"Ueber die Aufstellung der Differentialgleichungen der Bewegung für reibungslose Punktsysteme, die Bedingungsungleichungen unterworfen sind," Ber. der Verh. Kön. Sächs. Ges. Wiss. Leipzig **51** (1899), 224-244.

On the differential equations of motion for frictionless point-systems that are subject to condition inequalities

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A system of material points that is coupled with each other in any way, and at the same time can be subject to external restrictions, moves under the influence of given forces.

If the driving forces, couplings, and restrictions on the system, as well as the masses of its points, are given, and one knows, moreover, the *initial state* of the system – i.e., one knows the *position* and *velocity* that each system-point possesses at a given *initial moment* – then it is assumed in so doing that, other than the positions of the system-points, the given forces depend upon at most only the velocities and time, and that all possible *friction* can be neglected, and the *motion of the system* will be defined *uniquely* in a mechanical context. However, in order to also be able to carry out *calculations*, one must find the *differential equations* of the motion thus-defined; i.e., when one refers everything in space to a fixed rectangular system of axes and suggests the complete differentiations with respect to time by primes, one must know how to express the *accelerations* x'', y'', z'' that the system-points would achieve at an arbitrary moment t in the moment in terms of the *coordinates* x, y, z and the velocities x', y', z' of the points at the same moments.

As is known, that basic problem in dynamics can be solved very easily as long as the couplings and restrictions on the system are expressed by *condition equations* between the coordinates of its points or also between them and time. However, the problem will become much more difficult when the system is subordinate to the constraint of *condition inequalities*, e.g., when its points are coupled, not by rigid lines, but by inextensible strings whose weights can be neglected.

As far as I know, only **Ostragradsky** has addressed the latter case up to now, first in "Considérations générales sur les moments des forces," but then, above all, in the treatise "Sur les déplacements instantanées des systèmes assujettis à des conditions variables," which were papers that were submitted to the St. Petersburg Academy in 1834 and 1838

([1]). However, in that second treatise, **Ostragradsky** arrived at two different systems of inequalities that the solution to the problem must fulfill, and was content to assert that the one system could be replaced with the other one without in any way establishing that replaceability, which is entirely essential to the validity of the result. On first glance, his argument generally seems so natural and convincing that I have long believed that the gap in the method of proof must be eliminated in a purely mathematical way. However, from a correspondence that I had with **E. Study** on that question in 1889, I learned that it is not merely an oversight that is included in "Déplacements," but an actual fallacy. **Study** was then, above all, the first person we have to thank for the correct solution to the problem, which shall be summarized here, at least in principle. One will obtain it most simply and clearly when one starts from ***Gauss****'s principle of least constraint*, which also reduces to **Ostragradsky**'s concept of the equilibrium of lost forces.

§ 1. – The method of solution and the impossibility of a direct solution.

Let $m_1, m_2, \ldots, m_n$ be the masses of the system-points, and in general, at the moment t in the motion. Let x_i, y_i, z_i be the coordinates of the points m_i, and let X_i, Y_i, Z_i be the components of the driving forces that act upon them, which shall be given, single-valued functions of time, and the coordinates and velocities of the points in all of what follows. Finally, let the couplings and restrictions of the system be defined by *inequalities:*

$$f_1 \leq 0, \quad f_2 \leq 0, \quad \ldots, \tag{1}$$

in which $f_1, f_2, \ldots$ denote given single-valued functions of the coordinates that might possibly include time itself, as well. I assume that these functions, along with their first, second, and third partial differential quotients, will always remain continuous for all of the motions of the system that come under consideration.

The positions and velocities of all points are known at a given moment t. One next asks *what accelerations can those system points assume at that instant?*

The system allows only those motions for which the coordinates of its points continually satisfy the conditions (1). Any of those conditions might be represented by:

$$f \leq 0.$$

For any motion of the system, the coordinates x_i, y_i, z_i of its points m_i will be functions of time with continuous differential quotients. If we consider those coordinates to be such functions and let t go to $t + dt$, in order to focus on the following moment, then when the system condition $f \leq 0$ is developed in powers of dt, it will be converted into:

$$f + f' dt + f'' \frac{dt^2}{2} + r\, dt^3 \leq 0, \tag{a}$$

([1]) The latter seems to remain unknown to **Jacobi**. However, he cited the result of the former in a Berlin lecture, which unfortunately remained unconsidered in the publication of **Jacobi**'s *Vorlesungen über Dynamik*, although **Scheibner** has presented it in an excellent fashion.

in which $r\,dt^3$ denotes the remaining terms in the **Taylor** development.

Besides t, that development of f includes only the *coordinates* of the system-point, along with its *velocity* f', and finally, its *acceleration* f''.

If we now think of the coordinates x_i, y_i, z_i and the velocities x_i', y_i', z_i' as being set equal to the values, which are known, by assumption, that they possess at the moment t then f can only be < 0 or $= 0$, since otherwise the known *positions* of the points at time t would not indeed be possible positions of them.

