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Foundations of a Theory of Matter.

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Contents

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Introduction.

1. What the recent experimental facts about the behavior of atoms say is essentially in the negative, namely, that the laws of mechanics and the Maxwellian equations are no longer valid in their interiors. However, these experimental facts say nothing about just which equations must replace these equations or what viewpoint one must take in order to satisfy the most noteworthy facts that go by the name of quantum effects, much less the laws of atomic spectra, and so on. I believe that one must not merely wait for some appropriate sort of experiment to change all of this. Experiment and theory must work hand-in-hand, and that is not possible as long as the theory has no basis upon which one might found it.

It therefore seems to me that in order for our knowledge to advance it is necessary to create a new foundation for the theory of matter. In the work that follows, I have sought to make a start in this direction, but one must not expect from the difficulty of the subject that correspondingly tractable experimental results were involved. The next goals that I set for myself are to clarify the existence of indivisible electrons and to find an unavoidable connection between the facts of gravitation and the existence of matter. I believe that one must begin at this point because the electrical and gravitational effects are certainly the immediate statements of the forces that the existence of matter must rest upon. It is meaningless to think of matter whose smallest constituents do not have electrical charge, and just as meaningless to think of matter without gravitation. Only when both of the stated objectives are reached will one be able to think about the complexities, of which I spoke above, that appear in conjunction with bringing the theory into unified entity. However, there is another way to reach both of the aforementioned goals, and in the sequel I shall present only the preliminary work that might perhaps take us in that direction.

The basic assumption of my theory is that *electric and magnetic fields also exist in the interior of electrons.* According to this way of thinking, electrons, and therefore the smallest constituents of matter, are not distinct from the ether. They are not, as has been believed for twenty years, foreign particles in the ether, but *they are only places at which the ether takes on a particular state that we give the name of electrical charge.* Indeed, the enormous intensity of the field and charge at that point, which we have called an electron, in itself suggests that the usual Maxwell equations are no longer valid. The use of electromagnetic fields in the electron will presumably seem strange when one considers the laws of the "pure ether." However, if we are to speak of an electromagnetic field in the interior of an electron in any sense, then we must not understand this to mean that it is impossible to make a continuous transition from considering the "pure" ether to considering the ether inside an electron. Therefore, in my theory the electron is not a sharply defined region of the ether, but it possesses a kernel that goes continuously over into an atmosphere of electric charge that extends to infinity, although it is already so extraordinarily thin near the kernel that that one cannot imagine any experimental

verification of this. An atom is an amalgamation of a large number of electrons that are held together with a comparatively weak charge of opposite sign. Atoms are obviously surrounded with a powerful atmosphere that is nevertheless always so thin that no noticeable electrical field remains, although it might have some influence in gravitational effects.

One might perhaps think that one can make little progress past the aforementioned basic assumption. However, one is led to a general form for the fundamental equations of ether physics when one adds two more assumptions. The first one is that *the principle of relativity shall have a general validity*, and the second one is that *the hitherto known states of the ether, namely, electric field, magnetic field, electric charge, and current are completely sufficient to describe everything that one can observe in the material world.* The justification for the first assumption is completely beyond doubt. Whether the second one is equally justifiable is yet to be determined; one must examine it next. One is then left with a theory that correctly reproduces the material world, so it is therefore justified. In the contrary case, one must ask the question of how one is to extend the fundamental quantities.

In what follows, I will first present the considerations that led me from the three assumptions that were made above to a general form for the equations of the ether in a rather detailed fashion, in order to facilitate a discussion of whether the form that I assumed is possibly unique, or whether there are not perhaps other fundamental equations for ether physics that are consistent with the three assumptions. I confess that I have yet to find any other possibilities. The fact that I have assumed the validity of the principle of the conservation of energy and that energy is a localized quantity is self-explanatory.

Chapter One

The Field Equations

General Form of the field equations.

2. When one examines the Maxwell equations, which are best presented in the form that Minkowski gave to them, one immediately sees that the four-dimensional six-vector of "electromagnetic field strength" is not, by itself, sufficient to completely describe all phenomena in space and time. A self-explanatory four-vector, the "four-current," must therefore appear in Maxwell's equations, which is the least that must be added to them in order to make the description complete.

