should agree with equations (1) such that the assumption that quantities $a_1, a_2, \ldots, a_n$ are functions of t will imply no change in form of the differential equations then the equations:

$$\sum_{\mu=1}^{n} \frac{\partial x_\nu}{\partial a_\mu} \frac{da_\mu}{dt} = 0$$

must be true for $\nu = 1, 2, \ldots, n$.

Now, since the functional determinant $\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(a_1, a_2, \ldots, a_n)}$ does not vanish [since otherwise equations (2) would not represent the *general* solution of the differential equations (1)], it will then follow from those equations that:

$$\frac{da_\mu}{dt} = 0 \qquad \text{for} \qquad \mu = 1, 2, \ldots, n,$$

i.e., the quantities a_μ must be independent of t.

Thus, when a relation is established between the quantities a_μ and the independent variable t, the form of at least some of the equations (1) must change.

We now divide the n variables $x_1, x_2, \ldots, x_n$ into two groups (the first one might subsume, say, the variables $x_1, x_2, \ldots, x_m$, while the other might subsume the remaining $n - m$ variables) and demand that the first m of the differential equations (1), which refer to the variables of the first group, must keep their form.

That demand seems entirely arbitrary as long as we must also draw upon the remaining $n - m$ differential equations that change their forms in order to determine the n functions x. However, things will be essentially different when, to that end, different types of constraint equations are employed. We initially consider the simplest case: We set all of the variables of the second group equal to zero. The variables of the first group, about which nothing is required from the outset, might be referred to as "free" variables in order to distinguish them from the variables that are set equal to zero.

In order for the first m of equations (4) to have the same form as the first m of equations (1), it is necessary that:

$$(5) \qquad \sum_{\mu=1}^{n} \frac{\partial x_\nu}{\partial a_\mu} \frac{da_\mu}{dt} = 0 \qquad \text{for } \nu = 1, 2, \ldots, m.$$

The convention that the variables of the second group should vanish implies the following equations for the quantities $a_1, a_2, \ldots, a_n$:

$$(6) \qquad x_\nu = f_\nu(a_1, a_2, \ldots, a_n, t) = 0 \qquad \text{for } \nu = m + 1, m + 2, \ldots, n.$$

The quantities $a_1, a_2, \ldots, a_n$ are determined completely by equations (5) and (6) as long as the initial values for those m quantities are known. The type of those initial values can be specified in such a way that that the free variables assume prescribed initial values.

Equations (6) split into two groups: The first group subsumes the equations:

$$\left(\frac{dx_\nu}{dt}\right) = X_\nu \qquad \nu = 1, 2, \ldots, m, \tag{4.a}$$

while the second group consists of the partial (†) differential equations:

$$\frac{dx_\nu}{dt} = X_\nu \qquad \nu = m + 1, m + 2, \ldots, n. \tag{4.b}$$

In order to determine the free variables $x_1, x_2, \ldots, x_m$, equations (4.a) are sufficient, and for them to be true, it is not necessary to know the general solution to equations (1) or to draw upon equations (4.b).

I shall refer to a system of integrals for the differential equations (4.a) as a *singular system of integrals* for the differential equations (1).

2. – The definition of a singular system of integrals that was just proposed deviates somewhat from the usual terminology. The following argument will serve to justify it: We eliminate the variables $x_{m+1}, x_{m+2}, \ldots, x_n$ from equations (1) and the equations that can be derived from them by differentiation. We then arrive at a system (S) of higher-order differential equations that replace the differential equations (1) completely, to the extent that one deals with the determination of the variables $x_1, x_2, \ldots, x_m$. Therefore, the general system of integrals (2) of the differential equations (1), as well as the general system of integrals for the differential equations (S) and the singular system of integrals that is determined by equations (6) and (4.a), is also a singular system of integrals for the differential equations (S).

3. – The definition of singular system of integrals that was presented above can be easily extended.

To that end, we introduce the variables:

$$y_\nu = \varphi_\nu (x_1, x_2, \ldots, x_n), \qquad \nu = 1, 2, \ldots, n \tag{7}$$

into the differential equations (1) in place of the variables x_ν. The equations:

$$\frac{dy_\nu}{dt} = Y_\nu, \qquad \nu = 1, 2, \ldots, n, \tag{8}$$

in which:

(†) Translator: His use of the word "partial" in this context means only "some of the equations," and does not refer to partial differentiation in the usual sense,

(9) $$Y_\nu = \sum_{\lambda=1}^{n} X_\lambda \frac{\partial y_\nu}{\partial x_\lambda},$$

in place of equations (1).

We now consider the variables $y_1, y_2, \ldots, y_m$ to be free variables and set the variables $y_{m+1}, y_{m+2}, \ldots, y_n$ equal to zero. The singular system of integrals that corresponds to that classification is determined by the equations:

(10) $$\left(\frac{dy_\nu}{dt}\right) = Y_\nu, \qquad \nu = 1, 2, \ldots, m,$$

and

(11) $$y_{m+1} = 0, \qquad y_{m+1} = 0, \qquad \ldots, \qquad y_n = 0.$$

Since equations (8) are equivalent to equations (1), we would also like to refer to that system of integrals as a *singular system of integrals for the differential equations* (1).

In order for the singular system of integrals to be well-defined, one must give:

1. The "constraint equations": $\varphi_{m+1} = 0, \varphi_{m+2} = 0, \ldots, \varphi_n = 0$.

2. The "free functions: $\varphi_1, \varphi_2, \ldots, \varphi_n$, relative to which the validity of the originally-given differential equations (1) will remain unchanged.

Of course, the singular system of integrals will remain unchanged when we replace the $n - m$ functions $\varphi_{m+1}, \varphi_{m+2}, \ldots, \varphi_n$ with $n - m$ functions of those functions, because that means only a conversion of the constraint equations. However, it is easy to see that the singular system of integrals will also remain essentially unchanged when we replace the m free functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ with arbitrarily-chosen functions of those functions. In order for the singular system of integrals to be well-defined, only the system of constraint equations and the system of free functions must then be given.

4. – We have tacitly assumed that the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are mutually independent, as long as the quantities $x_1, x_2, \ldots, x_n$ can be considered to vary independently.

We set:

(12) $$x_\nu = \psi_\nu(y_1, y_2, \ldots, y_n), \qquad \nu = 1, 2, \ldots, n.$$

We now make the further assumption that this solution to equations (8) will also remain valid when the variables $y_{m+1}, y_{m+2}, \ldots, y_n$ vanish. In other words: We assume that not only does the functional determinant $\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}$ not vanish identically when the quantities $x_1, x_2, \ldots, x_n$ are considered to vary independently, but we will further assume that this determinant does not vanish for all values of the variables $x_1, x_2, \ldots, x_n$ that satisfy the constraint equations $\varphi_{m+1} = 0$, $\varphi_{m+2} = 0$, …, $\varphi_n = 0$, either.

