"Ueber eine Erweiterung der Lie'schen Integrationsmethode," Math. Ann. **8** (1875), 313-318.

On an extension of Lie's integration method *).

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Under the assumption that $H_1, H_2, \ldots, H_l$ are mutually independent functions of the $2n$ variables:

$$q_1, \ldots, q_n, p_1 = \frac{\partial V}{\partial q_1}, \ldots, p_n = \frac{\partial V}{\partial q_n},$$

and that these functions pair-wise fulfill the conditions:

$$[H_i, H_k] = 0,$$

the Lie method of integration yields the well-known theorem:

I. *If $H_1, H_2, \ldots, H_l$ are mutually independent relative to the differential quotients p then the simultaneous integrations of the r partial differential equations:*

(1) $$H_1 = h_1, H_2 = h_2, \ldots, H_l = h_l,$$

comes down to the complete integration of a single partial differential equation with only $n - r + 1$ independent variables.

By contrast, if the quantities p can be eliminated completely from equations (1) – always under the assumption that was given above – then these equations have no common solution at all, and therefore one can also no longer speak of a simultaneous integration of them. However, just the same, those of equations (1) that are mutually independent relative to the p define a Jacobi system in their own right, and if the Lie method may actually be performed on it, which my simplification of the Jacobi method affords, with its improvement by Lie **), then, with no recourse to extra considerations, this will show that in the latter case one can also reduce each Jacobi system that is contained in equations (1) with the help of the remaining equations (1), to a single partial differential equation that possesses only $n - r + 1$ independent variables.

The objective of the following communication ***) is to achieve this proof, and therefore to give the Lie method of integration the same generality that was Lie's purpose

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*) Cf., these Annalen, Bd. VI, pp. 162.

**) Cf., pp. 240.

***) Indeed, a previous Note (Göttinger Nachr., 1873, No. 11) already pursued a similar objective, but considered only the special case in which $r - 1$ of equations (1) define a Jacobi system.

in regard to the Jacobi method. The desire to extend my previous paper in volume VI of these Annalen in such a way that both of them together define a unified complete picture of Lie's method, so to speak, might be excused if I, in part, also reproduce known things anew, such as the aforementioned theorem I, in fact.

Along with the known symbol:

$$(F, \Phi) = \sum_{h=1}^{h=n} \left(\frac{\partial F}{\partial q_h} \frac{\partial \Phi}{\partial p_h} - \frac{\partial F}{\partial p_h} \frac{\partial \Phi}{\partial q_h} \right),$$

in the following, if F and Φ are regarded as functions of other variables $q_1', \ldots, q_n', p_1', \ldots, p_n'$, I will use the notation:

$$(F, \Phi)' = \sum_{h=1}^{h=n} \left(\frac{\partial F}{\partial q_h'} \frac{\partial \Phi}{\partial p_h'} - \frac{\partial F}{\partial p_h'} \frac{\partial \Phi}{\partial q_h'} \right).$$

From this definition, one immediately infers the theorem:

II. *If, in the functions F and Φ, one exchanges the variables:*

$$q_1, \ldots, q_m, q_{m+1}, \ldots, q_n, p_1, \ldots, p_m, p_{m+1}, \ldots, p_n$$

with:

$$q_1', \ldots, q_m', p_{m+1}', \ldots, p_n', -p_1', \ldots, -p_m', q_{m+1}', \ldots, q_n',$$

resp., then (F, Φ) *goes over to* – $(F, \Phi)'$.

Now, having established this, let $H_1, H_2, \ldots, H_m$ be functions of $q_1, \ldots, q_n, p_1, \ldots, p_n$ that pair-wise satisfy the conditions:

$$(H_i, H_k) = 0.$$

I assume that the variables $p_1, p_2, \ldots, p_m$ can be determined from the m equations:

(2) $$H_1 = h_1, H_2 = h_2, \ldots, H_m = h_m,$$

in which $h_1, h_2, \ldots, h_m$ refer to arbitrary constants, and denote the values thus obtained by:

(3) $$p_1 = F_1, p_2 = F_2, \ldots, p_m = F_m,$$

where only the variables $q_1, \ldots, q_n, p_{m+1}, \ldots, p_n$ enter into the functions $F_1, F_2, \ldots, F_m$.

