"Zur Regulierung der Stösse in reibungslosen Punktsystemen, die dem Zwange von Bedingungsungleichungen unterliegen," Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig **51** (1899), 245-264.

On the regularization of collisions in frictionless point-systems that are constrained by condition inequalities

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In recent times, the important problem of calculating the velocities that exist in a frictionless system of material points when one or more points suffer the same simultaneous collision has been treated only under the assumption that the system was subject to nothing but *condition equations*, and that the *duration* of the time during which those equations were valid was likewise known. One can also treat the problem in independent determining parts of the system, which is what APPELL did very beautifully and clearly in connection with some investigations of NIVEN and ROUTH in the note "Sur l'emploi des équations de LAGRANGE dans la théorie du choc et des percussions" ([1]). By contrast, to my knowledge, no one since OSTRAGRADSKY's "Mémoire sur la théorie générale de la percussion" ([2]) has further addressed the very interesting case in which the system is subject to the constraint of *condition inequalities*, and one does not know at all from the outset whether the system conditions that are fulfilled as *equations* at the moment of collision do or do not continue in the same form after the collision, and as significant as the results included in that treatise were, that still leaves the peculiar fact that the fundamental question of the duration of those equations still remains entirely untouched. Therefore, the following attempt to solve the problem for a point-system that is constrained by *condition inequalities* seems entirely justified. The solution is based upon the same conclusions that first led STUDY to the correct explanation and which also led to the presentation of the differential equations of motion for point systems of the kind considered ([3]), and indeed the argument is, in principle, actually simpler than the latter, but questions will arise in the present problem (cf., § **2**) that did not present themselves at all in the earlier work and which therefore also require a new handling.

([1]) Journal de Mathématiques (1896), 5-20. Cf., also APPELL, *Traité de mécanique rationelle*, t. II, 500-503.

([2]) Mémoires de l'Académie Impér. de Sci. de Saint-Petersbourg, sixth series, Sci. math. et phys., t. VI, 269-303.

([3]) Cf., the foregoing article (in this Journal).

§ 1. – Given external impulsive forces.

A system of n material points with masses m_1, m_2, …, m_n is in motion under the action of given forces. Let x_i , y_i , z_i be the coordinates of the point m_i at time t when referred to fixed rectangular axes, and let:

(1) $$f_1 \leq 0, \; f_2 \leq 0, \; \ldots$$

be the analytical expresses for the constraints and restrictions on the system. The left-hand sides of the conditions (1) are then single-valued functions of the coordinates of the system points and possibly time t, as well. I assume that those functions, along with their first and second partial differential quotients, are continuous at all times and at all positions of the system that come under consideration.

Although the system has moved completely unperturbed up to now, at the moment t, one or more points of it might suddenly be subjected to *impacts*, and therefore to very strong forces whose duration is, however, only exceptionally short, such that one can ignore them in all of the changes in position of the system that result from them, or one can neglect the *duration* of the impacts.

Let the positions of the points at the beginning of the impact be known, and let the velocities α_i, β_i, γ_i that each system point m_i has attained at the end of the impact when it was free at the beginning of it be given.

As a result of the constraints and restriction on the system, those impact velocities α_i, β_i, γ_i do not actually come about, but must be regularized in such a way that they satisfy the conditions on the system. When one neglects not only the duration of the impact, but also any *friction* that might develop due to, say, the restrictions on the system, one will be dealing with *the calculation of the velocities that have actually been produced in the system at the end of the impact*, and indeed that problem shall be solved here on the basis of *Gauss's principle of least constraint* because it leads to the solution in the clearest and most natural way, in my opinion.

Since the duration of the impact is to be neglected, the point m_i will possess the same coordinates x_i , y_i , z_i immediately after the impact that it had at the moment t. If it were free then it would arrive at a position B_i whose coordinates are:

$$x_i + \alpha_i \, dt, \qquad y_i + \beta_i \, dt, \qquad z_i + \gamma_i \, dt$$

after the subsequent infinitely small time interval dt . However, in reality, the point does not have the velocities α_i, β_i, γ_i at the end of the impact, but rather it has arrived at the *still-unknown velocities* x_i', y_i', z_i'. Therefore, at the moment $t + dt$, it does not reach the position B_i , but another position C_i whose coordinates are:

$$x_i + x_i' \, dt, \qquad y_i + y_i' \, dt, \qquad z_i + z_i' \, dt \, .$$

Now, from the principle of least constraint, among all of the positions C_i to which the points m_i can arrive during the time interval considered without violating the conditions on the system, their actual locations will be distinguished by the fact that:

$$\sum_{i=1}^{n} m_i \overline{C_i B_i}^2$$

must be a minimum for them, and with the coordinates of the points B_i and C_i, that requirement will reduce to this one:

$$\sum_{i=1}^{n} m_i \left\{(\alpha_i - x_i')^2 + (\beta_i - y_i')^2 + (\gamma_i - z_i')^2\right\} = \text{min.}\;; \tag{2}$$

i.e., the sum on the left must be smaller for the true final velocities x_i', y_i', z_i' than it is for all velocities that the points of the system can assume at the end of the impact, and therefore at the moment t, since one has neglected the duration of the impact ([1]).