Under that assumption, we can convert the differential equations (10) in a remarkable way.

Let:

$$\bar{\psi}\,(y_1, y_2, \ldots, y_n) \tag{13}$$

denote the expression to which the expression $\psi_\nu\,(y_1, y_2, \ldots, y_n)$ will go when one considers equations (11). One obviously has that:

$$\frac{\partial \bar{\psi}_\nu}{\partial y_\mu} \qquad (\mu = 1, 2, \ldots, m\,;\ \nu = 1, 2, \ldots, n)$$

will be the value that the derivatives $\partial \psi_\nu / \partial y_\mu$ will assume when one considers those equations. Introduce the notation:

$$\frac{\partial \psi_\nu}{\partial y_\lambda} = p_{\lambda\nu} \qquad (\lambda = m + 1, m + 2, \ldots, n\,;\ \nu = 1, 2, \ldots, n) \tag{14}$$

for the values that the derivatives $\partial \psi_\nu / \partial y_\lambda$ $(\lambda > m)$ when one considers equations (11). If the n functions $\bar{\psi}\,(y_1, y_2, \ldots, y_n)$ are given then the $n\,(n - m)$ functions $p_{\lambda\nu}\,(y_1, y_2, \ldots, y_n)$ can still be chosen arbitrarily. When one recalls (1), the following equations will exist:

$$\frac{d\bar{\psi}_\nu}{dt} = \sum_{\mu=1}^{m} \frac{\partial \bar{\psi}_\nu}{\partial y_\mu} \left(\frac{dy_\mu}{dt} \right) = \sum_{\mu=1}^{m} \frac{\partial \bar{\psi}_\nu}{\partial y_\mu} Y_\mu, \qquad \nu = 1, 2, \ldots, n. \tag{15}$$

On the other hand, when one considers the quantities $x_1, x_2, \ldots, x_n$ to vary independently and recalls (12), it will follow from equations (9) that:

$$X_\nu = \sum_{\mu=1}^{m} Y_\mu \frac{\partial \psi_\nu}{\partial x_\lambda}, \qquad \nu = 1, 2, \ldots, n.$$

When one recalls equations (11) and uses the relations (13) and (14), one can write those equations in the form:

$$\sum_{\mu=1}^{m} \frac{\partial \bar{\psi}_\nu}{\partial y_\mu} Y_\mu = X_\nu - \sum_{\lambda=m+1}^{n} Y_\mu\, p_{\lambda\nu}\,.$$

Now, since one is only dealing with the determination of the m functions:

$$y_\mu = \varphi_\mu\,(x_1, x_2, \ldots, x_n), \qquad \mu = 1, 2, \ldots, m,$$

the quantities $Y_\lambda = dy_\lambda / dt$ ($\lambda = m + 1, m + 1, \ldots, n$) will play the role of parameters. In order show that expressly, one sets $Y_\lambda = Y_{m+\kappa} = -\rho_\kappa$ ($\kappa = 1, 2, \ldots, n - m$).

We further set $\frac{d\psi_\nu}{dt} = \left(\frac{dx_\nu}{dt}\right)$, in agreement with the notation that was introduced above. With the use of that notation, the foregoing equations will take the form:

$$\left(\frac{dx_\nu}{dt}\right) = X_\nu + \sum_{\kappa=1}^{n-m} \rho_\kappa \, p_{m+\kappa,\nu} , \qquad \nu = 1, 2, \ldots, n. \tag{16}$$

Those n equations, in conjunction with the $n - m$ constraint equations $\varphi_{m+1} = 0$, $\varphi_{m+2} = 0, \ldots, \varphi_n = 0$, will suffice to determine the n functions $x_1, x_2, \ldots, x_n$ and the $n - m$ parameters $\rho_1, \rho_2, \ldots, \rho_n$.

The n ($n - m$) quantities $p_{\lambda\nu}$ ($\lambda = m + 1, m + 2, \ldots, n$; $\nu = 1, 2, \ldots, n$) will be determined completely when the m free functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ are given, in addition to the constraint equations. Namely, when the n quantities $x_1, x_2, \ldots, x_n$ can be considered to vary independently, the following equations will be valid:

$$\sum_{\mu=1}^{n} \frac{\partial y_\nu}{\partial x_\mu} \frac{\partial x_\mu}{\partial a_\lambda} = \delta_{\nu\lambda} , \qquad \lambda, \nu = 1, 2, \ldots, n.$$

If we now consider the constraint equations then, with the use of the notations (14), we will get:

$$\sum_{\mu=1}^{n} p_{\lambda\mu} \frac{\partial \varphi_\nu}{\partial x_\mu} = \delta_{\nu\lambda} , \qquad \lambda = m + 1, m + 2, \ldots, n ; \nu = 1, 2, \ldots, n.$$

Conversely, the system of free functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ will be determined when the n ($n - m$) quantities $p_{\lambda\nu}$ are given, because the following differential equations will be true for them:

$$\sum_{\mu=1}^{n} p_{\lambda\mu} \frac{\partial \varphi_\nu}{\partial x_\mu} = 0 , \qquad \lambda = m + 1, m + 2, \ldots, n ; \nu = 1, 2, \ldots, n.$$

In regard to that, it should be remarked: The quantities $p_{\lambda\nu}$ cannot be chosen arbitrarily as functions of the quantities $x_1, x_2, \ldots, x_n$. Rather, they must be satisfy the condition that the foregoing $n - m$ linear partial differential equations must define a complete system when one considers the constraint equations.

In the foregoing, the singular systems of integrals were defined for only first-order differential equations. However, that definition can be extended immediately to a system of higher-order differential equations since such a system can indeed be converted into a first-order system by adding new variables. For example, if the dependent variables $x_1, x_2, \ldots, x_n$ are determined by n second-order differential equations then we add the equations $\frac{dx_1}{dt} = x_1'$, $\frac{dx_2}{dt} = x_2'$, …, $\frac{dx_n}{dt} = x_n'$

to those equations and thus come to a system of $2n$ first-order differential equations with $2n$ undetermined functions.

II. – Application to the variational problem.

5. – Now that the concept of the singular system of integrals has been established, we turn to the resolution of the question that was posed to begin with.

It would seem appropriate to begin with a brief presentation of the line of reasoning in the usual calculus of variations.