By the substitutions (3), the equations $H_i = h_i$, along with the equation:

$$\frac{\partial H_i}{\partial x} + \sum_{\lambda=1}^{\lambda=m} \frac{\partial H_i}{\partial p_\lambda} \frac{\partial F_\lambda}{\partial x} = 0$$

will then be fulfilled immediately, as long as one understands x to mean one of the variables $q_1, \ldots, q_n, p_{m+1}, \ldots, p_n$. However, since each $\partial p_\lambda / \partial x = 0$ in this assumption, one can also write the latter equation thus:

$$\frac{\partial H_i}{\partial x} + \sum_{\lambda=1}^{\lambda=m} \frac{\partial H_i}{\partial p_\lambda} \frac{\partial (F_\lambda - p_\lambda)}{\partial x} = 0 \tag{4}$$

and in this form, one immediately sees that the equation also remains correct for $x = p_1, p_2, \ldots, p_m$. Now, the expression (H_i, H_k) is a linear, homogeneous function of the differential quotients of H_i . If one then substitutes the values of these differential quotients that follow from (4) then this yields:

$$(H_i, H_k) = -\sum_{\lambda=1}^{\lambda=m} \frac{\partial H_i}{\partial p_\lambda}(F_\lambda - p_\lambda, H_\lambda),$$

and when one also applies formula (4) to the differential quotients, one sees that from the substitutions (3), one will have identically:

$$(H_i, H_k) = \sum_{\lambda=1}^{\lambda=m} \frac{\partial H_i}{\partial p_\lambda} \sum_{\mu=1}^{\mu=m} \frac{\partial H_k}{\partial p_\mu}(F_\lambda - p_\lambda, F_\mu - p_\mu).$$

If one now imagines that the determinant:

$$\sum \pm \frac{\partial H_1}{\partial p_1} \frac{\partial H_2}{\partial p_2} \cdots \frac{\partial H_m}{\partial p_m},$$

since it is not in itself zero, can also not vanish when one sets the quantities h that do not enter into them at all equal to the given functions H, then this immediately implies, from the formula obtained, when one first applies the m identities:

$$(H_i, H_k) = 0, \ldots, (H_m, H_k) = 0,$$

and then considers that the same thing shall also be true for $k = 1, 2, \ldots, m$, that each of the:

$$(F_\lambda - p_\lambda, F_\mu - p_\mu)$$

is identically zero as a consequence of our assumptions. However, it follows from theorem IX of my previous paper that the m partial differential equations (3), and therefore, also the given ones (2), can be reduced to a single partial differential equation with only $n - m + 1$ independent variables, from which theorem I is proved.

However, we assume that we are given any $m + s$ independent functions $H_1, H_2, \ldots, H_{m+s}$, of the variables $q_1, \ldots, q_n, p_1, \ldots, p_n$, which fulfill the conditions:

$$(H_i, H_k) = 0,$$

but are so arranged that the differential quotients p can be eliminated completely from the $m + s$ equations:

(5) $$H_1 = h_1,\ H_2 = h_2,\ \ldots, H_{m+s} = h_{m+s}.$$

Since the case in which all $m + s$ functions H are free of the p makes no sense, I would like to assume that *perhaps the first m equations* (5) *can be solved for* $p_1, p_2, \ldots, p_m$, *while, by substitution, these solutions might make the last s equations* (5) *free of all p.*

In this assumption, from the foregoing, the equations:

(2) $$H_1 = h_1,\ H_2 = h_2,\ \ldots, H_m = h_m$$

define a Jacobi system, and one now asks how one can, without abandoning the Lie method, realize the integration of this Jacobi system for the s remaining equations (5).