By assumption, the time component of the four-current – the charge density ρ – represents a singular property of the universal ether that takes on a noticeable magnitude at only one point, and it brings with it the consequence that the electric field line $\mathfrak{d}$ simply vanishes at this point, in order for div $\mathfrak{d}$ to be non-zero. We can therefore take the value of div $\mathfrak{d}$ to be a measure for the new state of the ether:

$$\rho = \operatorname{div} \mathfrak{d}.$$

Likewise, the space components of the four-current — the electric current $\mathfrak{v}$ – describe a singular property of the ether that takes on noticeable values only at a single point, and that it brings with it the introduction of a vortex into the magnetic field $\mathfrak{h}$ that cannot be compensated for by a timelike variation of the electric field $\mathfrak{d}$. From this, we can use the difference rot $\mathfrak{h} - \dot{\mathfrak{d}}$ as a measure of the new state of the ether:

$$\operatorname{rot} \mathfrak{h} - \dot{\mathfrak{d}} = \mathfrak{v}.$$

3. We now make use of the basic assumptions that we introduced in **1**. In order for the "electromagnetic field" and "four-current" to describe collectively all of the phenomena in the material world, the causality principle entails that one must impose ten differential equations upon the ten components of the state variables $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$, whose left-hand side is always a differential quotient of the first order in time of one of these variables or a function of it, whereas a function of the variables and their spacelike differential quotients appears in the right-hand side. Only through such a system of equations does the distribution of the ether at *one* given moment always determine the distribution at the next moment after an infinitesimal time *dt* has elapsed, which thus satisfies the causality principle.

If the relativity principle is to still remain valid then the differential quotients in these equations must describe vectorial differential operators on four-dimensional variables; this reduces the number of possibilities considerably. One immediately sees e.g., that only differential quotients of the first order in the coordinates can appear, that all of the differential quotients appear to the first power, etc.

Ultimately, one must have that the equations must converge to the Maxwell equations in the "pure" ether, in order for the transition from matter to ether to be gradual. In order for the existence of true magnetic charges to be excluded, one must therefore characterize the magnetic field by means of a quantity $\mathfrak{b}$ that above all must have the property that div $\mathfrak{b} = 0$. We then come to the equations:

$$\frac{\partial \mathfrak{d}}{\partial t} = \text{rot } \mathfrak{h} - \mathfrak{v}, \tag{1}$$

$$\frac{\partial \mathfrak{b}}{\partial t} = -\text{rot } \mathfrak{e}, \tag{2}$$

and, indeed, one must have that in the pure ether $\mathfrak{b}$ is identical to $\mathfrak{h}$ and $\mathfrak{e}$ is identical to $\mathfrak{b}$. On the contrary, $\mathfrak{e}$ and $\mathfrak{b}$ can be complicated functions of $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$ in the interior of matter. Equations (1) and (2) can only superficially resemble the Maxwell equations. Since at least half of them are no longer linear the laws of the fields inside the atom are completely different from those of the pure ether, and one can give no electromagnetic waves there, by way of example, whose existence implies linear equations and the like.

In what follows, we shall therefore clearly distinguish between the two "intensity variables:" electric field strength $\mathfrak{e}$ and magnetic induction $\mathfrak{b}$, and the "quantity variables:" electrical displacement $\mathfrak{d}$ and the magnetic field strength $\mathfrak{h}$. The superposition principle for the electromagnetic field valid only in the pure ether, which we shall express by $\mathfrak{e} = \mathfrak{d}$, $\mathfrak{b} = \mathfrak{h}$.

In the nomenclature of four-dimensional vector analysis [1]) equations (1) and (2) take the following form:

$$\Delta \cdot \nu(\mathfrak{h}, -i\mathfrak{d}) = (\mathfrak{v}, i\rho), \tag{1a}$$

$$\Delta \cdot \nu(\mathfrak{e}, i\mathfrak{b}) = 0. \tag{2a}$$

All that remains for us now is to give the four corresponding equations for the four-vector $(\mathfrak{v}, i\rho)$. There are two kinds of first order differential operators in four dimensions, namely, the operators Div and $\mathfrak{Rot}$ [2]). Time components are differentiated by means of the first operator, whereas the three space components are differentiated with respect to t by means of the second. We must therefore use both of these operators in order to obtain the four missing differential equations. The operator Div appears in the well-known equation:

$$\frac{\partial \rho}{\partial t} + \text{div } \mathfrak{v} = 0. \tag{3}$$

In the four-dimensional notation this is written as:

[1] M. Laue, Das Relativitätsprinzip, pp. 70, Friedr. Vieweg & Sons, 1911.
[2] M. Laue, loc. cit., pp. 70.