Therefore, in order to ascertain the true regularized impact velocities x_i', y_i', z_i', first and foremost, it will important to find *those restrictions* that the system conditions impose upon the velocities of its points at that moment.

Now, by assumption, the system might exhibit only those motions for which the coordinates of its points will continually satisfy the conditions (1).

Any one of those conditions might be represented by:

$$f \leq 0.$$

Corresponding to any such possible motion, if one considers the coordinates x_i, y_i, z_i of each system point m_i to be continuous functions of time t, lets t go to $t + dt$, and develops the condition considered in powers of dt then it will go to:

$$f + f'\,dt + r\,dt^2 \leq 0, \tag{a}$$

where f' is the complete differential quotient of the function f with respect to time t, and $r\,dt^2$ denotes the remainder term in the TAYLOR development.

The value that the first term f possesses at the moment t is known, since it contains only time and the coordinates, and therefore quantities values are known at that moment, by assumption. By contrast, the first differential quotients of the coordinates – or the velocities x_i', y_i', z_i' of the system points – also enter into f'.

Furthermore, as a result of the system condition $f \leq 0$, the known value of f can only be < 0 or $= 0$. Therefore, if it is not precisely $= 0$ then for a sufficiently small dt, the condition (a) will already be fulfilled by itself and will not restrict the velocities in any way momentarily.

By contrast, if $f = 0$ then one can divide the condition (a) by the positive quantity dt and thus reduce it to:

$$f' + r\,dt \leq 0\,.$$

([1]) Or, in words: *If a system of material points is suddenly subjected to impacts that, in conjunction with the velocities that the individual points have attained, make them strive to attain given velocities, then those velocities can be regularized in such a way that the vis viva will be a minimum as a result of the conditions and restrictions on the system's lost velocities.*

However, that condition can be fulfilled for arbitrarily small dt only when one already has:

$$f' \leq 0 .$$

Of the system conditions (1), only the ones that exist as *equations* precisely will restrict the velocities of the system points at the moment t. Let them be the r *conditions:*

$$f_1 \leq 0, \quad f_2 \leq 0, \quad \ldots, \quad f_r \leq 0.$$

I then assume that the known position of the system at the moment t corresponds to the r equations:

$$f_1 = 0, \quad f_2 = 0, \quad \ldots, \quad f_r = 0,$$

while perhaps all of the remaining system conditions (1) exist momentarily with only the upper sign.

The system points m_i at that moment are then allowed to have all velocities x_i', y_i', z_i' that the conditions:

$$(3) \qquad f_1' \leq 0, \quad f_2' \leq 0, \quad \ldots, \quad f_r' \leq 0$$

will tolerate. If we now once more restrict the notations x_i', y_i', z_i' to the desired true velocities of the system points m_i at the end of the impact then we will, on the other hand, understand:

$$x_i' + \delta x_i', \qquad y_i' + \delta y_i', \qquad z_i' + \delta z_i'$$

to mean any other velocities of the system points that are possible at the same time and likewise deviate only slightly from the unknown true velocities, and on the grounds of the identities:

$$\frac{\partial f'}{\partial x_i'} \equiv \frac{\partial f}{\partial x_i}, \ldots,$$

if one introduces the abbreviation:

$$(4) \qquad \delta f' \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} x_i' + \frac{\partial f}{\partial y_i} y_i' + \frac{\partial f}{\partial z_i} z_i' \right),$$

then the conditions (3) will imply the following r conditions on the variations $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, of the velocities:

$$f_1' + \delta f_1' \leq 0, \quad f_2' + \delta f_2' \leq 0, \quad \ldots, \quad f_r' + \delta f_r' \leq 0,$$

and from (2), one must have:

$$(5) \qquad \sum_{i=1}^{n} m_i \{(\alpha_i - x_i')\, \delta x_i' + (\beta_i - y_i')\, \delta y_i' + (\gamma_i - z_i')\, \delta z_i'\} \leq 0$$

for all sufficiently small values of those variations fulfill those r conditions.

However, if:

$$f' \neq 0, \text{ but only} < 0$$

for the unknown true final velocities $x'_i, \; y'_i, \; z'_i$ then the condition:

$$f' + \delta f' \leq 0$$

will not restrict the velocities in any way, since it is then fulfilled by itself for all arbitrary, but sufficiently small, values of the variations.

Such a restriction will first come into play, moreover, when the actual final velocities satisfy the equation:

$$f' = 0,$$

which will reduce our condition to:

$$\delta f' \leq 0 .$$

However, the true final velocities are still yet-to-be-found. For the time being, they are still completely unknown, and we can then by no means decide *directly* which of the derivatives f' are = 0 for the true velocities at the moment considered t and which of them are < 0.

Hence, nothing else remains but to attempt to solve the problem by an *indirect* path, and indeed we can obviously proceed only as follows: We first assume, in a purely arbitrary way, that the unknown regularized impact velocities fulfill some of the conditions (3) as equations, and the others as only inequalities, determine the values of our unknowns from that arbitrary assumption, and thereafter look around for criteria that will show us whether the calculated values of the velocities will or will not be correct.