In order to answer this question, we must first examine to which of the $m + s$ of the $2n$ variables q and p the functions $H_1, H_2, \ldots, H_{m+s}$ are mutually independent, under our assumptions.

If we again denote the solutions of equations (2) by:

(3) $$p_1 = F_1,\ p_2 = F_2,\ \ldots, p_m = F_m$$

then we known from the foregoing that each:

$$(F_\lambda - p_\lambda, F_\mu - p_\mu) = 0.$$

Furthermore, if we understand $\Phi_{m+1}, \ldots, \Phi_{m+s}$ to mean the values that the functions $H_{m+1}, \ldots, H_{m+s}$ take on under the substitutions (3) then for $k = m + 1, \ldots, m + s$, and $x = q_1, \ldots, q_n, p_{m+1}, \ldots, p_n$, along with the equation $H_k = \Phi_k$, these substitutions also likewise fulfill the equation:

$$\frac{\partial \Phi_k}{\partial x} = \frac{\partial H_\lambda}{\partial x} + \sum_{\mu=1}^{\mu=m} \frac{\partial H_k}{\partial p_\mu}\frac{\partial F_\mu}{\partial x},$$

which, when one gives it the form:

$$\frac{\partial \Phi_k}{\partial x} = \frac{\partial H_\lambda}{\partial x} + \sum_{\mu=1}^{\mu=m} \frac{\partial H_k}{\partial p_\mu}\frac{\partial (F_\mu - p_\mu)}{\partial x},$$

also keeps its validity for $x = p_1, \ldots, p_m$. From this and (4), one will thus have by the substitutions (3) for $i = 1, 2, \ldots, m$ and $k = m + 1, \ldots, m + s$:

$$(H_i, H_k) = -\sum_{\lambda=1}^{\lambda=m} \frac{\partial H_i}{\partial p_\lambda}(F_\lambda - p_\lambda, \Phi_k) + \sum_{\lambda=1}^{\lambda=m}\sum_{\mu=1}^{\mu=m} \frac{\partial H_i}{\partial p_\lambda}\frac{\partial H_k}{\partial p_\mu}(F_\lambda - p_\lambda, F_\mu - p_\mu),$$

from which, it follows that the identities:

$$(H_i, H_k) = 0 \qquad \text{and} \qquad (F_\lambda - p_\lambda\,, F_\mu - p_\mu) = 0$$

imply the following one:

(6) $$(F_\lambda - p_\lambda\,, \Phi_k) = 0.$$

Now, by assumption, the functions $\Phi_{m+1}, \ldots, \Phi_{m+s}$, as the values that the $H_{m+1}, \ldots, H_{m+s}$ take on by way of the substitutions (3), are free of all the p and mutually independent functions of the variables q. From (6), therefore, each of them is a common solution of the m equations:

$$\frac{\partial \Phi}{\partial q_\lambda} - \sum_{h=m+1}^{h=n} \frac{\partial F_\lambda}{\partial p_h} \frac{\partial \Phi}{\partial q_h} = 0.$$

However, from this, it follows immediately that $\Phi_{m+1}, \ldots, \Phi_{m+s}$ are mutually independent relative to s of the variables:

$$q_{m+1}, \ldots, q_n\,.$$

The assumption:

$$\Phi_{m+s} = \varphi(\Phi_{m+1}, \ldots, \Phi_{m+s-1}, q_1, \ldots, q_m)$$

would then yield:

$$\frac{\partial \varphi}{\partial q_\lambda} = 0,$$

for $\lambda = 1, 2, \ldots, m$, and, as a result, it would contradict the independence of the functions $\Phi_{m+1}, \ldots, \Phi_{m+s}$.