(3a) $$\mathrm{Div}(\mathfrak{v}, i\rho) = 0.$$

The missing equations must then be include in a formula:

$$\mathfrak{Rot}(\mathfrak{f}, i\varphi) = \mathfrak{F},$$

in which $(\mathfrak{f}, i\varphi)$ is a four-vector that is related to $(\mathfrak{v}, i\rho)$ in the same way that the six-vector $(\mathfrak{b}, -i\mathfrak{e})$ is related to $(\mathfrak{h}, -i\mathfrak{d})$. Next, we only know that $\mathfrak{f}$ and φ are arbitrary functions of all of the state variables that collectively define a four-vector. The right-hand side of the equation, viz., $\mathfrak{F}$, is some six-vector that is likewise a function of the state variables, among which only one is known, namely, that the condition:

$$\Delta \cdot v\mathfrak{F}^* = 0$$

must be satisfied [1]), because nothing else can be obtained from applying the operator $\mathfrak{Rot}$ to a four-vector. This condition must further imply, when we do not assume as much, that $\mathfrak{F}$ = const., which then actually gives an identity that is identical with equation (2a). If this is not the case then, besides the ten differential equations that are required by the causality principle, we have three extra ones. The time evolution of the state in the ether is then overdetermined, which is naturally impossible. We must then necessarily either set $\mathfrak{F}$ = const. or $\mathfrak{F} = C \cdot (\mathfrak{b}, -i\mathfrak{e})$, in which C means an arbitrary constant factor. We can bring this factor to the other side of our equation $\mathfrak{Rot}(\mathfrak{f}, i\varphi) = \mathfrak{F}$ and absorb it into $\mathfrak{f}, i\varphi$, and we thus simply set $\mathfrak{F} = (\mathfrak{b}, -i\mathfrak{e})$. The three equations, which include a differential quotient with respect to time, then take the general form:

$$-\frac{\partial \mathfrak{f}}{\partial t} = \nabla\varphi + C \cdot \mathfrak{e} + \mathfrak{c},$$

in which C is either zero or one and $\mathfrak{c}$ means vector that is constant in all of spacetime. In a region of the pure ether where $\mathfrak{f} = 0$ and $\mathfrak{e} = 0$ one must have $\nabla\varphi = -\mathfrak{c}$. Although the state variables are thus a constant equal to zero here, as well, if φ, which shall be a function of the state variables, has a non-zero gradient then it must be non-constant. This is impossible, so we must therefore have $\mathfrak{c} = 0$. Otherwise, it is easy to show that C must be non-zero. If the state of the ether is in equilibrium in the neighborhood of an electron that moves with constant velocity then all of the differential quotients with respect to time must be zero. The equation then becomes:

$$\nabla\varphi + C \cdot \mathfrak{e} = 0.$$

[1] M. Laue, loc. cit., pp. 71.

Now, if $C = 0$ then we will also have $\nabla\varphi = 0$; hence, φ = const. The variable φ is therefore completely independent of the field magnitudes, and the same is also true for f, from the relativity principle, and the aforementioned equation collapses to an identity. One must therefore have $C = 0$. The last three equations of ether dynamics are thus:

$$-\frac{\partial \mathfrak{f}}{\partial t} = \nabla\varphi + \mathfrak{c}, \tag{4}$$

which can be written, in four-dimensional notation:

$$\mathfrak{Rot}(\mathfrak{f}, i\varphi) = (\mathfrak{b}, -i\mathfrak{e}). \tag{4a}$$

The expression (4a) includes the following three equations, in which no differential quotients with respect to time appear:

$$\mathfrak{rot}\ \mathfrak{f} = \mathfrak{b}. \tag{4b}$$

One easily sees that if one derives equations (4b) from (4) with the help of (2) then one obtains nothing new.