We then assume now that the unknown true final velocities $x'_i, \; y'_i, \; z'_i$ satisfy, say, the ρ equations:

$$(6) \qquad f'_\lambda \equiv \frac{\partial f_\lambda}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f_\lambda}{\partial x_i} x'_i + \frac{\partial f_\lambda}{\partial y_i} y'_i + \frac{\partial f_\lambda}{\partial z_i} z'_i \right) = 0 \qquad (\lambda = 1, 2, \ldots, \rho)$$

and the $r - \rho$ inequalities:

$$(6') \qquad f'_\lambda \equiv \frac{\partial f_\lambda}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f_\lambda}{\partial x_i} x'_i + \frac{\partial f_\lambda}{\partial y_i} y'_i + \frac{\partial f_\lambda}{\partial z_i} z'_i \right) < 0 \qquad (\mu = \rho + 1, 2, \ldots, r),$$

in which ρ can be any one of the numbers 0, 1, …, r, and then ask *what values of the desired velocities does that assumption imply?*

I shall make that more specific with the further assumption that none of the ρ equations (6) should be a mere consequence of the remaining ones in the known momentary position of the system. Otherwise, they would indeed contribute nothing to the determination of the unknowns $x'_i, \; y'_i, \; z'_i$, and could therefore be simply dropped. I shall then assume *that the* ρ *equations* (6) *determine* ρ *of the unknowns* x_i', y_i', z_i' *in terms of the* $3n - \rho$ *remaining ones.*

With those assumptions, the variations of the velocities are momentarily subject to only the ρ conditions:

$$\delta f_\lambda' \equiv \sum_{i=1}^{n} \left(\frac{\partial f_\lambda}{\partial x_i} \delta x_i' + \frac{\partial f_\lambda}{\partial y_i} \delta y_i' + \frac{\partial f_\lambda}{\partial z_i} \delta z_i' \right) \leq 0 \qquad (\lambda = 1, 2, \ldots, \rho), \tag{7}$$

and the desired velocities must then satisfy the requirement (5) for all values of their variations that are compatible with those ρ conditions.

In particular, one must then have:

$$\sum_{i=1}^{n} m_i \{(\alpha_i - x_i')\,\delta x_i' + (\beta_i - y_i')\,\delta y_i' + (\gamma_i - z_i')\,\delta z_i'\} = 0$$

as long as one subjects the $3n$ variations $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ to the ρ equations:

$$\delta f_\lambda' = 0 \qquad (\lambda = 1, 2, \ldots, \rho), \tag{7'}$$

and from the assumption that was introduced in regard to equations (6), those equations will determine ρ of the variations as functions of the remaining $3n - \rho$.

If one now multiplies the latter equations by the temporarily undetermined factors $-l_\lambda$ and then adds them to the previous equation then one will get the equation:

$$\sum_{i=1}^{n} m_i \{(\alpha_i - x_i')\,\delta x_i' + (\beta_i - y_i')\,\delta y_i' + (\gamma_i - z_i')\,\delta z_i'\} = \sum_{\lambda=1}^{\rho} l_\lambda\, \delta f_\lambda' . \tag{8}$$

However, one can determine the multipliers $l_1, l_2, \ldots, l_r$ in such a way that the coefficients of those of the ρ variations $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ that can be expressed in terms of the remaining ones by the ρ equations (7) will be equal to each other on the left and right. After dropping those equal terms, equation (8) will contain only entirely arbitrary variations. Therefore, the coefficients of the $3n - \rho$ variations that remain on both sides of it must be equal to each other. In that way, one will arrive at the $3n$ equations:

$$\left\{ \begin{aligned} m_i(\alpha_i - x_i') &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial x_i}, \\ m_i(\beta_i - y_i') &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial y_i}, \\ m_i(\gamma_i - z_i') &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial z_i}, \end{aligned} \right. \qquad (i = 1, 2, \ldots, n) \tag{9}$$

However, one still has the ρ equations (6) themselves, moreover, and it is easy to see that under our assumption, the determinant of the $3n + \rho$ linear equations (6) and (9) in the $3n + \rho$ unknowns:

$$x_i', \quad y_i', \quad z_i', \quad l_\lambda$$

cannot be zero.

Namely, if it were = 0 then one would be able to satisfy the $\rho + 3n$ homogeneous linear equations (7′) and:

$$(9')\qquad \begin{cases} m_i\,\delta x_i' + \sum_{\lambda=1}^{\rho} l_\lambda \dfrac{\partial f_\lambda}{\partial x_i} = 0, \\ m_i\,\delta y_i' + \sum_{\lambda=1}^{\rho} l_\lambda \dfrac{\partial f_\lambda}{\partial y_i} = 0, \\ m_i\,\delta z_i' + \sum_{\lambda=1}^{\rho} l_\lambda \dfrac{\partial f_\lambda}{\partial z_i} = 0 \end{cases}$$

with values of the $3n + \rho$ variables $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, l_λ that do not all vanish.

