"Ueber die Weiler'sche Integrationsmethode der partiellen Differentialgleichungen 1. Ordnung," Math. Ann. **9** (1875), 347-370.

On Weiler's method of integrating partial differential equations of first order.

By A. MAYER in Leipzig

Translated by D. H. Delphenich

In the year 1863, in Schlömilch's journal, Weiler described a method of integrating partial differential equations of first order that required noticeably fewer integrations than the method of Jacobi. However, whether one could not fail to recognize the healthy nucleus of Weiler's method, the derivation and results still remained misunderstood. Attracted to the importance of these results, Clebsch then sought to verify them, and he arrived, along a completely different path [*]), at a rigorous proof of the fact that, in general, one can, in fact, arrive at those results by a considerably smaller number of integrations. However, although Clebsch simply said of Weiler's simplification of the integration process: "This simplification…consists in the following," after examining and comparing these two papers one can still scarcely avoid the conclusion that Clebsch's simplification of the Jacobi method (which was later called the Jacobi-Weiler method) cannot possibly be identical to the actual Weiler process, and the latter, in fact, must rest on a completely different foundation. Furthermore, it is nowhere stated in the article by Clebsch in these annals (Bd. VII, p. 1) that there is an identity between these two methods. I therefore actually do not understand the extent to which matters discussed in it were incorrectly presented. However, if that is also the case then, in any event, one can only rejoice in the fact that the consideration of Weiler's work has allowed his method to be recently published in a more thorough treatment [**]).

In this most recent publication, the situation is generally treated in more detail than in the earlier one, although the presentation is nevertheless also such that it cannot be understood by my interpretation of things.

This comes down to the fact that one and the same symbol $(\varphi\,\psi)$ can be given completely different interpretations, such that there are very few places in which one can say with complete certainty which actual sense is being ascribed to the equations or operations in question. The equations that are otherwise thoroughly transcribed and enumerated (cf., § 5 of Weiler's treatise) change in form and meaning with each page, when one goes further in each direction, without once making any remark, in symbols or words, about this tacit alteration. However, all that actually appears are errors. Therefore, in particular, the theorem [***]) "If m partial differential equations define a

---

[*]) Borchardt's Journal, Bd. 65, pp. 263.
[**]) Schlömilch's Journal, Bd. XX, pp. 83 and pp. 271.
[***]) *Loc. cit.*, pp. 278.

complete system then any i of these m partial differential equations will also define a complete system, and for that reason, have $n - i$ common solutions" is obviously false, which emerges immediately from Clebsch's fundamental theorem, by which m linearly independent partial differential equations:

$$A_1(f) = 0, \qquad A_2(f) = 0, \ldots, \; A_m(f) = 0,$$

define a complete system when and only when each:

$$A_i(A_k(f)) - A_k(A_i(f))$$

can be represented as a linear combination of the $A(f)$. The theorem is, moreover, true only for the special case of a system in involution, in which all of the expressions have the value zero. Furthermore, the fact that this theorem does not merely possess the character of a casual aside implies that there is simply no factual basis for the words "this theorem…will find several uses in what follows" that Weiler immediately followed the previously cited statement with, although the explanation would lead me too far afield, here.

In my opinion, absolutely nothing is as self-explanatory as it appeared in Weiler's presentation. However, the method itself is correct *), and entirely worthy of being read and understood, and indeed not only for the results, but also especially because of its particular line of reasoning, which occasionally proceeds like a mere enumeration of the solutions and variables, and almost always disdains to travel along the customary path. On this basis, I do not consider it to be superfluous to also delve into the Weiler method, as I understand it, in these annals, which indeed already include the most recent investigations into partial differential equations of first order, in order to make it as clear and precise as possible.

The first three sections of the present article are devoted to this discussion, in which, for the sake of completeness, many known facts will also be recalled. The first section is concerned with Weiler's method of integration of complete systems, which Weiler himself had communicated notably only for a complete system of two equations. Whereas all additional methods for the purpose of integrating a given complete system come down to the integration of a system in involution, here the system will be brought into another special form, which I shall call the *Weiler form*. In the following section, the systems in involution that appeared in the Jacobi method will be replaced by such Weiler systems. The application of the method in § 1 then immediately yields in § 3 a method for the integration of partial differential equations of first order that does not differ from that of Jacobi in the number and difficulty of the integrations. As a consequence of a special property of any Weiler system, however, the composition of any two successive steps ultimately yields the actual Weiler simplification, which shows that one can, in general, be spared a large number of integrations when compared to the corresponding process of Clebsch, and therefore go about one's work more simply. If these simplifications of Weiler and Clebsch also implicitly require still more integrations than mine or Lie's method **) then one must, on the other hand, also once more draw one's

*) Cf., however, the remark at the conclusion.
**) Math. Annalen, Bd. V, pp. 448 and Bd. VI, pp. 162.

attention to the fact that in most cases these integrations will drop out more often then in the latter methods.

In the sequel, for the sake of brevity and clarity, I have only considered the simplest, and due to its relationship with dynamics and the calculus of variation, most important case of partial differential equations of first order, namely, the case in which the unknown function itself does not enter into the given equation. For this, corresponding to the conclusion of Weiler's article, an examination will be added in § 4 that has no necessary connection with the foregoing ones, and which shall abolish an old prejudice, which was also shared by myself, against the Jacobi reduction of the general case of partial differential equations of first order to the given simpler case [*]).

§ 1.

Integration of complete systems.

Let:

(1) $$A_1(f) = 0, \qquad A_2(f) = 0, \ \ldots, \ \ A_m(f) = 0,$$

where:

$$A_i(f) = \sum_{\lambda=1}^{n} a_\lambda^i \frac{\partial f}{\partial x_\lambda}$$

and $m < n$, be m given linear partial differential equations, of which none of them is merely an algebraic consequence of the others, which then determine m of the differential quotients as functions of the remaining ones and the independent variables. I assume that:

$$\frac{\partial f}{\partial x_1}, \ \frac{\partial f}{\partial x_2}, \ \ldots, \ \frac{\partial f}{\partial x_m}$$

are m such differential quotients. By solving these equations, one then obtains the system (1) in the form:

(2) $$B_1(f) = 0, \qquad B_2(f) = 0, \ \ldots, \ \ B_m(f) = 0,$$

where:

$$B_i(f) = \frac{\partial f}{\partial x_i} - \sum_{h=m+1}^{n} b_h^i \frac{\partial f}{\partial x_h}.$$

This solved form of system (1) immediately leads to the theorem that when the equations (1) possess several common solutions $f = f_1, f_2, \ldots, f_k$, of which, none of them is merely a function of the other ones, these solutions must necessarily be independent of each other relative to the $n - m$ variables $x_{m+1}, \ldots, x_n$.

