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On the most-general expression for the internal force potential of a system of moving material points (*).

A. Mayer

Translated by D. H. Delphenich

If one sets:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i\,(x_i'^2 + y_i'^2 + z_i'^2) \ ,$$

and W is equal to an arbitrary function of t, the $3n$ unknown functions x_i, y_i, z_i of t and their first differential quotients x_i', y_i', z_i' then the problem:

$$\delta \int_{t_0}^{t_1} (T + W)\, dt = 0$$

will lead to the $3n$ second-order differential equations:

$$m_i\, x_i'' = \frac{\partial W}{\partial x_i} - \frac{d}{dt}\frac{\partial W}{\partial x_i'} ,$$

$$m_i\, y_i'' = \frac{\partial W}{\partial y_i} - \frac{d}{dt}\frac{\partial W}{\partial y_i'} ,$$

$$m_i\, z_i'' = \frac{\partial W}{\partial z_i} - \frac{d}{dt}\frac{\partial W}{\partial z_i'} .$$

When one regards $m_1, m_2, \ldots, m_n$ as the masses of n points and the variables x_i, y_i, z_i as the coordinates of the point m_i in a fixed, rectilinear system of axes at time t, those equations are also the differential equations of the motion of a free system of material points for which the components of the force that acts upon the point m_i at time t have the values:

(*) Submitted to the printer on 23 April 1877.

$$(1)\qquad \begin{cases} X_i = \dfrac{\partial W}{\partial x_i} - \dfrac{d}{dt}\dfrac{\partial W}{\partial x_i'}, \\ Y_i = \dfrac{\partial W}{\partial y_i} - \dfrac{d}{dt}\dfrac{\partial W}{\partial y_i'}, \\ Z_i = \dfrac{\partial W}{\partial z_i} - \dfrac{d}{dt}\dfrac{\partial W}{\partial z_i'}. \end{cases}$$

Forces whose analytical expressions have those forms are what I call, as usual, *potential forces*, and the function W that determines them completely when it has been given is its *potential*. Furthermore, I understand *internal* forces in the system to mean ones that originate in just the mutual actions of the points in the system and would therefore preserve equilibrium in the system at each moment if the system were converted into a rigid body by introducing rigid connecting lines between its points at the same moment. The problem that this note is mainly concerned with solving is that of *finding the most-general possible analytical express for the internal forces of a system of material points that is found to be in motion when one establishes that those forces should possess a potential*.

By their definition, the internal forces in the system are characterized by the six constraint equations:

$$(\text{II})\qquad \sum_{i=1}^{n} X_i = 0, \qquad \sum_{i=1}^{n} Y_i = 0, \qquad \sum_{i=1}^{n} Z_i = 0,$$

$$(\text{III})\qquad \begin{cases} \sum_{i=1}^{n} (y_i Z_i - z_i Y_i) = 0, \\ \sum_{i=1}^{n} (z_i X_i - x_i Z_i) = 0, \\ \sum_{i=1}^{n} (x_i Y_i - y_i X_i) = 0. \end{cases}$$

When expressed analytically, one then deals with the question:

What are the most-general values of the forces X_i, Y_i, Z_i that equations (I) would imply when one demands that W should be a function of time, the coordinates, and the velocities that is free of accelerations that satisfies the six conditions that arise by substituting the values (I) in equations (II) and (III) identically?

On first glance, it might seem as though that problem can be expressed much more simply as:

Find the most-general value of the potential W that satisfies the requirements that were posed.

Although the solution of the latter problem, by itself, also implies a solution of the former *eo ipso*, nonetheless, as a result of a known property of the differential equations of the calculus of variations, the second problem is not completely identical to the original one, although it is much simpler.

Namely, if φ is any function of only time and the coordinates and one sets:

$$\psi = \frac{d\varphi}{dt}$$

then one will have:

$$\frac{\partial \psi}{\partial x_i} = \frac{d}{dt}\frac{\partial \varphi}{\partial x_i}, \qquad \frac{\partial \psi}{\partial x_i'} = \frac{\partial \varphi}{\partial x_i},$$

and as a result:

$$\frac{\partial \psi}{\partial x_i} - \frac{d}{dt}\frac{\partial \psi}{\partial x_i'} = 0,$$

which is a formula that is naturally also valid when one switches x with y or z in it.