However, from (7′), it follows from equations (9′) upon multiplying by $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, and adding that:

$$\sum_{i=1}^{n} m_i(\delta x_i'^2 + \delta y_i'^2 + \delta z_i'^2) = 0.$$

Equations (7′) and (9′) then require that all $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ will be = 0, and as a result of our assumption in regard to equations (6) to (9′), that will also imply the vanishing of all multipliers l_λ.

The determinant of equations (6) to (9) is then, in fact, $\neq 0$, and those equations will then determine their $3n + \rho$ unknowns x_i', y_i', z_i', l_λ uniquely in terms of the given impact velocities α_i, β_i, γ_i, the coordinates x_i, y_i, z_i, and possibly the time t, so in terms of nothing but quantities whose values will be known completely at the moment t.

Our assumption has then yielded a single completely-determined system of values for those velocities x_i', y_i', z_i' *that the points of the system can attain at the end of the impact.*

However, that assumption cannot by any means be established *a priori*. *Thus, whether or not the values of the velocities that are obtained from it are also the true regularized impact velocities is still entirely questionable.*

Meanwhile, our assumption itself, as well as the principle of least constraint, contains *more conditions* than the ones that are fulfilled already.

Namely, of the r conditions (3) that restrict the velocities of the system points at the moment t, up to now, we have only satisfied the first ρ, and indeed satisfied them by means of equations (6). Hence, if our assumption were correct then, above all, the last $r - \rho$ conditions (3) would have to be fulfilled by the values of the x_i', y_i', z_i' thus-obtained by themselves; i.e., *of the uniquely determined values that the expressions* (6′) *take on by substituting the solutions of equations* (6) *and* (9), *none of them can be* > 0. Therefore, if one of those values were found to be > 0 then our assumption would be false, and the calculated values of the velocities would not be their true values.

Moreover, the formulas (9) convert equation (8) into an identity and then reduce our original demand (5) to:

$$\sum_{\lambda=1}^{\rho} l_\lambda \, \delta f'_\lambda \leq 0.$$

That condition must also be fulfilled then for all variations $\delta x'_i$, $\delta y'_i$, $\delta z'_i$ that fulfill the ρ conditions (7), and that is identical to the condition that *none of the multipliers can satisfy:*

$$l_\lambda < 0.$$

The principle of least constraint then adds the ρ conditions:

$$l_\lambda > 0 \qquad (\lambda = 1, 2, \ldots, \rho) \tag{10}$$

to our equations (6) and (9), in which the > sign should not exclude equality.

Thus, whenever the solution to equations (6) and (9) for the unknowns x'_i, y'_i, z'_i, l_λ yields a *negative* value for some l_λ, our assumption will not, in turn, correspond to reality, and once more the calculated values of the velocities cannot be the correct ones.

Since those two criteria have only a *negative* nature, they will generally tell us nothing immediately except that *conversely, whenever the solution of equations* (6) *and* (9) *does not contain a negative l nor do any of the derivatives* (6′) *provide a positive value, that solution will also certainly represent the true velocities of the system points at the end of the impact.* In order to prove that, strictly speaking, one would first have to show that *one could not also arrive at other systems of values for the* x'_i, y'_i, z'_i *that likewise satisfy all of those conditions, as well as all of the demands of the principle of least constraint with any other decomposition of the conditions* (3). However, if one ignores the fact that one can actually carry out this proof only in the two simplest cases $r = 1$ and $r = 2$ ([1]), and very appreciable complications seem to present themselves for larger values of r, then one might, on the other hand, probably regard it as obvious for that reason that two different systems of velocities with the required behavior cannot exist, because if they did exist then there would be no means of deciding which of the two is the correct one. Namely, for the sake of dynamics, it is only essential that the demand (5) should be fulfilled, but otherwise it is entirely irrelevant whether the sum (2) is an actual minimum or not, and in addition, whether the value that the sum takes on by means of equations (6) and (9) is also, in fact, the smallest of all of the values that it might assume under the conditions (3) and the assumptions (6), as long as it does not imply that some $l < 0$ and some $f'_\mu > 0$.

In all cases, one can state with certainty that our method must surely yield the true velocities of the systems points at the end of the impact in all examples in which it leads to only a single solution. However, one likewise sees that for larger values of r, very

([1]) See pp. 237 of this volume.

many detailed investigations might be necessary until one arrives at those assumptions (6) that ultimately fulfill all requirements.

In the foregoing, we have sought the *total final velocities* that the system points have attained at the end of the impacts. However, instead of that, one might also wish to know the *changes in velocity* that the points of the system will suffer during the impacts.