Let the integral:

(1) $$\Omega = \int_{t_1}^{t_2} F(t, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n')\, dt$$

be given. The F in it means a single-valued function of the independent variable t, the n functions $x_1, x_2, \ldots, x_n$ of that variable, and their first derivatives x_1', x_2', …, x_n'. Let the n functions $x_1, x_2, \ldots, x_n$ be subject to the k constraint equations:

(2) $$\Phi_1 = 0\,, \qquad \Phi_2 = 0\,, \quad \ldots, \quad \Phi_k = 0\,.$$

We assume that "non-holonomic" constraints also appear among the constraint equations, i.e., equations in which a number of the derivatives x_1', x_2', …, x_n' occur, and they are not completely integrable. In order to avoid going too far afield, we would like to assume that among the constraint equations, there are none that do not include the derivatives, and that no such equation can be derived by converting the constraint equations. That assumption implies no essential restriction. Namely, if say the derivatives do not occur in the expression Φ_1 then we can replace the constraint equation $\Phi_1 = 0$ with the equation $d\,\Phi_1 / dt = 0$.

For mechanical problems, the only non-holonomic constraint equations that come under consideration are ones that are linear and homogeneous in the derivatives. That restriction is unnecessary for the general investigation that is being carried out here.

We now vary the functions $x_1, x_2, \ldots, x_n$. In that way, we assume that the values of the functions are given for the limiting values $t = t_1$ and $t = t_2$ such that the variations δx_1, δx_2, …, δx_n will then vanish at the limits.

(1) implies, in a known way, that:

$$\delta\Omega = \int_{t_1}^{t_2} \sum_{\nu=1}^{n} \left[\frac{\partial F}{\partial x_\nu'} \delta x_\nu' + \frac{\partial F}{\partial x_\nu} \delta x_\nu \right] dt\,,$$

and with an application of partial integration, it will then follow from this that:

(3) $$\delta\Omega = -\int_{t_1}^{t_2} \sum_{\nu=1}^{n} \left[\frac{d}{dt}\frac{\partial F}{\partial x'_\nu} - \frac{\partial F}{\partial x_\nu} \right] \delta x_\nu \, dt \,.$$

Moreover, we have:

(4) $$\delta\Phi_\kappa = \sum_{\nu=1}^{n} \left[\frac{\partial \Phi_\kappa}{\partial x'_\nu}\delta x'_\nu + \frac{\partial \Phi_\kappa}{\partial x_\nu}\delta x_\nu \right] .$$

We multiply that by an arbitrarily-chosen function ρ_κ of the independent variable t and integrate between the limits t_1 and t_2 . When we also apply partial integration to that, we will get:

(5) $$\begin{aligned} \int_{t_1}^{t_2} \rho_\kappa \, \delta\Phi_\kappa \, dt &= -\int_{t_1}^{t_2} \sum_{\nu=1}^{n} \left[\frac{d}{dt}\frac{\partial F}{\partial x'_\nu} - \frac{\partial F}{\partial x_\nu} \right] \delta x_\nu \, dt \\ &= -\int_{t_1}^{t_2} \sum_{\nu=1}^{n} \left[\frac{d\rho_\kappa}{dt}\frac{\partial \Phi_\kappa}{\partial x'_\nu} + \rho_\kappa \left(\frac{d}{dt}\frac{\partial \Phi_\kappa}{\partial x'_\nu} - \frac{\partial \Phi_\kappa}{\partial x_\nu} \right) \right] \delta x_\nu \, dt \,. \end{aligned}$$

Now pose the problem:

(*A*): Determine the functions $x_1, x_2, \ldots, x_n$ such that $\delta\Omega = 0$ for all variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ that satisfy the constraint equations:

$$\delta\Phi_1 = 0 \,, \quad \delta\Phi_2 = 0 \,, \ldots, \delta\Phi_k = 0$$

and vanish for $t = t_1$ and $t = t_2$.

That problem might be referred to as the general variational problem. When one recalls the conditions that were posed, it will follow from equations (3) and (5) that:

(6) $$\int_{t_1}^{t_2} \sum_{\nu=1}^{n} \left[\frac{d}{dt}\frac{\partial F}{\partial x'_\nu} - \frac{\partial F}{\partial x_\nu} + \sum_{\kappa=1}^{k} \left(\frac{d\rho_\kappa}{dt}\frac{\partial \Phi_\kappa}{\partial x'_\nu} + \rho_\kappa \frac{d}{dt}\frac{\partial \Phi_\kappa}{\partial x'_\nu} - \rho_\kappa \frac{\partial \Phi_\kappa}{\partial x_\nu} \right) \right] \delta x_\nu \, dt = 0 \,.$$

We will then satisfy the imposed constraints in any event when we set:

(7) $$\frac{d}{dt}\frac{\partial F}{\partial x'_\nu} - \frac{\partial F}{\partial x_\nu} + \sum_{\kappa=1}^{k} \left(\frac{d\rho_\kappa}{dt}\frac{\partial \Phi_\kappa}{\partial x'_\nu} + \rho_\kappa \frac{d}{dt}\frac{\partial \Phi_\kappa}{\partial x'_\nu} - \rho_\kappa \frac{\partial \Phi_\kappa}{\partial x_\nu} \right) = 0$$

for $\nu = 1, 2, \ldots, n$.

Those n equations, in conjunction with the k constraint equations (1), will suffice to determine the n functions $x_1, x_2, \ldots, x_n$, and the k multipliers $\rho_1, \rho_2, \ldots, \rho_k$.

One can convince oneself of the fact that equations (7) represent not only sufficient, but also necessary, conditions on the functions $x_1, x_2, \ldots, x_n$ by the following argument ([1]): The left-hand

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([1]) For its proof, see A. Mayer, Math. Ann., Bd. 27, pp. 74.

sides of equations (4) vanish on the basis of the constraints that were imposed. When one considers the equations:

$$\delta x'_\nu = \delta \frac{dx_\nu}{dx} = \frac{d\,\delta x_\nu}{dx},$$

one will get a system of k differential equations for the n variations $\delta x_1, \delta x_2, \ldots, \delta x_n$. $n - k$ of those variations (say, the variations $\delta x_{k+1}, \delta x_{k+2}, \ldots, \delta x_n$) can then be taken arbitrarily. If one then determines the multipliers $\rho_1, \rho_2, \ldots, \rho_k$, which are at one's disposal, by means of the first k of equations (7), in which the functions $x_1, x_2, \ldots, x_n$ are assumed to be given, then the variations $\delta x_1, \delta x_2, \ldots, \delta x_k$ will drop out of the integral, and the integral must vanish for arbitrary values of the remaining variations. Now, those variations can be chosen such that they are all everywhere equal to zero, except for one of them, while that one is non-zero only in an arbitrarily-small sub-interval. One can conclude from this that the coefficients of the individual variations $\delta x_{k+1}, \delta x_{k+2}, \ldots, \delta x_n$ must vanish. However, those coefficients are nothing but the left-hand sides of the last $n - k$ differential equations (7).