When everything is in equilibrium in the neighborhood of an electron at rest or a uniform velocity equation (4) becomes:

$$\nabla\varphi + \mathfrak{c} = 0.$$

We shall call this the *equilibrium condition* for the field in the neighborhood of an electron. If may clearly be interpreted as the statement that both of the forces $\mathfrak{c}$ and $=\varphi$ must be equal and opposite to each other. The electric field strength $\mathfrak{e}$ endeavors to draw the charge of the electron outward so it fills the largest possible space; it therefore represents a *body force* that lives in the matter. It acts against the *surface force* $\nabla\varphi$, which is computed as the gradient of the singular surface tension φ [1]) that acts on the electric charge. Body forces and surface forces are both effects upon which the existence of matter certainly rests, so they must enter into any possible theory of matter.

Equation (8) may be called equation of motion of the electric current. The vector $\mathfrak{f}$ is the *quantity of motion* [*]) that corresponds to the electric current $\mathfrak{v}$. In conventional mechanics the quantity of motion is known to be the mass times the velocity and is measured by means of the push that is necessary in order to bring the velocity up to its value. Since the quantity of motion and tension are to be thought of as "intensity variables," i.e., as quantities that one measures by means of force effects then we shall also regard φ and $\mathfrak{f}$ as "intensity variables" that correspond to the "quantity variables" ρ and $\mathfrak{v}$, respectively.

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[1] It is well known that a tension of this sort was first used by H. Poincaré (Compt. rend. **140**, pp. 1504, 1905). Cf., also H. Th. Wolff, Ann. d. Phys. **36**, pp. 1066, 1911.

[*] DHD: i.e., energy-momentum four-vector.

We can thus describe the state of the ether in terms of either the ten quantity variables ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$) or the ten intensity variables ($\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$).

4. The six differential equations (4) and (4b), which are summarized in formula (4a), are precisely the same as the differential equations of the so-called "four-potential" that one constructs from the scalar potential φ and the vector potential $\mathfrak{f}$. One can say with some justification the theory that was developed here simply says that the two potentials, φ and $\mathfrak{f}$, embody the physical state of the universal ether, namely the surface tension and quantity of motion.

We must therefore make a further important remark. It is known that the solution of equation (4a) for a given six-vector ($\mathfrak{b}$, $-i\mathfrak{e}$) is still undetermined when one makes no assumption about the value of Div($\mathfrak{f}$, $i\varphi$). In the theory of electricity, one defines both potentials by simply setting Div($\mathfrak{f}$, $i\varphi$) = 0. However, this equation is not valid for the ether state that was chosen in our theory, and they are thus generally not identical with the usual potentials. In place of the aforementioned equations of the theory of electricity, equation (3) appears in our ether dynamics: Div($\mathfrak{v}$, $i\rho$) = 0. This equation cannot be included with the other equations because then the time evolution of the ether state will be governed by eleven equations; hence it is over-determined, which is impossible. We thus have that Div($\mathfrak{v}$, $i\rho$) $\neq$ 0 in general. In a later section (pp. ?) we will find a simple meaning for the quantity Div($\mathfrak{v}$, $i\rho$).

In the rest case ($\mathfrak{v}$ = 0, $\mathfrak{h}$ = 0) the quantity φ is actually identical with the electrostatic potential because one then has the following equation:

$$\mathfrak{e} + \nabla\varphi = 0.$$

5. When we understand φ to mean a tension and ρ to mean a density then we can easily see that it would be advantageous for these quantities to always take on positive values, at least as far as the physics of gases is concerned.

We would like for them to have a constant positive value ρ_0 in the pure ether, where there are no fields, a value that we call the normal density. For any arbitrary choice of spacetime coordinate systems, it must naturally define a four-vector ($\mathfrak{v}_0$, $i\rho_0$) that is constant over all of some spacetime region. Electric and magnetic fields will enter into the picture only where ρ and $\mathfrak{v}$ take different values from ρ_0 and $\mathfrak{v}_0$, and equations (1) and (3) will thus take on the following form:

$$\Delta \cdot v(\mathfrak{h}, -i\mathfrak{d}) = ((\mathfrak{v}-\mathfrak{v}_0), i(\rho - \rho_0)),$$
$$\mathrm{Div}((\mathfrak{v}-\mathfrak{v}_0), i(\rho - \rho_0)) = 0.$$

One can naturally choose ρ_0 in such a way that quantity ρ, the "ether density," that enters into these equations is always positive. For that reason, in what follows, I will simply

write $\mathfrak{v}$ and ρ instead of $\mathfrak{v}-\mathfrak{v}_0$ and $\rho - \rho_0$, so we therefore compute with positive and negative densities, and I also set the pure ether density to zero.