Therefore, if the potential W possesses the form:

$$W = V + \frac{d\varphi}{dt}$$

then the part $d\varphi / dt$ will drop out of formulas (I) by itself, and it will then have no effect on the values of X_i, Y_i, Z_i .

Based upon that remark (which is by no means new), in what follows, we can and will neglect all of those terms in the potential W that are complete differential quotients with respect to time as being entirely superfluous to our question. In other words, when we follow **C. Neumann**'s procedure ([1]) and say *effective potential* to mean what remains of the total potential W after neglecting all of those terms, we will not need to determine the total potential, but only the effective potential.

We now split that problem into two mutually-independent parts by subjecting the function W, on the one hand, to only the conditions (II), and on the other, to only the conditions (III). A comparison of the most-general values of the effective potential that can be inferred from one and the other requirement will then imply the answer to our question immediately, which shall ultimately be combined with yet another, much-simpler, problem, but one that is (as far as I know) just as general as the foregoing one that was treated up to now, namely, with the problem of finding the most-general value of the force-potential (I) that fulfills the demand of the principle of *vis viva*, i.e., the condition that the expression:

$$\sum_{i=1}^{n} (X_i\, x_i' + Y_i\, y_i' + Z_i\, z_i')$$

([1]) **C. Neumann**, *Die Principien der Elektrodynamik*. Tübungen, 1868.

should be a complete differential quotient with respect to time ([1]).

In order to treat the subject more-or-less completely, some things that are known already will also be derived anew in the last §.

([1]) If one assumes that the forces X_i, Y_i, Z_i are independent of the velocities and the accelerations then, as is known, that demand will imply that one should have:

$$\sum_{i=1}^{n}(X_i\, x_i' + Y_i\, y_i' + Z_i\, z_i') = \frac{dU}{dt}$$

identically, since when that is solved by the equations:

$$0 = \frac{\partial U}{\partial t}, \qquad X_i = \frac{\partial U}{\partial x_i}, \qquad Y_i = \frac{\partial U}{\partial y_i}, \qquad Z_i = \frac{\partial U}{\partial z_i},$$

it will imply, conversely, that the forces must possess a potential, or that the principle of *vis viva* can never be valid unless **Hamilton**'s principle is, as well. However, that will not at all be the case when the forces also depend upon the velocities and the accelerations. For example, when V is any function of x, y, z, x', y', z' that is free of t, the equation:

$$X\,x' + Y\,y' + Z\,z' = \frac{d}{dt}\left(V - x'\frac{\partial V}{\partial x'} - y'\frac{\partial V}{\partial y'} - z'\frac{\partial V}{\partial z'}\right)$$

will be satisfied identically by the substitutions:

$$X = C\,z' - B\,y' + \frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial V}{\partial x'},$$
$$Y = B\,x' - A\,z' + \frac{\partial V}{\partial y} - \frac{d}{dt}\frac{\partial V}{\partial y'},$$
$$Z = A\,y' - C\,x' + \frac{\partial V}{\partial z} - \frac{d}{dt}\frac{\partial V}{\partial z'},$$

no matter what values the functions A, B, C might have. However, if one sets, say:

$$A = x', \quad B = y', \quad C = z'$$

in it then one will get formulas that can be put into the form:

$$X = \frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial V}{\partial x'}, \ldots$$

just like with the demand that the principle of *vis viva* can be satisfied by forces X_i, Y_i, Z_i whose analytical expressions include the velocities, but not, at the same time, the accelerations.

§ 1.

Determining the effective potential from the requirements (II).

If one substitutes the values (I) in the conditions (II) then one will get three condition equations for the potential *W*, the first of which is:

$$(1) \qquad \sum_{i=1}^{n} \left(\frac{\partial W}{\partial x_i} - \frac{d}{dt}\frac{\partial W}{\partial x_i'} \right) = 0 \, ,$$

and the other two of which will emerge from that by switching *x* with *y* (*z*, resp.). One next treats the problem of determining the potential *W* (up to the terms that are neglected) from equation (1) in the most general way. To that end, I set:

$$\sum_{i=1}^{n} \frac{\partial W}{\partial x_i'} = A,$$

and in that way, I will convert equation (1) into:

$$\sum_{i=1}^{n} \frac{\partial W}{\partial x_i} = \frac{dA}{dt} .$$

Now, by assumption, *W*, and therefore the left-hand side of the last equation, as well, is free of the second differential quotients x'', y'', z''. Therefore, the equation cannot be true identically unless the function *A* itself does not include any differential quotients of the *x*, *y*, *z* at all. As a result, the demand (1) implies the two linear first-order partial differential equations for the potential *W* :

$$(2) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial W}{\partial x_i'} = A, \\ \displaystyle\sum_{i=1}^{n} \frac{\partial W}{\partial x_i} = \frac{dA}{dt}, \end{cases}$$

in which *A* is an arbitrary function of *t* and the coordinates.

