“Ueber die Lies’schen Berührungstransformationen,” Göttinger Nachrichten **13** (1874), 317-331.

On Lie’s contact transformations

Prof. A. MAYER in Leipzig

Translated by D. H. Delphenich

In the paper “Zur analytischen Theorie der Berührungstransformationen [1]),” Lie has developed this new and important theory by regarding the problem of determining all contact transformations as a special case of the Pfaff problem and then applying the Clebsch reduction of the Pfaff problem to a system of simultaneous system of partial differential equation to it. Now, Clebsch has, however, given the stated reduction of the problem:

$$X_1\, dx_1 + X_1\, dx_1 + \ldots + X_{2n}\, dx_{2n} = 0$$

directly, only under the restricting assumption that the determinant that one constructs from the elements:

$$a_{ik} = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i},$$

namely:

$$\sum \pm a_{11}\, a_{22} \ldots a_{2n\,2n},$$

is non-zero, and this assumption does not necessarily need to be fulfilled in the case of contact transformations. Moreover, it is my opinion that either the manner by which Clebsch obtained a general reduction or the application of it to the problem of contact transformations are simple enough for one to accept them as a completely satisfactory basis for a theory that is of fundamental significance for partial differential equations, as well as in itself, and in particular, when one couples it with the Cauchy method the solution of the problem of integrating a given partial differential equation of first order takes on a simplicity and generality that has been hitherto unattained [2]).

Starting with these considerations, I have sought to derive the basic formulas for the theory of the contact transformations directly and independently of the Pfaff problem, and I hope that the derivation that is communicated in what follows is, in fact, simpler and more rigorous than the original basis that was given by Lie.

In order to not split off into neighboring considerations later on in the course of the investigation, I shall start by stating the following theorem on functional determinants:

Let $X_0, X_1, \ldots, X_n$ be $n + 1$ functions of $n + h + 1$ variables $x_0, x_1, \ldots, x_{n+h}$, and in general, let:

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[1]) Abhandl. d. Gesellsch. d. Wissensch. zu Christiania, 1873, pp. 237.

[2]) Cf., Lie, “Ueber eine Verbesserung der Jacobi-Mayer’schen Integrations-Methode,” Abh. d. G. d. W. zu Christiania, 1873, pp. 282.

$$a_i^k = \frac{\partial X_i}{\partial x_k}.$$

Therefore, if all of the determinants of the form:

$$(0\ k_1 \ldots k_n) = \sum \pm a_0^0 a_1^{k_1} \cdots a_n^{k_n}$$

vanish, in which $k_1, k_2, \ldots, k_n$ mean any n of the numbers 1, 2, …, $n+h$, then, assuming that not all of the $n + 1$ functions are free of x_0, all of the determinants of the form:

$$(k_1\ k_1 \ldots k_{n+1}) = \sum \pm a_0^{k_1} a_1^{k_2} \cdots a_n^{k_{n+1}},$$

vanish, in which $k_1, k_2, \ldots, k_{n+1}$ are $n+1$ arbitrary numbers from the sequence 0, 1, …, $n+h$, and the functions $X_0, X_1, \ldots, X_n$ are thus not independent of each other.

The proof is very simple. Namely, if one lets A denote the determinant:

$$A = \sum \pm a_{n+1}^0 a_0^{k_1} a_1^{k_2} \cdots a_n^{k_{n+1}},$$

in which the $a_{n+1}^0, a_{n+1}^{k_1}, \ldots, a_{n+1}^{k_{n+1}}$ are arbitrary new elements, then, by assumption, the theorem takes the form:

$$\frac{\partial A}{\partial a_{n+1}^{k_{n+1}}} = \frac{\partial A}{\partial a_{n+1}^{k_n}} = \ldots = \frac{\partial A}{\partial a_{n+1}^{k_1}} = 0,$$

while the quantities $a_0^0, a_1^0, \ldots, a_n^0$ are not simultaneously zero. From the identities:

$$0 = a_i^0 \frac{\partial A}{\partial a_{n+1}^0} + a_i^{k_i} \frac{\partial A}{\partial a_{n+1}^{k_1}} + \cdots + a_i^{k_{n+1}} \frac{\partial A}{\partial a_{n+1}^{k_{n+1}}},$$

in which $i = 0, 1, \ldots, n$, this immediately implies that:

$$0 = \frac{\partial A}{\partial a_{n+1}^0} = (k_1, k_2, \ldots, k_{n+1}),$$

which is just what was to be proved.