In fact, were:

$$f_k = \varphi(f_1, \ldots, f_{k-1}, x_1, \ldots, x_m)$$

then one would have:

[*]) Borchardt's Journal, Bd. 60, pp. 1.

$$B_i(f_k) = \frac{\partial \varphi}{\partial x_i},$$

and will thus follow from equations (2) under the assumption:

$$f_k = \varphi(f_1, \ldots, f_{k-1}).$$

From this theorem, one immediately infers that equations (1) can no longer possess more than $n - m$ mutually independent common solutions. If they admit $n - m$ such solutions then the system (1) would be called a *complete system*.

In the sequel, I will assume that the system (1) is a complete system, and will denote any $n - m$ independent solutions of it by $f_{m+1}, \ldots, f_n$. The functions $f_{m+1}, \ldots, f_n$ are then independent of each other relative to $x_{m+1}, \ldots, x_n$; any solution of the system (1) may be expressed as merely a function of $f_{m+1}, \ldots, f_n$, and conversely, any function of $x_1, x_2, \ldots, x_n$ that includes these variables only by way of $f_{m+1}, \ldots, f_n$ is a common solution of equations (1).

Since the m equations (1) should determine the m first differential quotients, so must the h equations:

$$A_1(f) = 0, \qquad A_2(f) = 0, \ldots, \quad A_h(f) = 0$$

be soluble for h of these m quantities for every $h < m$. If one assumes that:

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_h}$$

are h such differential quotients, and denotes the result of the substitution of their values in $A_{h+1}(f)$ by $A_{h+1}^{h}(f)$ then one can replace the system (1) with the following one:

$$(3) \qquad A_1(f) = 0, \qquad A_2^1(f) = 0, \ldots, \; A_m^{m-1}(f) = 0,$$

in which $A_k^{k-1}(f)$ generally has the form:

$$(4) \qquad A_k^{k-1}(f) = \sum_{\lambda=k}^{n} b_\lambda^k \frac{\partial f}{\partial x_\lambda},$$

and $A_k^{k-1}(x_k)$ is non-zero. When using the shorter expressions, this form of the complete system (1) will be called the *Weiler form*.

The system (3) possesses the important property that any of the last $m - i$ equations in it will again define a complete system.

In fact, the $m - i$ equations:

$$(5) \qquad A_{i+1}^{i}(f) = 0, \ldots, \; A_m^{m-1}(f) = 0,$$

which include only the $n - i$ differential quotients $\partial f / \partial x_{i+1}$, …, $\partial f / \partial x_n$, and can be solved for the last $m - i$ of them, possess the:

$$n - m = (n - i) - (m - i)$$

common solutions $f = f_{m+1}, \ldots, f_n$ that are independent functions of the variables $x_{m+1}, \ldots, x_n$.

Now, let $f = \varphi_{m+1}, \ldots, \varphi_n$ be any $n - m$ independent solutions of the complete system (5).

From the equations:

$$\varphi_{m+1} = x'_{m+1}, \ldots, \varphi_n = x'_n,$$

the $x_{m+1}, \ldots, x_n$ can then be expressed in terms of $x_1, \ldots, x_m, x'_{m+1}, \ldots, x'_n$. If one denotes the substitution of these values by [ ] and sets $[f] = f'$ then one has, consequently:

$$\left[\frac{\partial f}{\partial x_\lambda}\right] = \sum_{h=1}^{m} \frac{\partial f'}{\partial x_h}\frac{\partial x_h}{\partial x_\lambda} + \sum_{\mu=m+1}^{n} \frac{\partial f'}{\partial x_h}\left[\frac{\partial \varphi_\mu}{\partial x_\lambda}\right],$$

because, from (4), $A_k^{k-1}(f)$ is a linear function of $\partial f / \partial x_\lambda$, which vanishes for $f = x_1, x_2, \ldots, x_{k-1}$:

$$\left[A_k^{k-1}(f)\right] = \sum_{h=k}^{m}\left[A_k^{k-1}(x_h)\right]\frac{\partial f'}{\partial x_h} + \sum_{\mu=m+1}^{n}\left[A_k^{k-1}(\varphi_\mu)\right]\frac{\partial f'}{\partial x'_\mu}.$$

With the introduction of the new variables x' the complete system:

$$A_i^{i-1}(f) = 0,\ A_{i+1}^{i}(f) = 0,\ \ldots,\ A_m^{m-1}(f) = 0 \tag{6}$$

goes over to the following one:

$$\sum_{h=i}^{m}\left[A_i^{i-1}(x_h)\right]\frac{\partial f'}{\partial x_h} + \sum_{\mu=m+1}^{n}\left[A_i^{i-1}(\varphi_\mu)\right]\frac{\partial f'}{\partial x'_\mu} = 0,$$

$$\sum_{h=i+1}^{m}\left[A_i^{i-1}(x_h)\right]\frac{\partial f'}{\partial x_h} = 0,$$

………………………………

$$\left[A_m^{m-1}(x_m)\right]\frac{\partial f'}{\partial x_m} = 0.$$

However, the last $m - i$ equations yield:

$$\frac{\partial f'}{\partial x_m} = 0, \ldots, \frac{\partial f'}{\partial x_{i+1}} = 0.$$

Under the substitution [ ], any solution f of the system (6) then goes over to a solution f of the equation:

(7) $$\frac{\partial f'}{\partial x_i} + \sum_{\mu=m+1}^{n} \left[\frac{A_i^{i-1}(\varphi_\mu)}{A_i^{i-1}(x_i)} \right] \frac{\partial f'}{\partial x'_\mu} = 0$$

that is free of $x_{i+1}, \ldots, x_m$. Now, the complete system (6) possesses $n - m$ solutions that are mutually independent relative to the $x_{m+1}, \ldots, x_n$. As a consequence, equation (7) must admit $n - m$ solutions that are free of $x_{i+1}, \ldots, x_m$, and are independent functions of $x'_{m+1}, \ldots, x'_n$. However, when one understands x to mean any of the variables $x_{i+1}, \ldots, x_m$, each of these solutions satisfies, not only equation (7), but also the equation:

(8) $$\sum_{\mu=m+1}^{n} \frac{\partial f'}{\partial x'_\mu} \frac{\partial}{\partial x} \left[\frac{A_i^{i-1}(\varphi_\mu)}{A_i^{i-1}(x_i)} \right] = 0,$$

which one obtains when one differentiates equation (7) with respect to x, under the assumption that f' is free of x.

Equation (8) must then, in the even that it is not an identity, possess $n - m$ independent solutions relative to $x'_{m+1}, \ldots, x'_n$. However, this is impossible, since it only contains the $n - m$ differential quotients $\partial f' / \partial x'_{m+1}, \ldots, \partial f' / \partial x'_n$. They must then be identities, in and of themselves; i.e., either each:

$$\frac{\partial}{\partial x} \left[\frac{A_i^{i-1}(\varphi_\mu)}{A_i^{i-1}(x_i)} \right] = 0,$$

or all of the expressions:

$$\frac{A_i^{i-1}(\varphi_\mu)}{A_i^{i-1}(x_i)}$$

must represent merely functions of $x_1, \ldots, x_i, \varphi_{m+1}, \ldots, \varphi_n$. However, as such, they are again solutions of the system (5) into which $x_1, \ldots, x_i$ enter only as constants. One then has the theorem:

I. *Let:*

$$A_1(f) = 0, \ A_2^1(f) = 0, \ldots, \ A_m^{m-1}(f) = 0,$$

be a complete system, where, in general, $A_i^{i-1}(f)$ *possesses the form:*

$$A_i^{i-1}(f) = \sum_{h=i}^{n} b_h^i \frac{\partial f}{\partial x_h},$$

and $A_i^{i-1}(x_i)$ *is not zero. If* $f = \varphi$ *is then a common solution of the last* $m - i$ *equations of this system then:*

$$f = \frac{A_i^{i-1}(\varphi)}{A_i^{i-1}(x_i)}$$

is always one, as well.