If f *is only* < 0, *but not* $= 0$ then for a sufficiently-small dt, the condition (a) will already be fulfilled by itself, and the system will not be instantaneously restricted in any way.

However, if $f = 0$ then, after dividing by the positive quantity dt, the condition (a) will reduce to:

$$f' + f''\frac{dt}{2} + r\,dt^2 \leq 0, \tag{b}$$

and that condition can only be fulfilled for an arbitrarily-small dt when its first term is already itself ≤ 0.

Hence, as long as the position of the system at the moment t satisfies the assumption $f = 0$, the *velocities* of its points must already fulfill the condition:

$$f' \leq 0$$

by themselves.

However, if f' *is only* < 0 *and not* $= 0$ at the moment t for which the *positions* and *velocities* of the point are known then for a sufficiently-small dt, the condition (b) will once more be fulfilled by itself, and therefore the system will not be assigned any sort of restrictions instantaneously.

Rather, such a restriction will first occur *when one also has* $f' = 0$. The condition (b) will then reduce to:

$$f'' + 2r\,dt \leq 0$$

and can then exist for arbitrarily-small dt only when one already has:

$$f'' \leq 0\,. \tag{c}$$

Therefore, of the system conditions (1), the *accelerations* of a point are not restricted in any way at the moment t in question by:

1. All of the ones that are not fulfilled as *equations*, but only as *inequalities*, as well as:

2. All of the ones that exist precisely as equations whose complete first derivatives are however $\neq 0$, but only < 0, for the known position and velocity of the point at the moment t.

After dropping all of those *instantaneously ineffective* conditions, let:

(2) $$f_1 \leq 0, f_2 \leq 0, \ldots, f_r \leq 0$$

be the remaining system conditions (1). For them, the equations:

(3) $$f_1 = 0, \quad f_2 = 0, \ldots, f_r = 0$$

and

(4) $$f_1' = 0, \; f_2' = 0, \ldots, \; f_r' = 0$$

will then be fulfilled by the known state of the system, *and at that moment, the system will admit all accelerations that are compatible with the conditions:*

(5) $$f_1'' \leq 0, \; f_2'' \leq 0, \ldots, \; f_r'' \leq 0.$$

Among all of those instantaneously-possible accelerations, one must also include the *unknown true accelerations* that the points will attain at the moment t for the actual motion of the system. In order to find them, we turn to the aid of just the *principle of least constraint.*

The point m_i that is found *at the position* $a_i \equiv x_i, y_i, z_i$ at the moment t and already possesses the velocities x_i', y_i', z_i' is acted upon by the given accelerating forces:

$$\frac{X_i}{m_i}, \frac{Y_i}{m_i}, \frac{Z_i}{m_i}.$$

If the point were *free* then it would arrive *at a position* b_i with the coordinates:

$$\xi_i = x_i + x_i' \, dt + \frac{X_i}{m_i} \frac{dt^2}{2},$$

$$\eta_i = y_i + y_i' \, dt + \frac{Y_i}{m_i} \frac{dt^2}{2},$$

$$\zeta_i = z_i + z_i' \, dt + \frac{Z_i}{m_i} \frac{dt^2}{2}$$

at the next infinitely-small time-element dt. However, due to its coupling to the system, it will not actually reach that location, but rather, in the time dt, *it will come to another location* c_i, and *when we understand* x_i'', y_i'', z_i'' *to mean the unknown true accelerations of the point at the moment* t, it will possess the coordinates:

$$\bar{\xi}_i = x_i + x_i' \, dt + x_i'' \frac{dt^2}{2},$$

$$\bar{\eta}_i = y_i + y_i' \, dt + y_i'' \frac{dt^2}{2},$$

$$\bar{\zeta}_i = z_i + z_i' \, dt + z_i'' \frac{dt^2}{2}.$$

Now, from the principle of least constraint, the *actual positions* c_i of the system-points at time $t + dt$ are characterized by the fact that, among all positions c_i to which they can be brought from the position a_i in the time interval dt considered by any possible motion of the system of points m_i, they are the ones for which the sum:

$$\sum_{i=1}^{n} m_i \, \overline{c_i \, b_i}^2 \equiv \sum_{i=1}^{n} m_i \{(\xi_i - \bar{\xi}_i)^2 + (\eta_i - \bar{\eta}_i)^2 + (\zeta_i - \bar{\zeta}_i)^2\}$$

attains the least-possible value.

If one replaces the coordinates of the locations b_i and c_i in this with their values then one will see immediately that this principle comes down to:

For the desired true accelerations x_i'', y_i'', z_i'' *of the system-point* m_i *at the moment t, one must have that the sum:*

(6) $$\sum_{i=1}^{n} m_i \left\{ \left(\frac{X_i}{m_i} - x_i'' \right)^2 + \left(\frac{Y_i}{m_i} - y_i'' \right)^2 + \left(\frac{Z_i}{m_i} - z_i'' \right)^2 \right\} = \min;$$

i.e., it must be smaller for all other accelerations that are allowed for points of the system instantaneously.