Now, if *W* = *U* is any common solution to those two equations, and one sets:

$$W = U + V$$

then they will go to:

$$(3) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial V}{\partial x_i'} = 0, \\ \displaystyle\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} = 0. \end{cases}$$

We can then reduce equations (2) to the much-simpler (3) as soon as we have found any common solution to equations (2). However, one such solution is implied immediately by the assumption that:

$$W = \frac{d\varphi}{dt},$$

in which φ means an unknown function of time and the coordinates. Namely, that assumption will give:

$$\frac{\partial W}{\partial x_i'} = \frac{\partial \varphi}{\partial x_i}, \qquad \frac{\partial W}{\partial x_i} = \frac{d}{dt}\frac{\partial \varphi}{\partial x_i},$$

and equations (2) then then go to:

$$(4) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} = A, \\ \displaystyle\frac{d}{dt}\sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i} = \frac{dA}{dt}. \end{cases}$$

However, the second of those two equations is an immediate consequence of the first one. Thus, if φ is any solution of equation (4), and V is the general solution of the system (3) then:

$$W = V + \frac{d\varphi}{dt}$$

will be the general solution of the two equations (2), so from the prefatory remark:

$$W = V$$

will be the most-general value for the effective potential that one can get from the condition (1).

However, the two equations (3) say nothing beyond the fact that V can include the variables $x_1, \ldots, x_n, x_1', \ldots, x_n'$ only in the combinations:

$$x_2 - x_1, \quad \ldots, \quad x_n - x_1, \quad x_2' - x_1', \quad \ldots, \quad x_n' - x_1',$$

so when W is equal to an arbitrary function of those differences into which t, the y, z, and their first differential quotients can enter in an arbitrary way, it will be the most-general solution to our first problem.

Now, since the other two condition equation that arise from (I) and (II) differ from equation (1) only insofar as y (z, resp.) enters in place of x, we have then arrived at the theorem:

I. *The most-general expressions for the force-potentials (I) that satisfy the conditions (II) identically will be obtained when one sets the potential W equal to an arbitrary function of time, the relative coordinates, and the relative velocities of the points in the system.*

§ 2.

Determining the effective potential from the requirements (III).

The substitution of the formulas (I) in equations (III) produces three conditions for the potential W, the first of which is:

(1) $$\sum_{i=1}^{n}\left\{y_i\left(\frac{\partial W}{\partial z_i}-\frac{d}{dt}\frac{\partial W}{\partial z_i'}\right)-z_i\left(\frac{\partial W}{\partial y_i}-\frac{d}{dt}\frac{\partial W}{\partial y_i'}\right)\right\}=0\,,$$

and the other two will be obtained from it when one switches the symbols y, z with the symbols z, x (x, y, resp.).

We shall again initially consider only the condition (1).

If we add the following identity to it:

$$\sum_{i=1}^{m}\left\{y_i'\frac{\partial W}{\partial z_i'}+y_i\frac{d}{dt}\frac{\partial W}{\partial z_i'}-z_i'\frac{\partial W}{\partial y_i'}-z_i\frac{d}{dt}\frac{\partial W}{\partial y_i'}\right\}=\frac{d}{dt}\sum_{i=1}^{m}\left\{y_i\frac{\partial W}{\partial z_i'}-z_i\frac{\partial W}{\partial y_i'}\right\}$$

then it will be converted into:

$$\sum_{i=1}^{m}\left\{y_i\frac{\partial W}{\partial z_i}-z_i\frac{\partial W}{\partial y_i}+y_i'\frac{\partial W}{\partial z_i'}-z_i'\frac{\partial W}{\partial y_i'}\right\}=\frac{d}{dt}\sum_{i=1}^{m}\left\{y_i\frac{\partial W}{\partial z_i'}-z_i\frac{\partial W}{\partial y_i'}\right\}\,,$$

and can then decompose into the following two:

(2) $$\begin{cases}\displaystyle\sum_{i=1}^{m}\left(y_i\frac{\partial W}{\partial z_i'}-z_i\frac{\partial W}{\partial y_i'}\right)=M\,,\\[2ex] \displaystyle\sum_{i=1}^{m}\left(y_i\frac{\partial W}{\partial z_i}-z_i\frac{\partial W}{\partial y_i}+y_i'\frac{\partial W}{\partial z_i'}-z_i'\frac{\partial W}{\partial y_i'}\right)=\frac{dM}{dt}\,.\end{cases}$$

The second of those conditions shows that M can be only a function of t and the coordinates. We further set:

$$W=\frac{d\psi}{dt}\,,$$

in which ψ means a function such that equations (2) will then be converted into:

$$(3)\qquad \begin{cases} \sum_{i=1}^{m}\left(y_i\frac{\partial\psi}{\partial z_i}-z_i\frac{\partial\psi}{\partial y_i}\right)=M, \\ \frac{d}{dt}\sum_{i=1}^{m}\left(y_i\frac{\partial\psi}{\partial z_i}-z_i\frac{\partial\psi}{\partial y_i}\right)=\frac{dM}{dt}. \end{cases}$$

The substitution $W = d\psi / dt$ will then fulfill the two equations (2) as long as ψ is a solution of equations (3). However, with that assumption, if one assumes that:

$$W = V + \frac{d\psi}{dt}$$

then the two equations (2) will go to the following ones:

$$(4)\qquad \begin{cases} \sum_{i=1}^{m}\left(y_i\frac{\partial V}{\partial z_i'}-z_i\frac{\partial V}{\partial y_i'}\right)=0, \\ \sum_{i=1}^{m}\left(y_i\frac{\partial V}{\partial z_i}-z_i\frac{\partial V}{\partial y_i}+y_i'\frac{\partial V}{\partial z_i'}-z_i'\frac{\partial V}{\partial y_i'}\right)=0. \end{cases}$$

The determination of the effective potential then comes down to only the problem of finding the most-general common solution to the two equations (4).

Now, from the foregoing, the assumption that:

$$V = \frac{d\psi}{dt}$$

will simultaneously satisfy those two equations as long as ψ is a solution of the equation:

$$(5)\qquad \sum_{i=1}^{m}\left(y_i\frac{\partial\psi}{\partial z_i}-z_i\frac{\partial\psi}{\partial y_i}\right)=0\,.$$

However, that equation is equivalent to the system of $2n - 1$ ordinary differential equations:

$$dz_i : dy_h = y_i : -z_h\,,$$

whose $2n - 1$ integrals are:

$$(6)\qquad \begin{cases} 2u_i = y_i^2 + z_i^2 = \text{const.}, \\ u_{hk} = y_h\,y_k + z_h\,z_k = \text{const.}, \end{cases}$$

in which $i = 1, 2, \ldots, n$, k is any one of those numbers, and $h = 1, \ldots, k - 1, k + 1, \ldots, n$. As a result, the $2n - 1$ expressions:

$$(7)\qquad \begin{cases} u'_i = y_i\, y'_i + z_i\, z'_i\,, \\ u'_{hk} = y'_h\, y_k + y_h\, y'_k + z'_h\, z_k + z_h\, z'_k \end{cases}$$

will be common solutions of the two equations (4). However, those two equations will reduce to the one equation (5) in that way such that one can set V equal to a function ψ that is free of y', z' directly. Therefore, the $2n - 1$ expressions (6) will also be common solutions to equations (4) in their own right. Moreover, those two equations can have no more than $4n - 2$ mutually-independent solutions relative to the variables y, z, y', z', and that condition satisfies the $4n - 2$ expressions (6) and (7). As a result, the general solution V of the two equations (4) will have the following form:

$$(8)\qquad V = F\,(u_i, u_{hk}, u'_i, u'_{hk}, t, x_h, x'_h)\ ,$$

in which F denotes an arbitrary function, and of the various arguments that can include that arbitrary function, along with t, only one of them will ever give a representative. Moreover, from the foregoing, it will be, at the same time, the most-general value for the effective potential that can be inferred from the condition (1).