This immediately implies a method for arriving at a common solution of the last $m - i + 1$ of equations (3) from a given common solution to the last $m - i$ of equations (3) or (5).

Namely, if $f = \varphi_1$ is the given solution then, from I:

$$\varphi_2 = \frac{A_i^{i-1}(\varphi_1)}{A_i^{i-1}(x_i)}, \quad \varphi_3 = \frac{A_i^{i-1}(\varphi_2)}{A_i^{i-1}(x_i)}, \quad \ldots, \quad \varphi_{k+1} = \frac{A_i^{i-1}(\varphi_k)}{A_i^{i-1}(x_i)},$$

are also common solutions of equations (5). However, these equations possess only $n - m$ mutually independent common solutions. Therefore, if φ_{k+1} is the first of the functions $\varphi_1, \varphi_2, \ldots$ that can be expressed in terms of the foregoing ones and $x_1, x_2, \ldots, x_i$ alone then one must have:

$$k \leq n - m.$$

If one now sets:

$$f' = f'(x_1, x_2, \ldots, x_i, \varphi_1, \varphi_2, \ldots, \varphi_k)$$

then, when one divides it by $A_i^{i-1}(x_i)$, from (9), the equation:

$$A_i^{i-1}(f) = 0$$

goes to

$$\frac{\partial f'}{\partial x_i} + \sum_{\lambda=1}^{k} \varphi_{\lambda+1} \frac{\partial f'}{\partial \varphi_\lambda} = 0. \tag{10}$$

Only $x_1, x_2, \ldots, x_i, \varphi_1, \varphi_2, \ldots, \varphi_k$ enter into this equation, and thus, only those quantities that are solutions of the system (5). Any solution of equation (10) is therefore a common solution to the last $m - i + 1$ equations (3). One thus finds a solutions of these equations by the intermediary of an integral of the system of k ordinary differential equations:

$$dx_i : d\varphi_1 : \ldots : d\varphi_{k-1} : d\varphi_k = 1 : \varphi_1 : \ldots : \varphi_k : \varphi_{k+1},$$

or by an operation of order $k \leq n - m$.

From this, the following procedure serves to find a solution of the given complete system (1):

One first brings the system into the Weiler form (3) by successively solving its equations and substituting the solutions, which is always possible, possibly after a suitable permutation of variables.

One then determines a solution of the last equation in (3) by an operation of order $n - m$. By an operation of order at most $n - m$, one then obtains a common solution to the last two equations in (3). An operation that is generally of order at most $n - m$ then yields a common solution to the last three equations in (3), etc., such that in order to find a common solution to the entire complete system of m equations in n independent variables one requires an operation of order $n - m$ and $m - 1$ operations, each of which is of order at most $n - m$.

My method [*]) achieves the same objective by a single operation of order $n - m$.

§ 2.

The Weiler system that the general problem of integrating partial differential equations of first order comes down to.

Lie has shown [**]) that the complete integration of a given partial differential equation of first order:

$$\varphi_1(x_1, \ldots, x_n, p_1, \ldots, p_n) = c_1 ,$$

in which, $p_1, \ldots, p_n$ mean the partial differential quotients of the unknown function with respect to the independent variables $p_1, \ldots, p_n$, which can then come down to finding $n - 1$ functions $\varphi_1, \ldots, \varphi_n$ of the $2n$ independent variables $x_1, \ldots, x_n, p_1, \ldots, p_n$, which pair-wise satisfy the conditions:

$$(\varphi_k\ \varphi_h) = \sum_{i=1}^{n} \left(\frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_h}{\partial p_i} - \frac{\partial \varphi_k}{\partial p_i} \frac{\partial \varphi_h}{\partial x_i} \right) = 0,$$

be independent of each other, as well as φ_1 .

If one has already found the functions $\varphi_1, \ldots, \varphi_n$ then so is φ_{i+1} to be determined as a common solution to the i equations:

$$(1) \qquad (\varphi_1 f) = 0, \qquad (\varphi_2 f) = 0, \qquad \ldots \qquad (\varphi_i f) = 0,$$

that is independent of $\varphi_1, \varphi_2, \ldots, \varphi_n$. These i equations, of which, as one immediately realizes from the independence of the functions $\varphi_1, \ldots, \varphi_i$, none of them is merely an algebraic consequence of the remaining ones, is well-known to define a complete system, due to the assumptions $(\varphi_k\ \varphi_k) = 0$.

One can immediately convince oneself of this, when one employs the celebrated formula of Jacobi, by which, for any three arbitrary functions f, φ, ψ of the $2n$ variables x, p one has identically:

$$(f, (\varphi\ \psi)) + (\varphi, (\psi\ f)) + (\psi, (f\ \varphi)) = 0,$$

[*]) Math. Annalen, Bd. V, pp. 465.

[**]) Math. Annalen, Bd. VIII, pp. 245. Cf., also pp. 318 of the same volume.

with the well-known theorem that any *system in involution;* i.e., any system of mutually independent linear partial differential equations:

$$A_1(f) = 0, \qquad A_2(f) = 0, \ldots, \quad A_n(f) = 0,$$

whose left-hand sides satisfy the conditions:

$$A_h(A_k(f)) - A_h(A_k(f)) = 0,$$

is likewise a complete system, a theorem for which Lie (pp. 249 of this volume) has communicated a simple direct proof.

Namely, since $(\psi, (f\ \varphi)) = -(\psi, (\varphi\ f))$ the Jacobi identity, under the assumption that $(\varphi\ \psi) = 0$, yields:

$$(\varphi, (\psi\ f)) - (\psi, (\varphi\ f)) = 0.$$

Therefore, as long as $(\varphi\ \psi) = 0$ there exists between the operations:

$$A(f) = (\varphi\ f), \qquad B(f) = (\psi\ f),$$

the relation:

$$A(B(f)) - B(A(f)) = 0.$$

As a consequence of the assumptions $(\varphi_h\ \varphi_k) = 0$, however, it is just this case that applies to the system (1), so it is a system in involution *).