If we understand:

$$x_i'' + \delta x_i'', \qquad y_i'' + \delta y_i'', \qquad z_i'' + \delta z_i''$$

to mean any other instantaneously-possible accelerations of the system-points m_i that likewise deviate only slightly from the *unknown*, true accelerations of the system, and when we observe the identities:

$$\frac{\partial f''}{\partial x_i''} \equiv \frac{\partial f}{\partial x_i}, \ldots$$

and introduce the notation:

(7) $$\delta f'' \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \delta x_i'' + \frac{\partial f}{\partial y_i} \delta y_i'' + \frac{\partial f}{\partial z_i} \delta z_i'' \right),$$

the conditions (5), which instantaneous restrict the accelerations of the system exclusively, will also imply only the conditions:

$$(8) \qquad f_1'' + \delta f_1'' \le 0, \quad f_2'' + \delta f_2'' \le 0, \ \ldots, \ f_r'' + \delta f_r'' \le 0$$

for the variations of the accelerations. From (6), we will then get the following requirement for the determination of the true accelerations x_i'', y_i'', z_i'' that *one must have*:

$$(9) \qquad \sum_{i=1}^{n} \{(X_i - m_i\, x_i'')\,\delta x_i'' + (Y_i - m_i\, y_i'')\,\delta y_i'' + (Z_i - m_i\, z_i'')\,\delta z_i''\} \le 0$$

for all sufficiently-small values of the variations $\delta x_i''$, $\delta y_i''$, $\delta z_i''$ *that the conditions* (8) *fulfill.*

However, when the unknown, true accelerations x_i'', y_i'', z_i'' make:

$$f'' \neq 0, \text{ but only } < 0,$$

the condition:

$$f'' + \delta f'' \le 0$$

will in no way restrict the variations of the acceleration, since they will then be fulfilled by themselves for all arbitrary, but sufficiently-small, values of those variations.

Rather, a restriction will first occur when the true acceleration satisfies the equation:

$$f'' = 0,$$

which will make that condition reduce to:

$$\delta f'' \le 0 .$$

How can one nonetheless recognize which of the two derivatives f'' at the given moment t during the actual motion of the system possesses the value zero and which of them is only < 0?

On first glance, it would seem that this *cardinal question* must necessarily obviate the entire investigation. Indeed, we do not know the true acceleration itself at all, so how can we decide whether $f'' = 0$ or only < 0 for it, and yet we have to know that in order to even be able to calculate the true acceleration.

In general, **Ostragradsky** helped us avoid that complication by an argument that, in fact, initially seems very enlightening for the consideration of surfaces. Namely, he simply argued that ([1]): One replaces the unknown accelerations x_i'', y_i'', z_i'' of the system-points with the values:

$$\frac{X_i}{m_i}, \ \frac{Y_i}{m_i}, \ \frac{Z_i}{m_i}$$

([1]) Déplacements, § **8**.

that those accelerations would possess if the points were free, and then examines which of the conditions (5) are fulfilled by that and which are not. Only the latter conditions will restrict the accelerations instantaneously, while the former will express obstacles that do not presently stand in the way of the motion, and will therefore also not come under consideration.

However, the argument is correct only in the case of a single condition (5). For more than one condition, as the discussion of the case $r = 2$ in **§ 3** will show clearly, it can very well happen that a condition that poses no obstacle to the *free* motion of the point in time will define a real restriction for the *actual* motion of the system as a result of other conditions, and likewise a condition can, conversely, instantaneously limit the *free* motion, but pose no obstacle to the *actual* motion. In fact, **Ostragradsky** also came to *two systems of inequalities* by his argument, about which, he said (but totally forgot to prove): "We will see that the one of them is always fulfilled at the same time as the other, and that one can therefore replace the one with the other." However that is contradicted by the fact that, in reality, the one can be fulfilled without the other one being fulfilled.

This much is then clear in any case, that it is impossible to answer the basic question above *directly*. Therefore, nothing else remains but to try to make that decision in an *indirect* way. To that end, we will first have to establish *how the true accelerations of the system-point would be determined if we had already solved our cardinal question*, and we must then see *whether we cannot find some criterion from which we could then recognize whether the values that were found for the accelerations are correct or false*.

§ 2. – Indirect solution of the problem.

From the foregoing, I shall now assume that we already know that *during the unknown, true motion of our system*, which is instantaneously subordinate to only the r conditions (5), *the k equations:*

$$(10) \qquad f_g'' \equiv \sum_{i=1}^{n}\left(\frac{\partial f_g}{\partial x_i}x_i'' + \frac{\partial f_g}{\partial y_i}y_i'' + \frac{\partial f_g}{\partial z_i}z_i''\right) + F_g = 0, \quad (g = 1, 2, \ldots, k)$$

are valid, while the r – k expressions:

$$(11) \qquad f_\gamma'' \equiv \sum_{i=1}^{n}\left(\frac{\partial f_\gamma}{\partial x_i}x_i'' + \frac{\partial f_\gamma}{\partial y_i}y_i'' + \frac{\partial f_\gamma}{\partial z_i}z_i''\right) + F_\gamma, \qquad (\gamma = k + 1, 2, \ldots, r)$$

all possess negative values. Obviously, the F_g and F_γ denote the sums of all terms in the complete second differential quotients f_g'' and f_γ'' that are free of the second differential quotients of the coordinates, and k can be any number from the sequence $0, 1, \ldots, r$.