The general problem of contact transformations is connected with the following problem:

Determine $Z\ X_1, \ldots, X_n, P_1, \ldots, P_n$ *as functions of the* $2n+1$ *independent variables* $z\ x_1, \ldots, x_n, p_1, \ldots, p_n$ *in such a way that one has:*

$$dZ - \sum_{i=1}^{i=n} P_i dX_i = r\ (dz - \sum_{k=1}^{k=n} p_k dx_k),$$

where ρ is any function of each variable.

This requirement decomposes immediately into the $2n + 1$ equations:

1) $$\frac{\partial Z}{\partial z} - \sum_{i=1}^{i=n} P_i \frac{\partial X_i}{\partial z} = \rho,$$

2) $$\frac{\partial Z}{\partial x_k} - \sum_{i=1}^{i=n} P_i \frac{\partial X_i}{\partial x_k} = -\rho p_k,$$

3) $$\frac{\partial Z}{\partial p_k} - \sum_{i=1}^{i=n} P_i \frac{\partial X_i}{\partial p_k} = 0.$$

However, if, for brevity, one lets the symbol:

$$\frac{d}{dx_k} \text{ denote the operation } \frac{\partial}{\partial x_k} + p_k \frac{\partial}{\partial z}$$

then, from1), equations 2) can be replaced by the following ones:

$$\frac{dZ}{dx_k} - \sum_{i=1}^{i=n} P_i \frac{dX_i}{dx_k} = 0.$$

The present problem will then be solved by the $2n$ equations:

4) $$\begin{cases} A = \dfrac{\partial Z}{\partial x_k} - \sum\limits_{i=1}^{i=n} P_i \dfrac{dX_i}{dx_k} = 0, \\ B_k = \dfrac{\partial Z}{\partial p_k} - \sum\limits_{i=1}^{i=n} P_i \dfrac{dX_i}{dp_k} = 0, \end{cases}$$

when coupled with equation 1).

Now, if one understands f to mean an arbitrary function of z $x_1, \ldots, x_n, p_1, \ldots, p_n$ then one has the identities:

$$\frac{d}{dx_k}\frac{df}{dx_h} - \frac{d}{dx_k}\frac{df}{dx_h} = 0,$$

$$\frac{\partial}{\partial p_k}\frac{df}{dx_h} - \frac{d}{dx_k}\frac{\partial f}{\partial p_h} = \frac{\partial p_k}{\partial p_h}\frac{\partial f}{\partial z},$$

$$\frac{\partial}{\partial p_h}\frac{\partial f}{\partial p_k} - \frac{\partial}{\partial p_k}\frac{\partial f}{\partial p_h} = 0,$$

and by applying them to equations 4), with the use of equation 1), this yields the following conditions for the functions *X* and *P*:

a)
$$\begin{cases} \sum_{i=1}^{i=n}\left(\frac{dP_i}{dx_h}\frac{dX_i}{dx_k}-\frac{dP_i}{dx_k}\frac{dX_i}{dx_h}\right)=0, \\ \sum_{i=1}^{i=n}\left(\frac{dP_i}{dx_h}\frac{dX_i}{dx_k}-\frac{dP_i}{dx_k}\frac{dX_i}{dx_h}\right)=\rho\frac{\partial p_k}{\partial p_h}, \\ \sum_{i=1}^{i=n}\left(\frac{dP_i}{dp_h}\frac{dX_i}{dp_k}-\frac{dP_i}{dp_k}\frac{dX_i}{dp_h}\right)=0. \end{cases}$$