---

*) I will take this opportunity to publish the beautiful proof that Clebsch gave of the Jacobi identity in his lectures:

One immediately sees that as long as $A(f)$ and $B(f)$ mean any two expressions of the form:

$$A(f) = \sum_{h=1}^{m} a_h \frac{\partial f}{\partial y_h}, \qquad B(f) = \sum_{h=1}^{m} b_h \frac{\partial f}{\partial y_h},$$

under the operation:

$$A(B(f)) - B(A(f)),$$

the second partial differential quotients of the function f cancel out. From this, it follows immediately that the expression:

$$M = (f, (\varphi\ \psi)) + (\varphi, (\psi\ f)) + (\psi, (f\ \varphi))$$

includes no second differential quotients of the three functions f, φ, ψ. In fact, second differential quotients of f, for example, can only originate from the sum:

$$(\varphi, (\psi\ f)) + (\psi, (f\ \varphi)).$$

However, this is, as we showed above, an expression of the form:

$$A(B(f)) - B(A(f)),$$

and, in turn, is free of the second differential quotients of f. On the other hand, as a consequence, its inclusion in the expression M includes no terms in which a second differential quotient of f, φ, or ψ appears. Therefore, M can only have the value zero.

From the foregoing, one likewise also obtains the Poisson-Jacobi theorem, which I will express in the special form here that we will use later:

If $h < i$ and $f = \alpha$ is a solution of the system in involution:

$$(\varphi_1\, f) = 0, \qquad (\varphi_2\, f) = 0, \ldots, (\varphi_h\, f) = 0$$

then

$$f = (\varphi_i\; \alpha)$$

is always one, as well.

The complete system (1) now possesses i known systems $f = \varphi_1, \varphi_2, \ldots, \varphi_i$. When one introduces these i solutions as new variables, one can then convert equations (1) into a complete system of i equations in only $2n - i$ independent variables, and directly apply the method of the previous section to this reduced system, after one has brought it into the Weiler form by successive solutions of its equations and substitution of the solutions.

However, if one would like to present those equations on which the function φ_{i+1} is to be calculated according to any method, along this most obvious path, then one would scarcely detect the important property of these equations upon which the actual Weiler simplification of the integration process is based.

We would therefore like to perform the reduction of the complete system (1) to the Weiler form in another indirect way.

By assumption, the already known functions $\varphi_1, \varphi_2, \ldots, \varphi_i$, which have the mutual relationship $(\varphi_h\; \varphi_k) = 0$, are independent of each other. Thus, for each $h \leq i$, the h equations:

(2) $$\varphi_1 = c_1,\; \varphi_2 = c_2, \ldots, \varphi_h = c_h,$$

determine h of the $2n$ variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ as functions of the remaining ones and the c. These $2n$ variables are, in turn, pair-wise associated with each other. I would like to denote them, when taken in any sequence, $u_1, \ldots, u_n, v_1, \ldots, v_n$, in such a way that $u_1, \ldots, u_n$, shall mean any n different x, p, and v_k is associated with the variables u_k, in such a way that, for example, $v_k = x_k$, when one sets $u_k = p_k$.

For the sake of clarity, the substitution of the values of $u_1, \ldots, u_n$ that equations (2) provide might be indicated by the sign $[\;]^h$, or also briefly by the addendum of an upper index h^*. By this convention, for any $h \leq i$:

$$[f]^i = [f^h]^i = f^i$$

and $\partial\varphi_h^{h-1} / \partial u_h$ are non-zero. Finally, ω means an arbitrary one of the variables x, p.

From the identity $[f]^1 = f^1$ that emerges by substituting the values of u_1 in the equation $\varphi_1 = c_1$, it follows that:

$$\left[\frac{\partial f}{\partial \omega}\right]^1 = \frac{\partial f^1}{\partial \omega} + \frac{\partial f^1}{\partial c_1}\left[\frac{\partial \varphi_1}{\partial \omega}\right]^1.$$

If one substitutes these values for the $\left[\frac{\partial f}{\partial \omega}\right]^1$ in the expression $(\varphi_h\ f)^1$, which is a linear homogeneous function of these quantities, then, since $(\varphi_h\ \varphi_1) = 0$, this yields, for $h = 1, 2, \ldots, i$:

(3) $$(\varphi_h\ f)^1 = (\varphi_h\ f^1)^1,$$

a formula that, like many of the following ones, can also be inferred directly in the well-known way, in which one abbreviates a characteristic with the help of given solutions by just as many terms.

If one sets $f = \varphi_h$, where $h > 1$, and then applies this to the expression $(\varphi_h\ f^1)^1$ then it follows that:

$$(\varphi_h\ f^1)^1 = \left(\varphi_h^1\ f^1\right) + \frac{\partial \varphi_h^1}{\partial c_1}(\varphi_1\ f^1)^1 .$$

One thus has the identities:

(4) $$\begin{cases} (\varphi_1\ f)^1 = (\varphi_1\ f^1)^1, \\ (\varphi_h\ f)^1 = (\varphi_h^1\ f^1) + \dfrac{\partial \varphi_h^1}{\partial c_1}(\varphi_h\ f^1)^1. \end{cases}$$

As a consequence of the assumptions $(\varphi_1\ \varphi_\lambda) = (\varphi_h\ \varphi_\lambda) = 0$, one also has:

$$\left(\varphi_h^1 \varphi_\lambda^1\right) = 0.$$

In turn, the expression $(\varphi^1\ \psi^1)$ arises from the two functions φ^1 and ψ^1 in precisely the same way as the expression $(\varphi\ \psi)$ does from the φ and ψ. Therefore, for each $h \leq i$ the equations:

(5) $$\left(\varphi_2^1\ f^1\right) = 0, \quad \left(\varphi_3^1\ f^1\right) = 0, \ldots, \left(\varphi_h^1\ f^1\right) = 0$$

define a system in involution.

From the identity:

$$[\varphi_h^1]^{h-1} = \varphi_h^{h-1},$$

which arises when one substitutes the values for $u_2, \ldots, u_{h-1}$ in the function φ_h^1 that are obtained from the equations:

$$\varphi_2^1 = c_2, \ldots, \varphi_{h-1}^1 = c_{h-1} ,$$

one further obtains the formula:

$$\left[\frac{\partial \varphi_h^1}{\partial \omega}\right]^{h-1} = \frac{\partial \varphi_h^{h-1}}{\partial \omega} + \sum_{\lambda=2}^{h-1} \frac{\partial \varphi_h^{h-1}}{\partial c_\lambda}\left[\frac{\partial \varphi_\lambda^h}{\partial \omega}\right]^{h-1},$$

and by an application of it:

(6) $$(\varphi_h^1 f^1)^{h-1} = (\varphi_h^{h-1} f^1)^{h-1} + \sum_{\lambda=2}^{h-1} \frac{\partial \varphi_h^{h-1}}{\partial c_\lambda} (\varphi_\lambda^1 f^1)^{h-1}.$$

Likewise, it follows from the identity $[f^1]^i = f^i$ that comes from the equations $\varphi_2^1 = c_2, \ldots, \varphi_i^1 = c_i$ that:

$$\left[\frac{\partial f^1}{\partial \omega}\right]^i = \frac{\partial f^i}{\partial \omega} + \sum_{k=2}^{h-1} \frac{\partial f^i}{\partial c_k} \left[\frac{\partial \varphi_k^1}{\partial \omega}\right]^i,$$

and from this – since, from (4), one also has $(\varphi_1 \varphi_k^1)^1 = 0$ – one has:

(7) $$(\varphi_1 f^1)^i = (\varphi_1 f^i)^i,$$

a formula that has the same character as formula (3).