All of the above is likewise true for the surface tension φ. Since φ and $\mathfrak{f}$ appear in the fundamental equations for ether dynamics only when they are differentiated by time or space one can augment them with arbitrary quantities φ_0, $\mathfrak{f}_0$ that are completely independent of time and space, or else the description of the time evolution would take on a different form. One can, by way of example, take a value φ_0 that is large enough that $\varphi_0 - \varphi$ always remains positive. The equilibrium conditions then become:

$$\mathfrak{e} - \nabla(\varphi_0 - \varphi) = 0.$$

In pure ether we now have the positive tension φ_0, in the electron we have the smaller tension $(\varphi_0 - \varphi)$ and $\mathfrak{e}$ is the tension gradient $-\nabla(\varphi_0 - \varphi)$ that the ether exerts on the electron in equilibrium. In fact, H. Poincaré (loc. cit.) spoke of a tension that the electron exerted on the external space. However, I believe that it is simpler for the sake of representation when the null point of the tension lies in the pure ether, and may thus be computed in such a way that φ is set to zero at an infinite distance from the electron.

Likewise, we would also like for energy, which one can always augment with an additive constant, as is well-known, to have a null point that is so arranged that the energy density is zero in pure field-free ether. Similarly, just like ρ and φ, the energy density W, for that matter, can also take on negative values as well as positive ones; however, there is not the slightest reason that would compel us to always set W to be positive.

With these associations ρ, φ, W are now completely determined quantities with no further additive ambiguity.

Energy.

6. I will now assume that not only the principle of the conservation of energy, but also the principle of the localization of energy and energy transport [1]) are valid. In other words: if we denote the energy density by W and the energy current by $\mathfrak{s}$ then the following consequence must ensue from field equations (1) through (4):

$$\frac{\partial W}{\partial t} = -\operatorname{div} \mathfrak{s},$$

in which not only the scalar W, but also the vector $\mathfrak{s}$ are universal functions of the state associated with the chosen spacetime point. One can arrive at this energy equation from the field equations in only *one* way: one must determine factors $\mathfrak{k}$, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$ that are universal functions of the state variables, multiply equations (1) to (4) by them, and add the equations. It must therefore also be possible to determine the factors $\mathfrak{k}$, $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$ in such

[1] G. Mie, Wiener Sitzungsber. **107**, Vol. 11a, pp. 1117 and 1126, 1898.

a way that a complete differential quotient with respect to time appears in the left-hand side and a divergence appears on the right-hand side. We would now like to find the conditions for this to be true.

$$\mathfrak{k} \cdot \frac{\partial \mathfrak{d}}{\partial t} + \mathfrak{l} \cdot \frac{\partial \mathfrak{h}}{\partial t} + m \cdot \frac{\partial \mathfrak{j}}{\partial t} + \mathfrak{n} \cdot \frac{\partial \mathfrak{f}}{\partial t}$$
$$= \mathfrak{k} \cdot \mathfrak{rot}\ \mathfrak{h} - \mathfrak{k} \cdot \mathfrak{v} - \mathfrak{l} \cdot \mathfrak{rot}\ \mathfrak{e} - m \cdot \operatorname{div} \mathfrak{v} - \mathfrak{n} \cdot \nabla\varphi - \mathfrak{n} \cdot \mathfrak{e}.$$