Two groups of dependent variables occur in the differential equations (7): One of them consists of the variables $x_1, x_2, \ldots, x_n$ and their derivatives $x'_1, x'_2, \ldots, x'_n$, and the other consists of the multipliers $\rho_1, \rho_2, \ldots, \rho_k$. The problem that was posed demands only the determination of the variables $x_1, x_2, \ldots, x_n$. The multipliers are parameters that are introduced as only tools for calculation. That state of affairs is closely related to the idea of turning one's attention to not only the general system of integrals of the differential equations (7), but also to those singular systems of integrals that are characterized by constraint equations for the parameters. One easily convinces oneself that those singular systems of integrals are not solutions to the general variational problem (A). However, that raises the question of whether they can be considered to be at least solutions of suitably-defined "special" variational problems. It can be shown immediately that this is the case for at least some of the singular systems of integrals.

6. – We will get a singular system of integrals for the differential equations (5) when we consider the variables $x_1, x_2, \ldots, x_n$; $x'_1, x'_2, \ldots, x'_n$ to be free variables and set the parameters $\rho_1, \rho_2, \ldots, \rho_k$ equal to zero (see no. **1**). Under that assumption, we must substitute:

$$\rho_1 = 0, \rho_2 = 0, \ldots, \rho_k = 0, \qquad \frac{dx'_\nu}{dt} = \left(\frac{dx'_\nu}{dt}\right), \qquad \frac{dx_\nu}{dt} = x'_\nu = \left(\frac{dx_\nu}{dt}\right), \quad \nu = 1, 2, \ldots, n.$$

The quantities $d\rho_\kappa / dt$ ($\kappa = 1, 2, \ldots, k$) are considered to be parameters. We set $d\rho_\kappa / dt = \sigma_\kappa$, and we can then drop the parentheses around the differential quotients, since they will no longer be necessary. We will get:

$$\frac{d}{dt}\frac{\partial F}{\partial x'_\nu} - \frac{\partial F}{\partial x_\nu} + \sum_{\kappa=1}^{k} \sigma_\kappa \frac{\partial \Phi_\kappa}{\partial x'_\nu} = 0, \qquad \nu = 1, 2, \ldots, n. \tag{8}$$

We multiply that by δx_ν, add over all ν, and integrate between the limits t_1 and t_2. To abbreviate, we introduce the notation:

(9) $$\vartheta\,\Phi_\kappa = \sum_{\kappa=1}^{k} \frac{\partial \Phi_\kappa}{\partial x'_\nu}\,\delta x_\nu\,.$$

When we recall equation (3), it will follow that:

$$\delta\Omega = \int_{t_1}^{t_2} \sum_{\kappa=1}^{k} \sigma_\kappa\,\vartheta\,\Phi_\kappa\,dt\,.$$

With that, we have proved:

(*B*): Equations (8) represent the sufficient conditions for the variation $\delta\Omega$ to vanish for all values of the variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ that satisfy the conditions:

$$\vartheta\,\Phi_1 = 0, \qquad \vartheta\,\Phi_2 = 0, \qquad \ldots, \qquad \vartheta\,\Phi_k = 0$$

and vanish at the limits $t = t_1$ and $t = t_2$.

The argument from the calculus of variations that was discussed in the previous section shows that the given conditions are not only sufficient, but also necessary.

Equations (8) then represent the solution to a "special" variational problem. In the case where the constraint equations $\Phi_k = 0$ are linear and homogeneous in the derivatives $x'_1, x'_2, \ldots, x'_n$, the variational problem (*B*) will be nothing but Hölder's variational problem.

Instead of setting all k parameters equal zero, we can also count some of them (say, the parameters $\rho_1, \rho_2, \ldots, \rho_i$) among the free variables and set the last $k - i$ equal to zero. Under that assumption, we must set:

$$\rho_{i+1} = 0,\ \ \rho_{i+2} = 0,\ \ldots,\ \rho_k = 0\,,$$

$$\frac{d\rho_1}{dt} = \left(\frac{d\rho_1}{dt}\right), \qquad \frac{d\rho_2}{dt} = \left(\frac{d\rho_2}{dt}\right), \qquad \ldots, \qquad \frac{d\rho_i}{dt} = \left(\frac{d\rho_i}{dt}\right),$$

$$\frac{dx'_\nu}{dt} = \left(\frac{dx'_\nu}{dt}\right), \qquad \frac{dx_\nu}{dt} = x'_\nu = \left(\frac{dx_\nu}{dt}\right), \qquad \nu = 1, 2, \ldots, n$$

in equations (5). The quantities $\frac{d\rho_{\nu+1}}{dt}, \frac{d\rho_{\nu+2}}{dt}, \ldots, \frac{d\rho_k}{dt}$ are considered to be parameters. We set $\frac{d\rho_{\nu+\lambda}}{dt} = \sigma_\lambda$, $\lambda = 1, 2, \ldots, k - i$, and we can then once more drop the parentheses around the differential quotients. We will get:

$$(10)\qquad \frac{d}{dt}\frac{\partial F}{\partial x'_\nu}-\frac{\partial F}{\partial x_\nu}+\sum_{\kappa=1}^{i}\left(\frac{d\rho_\kappa}{dt}\frac{\partial\Phi_\kappa}{\partial x'_\nu}+\rho_\kappa\frac{d}{dt}\frac{\partial\Phi_\kappa}{\partial x'_\nu}-\rho_\kappa\frac{\partial\Phi_\kappa}{\partial x_\nu}\right)+\sum_{\lambda=1}^{k-i}\sigma_\lambda\frac{\partial\Phi_{i+\lambda}}{\partial x'_\nu}=0,\qquad \nu=1,2,\ldots,n.$$

We again multiply that by δx_ν, add over all ν, and integrate between the limits t_1 and t_2. When we recall equations (3), (5), and (9), we will get:

$$\delta\Omega=-\sum_{\kappa=1}^{i}\rho_\kappa\,\delta\Phi_\kappa+\sum_{\lambda=1}^{k-i}\sigma_\lambda\,\vartheta\Phi_{i+\lambda}\ .$$

It will then follow that:

(*C*): Equations (10) represent the necessary and sufficient conditions for the variation $\delta\Omega$ to vanish for all values of the variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ that satisfy the equations:

$$\delta\Phi_1=0,\ \delta\Phi_2=0,\ \ldots,\ \delta\Phi_i=0,\qquad \vartheta\Phi_{i+1}=0,\ \vartheta\Phi_{i+2}=0,\ldots,\ \vartheta\Phi_k=0$$

and vanish at the limits t_1 and t_2.

Equations can also be considered to be the solution to a "special" variational problem.