However, these conditions show that the $2n$ linear equations:

b)
$$\begin{cases} u_i=\sum_{k=1}^{k=n}\left(\frac{dX_i}{dx_k}y_k+\frac{\partial X_i}{\partial p_k}z_k\right), \\ v_i=\sum_{k=1}^{k=n}\left(\frac{dP_i}{dx_k}y_k+\frac{\partial P_i}{\partial p_k}z_k\right), \end{cases}$$

must imply the following $2n$ equations:

c)
$$\begin{cases} \sum_{k=1}^{k=n}\left(u_i\frac{\partial X_i}{\partial x_h}-v_i\frac{\partial X_i}{\partial p_h}\right)=\rho y_h, \\ \sum_{k=1}^{k=n}\left(u_i\frac{dP_i}{dx_h}-v_i\frac{dX_i}{dx_h}\right)=-\rho z_h. \end{cases}$$

This immediately brings to light the fact that the determinant:

5)
$$R=\sum\pm\frac{dX_1}{dx_1}\cdots\frac{dX_1}{dx_n}\frac{\partial P_1}{\partial p_1}\cdots\frac{\partial P_n}{\partial p_\mu}$$

can only vanish simultaneously with ρ, and thus, since it is in the nature of our problem that ρ cannot vanish identically, it must necessarily be non-zero. Moreover, one deduces, since, conversely, equations b) must be fulfilled under the substitutions c), that one can replace equations a) by the following ones:

6)
$$\begin{cases} [X_i, X_\lambda]=[X_\lambda, P_i]=[P_i, P_\lambda]=0, \\ [X_\lambda, P_\lambda]=\rho, \end{cases}$$

where $[F, \Phi]$ generally means the expression:

$$[F, \Phi] = \sum_{h=1}^{h=n} \left(\frac{dF}{dx_h} \frac{\partial \Phi}{\partial p_h} - \frac{\partial F}{\partial p_h} \frac{d\Phi}{dx_h} \right).$$

As a result of conditions (6), however, the expressions A_k and B_k that are defined by 4) satisfy the identities:

$$\sum_{k=1}^{k=n} \left(A_k \frac{\partial X_\lambda}{\partial p_k} - B_k \frac{dX_\lambda}{dx_k} \right) = [Z, X_\lambda],$$

$$\sum_{k=1}^{k=n} \left(A_k \frac{\partial P_\lambda}{\partial p_k} - B_k \frac{dP_\lambda}{dx_k} \right) = [Z, P_\lambda] - \rho\, P_\lambda\,.$$

Due to the property of the determinant R that was proved above, the $2n$ equations 4) may be replaced by the following $2n$ equations:

$$[Z, X_\lambda] = 0, \qquad [Z, X_\lambda] = \rho\, P_\lambda\,,$$

and we then obtain the theorem:

I. *In order to have:*

$$dZ - \sum_{i=1}^{i=n} P_i\, dX_i = r\,(dz - \sum_{k=1}^{k=n} p_k\, dx_k\,)$$

it is necessary and sufficient that:

$$\text{7)} \qquad \begin{cases} [Z, X_\lambda] = [X_i, X_\lambda] = [X_\lambda, P_i] = [P_i, P_\lambda] = 0 \\ \qquad [X_\lambda, P_\lambda] = \rho, \quad [Z, P_\lambda] = \rho P_\lambda, \end{cases}$$

where:

$$r = \frac{\partial Z}{\partial z} - \sum_{i=1}^{i=n} P_i \frac{\partial X_i}{\partial z}\,.$$

One easily sees:

II. *The functions Z* $X_1, \ldots, X_n, P_1, \ldots, P_n$ *that solve the problem considered are independent of each other.*

The functional determinant:

$$\Delta = \sum \pm \frac{\partial Z}{\partial z} \frac{\partial X_1}{\partial x_1} \cdots \frac{\partial X_n}{\partial x_n} \frac{\partial P_1}{\partial p_1} \cdots \frac{\partial P_n}{\partial p_n}$$

may, in fact, be written:

$$\Delta = \sum \pm \frac{\partial Z}{\partial z} \frac{dX_1}{dx_1} \cdots \frac{dX_n}{dx_n} \frac{\partial P_1}{\partial p_1} \cdots \frac{\partial P_n}{\partial p_n}\,.$$

However, if one set the elements:

$$\frac{\partial Z}{\partial z}\frac{dZ}{dx_1}\cdots\frac{dZ}{dx_n}\frac{\partial Z_1}{\partial p_1}\cdots\frac{\partial Z_n}{\partial p_n}$$

equal to their values in equations 1) and 4), then one obtains immediately:

$$\Delta = \rho R,$$

which, from the foregoing, proves the assertion above.