If one lets u_i, v_i, w_i *denote the velocities that the system points* m_i *possess immediately before the impacts, and lets* Δu_i, Δv_i, Δw_i *denote changes in velocity that the impacts actually confer to those points* then one will have:

$$\Delta u_i = x_i' - u_i\,, \quad \Delta v_i = y_i' - v_i\,, \quad \Delta w_i = y_i' - w_i\,, \tag{11}$$

when one denotes the regularized impact velocities by x_i', y_i', z_i', as always. On the other hand, the velocities α_i, β_i, γ_i that the points m_i would have attained at the end of the impacts if they were free at the beginning of it will be the resultants of the velocities u_i, v_i, w_i and the velocities a_i, b_i, c_i *that the impacts would impart to the free points* m_i *when starting from rest.* One will then have:

$$\alpha_i = u_i + a_i\,, \quad \beta_i = v_i + b_i\,, \quad \gamma_i = w_i + c_i\,, \tag{12}$$

such that in order to know the values of the α_i, β_i, γ_i, the individual values of the u_i, v_i, w_i and the a_i, b_i, c_i must be given.

Now, it follows from (11) and (12) that:

$$\alpha_i - x_i' = a_i - \Delta u_i\,, \quad \ldots$$

When one generally sets:

$$F_\nu \equiv \frac{\partial f_\nu}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f_\nu}{\partial x_i} u_i + \frac{\partial f_\nu}{\partial y_i} v_i + \frac{\partial f_\nu}{\partial z_i} w_i\right), \tag{13}$$

from (6) and (9), one will then have the following equations for the calculation of the changes in velocity Δu_i, Δv_i, Δw_i :

$$f_\lambda' \equiv \sum_{i=1}^{n}\left(\frac{\partial f_\nu}{\partial x_i}\Delta u_i + \frac{\partial f_\nu}{\partial y_i}\Delta v_i + \frac{\partial f_\nu}{\partial z_i}\Delta w_i\right) + F_\lambda = 0 \qquad (\lambda = 1, 2, \ldots, \rho) \tag{14}$$

$$\left\{\begin{aligned} m_i(a_i - \Delta u_i) &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial x_i}, \\ m_i(b_i - \Delta v_i) &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial y_i}, \\ m_i(c_i - \Delta w_i) &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial z_i}. \end{aligned}\right. \qquad (i = 1, 2, \ldots, n) \tag{15}$$

One will then get the desired changes in velocity when one solves the $\rho + 3n$ equations (14) and (15) for the $3n + \rho$ unknowns:

$$\Delta u_i\,,\ \Delta v_i\,,\ \Delta w_i\,,\ l_\lambda\,,$$

and in order for those solutions to represent the correct changes in velocity, none of the l can be < 0, and when they are substituted, no of the $r - \rho$ expressions:

$$f'_\mu \equiv \sum_{i=1}^{n}\left(\frac{\partial f_\mu}{\partial x_i}\Delta u_i + \frac{\partial f_\mu}{\partial y_i}\Delta v_i + \frac{\partial f_\mu}{\partial z_i}\Delta w_i\right) + F_\lambda \qquad (\mu = \rho + 1, \ldots, r) \tag{14'}$$

will be > 0.

In the foregoing, it was assumed that the point-system considered was subject to *condition inequalities* exclusively. If *condition equations:*

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \ldots$$

that its points should satisfy at every time are also prescribed, moreover, then it should be obvious that all that will change is that along with equations (6) and the equations:

$$\varphi'_1 = 0, \qquad \varphi'_2 = 0, \ldots,$$

and therefore equations (14), as well, one must add the ones that arise by introducing the Δu_i, Δv_i, Δw_i in place of the x'_i, y'_i, z'_i in the ones that were just written down. In that way, terms of the form:

$$p_1\frac{\partial\varphi_1}{\partial x_i} + p_2\frac{\partial\varphi_2}{\partial x_i} + \ldots,$$
$$p_1\frac{\partial\varphi_1}{\partial y_i} + p_2\frac{\partial\varphi_2}{\partial y_i} + \ldots,$$
$$p_1\frac{\partial\varphi_1}{\partial z_i} + p_2\frac{\partial\varphi_2}{\partial z_i} + \ldots$$

will appear on the right-hand sides of equations (9) and (15), but whose multipliers p_1, p_2, … are no longer subordinate to the conditions $p > 0$, but might be positive, as well as negative.

§ 2. – Internal collisions.

Up to now, we have always tacitly assumed that the collisions were produced by actual *external impacts*. In that way of looking at things, in the absence of impacts, the system would continue to move unperturbed, and the velocities u_i, v_i, w_i of the system points would already be compatible with the momentary system conditions (3) immediately before the impact. Due to the condition $f_\nu \leq 0$ and the assumptions that $f_\nu = 0$ at the moment t, from (13), one will also have $F_\nu \leq 0$ in each case then, and indeed one will have $F_\nu < 0$ when the constraint or restriction $f_\nu = 0$ proves to be rigorously true, but $= 0$ when it also persists for unperturbed motion.

However, the *sudden changes in velocity in the system also* arise *without any special impacts* in such a way that once the system has moved for a long enough time that one or more of its conditions $f_\lambda \leq 0$ consists of only an inequality, at the moment t, it will arrive at a position in which those conditions will be fulfilled as equalities. In that way, the functions f_λ in question will increase, so their complete differential quotients must also satisfy $f'_\lambda \geq 0$ at the moment t, and the new position can only be attained only with velocities u_i, v_i, w_i for the system points for which the associated $F_\lambda \geq 0$.