The variational problem (*C*) can be easily generalized. To that end, we replace the constraint equations $\Phi_\kappa = 0$ with arbitrary linear combinations of them, in which we set:

$$\Phi_\kappa = c_{\kappa1}\,\Psi_1 + c_{\kappa2}\,\Psi_2 + \ldots + c_{\kappa k}\,\Psi_k\,,\qquad \kappa = 1, 2, \ldots, k.$$

At the same time, we introduce be parameters in place of the parameters $\rho_1, \rho_2, \ldots, \rho_k$ by means of the equations:

$$\tau_\kappa = c_{\kappa1}\,\rho_1 + c_{\kappa2}\,\rho_2 + \ldots + c_{\kappa k}\,\rho_k\qquad (\kappa = 1, 2, \ldots, k),$$

such that the following identity will exist:

$$\rho_1\,\Phi_1 + \rho_2\,\Phi_2 + \ldots + \rho_k\,\Phi_k = \tau_1\,\Psi_1 + \tau_2\,\Psi_2 + \ldots + \tau_k\,\Psi_k\,.$$

The $c_{\kappa\lambda}$ mean arbitrarily-chosen functions of the quantities t ; $x_1, x_2, \ldots, x_n$; $x'_1, x'_2, \ldots, x'_n$. They are subject to only the condition that their determinant does not vanish. It is clear with no further explanation that equations (5) are equivalent to the equations that they will go to when one simultaneously replaces the functions $\Phi_1, \Phi_2, \ldots, \Phi_k$ with the functions $\Psi_1, \Psi_2, \ldots, \Psi_k$, and replaces the parameters $\rho_1, \rho_2, \ldots, \rho_k$ with the parameters $\tau_1, \tau_2, \ldots, \tau_k$. Therefore, the singular system of integrals of equations (5) that is characterized by the $k - i$ equations:

$$\tau_\kappa = c_{1\kappa}\,\rho_1 + c_{2\kappa}\,\rho_2 + \ldots + c_{k\kappa}\,\rho_k = 0\,,\qquad \kappa = i + 1, i + 2, \ldots, k$$

will correspond to a variational problem in which the allowable variations are defined by the equations:

$$\delta\Psi_1 = 0,\ \delta\Psi_2 = 0,\ \ldots,\ \delta\Psi_k = 0, \qquad \vartheta\Psi_{i+1} = 0,\ \vartheta\Psi_{i+2} = 0,\ \ldots,\ \vartheta\Psi_k = 0\ .$$

Since the quantities $c_{\lambda\kappa}$ can be chosen arbitrarily, it will follow that: Any singular system of integrals of the differential equations (5) that is characterized by linear and homogeneous relations between the parameters $\rho_1, \rho_2, \ldots, \rho_k$ will correspond to a special variational problem.

7. – In the case of the general variational problem, the allowable variations are determined by equations of the form:

$$\text{(11)} \qquad \delta\Phi_\kappa = \sum_{\nu=1}^{n} \left[\frac{\partial\Phi_\kappa}{\partial x'_\nu}\delta x'_\nu + \frac{\partial\Phi_\kappa}{\partial x_\nu}\delta x_\nu \right] = 0\ .$$

In the case of the special variational problem, either the only equations that appear will have the form:

$$\text{(12)} \qquad \vartheta\Phi_\kappa = \sum_{\nu=1}^{n} \frac{\partial\Phi_\kappa}{\partial x'_\nu}\delta x_\nu = 0$$

[case (B) of the previous section] or the two types of equations will occur alongside each other [case (C)].

We would like to establish the conditions under which those equations are equivalent.

Since the variations $\delta x_1, \delta x_2, \ldots, \delta x_n$ vanish for the limiting values $t = t_1$ and $t = t_2$, equation (12) will be fully represented by the equation $\frac{d\vartheta\Phi_\kappa}{dt} = 0$. When one recalls the identity $\frac{d\delta x_\nu}{dt} = \delta x'_\nu$, one will have:

$$\frac{d\vartheta\Phi_\kappa}{dt} = \sum_{\nu=1}^{n} \left(\frac{d}{dt}\frac{\partial\Phi_\kappa}{\partial x'_\nu} + \frac{\partial\Phi_\kappa}{\partial x'_\nu}\delta x'_\nu \right),$$

and as a result:

$$\frac{\delta\vartheta\Phi_\kappa}{dt} - \delta\Phi_\kappa = \sum_{\nu=1}^{n} \left(\frac{d}{dt}\frac{\partial\Phi_\kappa}{\partial x'_\nu} - \frac{\partial\Phi_\kappa}{\partial x_\nu} \right)\delta x'_\nu\ .$$

Should the equations (11) and (12) mean the same thing, then the following equations would have to exist:

$$\frac{d}{dt}\frac{\partial\Phi_\kappa}{\partial x'_\nu} - \frac{\partial\Phi_\kappa}{\partial x_\nu} = M\,\frac{\partial\Phi_\kappa}{\partial x'_\nu}, \qquad \text{for } \nu = 1, 2, \ldots, n,$$

in which M means an arbitrary function of the independent variable t. When we set $M = -\frac{1}{N}\frac{dN}{dt}$, those equations will take the form:

$$(13) \qquad \frac{d}{dt}\frac{\partial N\Phi_\kappa}{\partial x'_\nu} - \frac{\partial N\Phi_\kappa}{\partial x_\nu} = \frac{\partial N\Phi_\kappa}{\partial x'_\nu}.$$

The equations must either be fulfilled identically or at least when one recalls the given constraint equations:

$$\Phi_1 = 0, \qquad \Phi_2 = 0, \qquad \ldots, \qquad \Phi_k = 0.$$

If they are fulfilled identically then $N\,\Phi_\kappa$ can be represented in the form $N\,\Phi_\kappa = d\,\Psi / dt$, where Ψ is a function of the variables t, x_1, x_2, …, x_n . In that case, the constraint equation $\Phi_\kappa = 0$ is holonomic. If equations (13) are true only when one considers the constraint equations $\Phi_\kappa = 0$ then the equation will be at least equivalent to a holonomic equation. In any other case, equations (11) and (12) will have different meanings.

If all of the given constraint equations are holonomic then each equation $\delta\,\Phi_\kappa = 0$ will be equivalent to the corresponding one $\vartheta\,\Phi_\kappa = 0$, and the difference between the general variational problem and the special one will drop away. In that case, one will have:

$$\frac{d}{dt}\frac{\partial\Phi_\kappa}{\partial x'_\nu} - \frac{\partial\Phi_\kappa}{\partial x_\nu} = 0 \qquad \text{for} \qquad \begin{cases} \nu = 1, 2, \ldots, n, \\ \kappa = 1, 2, \ldots, k. \end{cases}$$

The quantities ρ_1, ρ_2, …, ρ_k will then drop out of equations (5), and those equations will differ equations (8), which are true for the special variational problem (*B*), by only the notations for the multipliers. By contrast, the special variational problem will differ essentially from the general one when the equations $\Phi_\kappa = 0$ are not all holonomic, and they are also not equivalent to a system of nothing but holonomic equations.