On the other hand, one always has:

$$\sum_{k=1}^{k=n}\left(A_k\frac{\partial Z}{\partial p_k}-B_k\frac{dZ}{dx_k}\right) = -\sum_{i=1}^{i=n}P_i[Z, X_i],$$

$$\sum_{k=1}^{k=n}\left(A_k\frac{\partial X_\lambda}{\partial p_k}-B_k\frac{dX_\lambda}{dx_k}\right) = -\sum_{i=1}^{i=n}P_i[Z, X_i].$$

As long as one has found the $n + 1$ functions $Z, X_1, \ldots, X_n$, which fulfill the $n(n + 1)/2$ equations:

8) $\qquad [Z, X_\lambda] = 0, \qquad [X_i, X_\lambda] = 0,$

and for which, moreover, not all of the determinants of the forms:

$$\sum \pm \frac{dX_1}{dx_{k_1}}\cdots\frac{dX_n}{dx_{k_i}}\frac{\partial X_{i+1}}{\partial p_{k_{i+1}}}\cdots\frac{\partial X_n}{\partial p_{k_n}},$$

or:

$$\sum \pm \frac{dZ}{dx_{k_1}}\frac{dX_{\lambda_2}}{dx_{k_2}}\cdots\frac{dX_{\lambda_i}}{dx_{k_i}}\frac{\partial X_{\lambda_{i+1}}}{\partial p_{k_{i+1}}}\cdots\frac{\partial X_{\lambda_n}}{\partial p_{k_n}}$$

are null, one can always deduce n of the $2n$ equations 4) from the remaining n.

Now, all of the determinants cannot be zero, as long as every determinant of the form:

9) $$\sum \pm \frac{dZ}{dz}\frac{dX_1}{dx_{k_1}}\cdots\frac{dX_i}{dx_{k_i}}\frac{\partial X_{i+1}}{\partial p_{k_{i+1}}}\cdots\frac{\partial X_n}{\partial p_{k_n}} = \sum \pm \frac{dZ}{dz}\frac{dX_1}{dx_{k_1}}\cdots\frac{dX_i}{dx_{k_i}}\frac{\partial X_{i+1}}{\partial p_{k_{i+1}}}\cdots\frac{\partial X_n}{\partial p_{k_n}}$$

is non-zero, and the n equations:

$$A_{k_1} = 0, \qquad A_{k_i} = 0, \qquad B_{k_{i+1}} = 0, \qquad \ldots,\ B_{k_n} = 0,$$

when coupled with equation 1) uniquely determines the $n + 1$ remaining unknowns of the problem $P_1, \ldots, P_n$, and ρ. The determinants of the form 9), however, might not all be zero. As a result of equation 1), at least one of the functions Z, X_1, …, X_n must then include the variable z. From the theorem presented, the vanishing of all determinants of the form 9 identically emphasizes the independence of the functions Z, X_1, …, X_n .

The following theorem is thus achieved:

III. *In order to solve the problem:*

$$dZ - \sum_{i=1}^{i=n} P_i\, dX_i = \rho\,(dz - \sum_{k=1}^{k=n} p_k\, dx_k),$$

it is necessary and sufficient that one find $n + 1$ *mutually independent functions* Z, X_1, …, X_n *of the variables* z, x_1, …, x_n, p_1, …, p_n , *which are not all free of* z, *and pair-wise satisfy the equations:*

$$[Z, X_\lambda] = 0, \qquad [X_i, X_\lambda] = 0.$$

The remaining $n + 1$ *unknowns* P_1, …, P_n , *and* ρ *are determined from this uniquely by equation* 1) *and* n *of the equations* 4).