Finally, if one makes the substitution $[\ ]^i$ in the identity (6), under the assumption that $h \leq i$, and then applies the penultimate formula to $\left(\varphi_h^{h-1} f^1\right)^i$, this gives:

$$\left(\varphi_h^1 f^1\right)^i = \left(\varphi_h^{h-1} f^i\right)^i + \sum_{k=2}^{i} \frac{\partial f^i}{\partial c_k} \left(\varphi_h^{h-1} \varphi_k^1\right)^i + \sum_{\lambda=2}^{h-1} \left[\frac{\partial \varphi_h^{h-1}}{\partial c_\lambda}\right]^i \left(\varphi_\lambda^1 f^1\right)^i.$$

Since $(\varphi_\lambda^1 \varphi_k^1) = 0$, however, from (6), one also has $(\varphi_h^{h-1} \varphi_k^1)^{h-1} = 0$, it follows that all that remains is:

(8) $$\left(\varphi_h^1 f^1\right)^i = \left(\varphi_h^{h-1} f^i\right)^i + \sum_{\lambda=2}^{h-1} \left[\frac{\partial \varphi_h^{h-1}}{\partial c_\lambda}\right]^i \left(\varphi_\lambda^1 f^1\right)^i.$$

The coupling of formulas (7) and (8) with formulas (4) shows only, when one sets $h = 2, 3, \ldots, i$ in sequence, that by the substitution $[\ ]^i$, each solution f of the system (1) that is independent of $\varphi_1, \varphi_2, \ldots, \varphi_i$, will go to an a solution f^i of the system with only the $2n - i$ independent variables:

(9) $$(\varphi_1 f^i)^i = 0, \quad (\varphi_2^1 f^i)^i = 0, \ldots, (\varphi_i^{i-1} f^i)^i = 0,$$

while conversely, by the substitution $c_1 = \varphi_1, c_2 = \varphi_2, \ldots, c_i = \varphi_i$, each solution f^i of the latter system again goes to a solution of the given system (1) that is independent of $\varphi_1, \varphi_2, \ldots, \varphi_i$, such that one can replace this with the system (9).

From this, one obtains, with no further assumptions, that the system (9) is also, in turn, a complete system.

However, the system (9) has the Weiler form. Then, since φ_h^{h-1} is free of $u_1, u_2, \ldots, u_{h-1}$, and the differential quotients $\partial f^i / \partial v_k$ appear in the expressions $\left(\varphi_h^{h-1} f^i\right)^i$, only in the form:

$$\pm\left[\frac{\partial\varphi_h^{h-1}}{\partial u_k}\right]^i \frac{\partial f^i}{\partial v_k},$$

then each successive equation in the system (9) is fewer by one of the differential quotients:

$$\frac{\partial f^i}{\partial v_1}, \ \frac{\partial f^i}{\partial v_2}, \ldots, \ \frac{\partial f^i}{\partial v_{i-1}},$$

and one has that:

$$\left(\varphi_h^{h-1} v_h\right)^i = \pm\left[\frac{\partial\varphi_h^{h-1}}{\partial u_h}\right]^i$$

is then non-zero.

Therefore, as long as the functions $\varphi_2, \ldots, \varphi_i$ are already known, we can immediately define the Weiler system from them and the given function φ_1, on which, the next of the desired functions depends.

However, it still remains for us to find the important property of equations (9) that we spoke of.

For this, we again resort to formulas (7) and (8).

Namely, they also teach us that by the substitution $[\ ]^i$, the system in involution (5) will be converted into the following system:

$$(10) \qquad \left(\varphi_2^1 f^i\right)^i = 0, \quad \left(\varphi_3^2 f^i\right)^i = 0, \ldots, \left(\varphi_h^{h-1} f^i\right)^i = 0,$$

which is then a complete system in any case.

Conversely, if $f^i = u^i$ is a solution of the system (10), and, by the substitution $c_2 = \varphi_2^1$, …, $c_i = \varphi_i^1$, the function α^i goes to α^1 then $f^1 = \alpha^1$ is a solution of the system in involution (5). Under the assumption that $h < i$, however, from the Poisson-Jacobi theorem with $f^1 = \alpha^1$, one likewise also has that:

$$f^1 = \left(\varphi_i^1 \alpha^1\right)$$

is a solution of this system, and that each solution f^1 of it goes to a solution f^i of the complete system (10) under the substitution $[\ ]^i$. With $f^i = \alpha^i$, then:

$$f^i = \left(\varphi_i^1 \alpha^1\right)^i$$

is also a solution (10). From (8), however, one has:

$$\left(\varphi_i^1 \alpha^1\right)^i = \left(\varphi_i^{i-1} \alpha^i\right)^i.$$

One then has the theorem:

II. *If* $f^i = \alpha^i$ *is a solution of the complete system* (10) *and* $h < i$ *in this then one also always has that:*

$$f^i = \left(\varphi_i^{i-1}\alpha^i\right)^i$$

is a solution of this system,

and this is the desired property of equations (9).

Finally, if one has found one solution $f^i = \varphi_{i+1}^i$ of the system (9) and suggests the substitution of values of $u_1, u_2, \ldots, u_{i+1}$ that follows from the equations:

$$\varphi_1 = c_1,\ \ \varphi_2^1 = c_2,\ \ldots,\ \varphi_{i+1}^i = c_{i+1}\,,$$

by the upper index $i + 1$ then one has:

$$\left(\varphi_h^{h-1} f^i\right)^{i+1} = \left(\varphi_h^{h-1} f^{i+1}\right)^{i+1}.$$

This formula, which, in turn, has the same character as formula (3), flows into the theorem:

III. *If* $f^i = \alpha^i$ *is any solution of the system:*

$$(\varphi_2^1 f^i)^i = 0, \quad (\varphi_3^2 f^i)^i = 0,\ \ldots,\ (\varphi_i^{i-1} f^i)^i = 0$$

that is independent of v_1 *and* φ_{i+1}^i *then* $f^{i+1} = \alpha^{i+1}$ *is a solution of the system:*

$$(\varphi_2^1 f^{i+1})^{i+1} = 0,\ \ (\varphi_3^2 f^{i+1})^{i+1} = 0,\ \ldots,\ (\varphi_i^{i-1} f^{i+1})^{i+1} = 0.$$

§ 3.

Integration of partial differential equations of first order.

On the basis of the theorem of Lie, along with the conversion of the system in involution (1) to the Weiler system (9), this likewise points directly to a path, along which one can employ the method of § 1 for the integration of the given partial differential equation $\varphi_1 = c_1$.