8. – We would like to clarify the foregoing general discussion with an example.

We address the problem of determining the motion of a two-wheeled cart that rolls without slipping on a horizontal plane. For the sake of simplicity, we assume that the center of mass is found at the center of the axle. We can ignore forces since the effect of gravity on the motion will not come under consideration.

We lay the *xy*-plane horizontally through the axle and let x, y denote the coordinates of the center of mass, while ω denotes the angle that a horizontal normal to the axle makes with the direction of increasing x, and finally M will denote the mass of the cart, while $M\,l^2$ is its moment of inertia relative to a vertical that goes through the center of mass. The *vis viva* of the cart is:

$$T = \tfrac{1}{2} M\,[x'^2 + y'^2 + l^2\omega'^2].$$

Since the cart should roll without slipping, a translation can result only perpendicular to the direction of the axle. The non-holonomic constraint equation will then exist:

(1) $$\cos\omega\, y' - \sin\omega\, x' = 0 .$$

The differential equations for the motion of the cart are implied by the condition:

$$\delta \int_{t_1}^{t_2} T\, dt = 0$$

for all variations that satisfy the condition $\cos\,\omega\,\delta y - \sin\,\omega\,\delta x = 0$ and vanish for $t = t_1$ and $t = t_2$. We will then get:

(2) $$M\, x'' = -\sigma \sin\,\omega , \qquad M\, y'' = \sigma \cos\,\omega, \qquad \omega'' = 0 .$$

It then follows from this that:

$$\omega' = \gamma, \qquad \omega = \gamma\,(t - t_0),$$

and furthermore, when we recall (1):

$$x' = c\,\cos\,\omega\,, \quad y' = c\,\sin\,\omega\,.$$

The following equations are then true for the motion of the cart:

(3) $$x - x_0 = \frac{c}{\gamma}\sin\omega\,, \qquad y - y_0 = \frac{c}{\gamma}(1 - \cos\omega)\,, \qquad \omega = \gamma\,(t - t_0)\,.$$

We now determine the functions x, y, ω by the constraints on the general variational problem:

$$\delta \int_{t_1}^{t_2} T\, dt = 0$$

for all variations $\delta x'$, $\delta y'$, $\delta z'$ that satisfy the condition:

$$\cos\omega\,\delta x' - \sin\omega\,\delta y' - (\sin\omega\,\delta y' + \cos\omega\,\delta x') = 0$$

and vanish at the limits.

In that case, we will get the differential equations:

(4) $$\begin{aligned} M\, x'' &= \rho' \sin\omega + \rho\cos\omega\,, \\ M\, y'' &= -\rho' \cos\omega + \rho\sin\omega\,, \\ l^2 M\,\omega'' &= -\rho'\,(\sin\omega\, y' + \cos\omega\, x')\,. \end{aligned}$$

The integration of the first two equations gives:

$$M\,x' = \rho \sin\omega + a\cos\alpha, \qquad M\,y' = -\rho\cos\omega + a\sin\alpha,$$

and when one recalls (1), it will then follow from this that:

$$\rho = a\sin(\alpha-\omega)\,, \qquad M\,(x'\cos\omega + y'\sin\omega) = a\cos(\alpha-\omega)\,.$$

If one introduces those values in the last of equations (4) then one will get:

$$2l^2M^2\,\omega'' = -\,a^2\sin 2(\alpha-\omega)\,,$$

and it will then follow that:

$$l^2M^2\,\omega'^2 = -\tfrac{1}{2}a^2\cos 2(\alpha-\omega) + \text{const.} = -\tfrac{1}{2}a^2\cos 2(\alpha-\omega) + \tfrac{1}{2}a^2\left(\frac{2}{k^2}-1\right),$$

so

$$\omega' = \frac{a}{k\,l\,M}\sqrt{1-k^2\cos^2(\alpha-\omega)}\,.$$

Eliminating the time differential will yield the equations:

$$\frac{dx}{d\omega} = \frac{k\,l}{a}\frac{\rho\sin\omega + a\cos\alpha}{\sqrt{1-k^2\cos^2(\alpha-\omega)}} = k\,l\frac{\cos(\alpha-\omega)\cos\omega}{\sqrt{1-k^2\cos^2(\alpha-\omega)}}\,,$$

$$\frac{dy}{d\omega} = \frac{k\,l}{a}\frac{-\rho\cos\omega + a\sin\alpha}{\sqrt{1-k^2\cos^2(\alpha-\omega)}} = k\,l\frac{\cos(\alpha-\omega)\sin\omega}{\sqrt{1-k^2\cos^2(\alpha-\omega)}}\,.$$

We will ultimately obtain the integral equations:

$$(5) \qquad \begin{aligned} x - x_0 &= k\,l\int_0^\omega \frac{\cos(\alpha-\omega)\cos\omega\,d\omega}{\sqrt{1-k^2\cos^2(\alpha-\omega)}}\,,\\ y - y_0 &= k\,l\int_0^\omega \frac{\cos(\alpha-\omega)\sin\omega\,d\omega}{\sqrt{1-k^2\cos^2(\alpha-\omega)}}\,,\\ t - t_0 &= \frac{k\,l\,M}{a}\int_0^\omega \frac{d\omega}{\sqrt{1-k^2\cos^2(\alpha-\omega)}}\,. \end{aligned}$$

We will arrive at the singular system of integrals of the differential equations (4) that is characterized by the constraint equation $\rho = 0$ when we consider the integration constants a, k, x_0, y_0 to be constants, as before, while the integration constant α is defined to be a function of time.

The integral equations (5) will then be identical to the integral equations (3), except for the notation of the integration constants.

III. – The dynamical equations for unfree systems.

9. – In conclusion, we would like to show that the concept of a singular system of integrals will lead to a new conception of the equations of motion for unfree systems.

To that end, we start from the second form of the Lagrange equations for a *free* system:

$$(1) \qquad \frac{d}{dt}\frac{\partial T}{\partial x'_\nu} - \frac{\partial T}{\partial x_\nu} = X_\nu, \qquad \nu = 1, 2, \ldots, n.$$

Here, $x_1, x_2, \ldots, x_n$ mean general coordinates that determine the configuration of the system. X_ν is the component of the forces that strives to increase x_ν. T means the *vis viva* of the system.

We assume that the coefficients of the quadratic form T and the force components X_ν are functions of the coordinates, but they do not include time explicitly.