Next, we see that both of the terms $-\mathfrak{k} \cdot \mathfrak{v}$ and $-\mathfrak{n} \cdot \mathfrak{e}$, which are pure universal functions of the state variables, must drop out, because div $\mathfrak{s}$ can only depend on terms that include differential quotients with respect to the coordinates. One must therefore have:

$$\mathfrak{k} = u \cdot \mathfrak{e},$$
$$\mathfrak{n} = -u \cdot \mathfrak{v},$$

in which u is again a universal function of the state variables. A simple computation then gives, for the right-hand side of the equation:

$$\operatorname{div}(u \cdot [\mathfrak{h} \cdot \mathfrak{e}]) + \operatorname{div}(u \cdot \varphi \cdot \mathfrak{v}) + (u \cdot \mathfrak{h} - \mathfrak{l}) \cdot \mathfrak{rot}\ \mathfrak{e} - \mathfrak{h} \cdot [\mathfrak{e} \cdot \nabla u] - (m + u \cdot \varphi) \cdot \operatorname{div} \mathfrak{v} - \varphi \cdot (\mathfrak{v} \cdot \nabla u).$$

In general, this expression can only be a divergence when the last summands drop out, and thus:

$$\nabla u = 0,$$
$$u \cdot \mathfrak{h} - \mathfrak{l} = 0,$$
$$m + u \cdot \varphi = 0.$$

The first of these equations gives u = const., and indeed this constant is determined in such a way that the expression for the energy current in pure ether must becomes the well-known Poynting expression $[\mathfrak{d} \cdot \mathfrak{h}] = [\mathfrak{e} \cdot \mathfrak{h}]$. From this, it follows:

$$u = 1, \quad \mathfrak{k} = \mathfrak{e}, \quad \mathfrak{l} = \mathfrak{h}, \quad m = -\varphi, \qquad \mathfrak{n} = -\mathfrak{v}.$$

We have thus found the energy equation:

$$\mathfrak{e} \cdot \frac{\partial \mathfrak{d}}{\partial t} + \mathfrak{h} \cdot \frac{\partial \mathfrak{h}}{\partial t} - \varphi \cdot \frac{\partial \mathfrak{j}}{\partial t} - \mathfrak{v} \cdot \frac{\partial \mathfrak{f}}{\partial t} = -\operatorname{div}([\mathfrak{e} \cdot \mathfrak{h}]) - \varphi \cdot \mathfrak{v}).$$

The expression for the *energy current* in general ether dynamics is then:

(5) $$\mathfrak{s} = [\mathfrak{e} \cdot \mathfrak{h}]) - \varphi \cdot \mathfrak{v}.$$

7. The energy principle now tells us more: that the expression on the left-hand side of the energy equation is a complete differential. We must therefore state the condition for this, that:

(6) $$\mathfrak{e} \cdot d\mathfrak{d} + \mathfrak{h} \cdot d\mathfrak{b} - \varphi \cdot d\rho - \mathfrak{v} \cdot d\mathfrak{f} = dW$$

is a complete differential, and that W may therefore be determined as a function of ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$). Just as we did for W, we can initiate a search for a quantity H that is determined by means of the following equation:

(7) $$W = H + \mathfrak{h} \cdot \mathfrak{b} - \mathfrak{v} \cdot \mathfrak{f}.$$

If W is a function of ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$) then so is H, and vice versa. From (6) and (7), we obtain the following expression for the differential of H:

(8) $$dH = \mathfrak{e} \cdot d\mathfrak{d} - \mathfrak{b} \cdot d\mathfrak{h} - \varphi \cdot d\rho + \mathfrak{f} \cdot d\mathfrak{v},$$

in which $\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$ are functions of ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$). We would now like to abbreviate the notation for a vector whose components are:

$$\frac{\partial H}{\partial \mathfrak{d}_x}, \quad \frac{\partial H}{\partial \mathfrak{d}_y}, \quad \frac{\partial H}{\partial \mathfrak{d}_z},$$

by simply saying $\partial H/\partial \mathfrak{d}$, and analogous expressions in all other cases. It then follows from (8), with no further assumptions, that:

(9) $$\mathfrak{e} = \frac{\partial H}{\partial \mathfrak{d}}, \qquad \mathfrak{b} = -\frac{\partial H}{\partial \mathfrak{h}}, \qquad \varphi = -\frac{\partial H}{\partial \rho}, \qquad \mathfrak{f} = \frac{\partial H}{\partial \mathfrak{v}}.$$