Now, the second condition equation that arises from the demands (III) on the potential W differs from (1) only by the fact that z, x have taken the places of y, z, resp. The most-general solution to that second equation that can come under consideration is the following one:

$$W = \Phi\,(v_i, v_{hk}, v'_i, v'_{hk}, t, y_h, y'_h)\ ,$$

in which Φ is an arbitrary function, and:

$$\begin{aligned} 2\, v_i &= z_i^2 + x_i^2\,, \\ v_{hk} &= z_h\, z_k + x_h\, x_k\,, \\ v'_i &= z_i\, z'_i + x_i\, x'_i\,, \\ v'_{hk} &= z'_h\, z_k + z_h\, z'_k + x'_h\, x_k + x_h\, x'_k\ . \end{aligned}$$

However, if one compares that function with the formulas (6), (7), (8) then one will see immediately that the most-general value of the effective potential W that satisfies the first two conditions (III) simultaneously will have the form:

$$(9)\qquad W = \Psi\,(p_i, p_{hk}, p'_i, p'_{hk}, t)\ ,$$

in which Ψ is an arbitrary function:

$$(10)\quad \begin{cases} 2p_i = x_i^2 + y_i^2 + z_i^2, \\ p_{hk} = x_h\, x_k + y_h\, y_k + z_h\, z_k, \\ p_i' = x_h\, x_k' + y_h\, y_k' + z_h\, z_k', \\ p_{hk}' = x_h'\, x_k + x_h\, x_k' + y_h'\, y_k + y_h\, y_k' + z_h'\, z_k + z_h\, z_k', \end{cases}$$

and since those arguments will remain unchanged when one permutes the symbols x, y, z with each other, that will likewise shed light upon the fact that this value of W already satisfies the third condition by itself, which the requirements (III) imply for that function.

The last remark shows that one will always have the theorem for a force-potential that for the motion of a system of material points under the influence of a potential that is independent of velocities, two of the three laws of areas can never be true without the third one also being true, namely, the theorem that when forces with the analytical form (I) fulfill two of the three conditions (III), they will always necessarily satisfy the third one, as well ([1]).

Finally, if we set:

$$r_i^2 = x_i^2 + y_i^2 + z_i^2,$$
$$r_{hk}^2 = (x_h - x_k)^2 + (y_h - y_k)^2 + (z_h - z_k)^2$$

then it will follow upon differentiating (10) with respect to t that:

$$r_i^2 = 2\, p_i\,,$$
$$r_{hk}^2 = 2\, p_h + 2\, p_k - 2\, p_{hk}\,,$$

([1]) That theorem can be proved without any integration in the following way:

For the effective potential $W = V$, the condition (1) reduces to the two equations (4). If one then denotes the left-hand sides of those two equations by $X_1(V)$ and $X_2(V)$ and understands $Y_1(V)$ and $Y_2(V)$ [$Z_1(V)$ and $Z_2(V)$, resp.] to mean the expressions that arise by permuting the symbols z, x (x, y, resp.) then the requirements (III) for the effective potential V will successively imply the conditions:

$$\begin{array}{ccc} X_1(V) = 0 & \text{and} & X_2(V) = 0\,, \\ Y_1(V) = 0 & '' & Y_2(V) = 0\,, \\ Z_1(V) = 0 & '' & Z_2(V) = 0\,. \end{array}$$

However, one has:

$$Y_1(X_1(V)) - X_1(Y_1(V)) = Z_1(V)\,,$$
$$Y_1(X_2(V)) - X_2(Y_1(V)) = Z_2(V)\,,$$

identically. Every common solution V of the four equations:

$$\begin{array}{cc} X_1(V) = 0\,, & X_2(V) = 0\,, \\ Y_1(V) = 0\,, & Y_2(V) = 0\,, \end{array}$$

will therefore always simultaneously satisfy the two equations:

$$Z_1(V) = 0\,, \qquad Z_2(V) = 0\,.$$

$$\begin{aligned} r_i\, r_i' &= p_i', \\ r_{hk}\, r_{hk}' &= p_h' + p_k' - p_{hk}'. \end{aligned}$$

We can also replace formula (9) with this one then:

$$W = F\,(r_i, r_i', r_{hk}, r_{hk}', t),$$

with which, we have arrived at the theorem:

II. *The most-general expressions for the force-potential (I) that fulfill the requirements (III) will arise when one replaces W with an arbitrary function of time, the distances from the points of the system to the coordinate origin, their mutual distances, and the first differential quotients of those two types of distances with respect to time.*

§ 3.

The principle of *vis viva*.