One next asks: *What values will this assumption (which is still entirely arbitrary, for the moment, mind you) yield for the true accelerations of the system-point at time t* ?

I shall make that assumption even more precise by establishing that for a given state of motion of the system at the moment t, none of equations (10) should be a mere consequence of the remaining ones. Indeed, such an equation in (10) would not contribute to the determination of the unknown accelerations and would therefore be simply dropped. I shall then assume that *the k equations* (10) *determine k of the* 3*n unknowns* x_i'', y_i'', z_i'' *in terms of the remaining* 3*n* – *k*.

With those assumptions, the variations of the acceleration are instantaneously subject to only the *k* conditions:

$$\delta f_g'' \equiv \sum_{i=1}^{n} \left(\frac{\partial f_g}{\partial x_i} \delta x_i'' + \frac{\partial f_g}{\partial y_i} \delta y_i'' + \frac{\partial f_g}{\partial z_i} \delta z_i'' \right) \leq 0 \qquad (g = 1, 2, \ldots, k), \tag{12}$$

and the unknown, true accelerations x_i'', y_i'', z_i'' must then satisfy the demand (9) for all values of their variations $\delta x_i''$, $\delta y_i''$, $\delta z_i''$ that fulfill those *k* conditions.

In particular, one must then have:

$$\sum_{i=1}^{n} \{(X_i - m_i x_i'')\,\delta x_i'' + (Y_i - m_i y_i'')\,\delta y_i'' + (Z_i - m_i z_i'')\,\delta z_i''\} = 0$$

for all $\delta x_i''$, $\delta y_i''$, $\delta z_i''$ that satisfy the *k* equations:

$$\delta f_h'' = 0, \qquad (h = 1, 2, \ldots, k), \tag{12′}$$

and from our assumption on the nature of equations (10), equations (12′) will determine *k* of those variations as functions of the remaining 3*n* – *k*.

If one multiplies the last equations by the temporarily undetermined factors – λ_h and then adds them to the foregoing conditions then one will get the equation:

$$\sum_{i=1}^{n} \{(X_i - m_i x_i'')\,\delta x_i'' + (Y_i - m_i y_i'')\,\delta y_i'' + (Z_i - m_i z_i'')\,\delta z_i''\} - \sum_{h=1}^{k} \lambda_h\, \delta f_h' = 0. \tag{13}$$

Now, one can always determine the multipliers λ_1, λ_2, …, λ_k in such a way that the coefficients of the *k* variations in that equation that are expressed in terms of the other ones from equations (12′) will be zero, and all that will remain then in equation (13) will be completely-arbitrary variations. Their coefficients must also vanish then, and one will thus get the following 3*n* equations from (13):

$$\left\{ \begin{aligned} X_i - m_i x_i'' &= \sum_{h=1}^{k} \lambda_h \frac{\partial f_h}{\partial x_i}, \\ Y_i - m_i z_i'' &= \sum_{h=1}^{k} \lambda_h \frac{\partial f_h}{\partial y_i}, \qquad (i = 1, 2, \ldots, n) \\ Z_i - m_i z_i'' &= \sum_{h=1}^{k} \lambda_h \frac{\partial f_h}{\partial z_i}. \end{aligned} \right. \tag{14}$$

However, by assumption, the components X_i , Y_i , Z_i of the driving force that acts at the point m_i are given, single-valued functions of time t, the coordinates, and the velocity of the system-point, so they are functions whose values at the moment t will be, at the same time, determined completely by the state of motion of the system. Equations (14) then express the 3n unknowns x_i'', y_i'', z_i'' in terms of the k multipliers uniquely.

If one further sets, in general:

$$\Phi_{lh} \equiv \sum_{i=1}^{n} \frac{1}{m_i}\left(\frac{\partial f_l}{\partial x_i}\frac{\partial f_h}{\partial x_i} + \frac{\partial f_l}{\partial y_i}\frac{\partial f_h}{\partial y_i} + \frac{\partial f_l}{\partial z_i}\frac{\partial f_h}{\partial z_i}\right), \tag{15}$$

such that $\Phi_{lh} \equiv \Phi_{hl}$ and each $\Phi_{ll} > 0$, and if:

$$\Phi_l^0 \equiv \sum_{i=1}^{n} \frac{1}{m_i}\left(X_i \frac{\partial f_l}{\partial x_i} + Y_i \frac{\partial f_l}{\partial y_i} + Z_i \frac{\partial f_l}{\partial z_i}\right) + F_l \tag{16}$$

denotes the values that the complete second derivative of f_l would assume for the *free* motion of the point, so Φ_{lh} and Φ_l^0 will also be quantities whose instantaneous values are known completely, then substituting the values of the x_i'', y_i'', z_i'' in equations (13) in the k equations (10) will yield the following k linear equations for the determination of the multipliers λ_h themselves:

$$\sum_{h=1}^{k} \Phi_{gh}\,\lambda_h = \Phi_g^0 \qquad (g = 1, 2, ..., k)\,. \tag{17}$$

One also sees directly that these k equations will actually determine the k unknown λ_1, λ_2, …, λ_k, so their determinant:

$$\Delta_k \equiv \sum \pm \Phi_{11}\,\Phi_{22}\cdots\Phi_{kk} \tag{18}$$

will not be zero.