With this, we have achieved the theorem:

III. *Let $H_1, H_2, \ldots, H_{m+1}$ be mutually independent functions of the variables $q_1, \ldots, q_n, p_1, \ldots, p_n$ that pair-wise satisfy the conditions:*

$$(H_i, H_k) = 0.$$

If the quantities $p_1, p_2, \ldots, p_m$ can be determined from m of the equations:

$$H_1 = h_1, H_2 = h_2, \ldots, H_{m+s} = h_{m+s}\,,$$

and by substitution of these values, the s remaining equations will be free of all the variables p then the functions $H_1, H_2, \ldots, H_{m+s}$ are independent of each other relative to the $p_1, \ldots, p_m$ and s of the variables $q_{m+1}, \ldots, q_m$.

In order to arrive at the answer to the question that we posed, moreover, we need only to apply the following theorem to our Jacobi system (2), which immediately appears to be a special case of theorem II in my previous treatise when one takes:

$$\varphi = c_m\, q_m + \ldots + c_n\, q_n$$

and suitably alters the notation of the variables:

IV. *If the m equations:*

$$H_1 = h_1,\ H_2 = h_2,\ \ldots,\ H_m = h_m, \qquad \left(p_i = \frac{\partial V}{\partial q_1}\right),$$

which are soluble in terms of $p_1, p_2, \ldots, p_m$, *define a Jacobi system then the same thing is also true for the m equations:*

$$H_1' = h_1,\ H_2' = h_2,\ \ldots,\ H_m' = h_m, \qquad \left(p_i' = \frac{\partial V}{\partial q_1'}\right),$$

into which the foregoing are converted when one exchanges:

$$q_1, \ldots, q_m, q_{m+1}, \ldots, q_n, p_1, p_2, \ldots, p_m, p_{m+1}, \ldots, p_n,$$

with:

$$q_1', \ldots, q_m', p_{m+1}', \ldots, p_n', -p_1', \ldots, -p_m', q_{m+1}', \ldots, q_n',$$

resp., and one can obtain a complete solution to the latter system by just algebraic operations on an arbitrary complete solution of the former one.

The functions $H_1', H_2', \ldots, H_{m+s}'$, which, in fact, arise from our functions $H_1, H_2, \ldots, H_{m+s}$ by the given exchange, are, from theorem III, independent of each other relative to the differential quotients p'. Due to the identities $(H_i, H_k) = 0$, one also has that each $(H_i', H_k') = 0$ from theorem II, moreover. From theorem I, therefore, the $m + s$ *equations:*

$$H_1' = h_1,\ H_2' = h_2,\ \ldots,\ H_{m+s}' = h_{m+s}$$

define a Jacobi system whose integration can be reduced to the complete integration of a single partial differential equation with only $n - m - s + 1$ independent variables. Every complete solution of this Jacobi system is, however, at the same time a complete solution of the system:

$$H_1' = h_1,\ H_2' = h_2,\ \ldots,\ H_m' = h_m,$$

and the given system (2) can be reduced to the latter one by theorem IV.

We thus obtain the following theorem as our ultimate result:

In order to be able to reduce the given Jacobi system of m partial differential equations with n independent variables:

$$H_i(q_1, \ldots, q_n, p_1, \ldots, p_n) = h_i, \qquad i = 1, 2, \ldots, m$$

to a single partial differential equation with only $n - m - s + 1$ *independent variables, it suffices to have found any s functions* $H_{m+1}, \ldots, H_{m+s}$ *that are independent of each other, as well as* $H_1, \ldots, H_m$*, and which fulfill all of the equations* $(H_i, H_k) = 0$.

If one would like to apply this theorem to the most important case of a single partial differential equation then one need only take $m = 1$. If one sets $s = n - 1$, moreover, then one obtains the theorem of Lie:

The complete integration of the given partial differential equation:

$$H_1(q_1, \ldots, q_n, p_1, \ldots, p_n) = h_1$$

always requires just one quadrature, as long as one has found any $n - 1$ *functions* $H_2, \ldots, H_n$ *that are independent of each other, as well as* H_1*, and which satisfy the demands that* $(H_i, H_k) = 0$.

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