The system (9), whose presentation assumes that one already knows $i - 1$ functions $\varphi_2, \ldots, \varphi_i$ that are independent of each other, as well as φ_1, and satisfy the conditions $(\varphi_h\, \varphi_k) = 0$, includes i equations and $2n - i$ independent variables.

From § 1, one then finds a solution $f^i = \varphi_{i+1}^i$ of it by an operation of the order $2n - 2i$ and $i - 1$ operations, of which, at most one of them has the same order.

Thus, as long as the functions $\varphi_2, \ldots, \varphi_i$ are already known – and from the partial differential equations of dynamics one can indeed mostly obtain a sequence of functions from general principles [*]) – requires the next step in the integration of the given differential equation by the Weiler method, as well as the Jacobi method; one may also not decide, *a priori*, whether the order of these integrations by the one method will turn out to be less than it is by the other one. This first step then shows us nothing in regard to the advantage of the Weiler method over the Jacobi one.

However, this situation changes when one goes further. Whereas, by the Jacobi method, one must also repeat essentially the same process to ascertain φ_{i+2} that already gave φ_{i+1} , in general, by evaluating the results that, in a sense, were byproducts of obtaining φ_{i+1}, the Weiler method can now admit a very remarkable simplification, which is just the defining characteristic of this method.

In order to correctly understand this simplification, we must next present the actual operations by which the function φ_{i+1}^i was found in the foregoing.

For the determination of this function, one must first look for a solution of the equation:

$$\left(\varphi_i^{i-1} f^i\right)^i = 0,$$

and then, a common solution to the two equations:

$$\left(\varphi_{i-1}^{i-2} f^i\right)^i = 0, \quad \left(\varphi_i^{i-1} f^i\right)^i = 0,$$

etc. Finally, once one has found a common solution of the i – 1 equations:

$$\left(\varphi_2^1 f^i\right)^i = 0, \quad \left(\varphi_3^2 f^i\right)^i = 0, \ldots, \left(\varphi_i^{i-1} f^i\right)^i = 0, \tag{11}$$

a common solution of all i equations (9) in the following way:

Let $f^i = \alpha_1^i$ be the solution to the system (11) that was already found. From theorem I, one computes the new solutions:

$$\alpha_2^i = \frac{(\varphi_1\alpha_1^i)^i}{(\varphi_1 v_1)^i}, \qquad \alpha_3^i = \frac{(\varphi_1\alpha_1^i)^i}{(\varphi_1 v_1)^i}, \ldots,$$

up to the first function α_{k+1}^i, that can be expressed in terms of just the previous ones and v_1, and which must necessarily enter in for $k \leq 2n - 2i$.

When one then sets:

$$f^i = F^i(x_1, \alpha_1^i, \alpha_2^i, \cdots, \alpha_k^i),$$

one converts the equations $(\varphi_1 f^i)^i = 0$ into the following one:

---

[*]) Cf., Lie, Math. Annalen, Bd. VIII, pp. 282.

(12) $$\frac{\partial F^i}{\partial v_1} + \sum_{h=1}^{k} \alpha_{h+1}^i \frac{\partial F^i}{\partial \alpha_h^i} = 0.$$

Any solution of this equation whose independent variables are v_1, α_1^i, …, α_k^i delivers a common solution $f^i = \varphi_{i+1}^i$ of the entire system (9).

Once one has thus found φ_{i+1}^i, one must, in turn, find a solution $f^{i+1} = \varphi_{i+2}^{i+1}$ to the Weiler system:

(13) $$(\varphi_1 f^{i+1})^{i+1} = 0, \qquad (\varphi_2^1 f^{i+1})^{i+1} = 0, \ldots, (\varphi_{i+1}^i f^{i+1})^{i+1} = 0,$$

in which the upper index $i + 1$ refers to the values of the variables u_1, u_2, …, u_{i+1} that follow from the substitutions defined by the equations:

$$\varphi_1 = c_1, \ \varphi_2^1 = c_1, \ \ldots, \ \varphi_{i+1}^i = c_{i+1}.$$

Now, in general, by the statement of equation (12), one will find yet another solution $f^1 = \beta_1^i$ of the system (11), in addition to φ_{i+1}^i, which does not go to a mere function of v_1 under the substitution of values for u_{i+1} that were obtained from the equation $\varphi_{i+1}^i = c_{i+1}$. In fact, this is always the case, as long as $k > 1$ in (12). From III, this solution is converted by the stated substitution into a solution $f^{i+1} = \beta_1^{i+1}$ of the system:

(14) $$\left(\varphi_2^1 f^{i+1}\right)^{i+1} = 0, \left(\varphi_3^2 f^{i+1}\right)^{i+1} = 0, \ldots, \left(\varphi_i^{i-1} f^{i+1}\right)^{i+1} = 0.$$

However, from theorem II, when one exchanges i for $i + 1$, along with β_1^{i+1}, one simultaneously has:

(15) $$\beta_2^{i+1} = \left(\varphi_{i+1}^i \beta_1^{i+1}\right)^{i+1}, \ \beta_3^{i+1} = \left(\varphi_{i+1}^i \beta_2^{i+1}\right)^{i+1}, \ldots$$

as solutions of system (14). Let $\beta_{\lambda+1}^{i+1}$ be the first of the functions β_1^{i+1}, β_2^{i+1}, … that can be expressed in terms of the previous ones and v_1 alone. One must then have:

$$\lambda \leq 2n - 2i - 1,$$

because the complete system (14) includes $i - 1$ equations, but only $2n - i - 2$ actual variables, which allow only:

$$2n - i - 2 - (i - 1) = 2n - 2i - 1$$

independent solutions.

If one now takes:

$$f^{i+1} = F^{i+1}(v_1, \beta_1^{i+1}, \cdots, \beta_\lambda^{i+1})$$

then the equation:

$$\left(\varphi_{i+1}^i f^{i+1}\right)^{i+1} = 0$$

goes to the following one:

$$\sum_{h=1}^{\lambda} \left(\varphi_{i+1}^{i} \beta_h^{i+1}\right) \frac{\partial F^{i+1}}{\partial \beta_h^{i+1}} = 0,$$

which, from (15), can be written as:

$$\sum_{h=1}^{\lambda} \beta_{h+1}^{i+1} \frac{\partial F^{i+1}}{\partial \beta_h^{i+1}} = 0, \tag{16}$$

and consequently include only such quantities that are solutions of the system (14).

Any solution of this equation – and the discovery of one demands an operation of order at most $2n - 2i - 2$ – provides a solution of the system:

$$\left(\varphi_2^1 f^{i+1}\right)^{i+1} = 0, \ \left(\varphi_3^2 f^{i+1}\right)^{i+1} = 0, \ \ldots, \ \left(\varphi_{i+1}^{i} f^{i+1}\right)^{i+1} = 0. \tag{17}$$

Thus, in the event that not just $\beta_2^{i+1} = 0$, but then β_1^{i+1} itself, is already a solution of this system, one must have $\lambda > 1$ if equation (16) is to produce F^{i+1} = (a function of just v_1) as the obvious and unneeded solution.