Namely, due to equations (14), and from (15), the sum (6) will obtain the value:

$$2\Omega \equiv \sum_{g=1}^{k}\sum_{h=1}^{k} \Phi_{gh}\,\lambda_g\,\lambda_h\,.$$

However, that sum is always positive and will vanish only when all 3n differences:

$$\frac{X_i}{m_i} - x_i'', \qquad \frac{Y_i}{m_i} - y_i'', \qquad \frac{Z_i}{m_i} - z_i''$$

vanish simultaneously. Moreover, among the 3n equations (14), there will already be k of them whose derivatives determine the k multipliers λ_h as linear, homogeneous functions of k of those differences.

The values of those $3n$ differences that are implied by equations (14) can then vanish simultaneously only when all $\lambda_h = 0$. Therefore, the value of 2Ω that the sum (6) assumes because of equations (14) will be a positive-definite form of the variables λ_1, λ_2, …, λ_k, and that is known to involve, *eo ipso*, the fact that Δ_k *is non-zero*, and indeed *will necessarily be positive*, which will be used in the following §.

The k linear equations (17) then, in fact, determine their k unknowns λ_h, and when one substitutes the solutions in equations (14), one will get the $3n$ unknowns x_i'', y_i'', z_i'' themselves uniquely and expressed in terms of nothing but quantities whose values are known at time t.

With that, the question that was posed at the beginning of this § is answered, and one sees *that it always admits only one solution.*

However, we must now ask: Were the assumptions that started with in the foregoing in a completely arbitrary way correct or not?

Certain criteria can be obtained for them, as well.

In fact, we must impose *even more conditions* upon our assumptions, as well as in the principle of least constraint, than the ones that we have fulfilled by way of equations (14) and (17).

Namely, for one thing, of the r conditions (5) that restrict the motion of our system at the moment t, up to now only the first k were satisfied, and indeed, they were satisfied as a result of the k equations (10). If our assumptions were applicable then the last $r - k$ conditions (5) would be fulfilled by our solution automatically.

However, by substituting the values of the x_i'', y_i'', z_i'' in equations (14), from (15), (16), and (17), the $r - k$ expressions (11) would take on the values:

$$f_\gamma'' = \Phi_\gamma^0 - \sum_{h=1}^{k} \Phi_{\gamma h}\, \lambda_h \equiv \Phi_\gamma^k \qquad (\gamma = k+1, \ldots, r), \tag{19}$$

in which Φ_γ^k shall denote the value that the linear function of the λ_h on the left-hand side would assume if one were to substitute the solutions of equations (17). These values of Φ_γ^k are also once more merely functions of time, the coordinates and the velocities, and as such, can likewise be determined completely at the moment t.

Our assumptions can therefore next be true in any case only when every $\Phi_\gamma^k \le 0$. *By contrast, whenever any* $\Phi_\gamma^k > 0$ *that assumption would be false*, and the last $r - k$ conditions (5) would instantaneously prevent the motion that we have calculated from taking place.

Furthermore, the formulas (14) will convert equation (13) into an identity that is valid for all arbitrary values of the variations $\delta x_i''$, $\delta y_i''$, $\delta z_i''$, and will thus reduce our original requirement (9) to:

$$\sum_{h=1}^{k} \lambda_h\, \delta f_h'' \le 0.$$

In addition, this condition must then be fulfilled for all values of the variations that fulfill the k conditions (12), so the ones that make each:

$$\delta f_h'' \leq 0.$$

However, it is necessary and sufficient for this to be true that no $\lambda_h < 0$. The principle of least constraint then appends the following k conditions to our equations (14) and (17):

(20) $$\lambda_h > 0 \qquad (h = 1, 2, \ldots, k)$$

in which the > sign should not exclude equality.

Our assumptions will, in turn, be false whenever equations (17) *imply a negative value for any* λ_h , and once more the system cannot be instantaneously capable of performing the motion that is determined by equations (14) and (17).

Conversely, however, *when the accelerations that are calculated from* (14) *and* (17) *are not associated with negative* λ_h *or positive* Φ_γ^k, one can conclude that *they will give the true instantaneous accelerations of the system points* m_i .

Strictly speaking, in order to leave no doubt in regard to this conclusion, one must obviously first show that *it is only by that one way of fulfilling the r instantaneous system conditions* (5) *that one will arrive at values for the* x_i'', y_i'', z_i'' *that fulfill not only those conditions, but also all of the demands of the principle of least constraint.* The proof of that will be provided in the following § for the two simplest cases of $r = 1$ and $r = 2$. By contrast, its implementation can present very great difficulties for arbitrary r. On the other hand, for that reason, one might also regard it as obvious that two different systems of accelerations with the given properties cannot exist, since if they did exist, there would exist no means whatsoever of establishing which of the two systems is the correct one, because whether or not the sum (6) is an actual minimum is entirely irrelevant for mechanics, and anyway, due to the conditions (5) and as a result of equations (10) and (14), that sum will, in fact, actually attain a smallest value whenever it leads to either a negative λ_h *or positive* Φ_γ^k.