Along with the coordinates $x_1, x_2, \ldots, x_n$, we also consider the components of the velocity x'_1, $x'_2, \ldots, x'_n$ to be autonomous dependent variables. The two systems of variables are coupled with each other by the relation:

$$(2) \qquad \frac{dx_\nu}{dt} = x'_\nu, \qquad \nu = 1, 2, \ldots, n.$$

We now assume that the system that was considered to be free up to now is subject to the constraints:

$$(3) \qquad x'_{m+1} = 0, \qquad x'_{m+2} = 0, \qquad \ldots, \qquad x'_n = 0.$$

The motion of that unfree system will be determined by the equations:

$$(4) \qquad \frac{d}{dt}\frac{\partial \bar{T}}{\partial x'_\nu} - \frac{\partial \bar{T}}{\partial x_\nu} = X_\nu, \qquad \nu = 1, 2, \ldots, n,$$

in conjunction with equations (2) and (3).

Here, $\bar{T}$ means the expression that the quadratic form T goes to when one sets x'_{m+1}, $x'_{m+2}, \ldots$, x'_n equal to zero.

For the moment, we shall assume that the products of the variables $x'_1, x'_2, \ldots, x'_m$ with the variables $x'_{m+1}, x'_{m+2}, \ldots, x'_n$ do not occur in the quadratic form T, so T will be the sum of a quadratic form in the m variables x_1', x_2', …, x_m' and a quadratic form in the $n - m$ variables x_{m+1}', x_{m+2}', …, x_n'. Under that assumption, we will obviously have:

$$\frac{\partial \bar{T}}{\partial x_1'} = \frac{\partial T}{\partial x_1'}, \qquad \frac{\partial \bar{T}}{\partial x_2'} = \frac{\partial T}{\partial x_2'}, \qquad \ldots, \qquad \frac{\partial \bar{T}}{\partial x_m'} = \frac{\partial T}{\partial x_m'}.$$

In that case, equations (4) and (2), together with equations (3), will then determine a singular system of integrals of the differential equations (1) and (2), and indeed the variables $x_1, x_2, \ldots, x_n$; x_1', x_2', …, x_m' are considered to be free variables (see no. **1**).

Nothing about this relationship between the equations of motion for the free and unfree system will change essentially when we drop our assumption that the quadratic form T is composed of two forms, except that the free variables x_1', x_2', …, x_m' will be replaced with linear and homogeneous functions of the n velocity components as free functions (see no. **3**).

In the foregoing, we assumed that the constraint equations for the unfree system were holonomic; that assumption is not essential, either. If the constraint equations are linear and homogeneous in only the velocity components then the integral equations that determine the motion of the unfree system will always be a singular system of integrals of the differential equations that are true for the free system.

We must preface the proof of that assertion with some remarks about the quadratic form T.

10. – When one regards the velocity components as autonomous variables that are on an equal footing with the coordinates, that is closely related to the idea that one can also transform themselves among themselves with no regard for the coordinates. However, in that way one will restrict oneself to transformations that are linear and homogeneous in the quantities x_ν' in order for the characteristic form of the expression for the *vis viva* to remain preserved. One sets:

$$u_\nu = \sum_{\mu=1}^{n} p_{\nu\mu}\, x_\mu', \qquad \nu = 1, 2, \ldots, n. \tag{5}$$

The $p_{\nu\mu}$ in that mean functions of the quantities $x_1, x_2, \ldots, x_n$ that are subject to only the condition that their determinant should not vanish. We refer to the quantities u_ν as general velocity parameters ([1]). Along with the velocity parameters x_ν' and u_ν, it is also convenient to introduce the "impulses" that are associated with them:

([1]) The components of the translational velocity and the angular velocity of a rigid body, relative to a coordinate system that is fixed in the body, are to be regarded as "general velocity parameters," in a sense. Volterra was probably the first to propose the transformation of the equations of motion by introduction of general velocity parameters [Atti di Torino **33** (1898)].

$$\xi_\nu = \frac{\partial T}{\partial x'_\nu}, \qquad \eta_\nu = \frac{\partial T}{\partial u_\nu}.$$

When one recalls (5), the identity:

$$\xi_1\, x_1 + \xi_2\, x_2 + \ldots + \xi_n\, x_n = \eta_1\, u_1 + \eta_2\, u_2 + \ldots + \eta_n\, u_n$$

will imply the equations:

$$(6) \qquad \xi_\nu = \sum_{\mu=1}^{n} p_{\nu\mu}\, \eta_\mu\,, \qquad \nu = 1, 2, \ldots, n.$$

We introduce the variables $u_1, u_2, \ldots, u_n$ into the quadratic form T in place of the variables x'_1, $x'_2, \ldots, x'_m$.

One can choose the substitution coefficients such that the transformed form splits into two parts, one of which includes only the variables $u_1, u_2, \ldots, u_m$, while the other one includes only the remaining variables $u_{m+1}, u_{m+2}, \ldots, u_n$. In that case, the two systems of variables might be referred to as *conjugate* systems of velocity parameters.

Now, it will be important in what follows to point out: One of the two conjugate systems can be chosen arbitrarily, while the other one then essentially determined.

In order to prove that assertion, we would like to write down the transformation equations in detail. Let:

$$T = \tfrac{1}{2}\sum a_{\lambda\mu}\, x'_\lambda\, x'_\mu = \tfrac{1}{2}\sum b_{\lambda\mu}\, u_\lambda\, u_\mu\,.$$

If we introduce the associated impulses in place of the velocity parameters then that will give the equations:

$$\mathrm{T} = \tfrac{1}{2}\sum \alpha_{\lambda\mu}\, \xi_\lambda\, \xi_\mu = \tfrac{1}{2}\sum \beta_{\lambda\mu}\, \eta_\lambda\, \eta_\mu\,.$$

If $b_{\lambda\mu} = 0$ for $\lambda \le m$ and $\mu > m$ then the corresponding equations $\beta_{\lambda\mu} = 0$ for $\lambda \le m$ and $\mu > m$ will also be true for the adjoint form, and *vice versa*. Now, as a result of (6), one has:

$$\beta_{\lambda\mu} = \sum_\kappa \sum_\nu \alpha_{\kappa\nu}\, p_{\kappa\lambda}\, p_{\kappa\mu}\,.$$

The equations:

$$\sum_\kappa \sum_\nu \alpha_{\kappa\nu}\, p_{\kappa\lambda}\, p_{\kappa\mu} = 0 \qquad \text{for} \qquad \lambda \le m \text{ and } \mu > m$$

will imply, when the substitution coefficients are given:

$$p_{\nu 1}, p_{\nu 2}, \ldots, p_{\nu n}\,, \qquad \nu = m+1, m+2, \ldots, n,$$

a system of $n - m$ linear and homogeneous equations that each of the systems of quantities:

$$p_{\kappa 1}, p_{\kappa 2}, \ldots, p_{\kappa n}, \qquad \nu = 1, 2, \ldots, m,$$

must satisfy. In that way, those m systems of quantities, and therefore the velocity parameters u_1, u_2, …, u_m, are not in fact completely determined. If we let $\bar{u}_1$, $\bar{u}_2$, …, $\bar{u}_n$ denote a special system of those quantities that satisfy the requirement that was imposed then that can be represented in the most-general form:

$$u_\mu = c_{\mu 1}\,\bar{u}_1 + c_{\mu 2}\,\bar{u}_2 + \cdots + c_{\mu n}\,\bar{u}_n .$$

11. – We can now formulate the theorem that was stated at the conclusion of no. **9** more precisely.