One then assumes in this way that in order to find a common solution of equations (17), the operations (15) have produced, in addition to β_1^{i+1}, at least a second solution β_2^{i+1} of the system (14) that is independent of it and v_1, or perhaps that one already has found two solutions of the system (11) that are independent of each other, as well as φ_{i+1}^{i} and v_1 by the definition of equation (12).

If neither of these two assumptions apply then one must also once more apply the same process for the determination of φ_{i+2}^{i+1} as one did for the discovery of φ_{i+1}^{i}. Then, however, by this previous problem, one has already entered into the very fortunate situation in which the last step for ascertaining φ_{i+1}^{i} – viz., the discovery of a solution of equation (12) – requires an operation of only at most second order.

If one has found a solution $f^{i+1} = \gamma_1^{i+1}$ to the system (17) in one way or the other then one constructs from it the nine:

$$\gamma_2^{i+1} = \frac{(\varphi_1 \gamma_1^{i+1})^{i+1}}{(\varphi_1 v_1)^{i+1}}, \quad \gamma_3^{i+1} = \frac{(\varphi_1 \gamma_2^{i+1})^{i+1}}{(\varphi_1 v_1)^{i+1}}, \ \ldots$$

up to the first function $\gamma_{\mu+1}^{i}$ that can be expressed in terms of the previous ones and v_1 alone, and then obtains, by the discovery of a solution F^{i+1} to the equation:

$$\frac{\partial F^{i+1}}{\partial v_1} + \sum_{h=1}^{\mu} \gamma_{h+1}^{i+1} \frac{\partial F^{i+1}}{\partial \gamma_h^{i+1}} = 0, \tag{17}$$

and thus, by an operation of the order $m \leq 2n - 2i - 2$, a common solution $f^{i+1} = \varphi_{i+2}^{i+1}$ of the entire system (13).

After one has already found the function φ_{i+1} for a given $\varphi_1, \varphi_2, \ldots, \varphi_i$ by the Weiler method, one then generally needs only two more operations in order to ascertain φ_{i+2}, each of which is of order at most $2n - 2i - 2$, which is obviously a very substantial simplification compared to the Jacobi method.

Clebsch also achieved the same savings in integrations by his modification of the Jacobi method. Meanwhile, one must then concede that the Clebsch process is more circumstantial than the Weiler one, and can be applied, moreover, only when one has found two more solutions, in addition to the desired function φ_{i+1}, to the system that Clebsch used in place of the Weiler system (11).

Furthermore, as was suggested already in the introduction, in comparison to my method one may make a point that, in a certain sense, can be regarded as an advantage over the Weiler method. Namely, in order to determine φ_{i+2}, after one has already found $\varphi_2, \varphi_3, \ldots, \varphi_{i+1}$, my method requires just one more operation of order $2n - 2i - 2$, while, from the foregoing, the two operations through which the function φ_{i+2} is generally determined after the Weiler simplification can be performed only in the most pathological cases of this order. On the other hand, one may, by comparison, not overlook the fact that my method has the advantage of being completely independent of whether the case is favorable or not, and that its first step – viz., the determination of φ_{i+1} for a given $\varphi_2, \varphi_3, \ldots, \varphi_i$ – always requires only a single operation of order $2n - 2i$, where the Weiler method requires that, in addition to an operation of order $2n - 2i$, one must perform $i - 1$ further operations that are of at most the same order.

Weiler himself, in the selfsame way as in the discussion of my method, emphasized not the aforementioned, but an entirely different point as an advantage of my method. Namely, he raised the objection to my method that the linear partial differential equation that appeared for me in place of the two Weiler equations (16) and (18) included, in addition to its $2n - 2i - 1$ independent variables, i other variables as undetermined constants, which is not the case for equations (16) and (18). Now, except for the question of when one may attach definite values to the undetermined constants that appear in one and the same partial differential equation, there is generally a substantial lightening of the task of integrating the equation can be attained by this means. However, the fact that, for that reason, it only becomes more difficult to find a solution to one linear partial differential equation than it is for another, because the former one includes a number of undetermined constants that do not enter into the latter one, is then obviously (if one completely ignores the fact that in the present case two equations do not confront each other, but one equation confronts two others), when the two equations are also defined by anything completely different, only one claim!

§ 4.

On the Jacobi treatment of those partial differential equations of first order in which the unknown function itself appears.

At the conclusion of his article, Weiler also spoke of the way by which Jacobi also converted partial differential equations in which the unknown function itself occurred into ones in which the dependent variable was no longer explicitly included, and thus repeated the old objection that had been made against this Jacobi reduction.

Namely, when:

$$z = F\left(x_1, \cdots, x_n, \frac{\partial z}{\partial x_1}, \cdots, \frac{\partial z}{\partial x_n}\right) \tag{1}$$

is the given partial differential equation, Jacobi converted it, when he introduced the new independent variable t by means of the substitution:

$$V = tz, \tag{2}$$

and introduced a new unknown variable V into the following equation:

$$\frac{\partial V}{\partial t} = F\left(x_1, \cdots, x_n, \frac{1}{t}\frac{\partial V}{\partial x_1}, \cdots, \frac{1}{t}\frac{\partial V}{\partial x_n}\right). \tag{3}$$

However, this transformation immediately teaches us that each solution z of equation (1) can give us a solution V of equation (3), and one knows of no means for conversely deriving a solution z of equation (1) from an arbitrary solution V of equation (3) that would follow from it directly. Therefore, one rejects this Jacobi reduction and replaces it with another transformation that is known from the theory of linear partial differential equations, which seeks, in place of z, a finite equation of the form:

$$W(z, x_1, \ldots, x_n) = \text{const.}, \tag{4}$$

whose solution provides a function z that satisfies the equation (1). One then obtains the following partial differential equation for the new unknown function W:

$$z = F\left(x_1, \cdots, x_n, -\frac{\frac{\partial W}{\partial x_1}}{\frac{\partial W}{\partial z}}, \cdots, -\frac{\frac{\partial W}{\partial x_n}}{\frac{\partial W}{\partial z}}\right). \tag{5}$$

By a closer examination of the problem, whose basic notions I have Lie to thank for his written communication – I have convinced myself – however, that there is no basis whatsoever for abandoning Jacobi's reduction. Namely, one has the following simple theorem, through which the problem of which reduction one should use is generally solved:

If $V = \varphi(t, x_1, \ldots, x_n)$ *is a solution of equation* (3) *then:*

$$z = \frac{\partial \varphi}{\partial t} \tag{6}$$

is always a solution of equation (1), *assuming that, as long as it is possible, one substitutes for t, the value that it takes on from the equation:*

$$-\varphi + t\,\frac{\partial \varphi}{\partial t} = \text{const.} = c.$$

Of the possible exceptions for equation (3) that relate to only a completely special form, a solution V is this equation that is free of t will naturally be omitted.