§ 3. – Proof of the uniqueness of the solution in the two simplest cases $r = 1$ and $r = 2$.

Although, on the grounds that were just cited, one can probably consider the proof of the uniqueness of the solution to be a much-too-tedious point of rigor, on the other hand, it is still important to recall the different steps that one must take from the foregoing in order to ascertain the true instantaneous motion in the various possible cases. Therefore, the following complete discussion of the two simplest cases $r = 1$ and $r = 2$ might not be superfluous in its own right.

Therefore, first let $r = 1$, so *only one condition* (5) *is present*, such that the number k of equations (10) can possess only the values 0 and 1. At the moment t considered, the accelerations are now subordinate to only one condition:

(5′) $$f_1'' \leq 0.$$

In order to find the true instantaneous accelerations then, one must first calculate the value Φ_1^0 that f_1'' assumes for the accelerations:

$$(0)\qquad x_i'' = \frac{X_i}{m_i},\ y_i'' = \frac{Y_i}{m_i},\qquad z_i'' = \frac{Z_i}{m_i}$$

that the points m_i would possess instantaneously for *free* motion.

If that yields $\Phi_1^0 \leq 0$ then there would be nothing present instantaneously that could prevent the point from exhibiting that free motion, and that will also occur necessarily.

By contrast, if $\Phi_1^0 > 0$ then the condition (5′) will not allow the free motion to take place, and the true accelerations must then satisfy the equation:

$$(10')\qquad f_1'' = 0.$$

From (14) and (17), one can then determine those accelerations from the equations:

$$(1)\qquad \begin{cases} m_i\, x_i'' = X_i - \lambda_1 \dfrac{\partial f_1}{\partial x_i}, \quad m_i\, y_i'' = Y_i - \lambda_1 \dfrac{\partial f_1}{\partial y_i}, \quad m_i\, z_i'' = Z_i - \lambda_1 \dfrac{\partial f_1}{\partial z_i}, \\ \Phi_{11}\lambda_1 = \Phi_1^0, \end{cases}$$

and due to that fact that $\Phi_{11} > 0$ and the assumption $\Phi_1^0 > 0$, those equations will, in fact, also imply that $\lambda_1 > 0$.

By contrast, in the case of $\Phi_1^0 < 0$, they would imply that $\lambda_1 < 0$, and that would directly characterize the accelerations (1) as incorrect, while for $\Phi_1^0 = 0$, one would also have $\lambda_1 = 0$, and the *constrained motion* (1) would coincide with the *free* motion (0).

Now let $r = 2$, such that the two conditions to be fulfilled at the moment t are:

$$(5'')\qquad f_1'' \leq 0, \qquad f_2'' \leq 0,$$

and k can assume the values 0, 1, 2.

We then first calculate the values Φ_1^0 and Φ_2^0 that f_1'' and f_2'' assume for the free accelerations (0).

If both values are ≤ 0 then neither of the two conditions (5′) will pose an obstacle to the free motion, which it will already be the true instantaneous motion then.

By contrast, if Φ_1^0 and Φ_2^0 are not both ≤ 0 *the let:*

$$\Phi_1^0 > 0$$

in any case. The first condition (5″) will oppose the free motion then, and we must once more examine the motion (1), which satisfies equation (10′), and only that equation. For that motion, we will have:

(1) $$f_2'' = \Phi_2^0 - \Phi_{21}\,\lambda_1 = \Phi_2^0 - \Phi_1^0\,\frac{\Phi_{12}}{\Phi_{11}} \equiv \Phi_2^1 .$$

Therefore, if $\Phi_2^1 \le 0$ then nothing prevents the system from exhibiting the motion (1), and that will therefore also occur presently.

By contrast, if $\Phi_2^1 > 0$ then the second equation (5″) will also prevent the acceleration (1) from taking place, and one will then have to discuss *the motion* (2) that would occur if only that second condition were in effect at the moment considered.

However, that motion (2) will itself be once more different according to whether one has:

$$\Phi_2^0 \le 0 \qquad \text{or} \qquad > 0 .$$

In the first case, the motion (2) is nothing but the free motion (0), and therefore already impossible, due to the first condition (5″).