From the demand that the sum:

$$\sum_{i=1}^{n} (X_i\, x_i' + Y_i\, y_i' + Z_i\, z_i')$$

should be a complete differential quotient with respect to time, one will get the condition:

$$\text{(1)} \qquad \sum_{i=1}^{n} \left\{ x_i' \left(\frac{\partial W}{\partial x_i} - \frac{d}{dt} \frac{\partial W}{\partial x_i'} \right) + y_i' \left(\frac{\partial W}{\partial y_i} - \frac{d}{dt} \frac{\partial W}{\partial y_i'} \right) + z_i' \left(\frac{\partial W}{\partial z_i} - \frac{d}{dt} \frac{\partial W}{\partial z_i'} \right) \right\} = \frac{dU}{dt}$$

when one denotes those differential quotients by dU / dt and replaces the forces X_i, Y_i, Z_i with their values (I). When one adds the identity:

$$\sum_{i=1}^{n} \left\{ x_i'' \frac{\partial W}{\partial x_i'} + x_i' \frac{d}{dt} \frac{\partial W}{\partial x_i'} + \cdots \right\} - \frac{d}{dt} \sum_{i=1}^{n} \left\{ x_i' \frac{\partial W}{\partial x_i'} + \cdots \right\} = 0$$

to that, one will convert that condition into the following one:

$$\text{(2)} \qquad \frac{d}{dt} \left\{ W - \sum_{i=1}^{n} \left(x_i' \frac{\partial W}{\partial x_i'} + y_i' \frac{\partial W}{\partial y_i'} + z_i' \frac{\partial W}{\partial z_i'} \right) \right\} - \frac{\partial W}{\partial t} = \frac{dU}{dt}.$$

In order for that to be true identically, it is then necessary and sufficient that the partial differential quotient $\partial W / \partial t$ should be a complete differential quotient with respect to time. Now, the potential W includes only the first differential quotients of the coordinates, and we can therefore always regard any function for which $\partial W / \partial t$ is supposed to be a complete differential quotient as the partial differential quotients with respect to t of a function φ that depends upon only time and the coordinates. However, the formula:

$$\frac{\partial W}{\partial t} = \frac{d}{dt}\frac{\partial \varphi}{\partial t}$$

implies that:

$$W = \frac{d\varphi}{dt} + V,$$

in which V is free of t and is a function of just x, y, z, x', y', z'. Thus, the effective potential V cannot include time, i.e.:

III. *One will get the most-general expressions for the force-potentials (I) that satisfy the demand of the principle of vis viva when one assumes that the potential W is free of time t.*

§ 4.

Summary of the results obtained and their consequences.

If we now combine the three theorems that we obtained then that will give the following result:

IV. *The most-general analytical expressions for the force-potentials for a system of material points in motion that satisfy the demand of the principle of vis viva will be obtained from formula (I) when one sets the potential W equal to an arbitrary function of the mutual distances between the points of the system and the first differential quotients of those distances*.

If one drops the requirement of the principle of *vis viva* in order to obtain the answer to our original question then the potential W can even be obtained in yet another way, moreover.

If we further subject the system, which was free up to now, to the conditions:

(1) $$\varphi_1 = 0, \qquad \varphi_2 = 0, \qquad \ldots,$$

in which $\varphi_1, \varphi_2, \ldots$ are given functions of t and the coordinates of the points of the system then the differential equations of its motion will now be:

$$
(2) \qquad \begin{cases} m_i\, x_i'' = X_i + \lambda_1 \dfrac{\partial \varphi_1}{\partial x_i} + \lambda_2 \dfrac{\partial \varphi_2}{\partial x_i} + \cdots, \\ m_i\, y_i'' = Y_i \;+ \lambda_1 \dfrac{\partial \varphi_1}{\partial y_i} + \lambda_2 \dfrac{\partial \varphi_2}{\partial y_i} + \cdots, \\ m_i\, z_i'' = Z_i + \lambda_1 \dfrac{\partial \varphi_1}{\partial z_i} + \lambda_2 \dfrac{\partial \varphi_2}{\partial z_i} + \cdots \end{cases}
$$