By the way, one can also drop the condition "*which are not all free of* z." There are then no $n + 1$ mutually independent functions Z, X_1, …, X_n that satisfy the equations 8), and are free of z. In fact, one considers Z, X_1, …, X_n to be merely functions of x_1, …, x_n, p_1, …, p_n , and assumes that X_1, …, X_n pair-wise satisfy the conditions:

$$[X_i, X_\lambda] = 0,$$

and are independent of each other, then the n equations:

$$[Z, X_1] = 0, \ldots, [Z, X_n] = 0,$$

define a system of n linear partial differential equations for the function Z that include only $2n$ independent variables, and in which no equation is merely an algebraic consequence of the remaining ones. This system can therefore possess no more that n independent solutions, and they are X_1, …, X_n .

If one considers X_1 to be an arbitrary given functions of z, x_1, …, x_n, p_1, …, p_n , and z to be a function of x_1, …, x_n , and p_1, …, p_n to be the partial differential quotients of z with respect to x_1, …, x_n then the equation:

$$X_1 = \text{const.} = h$$

becomes a partial differential equation of first order between the unknown function z and the n independent variables x_1, …, x_n .

From the Cauchy method, for the complete integration of this equation, it suffices to know all solutions of the linear partial differential equation with $2n + 1$ independent variables z, x_1, …, x_n, p_1, …, p_n :

10) $$[X_1, Z] = 0.$$

However, if one has found, like the given function X_1, n mutually independent functions $X_2, \ldots, X_n$, Z that satisfy the $n(n + 1)/2$ equations 8), one can, from the last theorem, find n other functions $P_1, \ldots, P_n$ by merely solving linear equations, by which the equation:

$$dZ - \sum_{i=1}^{i=n} P_i \, dX_i = \rho \, (dz - \sum_{k=1}^{k=n} p_k \, dx_k)$$

becomes an identity. However, since, from theorem I, one has, in turn:

$$[X_1, Z] = [X_1, X_i] = [X_1, P_i] = 0,$$

one sees, with consideration of II, that:

$$F = X_1, X_2, \ldots, X_n, Z, P_2, \ldots, P_n,$$

are the $2n$ independent solutions of equation 10). One thus has the theorem, which defines the foundation for the important generalization that Lie gave for the recent integration method of the partial differential of first order [1]):

In order to be able to complete integrate the given partial differential equation of first order:

$$H_0(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = h_0 ,$$

one needs only to find any n functions $H_1, \ldots, H_n$ of the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ that are independent of each other, as well as H_0, that pair-wise satisfy the conditions:

$$[H_i, H_k] = 0.$$

The most important class of contact transformations is the one for which the variable z does not enter into the functions $X_1, \ldots, X_n, P_1, \ldots, P_n$ at all. From the foregoing, it follows immediately that in this case Z must have the form:

$$Z = Az + \Pi,$$

where A is a constant and P is merely of a function of $x_1, \ldots, x_n, p_1, \ldots, p_n$. In fact, if the functions X and P do not contain z then:

$$\rho = [X_\lambda, P_\lambda]$$

is free of z, and since equation 1) reduces to:

$$\frac{\partial Z}{\partial z} = \rho$$

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[1]) Cf., the 2nd of the aforementioned treatises of Lie.

under the assumption that was made, it follows that:

$$Z = \rho z + H.$$

However, by this substitution, equations 2) and 3) become:

$$\frac{\partial \rho}{\partial x_k} z + \frac{\partial \Pi}{\partial x_k} - \sum_{i=1}^{i=n} P_i \frac{\partial X_i}{\partial x_k} = -\rho\, p_k\,,$$

$$\frac{\partial \rho}{\partial p_k} z + \frac{\partial \Pi}{\partial p_k} - \sum_{i=1}^{i=n} P_i \frac{\partial X_i}{\partial p_k} = 0,$$

and thus show, since the variable z enters into them only in the first term, that one must have ρ = const.

With that, the goal of this communication – viz., to show how one can base the theory of contact transformation directly – is achieved, and all that remains for me to do is to refer the further construction of this theory to the Lie treatise itself.

Leipzig, 8 April 1874.