With our present assumption that:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 \le 0, \qquad \Phi_2^1 > 0,$$

the true accelerations must necessarily satisfy the two conditions:

(10″) $$f_1'' = 0, \qquad f_2'' = 0,$$

then. From (14) and (17), the accelerations can now be determined from the equations:

(1, 2) $$\left\{\begin{aligned} m_i\,x_i'' &= X_i - \lambda_1\frac{\partial f_1}{\partial x_i} - \lambda_2\frac{\partial f_2}{\partial x_i}, \\ m_i\,y_i'' &= Y_i - \lambda_1\frac{\partial f_1}{\partial y_i} - \lambda_2\frac{\partial f_2}{\partial y_i}, \\ m_i\,z_i'' &= Z_i - \lambda_1\frac{\partial f_1}{\partial z_i} - \lambda_2\frac{\partial f_2}{\partial z_i}, \end{aligned}\right.$$

and

(1, 2)′ $$\left\{\begin{aligned} \Phi_{11}\,\lambda_1 + \Phi_{12}\,\lambda_2 &= \Phi_1^0, \\ \Phi_{21}\,\lambda_1 + \Phi_{22}\,\lambda_2 &= \Phi_2^0. \end{aligned}\right.$$

In fact, if one sets:

$$\Delta_2 \equiv \Phi_{11}\,\Phi_{22} - \Phi_{12}\,\Phi_{21} ,$$

from (18), then since $\Phi_{21} \equiv \Phi_{12}$, the last two equations will give:

(1, 2)″ $$\left\{\begin{aligned} \Delta_2\,\lambda_1 &= \Phi_1^0\,\Phi_{22} - \Phi_2^0\,\Phi_{12}, \\ \Delta_2\,\lambda_2 &= -\Phi_1^0\,\Phi_{12} + \Phi_2^0\,\Phi_{11}. \end{aligned}\right.$$

However, from (1)′, the last equation can be written:

$$\Delta_2\,\lambda_2 = \Phi_{11}\,\Phi_2^1,$$

and therefore, since $\Delta_2 > 0$, $\Phi_{11} > 0$, along with $\Phi_2^1 > 0$, it will likewise also imply that $\lambda_2 > 0$.

It follows, moreover, from our assumptions:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 \leq 0, \qquad \Phi_2^1 \equiv \Phi_2^0 - \Phi_1^0 \frac{\Phi_{12}}{\Phi_{11}} > 0$$

that Φ_{12} must necessarily be < 0. Due to the fact that $\Phi_{22} > 0$, the first equation (1, 2)″ will then tell us that we also have $\lambda_1 > 0$.

One sees here how, in agreement with the objection that was made against **Ostrogradsky**'s argument at the conclusion of § **1**, although the system condition $f_2'' \leq 0$ in itself presents no obstacle to the *free* motion, nevertheless, due to the assumption $\Phi_2^0 \leq 0$, the *actual* motion will be subject to the restriction that $f_2'' = 0$, as a result of the other condition that $f_1'' \leq 0$.

By contrast, if $\Phi_2^0 > 0$ then the accelerations (2) can be determined from the equations:

$$(2)\qquad \begin{cases} m_i\,x_i'' = X_i - \lambda_2 \dfrac{\partial f_2}{\partial x_i}, \quad m_i\,y_i'' = Y_i - \lambda_2 \dfrac{\partial f_2}{\partial y_i}, \quad m_i\,z_i'' = Z_i - \lambda_2 \dfrac{\partial f_2}{\partial z_i}, \\ \Phi_{22}\lambda_2 = \Phi_2^0, \end{cases}$$

and one will have:

$$(2)'\qquad f_1'' = \Phi_1^0 - \Phi_{12}\,\lambda_2 = \Phi_1^0 - \Phi_2^0 \frac{\Phi_{12}}{\Phi_{22}} \equiv \Phi_1^2$$

for them.

Therefore, if $\Phi_1^2 \leq 0$, or the case:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 > 0, \qquad \Phi_2^1 > 0, \qquad \Phi_1^2 \leq 0$$

occurs, then the accelerations (2), in which $\lambda_2 > 0$, will already be the true accelerations.

Since $\Phi_1^0 > 0$, now, as opposed to before, the condition $f_1'' \leq 0$ will be a real obstacle to the *free* motion, and therefore it will imply no restriction on the *actual* motion.

By contrast, if $\Phi_1^2 > 0$ then one will have the inequalities:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 > 0, \qquad \Phi_2^1 > 0, \qquad \Phi_1^2 > 0,$$

so the first condition (5″) will allow the accelerations (2) to take place just as little as it will the second of the accelerations (1). The actual accelerations must then once more necessarily satisfy equations (1, 2), and since, from (1)′ and (2)′, equations (1, 2)″ can be written:

$$\Delta_2\, \lambda_1 = \Phi_{22}\, \Phi_1^2\,, \qquad \Delta_2\, \lambda_2 = \Phi_{11}\, \Phi_2^1\,,$$

λ_1 and λ_2 will, in fact, both be < 0 now, as well.

With that, the case of $r = 2$ is also dealt with, and one sees quite clearly how among the various possible motions of the system, ultimately there will always be *just one of them that is determined completely* to be the one that represents the true instantaneous motion.

However, at the same time, the foregoing argument also illuminates the fact that for larger values of r, in some situations, a good number of detailed investigations might be necessary before one is fortunate enough to discover which of equations (10) and (14) determines the true instantaneous motion of the system.

§ 4. – Determining the motion of the system during an arbitrary finite time interval.