From them and equations (1), one will then get:

$$
\begin{aligned}
&\frac{dT}{dt} = \sum_{i=1}^{n} (X_i\, x_i' + Y_i\, y_i' + Z_i\, z_i') - \lambda_1 \frac{\partial \varphi_1}{\partial t} - \cdots, \\
&\frac{d}{dt} \sum_{i=1}^{n} m_i\, x_i' = \sum_{i=1}^{n} X_i + \lambda_1 \sum_{i=1}^{n} \frac{\partial \varphi_1}{\partial x_i} + \cdots, \\
&\frac{d}{dt} \sum_{i=1}^{n} m_i\, (y_i\, z_i' - z_i\, y_i') = \sum_{i=1}^{n} (y_i\, Z_i - z_i\, Y_i) + \lambda_1 \sum_{i=1}^{n} \left(y_i \frac{\partial \varphi_1}{\partial z_i} - z_i \frac{\partial \varphi_1}{\partial y_i} \right) + \cdots
\end{aligned}
$$

However, the terms in the latter equations that are multiplied by λ will vanish completely when one subjects each of the functions φ to the conditions:

$$
\frac{\partial \varphi}{\partial t} = 0, \qquad \sum_{i=1}^{n} \frac{\partial \varphi_1}{\partial x_i} = 0, \qquad \sum_{i=1}^{n} \left(y_i \frac{\partial \varphi_1}{\partial z_i} - z_i \frac{\partial \varphi_1}{\partial y_i} \right) = 0,
$$

and under that assumption and when one assigns the values to the forces X_i, Y_i, Z_i that follow from Theorem IV, so just as if the system were free, one will then get the following integrals of the equations of motion:

$$
\begin{aligned}
&T - W + \sum_{i=1}^{n} \left(x_i' \frac{\partial W}{\partial x_i'} + y_i' \frac{\partial W}{\partial y_i'} + z_i' \frac{\partial W}{\partial z_i'} \right) = \text{const.}, \\
&\sum_{i=1}^{n} m_i\, x_i' = \text{const.}, \\
&\sum_{i=1}^{n} m_i\, (y_i\, z_i' - z_i\, y_i') = \text{const.}
\end{aligned}
$$

From the foregoing, that will imply the theorem:

V. *If a system of material points is subject to only forces whose analytical expressions emerge from formulas (I) when one sets W in them equal to a function of merely the mutual distances* r_{hk} *between the points of the systems and first differential quotients* r_{hk}' *of those distances with respect to time then the motion of the system will obey the principle of the conservation of areas and the principle of vis viva, which will assume the form:*

$$T - W + \sum_{h,k} \frac{\partial W}{\partial r'_{hk}} r'_{hk} = \text{const.},$$

and in fact regardless of whether the system is free or subjected to the constraint of conditions in which only the mutual distances play a role.

Finally, if we let the system reduce to two points m_1 and m_2 and call the distance between them r then based upon Theorem IV, that will give the following values for the components of the force that acts between the points:

$$X_1 = \frac{\partial W}{\partial x_1} - \frac{d}{dt}\frac{\partial W}{\partial x'_1}, \quad \ldots,$$

in which W is a function of just r and r'. Upon performing the differentiations, we can then represent those values thusly:

$$X_1 = \frac{\partial W}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial W}{\partial r'}\frac{\partial r'}{\partial x_1} - \frac{\partial r'}{\partial x'_1}\frac{d}{dt}\frac{\partial W}{\partial r'} - \frac{\partial W}{\partial r'}\frac{d}{dt}\frac{\partial r'}{\partial x'_1}, \ldots$$

However, one has:

$$\frac{\partial r'}{\partial x_1} = \frac{\partial r}{\partial x_1}, \qquad \frac{\partial r'}{\partial x_1} = \frac{d}{dt}\frac{\partial r}{\partial x_1},$$

so they will reduce to ([1]):

$$X_1 = \left(\frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'}\right)\frac{\partial r}{\partial x_i}, \quad \ldots,$$

and we will then obtain the theorem:

VI. *If one establishes the axiom that the interaction between two points that takes place during their motion must possess a potential and satisfy the demand of the principle of vis viva then it will follow that this interaction must consist of a force R that acts between the two points in the direction of their connecting line and its analytical expression will have the form:*

$$R = \frac{\partial W}{\partial r} - \frac{d}{dt}\frac{\partial W}{\partial r'},$$

or what amounts to the same thing:

$$R = \frac{1}{r'}\frac{d}{dt}\left(W - r'\frac{\partial W}{\partial r'}\right),$$

in which W is a function of just the distance r between the two points and its differential quotient r'.

([1]) Cf., **C. Neumann**, *loc. cit.*, pp. 27.