In order to prove this theorem, I first remark that one always has that $z = \partial V / \partial t$ is, in principle, a solution of equation (1), as long as the variable t enters into the given solution V of equation (3) only in a linear way. In fact, when equation (3) is satisfied for:

$$V = t\,\psi + c,$$

where the functions ψ and χ are free of t, then one has identically for each value of t:

$$y = F\left(x_1, \cdots, x_n, \frac{\partial \psi}{\partial x_1}, \cdots, \frac{\partial \psi}{\partial x_n}\right),$$

then $z = \psi$, in turn, satisfies equation (1). We thus need only to examine the case where the given solution $V = \varphi$ to equation (3) is not a linear function of t.

In this case, the expression:

$$-\varphi + t\,\frac{\partial \varphi}{\partial t}$$

is not free of t. The assumption:

$$-\varphi + t\,\frac{\partial \varphi}{\partial t} = P,$$

where P is free of t, then yields, by integration:

$$\varphi = t\,Q - P,$$

where Q is also merely a function of x, and thus contradicts our assumption.

Therefore, as long as φ is not linear in t, equation (7) can always be used for the determination of t. If one now denotes the substitution of values for t that follow from it by [ ], and sets:

$$z = \left[\frac{\partial\varphi}{\partial t}\right] = \left[\frac{\varphi + c}{t}\right]$$

then one has:

$$\frac{\partial z}{\partial x_h} = \left[\frac{1}{t}\frac{\partial\varphi}{\partial x_h}\right],$$

and thus, under the substitution of these values for t and z, the identity:

$$\frac{\partial\varphi}{\partial t} = F\left(x_1, \cdots, x_n, \frac{1}{t}\frac{\partial\varphi}{\partial x_1}, \cdots, \frac{1}{t}\frac{\partial\varphi}{\partial x_n}\right)$$

goes to the following one:

$$z = F\left(x_1, \cdots, x_n, \frac{\partial z}{\partial x_1}, \cdots, \frac{\partial z}{\partial x_n}\right).$$

With this, the given theorem is proved and the justification for Jacobi's reduction, along with it.

I then add that the same manner of reduction that is included in formulas (4) and (5) not only lends support to the other one, but also, along with this, it possesses a not entirely insubstantial advantage that generally first takes on significance when one has to consider not just one, but several, associated partial differential equations between the same variables.

Namely, whereas both processes imply the same thing for complete or particular solutions of it, they differ essentially from each other in their behavior relating to singular solutions. By the second type of reduction, one excludes the latter from now on, and thus the same solutions can be lost completely and irretrievably, which is impossible for the Jacobi type of reduction.

In fact, any solution z of the given system of partial differential equations:

$$F\left(z, x_1, \cdots, x_n, \frac{\partial z}{\partial x_1}, \cdots, \frac{\partial z}{\partial x_n}\right) = 0 \tag{8}$$

that one can obtain by solving the equation W = const. from a solution W of the transformed system:

$$F\left(z, x_1, \cdots, x_n, -\frac{\dfrac{\partial W}{\partial x_1}}{\dfrac{\partial W}{\partial z}}, \cdots, -\frac{\dfrac{\partial W}{\partial x_n}}{\dfrac{\partial W}{\partial z}}\right) = 0, \tag{9}$$

includes a arbitrary constant.

Thus, when the given system (8) possesses such a solution, which either itself contains an arbitrary constant or also a solution with arbitrary constants, can be obtained in such a way that one attaches definite values to these constants, so it is impossible to derive this solution from a solution W of the system (9) in the manner described.

Moreover, if, in particular, the system (8) is nothing else but one that admits such singular solutions then equations (9) can possess no common solution at all – at least, none that includes z – and the solution of the system (8) is then lost completely when one substitutes equations (9) for equations (8). A system of this nature is defined by, for example, the two equations:

$$F = 0, \qquad F + z - \varphi = 0,$$

under the assumption that the first one represents a given partial differential equation, for which, $z = \varphi$ is any of its solutions.

By contrast, any system that arises from the given one (8) by the substitution $V = zt$ always has a solutions that includes t, as long as equations (8) admit no common solution z at all, regardless of whether this solution includes arbitrary constants or not. Thus, such loss of solutions can never enter into the Jacobi type of reduction.

Remark. In my quest to make the Weiler method clearer, from the outset, I have always had in mind only the case that is familiar to me, of a partial differential equation in which the desired function does not appear explicitly, and as a consequence of this, a point in the Weiler treatise escaped me completely, a fact that first appeared in the proofs, so it now seems, in retrospect, necessary to rectify a previous assertion.

Weiler treated the general case in which the unknown function itself appeared in the given partial differential equation, and, in turn, placed such a great weight upon his last section that he took into consideration the presence of the dependent variables. When taken in this generality, however, his results are presented not merely unclearly, but also downright falsely, and indeed, false for that reason, because the title on Weiler's § 4 is likewise incorrect, like the argument by which he proved it.

Namely, if one understands the symbol d / dx_h to mean the operation:

$$\frac{d}{dx_h} = \frac{\partial}{\partial x_h} + p_h \frac{\partial}{\partial z}$$

and sets:

$$[A\ B] = \sum_{h=1}^{n} \left(\frac{dA}{dx_h}\frac{\partial B}{\partial p_h} - \frac{dA}{dp_h}\frac{\partial B}{\partial x_h} \right)$$

then, when the given partial differential equation $\varphi_1 = c_1$ includes the unknown function z itself, in place of the previous equations $(\varphi_i\ \varphi_k) = 0$, one finds, from then on, the equations:

$$[\varphi_i\ \varphi_k] = 0.$$

Weiler now sought to prove in the stated section that the Poisson-Jacobi theorem also has validity for these general equations; i.e., that when $f = \psi$ and $f = \chi$ satisfy the equation:

$$[\varphi\ f] = 0,$$

one always has that $f = [\psi\ \chi]$ is a solution of this equation, as well.

However, that is false. Namely, from the Jacobi identity, by the application of the Jacobi reduction, one obtains with no difficulty that for any three functions φ, ψ, χ of the $2n + 1$ variables z, x_1, …, x_n, p_1, …, p_n the identity formulas must exist:

$$[\varphi, [\psi\,\chi]] + [\varphi, [\psi\,\chi]] + [\varphi, [\psi\,\chi]] = \frac{\partial\varphi}{\partial z}[\psi\chi] + \frac{\partial\psi}{\partial z}[\chi\varphi] + \frac{\partial\chi}{\partial z}[\varphi\psi],$$

and from this, it follows, when one assumes that $[\varphi\,\chi] = [\varphi\,\psi] = 0$, that:

$$[\varphi, [\psi\,\chi]] = \frac{\partial\varphi}{\partial z}[\psi\chi],$$

i.e., when $f = \psi$ and $f = \chi$ are any two solutions of the equation:

$$[\varphi f] = 0$$

and $[\psi\,\chi]$ is not zero then $f = [\psi\,\chi]$ is, in turn, a solution of this equation when and only when the function φ is free of z.