Therefore, whenever one of the expressions $F_\lambda > 0$, *an impact will occur in the system*, and the possibility of such impacts, which are most simply illustrated by loose connecting threads that are suddenly tensed violently or the impinging of system points on rigid surfaces, shows quite clearly that one cannot attach the constraints and restrictions on the system to any *insurmountable obstacle* without contradicting the continuity of the changes in velocity beyond resolution.

Namely, whenever $f_\lambda = 0$, the system condition $f_\lambda \leq 0$ will forbid any increase in the function f_λ, and will also demand that $f'_\lambda \leq 0$ then. However, if $F_\lambda > 0$ then the moment at which the equation $f_\lambda = 0$ is established will be when the value of f'_λ just becomes > 0. Now, the velocities of the system points cannot unexpectedly jump from values for which f'_λ possesses a finite positive value to values that will make $f'_\lambda \leq 0$. It must then be necessary (if also only minimally and during an exceptionally short time) to overcome the obstacle of the condition $f_\lambda \leq 0$ and for f_λ to increase until the exceptionally turbulent, but also continuously varying, velocities have been regularized in such a way that the equation $f'_\lambda = 0$ has been established, *at which point, the impact considered has reached its conclusion* ([1]).

In that argument, it should not at all be said that the equation $f_\lambda = 0$ must now necessarily persist after the impact. Rather, as the second example in the following § will show, in some situations, that equation might be in force only during the impact itself, but once more cease to apply afterwards, just as it did beforehand.

For the sake of brevity, I would like to call the impacts that arise within the system itself *internal collisions*, in contrast to the ones that are produced by actual impulsive forces, although naturally they can arise from sudden tensions in connecting threads, as

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([1]) OSTRAGRADSKY, pp. 287. Naturally, the process does not play out in the same way when the system constraint or restriction $f_\lambda \leq 0$ is *elastic*. The turbulent changes in velocity will then endure beyond the moment at which the equation $f'_\lambda = 0$ has been established, and at the end of the impact one will already have $f'_\lambda < 0$. In that case, one will then no longer have any right to subject the desired velocities to the equation $f'_\lambda = 0$, and one will then also no longer have the sufficient number of equations for determining those unknowns, but one must (as in the collision of moving bodies) appeal to the *theory of elasticity* for the missing equations. However, in that way, the entire problem will lose its purely *mechanical* character, and will become a *physical* problem, and in a certain sense, the same thing will also be true whenever any sort of *friction* should be considered.

well as from collisions with external obstacles. Indeed, the latter must also be included among the system conditions as long as the system falls within their scope.

What are the values of the velocities α_i, β_i, γ_i that such internal collisions would impart to the points m_i that are already equipped with the velocities u_i, v_i, w_i when they were free immediately before them and obviously also remain free afterward? Now, by assumption, the impacts now originate from only the fact that one or more constraints (restrictions, resp.) $f_\lambda = 0$ suddenly come about that did not exist before. Hence, if the points were free and remained free then they would experience no impacts at all, and the impact velocities α_i, β_i, γ_i would then reduce to simply the velocities u_i, v_i, w_i that the points possessed before the equations $f_\lambda = 0$ became valid in the system. From (12), that coincidence of the α_i, β_i, γ_i with the u_i, v_i, w_i will simultaneously show immediately that the velocities a_i, b_i, c_i will all have the value zero, moreover.

Since the values of the α_i, β_i, γ_i, as well as the a_i, b_i, c_i, are also known for internal collisions then, we can also seek to determine the final velocities or changes in velocity from the previous type that are produced by such collisions in the system. To that end, among equations (6), we must now absorb those of the equations $f'_\lambda = 0$ that correspond to the equations $f_\lambda = 0$ that are established at the moment t when F_λ has positive values, and once more construct equations (9) or (15) with the equations (6) that are obtained in that way, and set (1):

$$(16) \qquad \left\{ \begin{array}{lll} \alpha_i = u_i, & \beta_i = v_i, & \gamma_i = w_i \quad \text{or} \\ a_i = 0, & b_i = 0, & c_i = 0 \end{array} \right.$$

in them everywhere. Moreover, the further conditions that were cited in § **1** must be fulfilled if the velocities or changes in velocity that are calculated in that way are to be the correct ones.

However, if sudden changes of velocity in the system are produced, in particular, in such a way that one imposes entirely new condition equations $f_\lambda = 0$ at the moment t, such as, e.g., when system points are suddenly obliged (or rather seized and forced) to move in a given way, then the conditions $l_\lambda > 0$ will, in turn, drop away for the corresponding multipliers l_λ (2).

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(1) For internal collisions, from (16), the general formula (2) will go to:

$$\sum_{i=1}^{n} m_i \left\{ (x'_i - u_i)^2 + (y'_i - v_i)^2 + (z'_i - w_i)^2 \right\} = \min.,$$

and one will then get the theorem:

Any sudden appearance of new constraints or restrictions in a system of material points will change the velocities of the points in such a way that the vis viva of the changes in velocities that come about will be a minimum.