The equations of motion for the unfree systems are a singular system of integrals of the differential equations that are true for the corresponding free systems, and indeed they are to be regarded as free functions of the coordinates x_1, x_2, …, x_n, and those velocity parameters that are conjugate to the velocity parameters that vanish as a result of the equations of constraint.

In order to prove that we assume that the constraint equations ([1]):

$$(7) \qquad p_{\lambda 1}\,x_1' + p_{\lambda 2}\,x_2' + \cdots + p_{\lambda n}\,x_n' = 0 ,$$

in which $p_{\lambda 1}, p_{\lambda 2}, \ldots, p_{\lambda n}$ mean functions of the coordinates x_ν. We initially ignore those constraint equations and introduce the general velocity parameters u_1, u_2, …, u_n into T by the substitution (5). We choose the available coefficients:

$$p_{\mu 1}, p_{\mu 2}, \ldots, p_{\mu n} \qquad (\mu = 1, 2, \ldots, m)$$

such that the systems of parameters:

$$u_1, u_2, \ldots, u_m \quad \text{and} \quad u_{m+1}, u_{m+2}, \ldots, u_n$$

are conjugate. The impulses η_1, η_2, …, η_m that are associated with the velocity parameters u_1, u_2, …, u_m depend upon only those parameters, but not upon u_{m+1}, u_{m+2}, …, u_n, and correspondingly, the impulses η_{m+1}, η_{m+2}, …, η_n depend upon only u_{m+1}, u_{m+2}, …, u_n, but not upon u_1, u_2, …, u_m. It then follows that: We can replace the constraint equations (7), which we can also write in the form:

$$u_{m+1} = 0 , \qquad u_{m+2} = 0 , \qquad \ldots, \qquad u_n = 0 ,$$

with the equations:

$$\eta_{m+1} = 0 , \qquad \eta_{m+2} = 0 , \qquad \ldots, \qquad \eta_n = 0 ,$$

([1]) We can replace a constraint equation $\varphi = 0$ in which only the coordinates x_ν occur with the equation $d\varphi / dt = 0$, and the initial condition $\varphi = 0$ for $t = t_0$. Therefore, the assumption that we made in regard to the form of the constraint equations comes down to the fact that they either do not include the velocity components at all or they are linear and homogeneous in those quantities.

and we can consider the associated impulses $\eta_1, \eta_2, \ldots, \eta_m$ to be free functions, instead of the velocity parameters $u_1, u_2, \ldots, u_m$. (Cf., no. **3**, conclusion)

When we introduce the impulses in place of the velocity components, equations (1), which are true for the free system, will take the form:

$$\frac{d\xi_\nu}{dt} - \frac{\partial T}{\partial x_\nu} = X_\nu , \qquad \nu = 1, 2, \ldots, n.$$

In order to determine the singular systems of integrals that imply the equations of motion for the unfree system, we return to what we did in no. **4**.

In place of the n variables x_ν that were used there, the $2n$ variables:

$$x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n$$

will now appear, and the n variables:

$$y_1, y_2, \ldots, y_m \,;\, y_{m+1}, y_{m+1}, \ldots, y_n$$

will be replaced with the variables:

$$x_1, x_2, \ldots, x_n, \eta_1, \eta_2, \ldots, \eta_m \,;\, \eta_{m+1}, \eta_{m+2}, \ldots, \eta_n .$$

When one recalls equations (6) (no. **10**), equations (14) in no. **4**:

$$\frac{\partial \psi_\nu}{\partial y_\lambda} = \frac{\partial x_\nu}{\partial y_\lambda} = p_{\lambda\nu} \quad (\lambda = m + 1, m + 2, \ldots, n \,;\, \nu = 1, 2, \ldots, n)$$

will be replaced with the equations:

$$\frac{\partial x_\nu}{\partial \eta_\lambda} = \frac{\partial \xi_\nu}{\partial \eta_\lambda} = p_{\lambda\nu} \quad (\lambda = m + 1, m + 2, \ldots, n \,;\, \nu = 1, 2, \ldots, n).$$

The equations:

$$\left(\frac{d\xi_\nu}{dt}\right) - \frac{\partial T}{\partial x_\nu} = X_\nu + \sum_{\kappa=1}^{n-m} \rho_\kappa \, p_{m+\kappa,\nu} ,$$

$$\left(\frac{dx_\nu}{dt}\right) = x'_\nu \qquad \nu = 1, 2, \ldots, n,$$

will then enter in place of equations (16) (no. **4**), and here one has:

$$\xi_\nu = \frac{\partial T}{\partial x_\nu}.$$

The foregoing equations agree with the second form of the Lagrange equations for the unfree system. Our theorem is proved with that.

We have made two assumptions in regard to the constraint equations: They are linear and homogeneous in the velocity components, and they do not include time.

We made the latter assumption for the sake of simplicity, while the former assumption is essential because the possibility of splitting the quadratic form T into two parts is based upon it [cf., the remark in regard to (7)].

12. – Instead of proving the theorem that was expressed in the previous section with the help of the Lagrange equations for the unfree systems, we can also take the opposite route and regard that theorem as a principle that is inferred from experiments. With the help of that principle, we can then derive the Lagrange equations for the unfree systems, as well as the principle of virtual displacements and Hamilton's principle, in the formulation that Hölder gave it.

The principle that was presented here has the advantage over the principle of virtual displacements that Lagrange used that it is not based upon a mathematical expedient but makes use of only quantities that are mechanically well-defined. It is also not lacking in intuitive appeal.

For example, let us assume that a body rolls on a fixed surface without slipping. The motion at each moment in time will be the same as that of a freely-moving body that contacts the fixed surface in its initial configuration and is endowed with an initial velocity that creates a rotation around an axis that goes through the point of contact. In order for the agreement between the motion that actually results and the fictitious free motion to be preserved, the initial velocities that determine the free motion must be changed at each moment in time. However, mathematically speaking, that means nothing but the statement: The equations that determine the motion of the body are to be regarded as singular solutions to the differential equations that are true for the free motion.

Tübingen, February 1905.