However, once one has found those equations, one can also try to pursue the further motion of the system during an arbitrary finite time interval with the help of them ([1]).

To that end, one must integrate the system of $3n$ second-order differential equations in the $3n$ coordinates x_i , y_i , z_i , and time t that equations (14) go to upon substituting the values of λ from equations (17), which is an integration in which one must also appeal to the $2k$ equations:

$$f_g' = 0 \qquad \text{and} \qquad f_g = 0 \qquad (g = 1, 2, \ldots, k).$$

In other words, one must look for the motion that the system would execute if it were subject to the k condition equations $f_g = 0$, and no further restrictions. The integration constants are determined from the known initial state of the system at time t, which, by assumption, will fulfill the $2\,(r - k)$ equations:

$$f_\gamma' = 0 \qquad \text{and} \qquad f_\gamma = 0 \qquad (\gamma = k + 1, 2, \ldots, r),$$

in addition to those $2k$ equations. If one has succeeded in calculating the coordinates of the system-point, and therefore the multipliers λ, as well, in that way then if the system is subordinate to only the k conditions:

$$f_1 \le 0, \quad \ldots, f_k \le 0$$

from the outset (which assumes that $r = k$, in particular), those functions will represent the *true* motion of the system *whenever none of the multipliers λ are negative.*

([1]) Déplacements, § 13.

By contrast, if those k conditions define only a part of the system conditions (1) then the fact that the multipliers remain positive alone will still not suffice to ensure that the calculated motion is the true system motion, since one or more of the other system conditions that do not restrict the motion of the system at the initial moment t can come into play in some situations that would prevent the system from proceeding with its initial motion.

In order to get information about that, so to decide whether any of the system conditions that are in effect might meanwhile make the persistence of the motion that is determined by equations (10) and (14) impossible, by substituting the functions of time that are obtained for the coordinates and multipliers, one must further examine:

First of all, whether the $r - k$ quantities Φ_γ^k that are defined by equations (19) and (17) are all ≤ 0 at the initial moment t (viz., the assumption that $f_\gamma = 0$, $f'_\gamma = 0$), and

Secondly, whether the functions f_σ in the original system conditions $f_\sigma \leq 0$ either possess only negative values (the case of $f_\sigma < 0$) or go straight through zero to negative values (the case of $f_\sigma = 0$, $f'_\sigma = 0$) at time t, since none of them are positive.

The motion that is calculated from (10) *and* (14) *will coincide with the true motion whenever either one of all those functions or any of the multipliers changes signs.*

By contrast, as soon as one or more of the aforementioned quantities changes its sign by going through zero at a certain moment t_1 , that coincide will persist from that moment on, and one must then once more address the problem from the beginning with those initial values of the coordinates and velocities that the calculated motion implies for $t = t_1$.

However, when such a later deviation of the calculated motion of the system from the true one is not predicted by merely the sign change of any multiplier λ, the tool that was obtained in the foregoing might become *inadequate*; i.e., by itself, it would not succeed in solving the new problem for the new values that the accelerations of the system points achieve at the moment $t = t_1$. Namely, whenever the point that has moved according to any system condition $f_\sigma \leq 0$ up to now such that one continues to have $f_\sigma < 0$, and with a velocity for which f'_σ is not exactly zero, but possesses a finite positive value, gets into a position in which $f_\sigma = 0$ (so whenever, e.g., a connecting string that was loose up to now suddenly tensed violently or a system-point impinges upon a rigid wall), a *shock* to the system will always arise. The velocities of the old motion will then suddenly come into conflict with the system conditions, and must first be regulated in such a way that those conditions are once more obeyed, and those regularized shock velocities will be the initial velocities of the new motion. One then sees how it can become necessary to appeal to the *theory of shocks* in order to determine the system motion during a finite time interval from time to time ([1]).

Finally, it is self-explanatory that when the constraints and restrictions on the point-system in question are defined by not only inequalities, but partly by *inequalities* and partly by *condition equations*, only the condition equations on the system:

([1]) Cf., the following article.

$$\varphi_\rho = 0 \qquad (\rho = 1, 2, \ldots)$$

which are always valid, will have to be appended to those condition equations:

$$f_g \leq 0 \qquad (g = 1, 2, \ldots, k)$$

that actually restrict the mobility of the system at the moment t considered. In that way, terms of the form:

$$\mu_1 \frac{\partial \varphi_1}{\partial x_i} + \mu_2 \frac{\partial \varphi_2}{\partial x_i} + \cdots,$$

$$\mu_1 \frac{\partial \varphi_1}{\partial y_i} + \mu_2 \frac{\partial \varphi_2}{\partial y_i} + \cdots,$$

$$\mu_1 \frac{\partial \varphi_1}{\partial z_i} + \mu_2 \frac{\partial \varphi_2}{\partial z_i} + \cdots$$

will occur in the right-hand sides of equations (14), and equations (17) will be altered accordingly. However, the new multipliers μ_1, μ_2, … can have arbitrary signs now and will no longer be subject to the condition that they must be positive, unlike the λ.

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