See pp. 216 of this volume.

(2) Should impacts from outside the system points also be simultaneously exerted at the moment of the internal collisions, then one would obviously preserve the a_i, b_i, c_i in (15) and give the α_i, β_i, γ_i the values (12), in which, as before, one understands the a_i, b_i, c_i to mean the velocities that the external impulsive forces would impart upon the free points m_i from a state of rest.

§ 3. – Examples.

In order to explain the method that was deduced, allow me to pursue it in two simple examples.

I. *Two material points m_1 and m_2 are linked by an inextensible string, and when the string is tensed, they move from the side $f_1 < 0$ of a fixed surface:*

$$f_1\ (x, y, z) = 0$$

until the point m_1 impinges upon it at the moment t. What changes in velocity will the two points experience then?

If we call the length of the connecting thread L then we will have the following equations here:

$$f_1\ (x_1, y_1, z_1) = 0,$$

$$f_2 \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - L^2 = 0,$$

$$F_2 \equiv (x_1 - x_2)(u_1 - u_2) + (y_1 - y_2)(v_1 - v_2) + (z_1 - z_2)(w_1 - w_2) = 0,$$

while preserving the previous notations; at the same time:

$$f_1\ (x_2, y_2, z_2) \neq 0,$$

$$F_1 \equiv \frac{\partial f_1}{\partial x_1} u_1 + \frac{\partial f_1}{\partial y_1} v_1 + \frac{\partial f_1}{\partial z_1} w_1 > 0,$$

while the system conditions themselves are:

$$f_1\ (x_1, y_1, z_1) \leq 0, \qquad f_2 \leq 0\ .$$

If we next assume that *the string still remains tensed after the collision* then we will get the following eight equations for the determination of the changes in velocity of the two points from (14), (15), and (16):

$$f_1' \equiv \frac{\partial f_1}{\partial x_1} \Delta u_1 + \frac{\partial f_1}{\partial y_1} \Delta v_1 + \frac{\partial f_1}{\partial z_1} \Delta w_1 + F_1 = 0,$$

$$f_2' \equiv (x_1 - x_2)(\Delta u_1 - \Delta u_2) + (y_1 - y_2)(\Delta v_1 - \Delta v_2) + (z_1 - z_2)(\Delta w_1 - \Delta w_2) = 0,$$

$$m_1\ \Delta u_1 + l_1 \frac{\partial f_1}{\partial x_1} + l_2\ (x_1 - x_2) = 0,\ \ldots$$

$$m_2\,\Delta u_2 - l_2\,(x_1 - x_2) = 0,\ \ldots,$$

and when we substitute the values of the Δu, Δv, Δw from the last six equations into the first two and employ the abbreviations:

$$\text{(A)}\qquad \begin{cases} B_1 \equiv +\sqrt{\left(\dfrac{\partial f_1}{\partial x_1}\right)^2 + \left(\dfrac{\partial f_1}{\partial y_1}\right)^2 + \left(\dfrac{\partial f_1}{\partial z_1}\right)^2}, \\ B_{12} \equiv (x_1 - x_2)\dfrac{\partial f_1}{\partial x_1} + (y_1 - y_2)\dfrac{\partial f_1}{\partial y_1} + (z_1 - z_2)\dfrac{\partial f_1}{\partial z_1}, \end{cases}$$

they will come down to these two:

$$\text{(B)}\qquad \begin{cases} B_1^2\, l_1 + B_{12}\, l_2 = m_1\, F_1, \\ m_2\, B_{12}\, l_1 + (m_1 + m_2) L^2\, l_2 = 0. \end{cases}$$

However, if one lets λ_1, μ_1, ν_1 denote the direction cosines of the normal to the surface $f_1 = 0$ that points upwards from the side $f_1 > 0$ at the point x_1, y_1, z_1, and lets φ_1 denote the angle that the normal makes with the line $\overrightarrow{m_2\, m_1}$ then one will have:

$$\text{(C)}\qquad \frac{1}{B_1}\frac{\partial f_1}{\partial x_1} = \lambda_1, \quad \frac{1}{B_1}\frac{\partial f_1}{\partial y_1} = \mu_1, \quad \frac{1}{B_1}\frac{\partial f_1}{\partial z_1} = \nu_1,$$

and as a result:

$$B_{12} = L\, B_1 \cos \varphi_1 .$$

The two equations (B) then show that l_1 and l_2 cannot both be > 0 whenever φ_1 is an acute angle. On the other hand, the line $\overrightarrow{m_2\, m_1}$ meets the surface $f_1 = 0$ on the side $f_1 < 0$. If we ignore the latter case then equations (B) will require that one must have $l < 0$, in any event, and that will prove that *the string cannot remain tensed after the collision* (1).

The actual changes in velocity cannot satisfy the equation $f_2' = 0$ then. We must drop that equation, and then set $l_2 = 0$, so we will get only the equations:

$$f_1' = 0,$$

$$\text{(D)}\qquad \begin{cases} m_1\, \Delta u_1 + l_1 \dfrac{\partial f_1}{\partial x_1} = 0, & \cdots \\ m_2\, \Delta u_2 \qquad\qquad = 0, & \cdots \end{cases}$$

for the true values of the Δu, Δv, Δw . In fact, from (A), they imply that:

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(1) Moreover, it will remain tensed only in the case $\varphi_1 = \pi / 2$ or $B_{12} = 0$, where equations (B) will imply that $l_1 > 0$, $l_2 = 0$, as well as in the case $F_1 = 0$, in which $l_1 = l_2 = 0$ and there will be no collision at all.

$$B_1^2\, l_1 = m_1 F_1 > 0\,,$$

from which, (C) will imply that:

$$\Delta u_1 = -\frac{F_1}{B_1}\lambda_1\,, \qquad \Delta v_1 = -\frac{F_1}{B_1}\mu_1\,, \qquad \Delta w_1 = -\frac{F_1}{B_1}\nu_1\,,$$

and due to the fact that $\Delta u_2 = \Delta v_2 = \Delta w_2 = 0$, one will have:

$$f_2' = -\frac{L\,F_1}{B_1}\cos\varphi_1 < 0$$

with those values, so they fulfill all of the requirements that the true changes in velocity are subject to and are, at the same time, the unique solutions to our problem.

II. *One finds a rigid circular ring in a state of forced rectilinear translational motion on the fixed horizontal xy-plane, such that its center advances with constant positive velocity ϖ along the x-axis. At the moment t, the ring collides with a point of mass m that rests upon the xy-plane. What velocity does the point attain by the collision with the ring when the plane and ring are assumed to be completely smooth?*

The condition equation:

$$\text{(a)} \qquad 2f(x, y, z) \equiv (x - \varpi\, t)^2 + y^2 - r^2 = 0$$

between the coordinates x, y of the point and the time t is suddenly established at the moment t, but $2f \leq 0$ or $-2f \leq 0$ according to whether the ring meets the point *on the inside* or *on the outside*, resp.

In order to calculate the initial velocities x', y' that the collision of the ring and the point brings about, since the point was at rest beforehand, we will then get the following equations from (9) and (16):

$$f' \equiv (x - \varpi\, t)\,(x' - \varpi) + y\, y' = 0, \qquad -m\, x' = l\,(x - \varpi\, t), \qquad -m\, y' = l\, y,$$

in the *first* case, and when we consider (a), they will imply that:

$$l\, r^2 = -m\,\varpi\,(x - \varpi\, t)\,.$$

Therefore (in agreement with the impact condition $F_\lambda > 0$ in § **2**, which reduces to $\partial f / \partial t > 0$ here), one must have:

$$x - \varpi\, t < 0$$

at the moment t of the collision, which should be obvious *a priori*, since otherwise no collision would even happen. If equation (a) is fulfilled by the substitutions:

$$\text{(b)} \qquad x - \varpi\, t = r\cos\varphi, \qquad y = r\sin\varphi$$

then one will get the following values for the initial velocities of m :

(c) $$x' = \varpi \cos^2 \varphi, \qquad y' = \varpi \cos \varphi \sin \varphi .$$

By contrast, in the *second* case, as a result of the condition $-2f \leq 0$, one will have:

$$m\,x' = l\,(x - \varpi\,t), \qquad m\,y' = l\,y,$$

and one will get:

$$l\,r^2 = +\,m\;\varpi\,(x - \varpi\,t) .$$

Hence, the condition $l > 0$ now demands that:

$$x - \varpi\,t > 0,$$

which should be, in turn, obvious, and one again comes back to the same initial velocities (c).

By contrast, the further motion is entirely different in both cases.

Namely, if the point remains on the advancing circular ring then its motion will obey the differential equations:

$$m\,x'' = \lambda\,(x - \varpi\,t), \qquad -\,m\,y'' = \lambda\,y,$$

and therefore, one must have either $\lambda > 0$ or $\lambda < 0$ according to whether the motion proceeds on the *inner* or *outer* side of the ring, resp. ([1]) However, in conjunction with the equation:

$$(x - \varpi\,t)\,x'' + y\,y'' + (x - \varpi\,t)^2 + y'^2 = 0,$$

which follows from (a), that will yield the differential equations:

$$\lambda\,r^2 = m\{(x - \varpi\,t)^2 + y'^2\}.$$

Therefore, λ can never be negative and can vanish only for $x' = \varpi$, $y' = 0$. Thus, whereas the point will remain on the ring as long it meets the ring from the *inside*, by contrast, it will always collide with the ring when it meets it from the *outside*, except when $\varphi = 0$; i.e., when the point is found along the x-axis itself. In the former case, the point will traverse the inner circular ring with the constant angular velocity:

$$\varphi' = \frac{\varpi \sin \varphi}{r},$$

while in the latter case, it will separate from the ring with the constant velocity:

$$V = \varpi \cos \varphi$$

([1]) Cf., pp. 236 of the volume.

in the direction of the line that goes through the point and the center of the circle at the moment of impact without being overtaken by the ring again. Only when the point lies on the x-axis itself will the same motion occur in both cases, namely, the point will be at rest relative to the ring.