The condition for this, that the energy principle is valid, is, since all of the intensity variables $\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$ *can be computed by means of a single function the magnitude variables* $H(\mathfrak{d}, \mathfrak{h}, \rho, \mathfrak{v})$, *which we would like to use for a Hamiltonian function. Indeed, each intensity variable is obtained as a differential quotient of H with respect to the corresponding magnitude variable, in two cases* ($\mathfrak{b}$ *and* φ) *with the negative sign.*

One can also find the energy density W from just the Hamiltonian function. If we use (9) then (7) gives:

(10) $$W = H - \frac{\partial H}{\partial \mathfrak{d}} \cdot \mathfrak{h} - \frac{\partial H}{\partial \mathfrak{v}} \cdot \mathfrak{v},$$

From the form of fundamental equations of ether dynamics, (1) through (4), one immediately obtains the following theorem, when one considers equation (9):

The relativity principle is valid for all physical motions, as long as the Hamiltonian function H($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$) *is invariant under Lorentz transformations.*

We will have therefore completely exhibited the equations for ether dynamics when we know sort of form the universal function *H* takes. Finding this form is, however, an extremely difficult problem.

The problem of a theory of matter goes back to the problem of finding the universal function H($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$).

Certainly, we know one thing about *H*: In pure ether, the superposition principle for electromagnetism is known with considerable accuracy; if one then substitutes a summand $(\mathfrak{b}^2 - \mathfrak{h}^2)/2$ for *H*:

$$H = \tfrac{1}{2}(\mathfrak{b}^2 - \mathfrak{h}^2) + H_1,$$

then the remaining term H_1 must be vanishingly small compared to the first term anywhere that ρ is very small. On the other hand, in the interior of the atom, where ρ is large, H_1 will far outweigh that term, such that the laws of fields are completely different here from what they are in pure ether.

8. For the sake of computation, it is generally more convenient to take the intensity variables ($\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$) to be the independent variables in terms of which the state of the ether is determined, and the magnitude variables ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$) to be functions of them.

We would now like to define the following function Φ:

$$\Phi(\mathfrak{e}, \mathfrak{b}, \varphi, \mathfrak{f}) = H - (\mathfrak{e} \cdot \mathfrak{d} - \mathfrak{b} \cdot \mathfrak{h}) + (\varphi \cdot \rho - \mathfrak{f} \cdot \mathfrak{v}), \tag{11}$$

and we next compute the quantities $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$ as functions of $\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$, using equations (9), and then substitute the expression so obtained in the right-hand side of equation (11). If we refer to (8) then we obtain the following expression for the differential of Φ:

$$d\Phi = -\mathfrak{d} \cdot d\mathfrak{e} + \mathfrak{h} \cdot d\mathfrak{b} + \rho \cdot d\varphi - \mathfrak{v} \cdot d\Phi. \tag{12}$$

From this, it follows that:

$$\mathfrak{d} = -\frac{\partial \Phi}{\partial \mathfrak{e}}, \qquad \mathfrak{h} = \frac{\partial \Phi}{\partial \mathfrak{b}}, \qquad \rho = \frac{\partial \Phi}{\partial \varphi}, \qquad \mathfrak{v} = -\frac{\partial \Phi}{\partial \mathfrak{f}}. \tag{13}$$

The magnitude variables $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$ *may be then computed with the help of a single function of the intensity variables* $\Phi(\mathfrak{e}, \mathfrak{d}, \varphi, \mathfrak{f})$ *when one differentiates this function with respect to the corresponding intensity variables. In two cases* ($\mathfrak{d}$ *and* $\mathfrak{v}$) *one must give the differential quotients the negative sign.*

The energy density *W* may then be obtained from Φ by the following equation:

$$W = \Phi + \mathfrak{e} \cdot \mathfrak{d} - \varphi \cdot \rho = \Phi - \frac{\partial \Phi}{\partial \mathfrak{e}} \cdot \mathfrak{e} - \frac{\partial \Phi}{\partial \varphi} \cdot \varphi . \tag{14}$$

The Hamiltonian function *H* is then computed from (11):

$$H = \Phi - \frac{\partial \Phi}{\partial \mathfrak{e}} \cdot \mathfrak{e} - \frac{\partial \Phi}{\partial \mathfrak{b}} \cdot \mathfrak{b} - \frac{\partial \Phi}{\partial \varphi} \cdot \varphi - \frac{\partial \Phi}{\partial \mathfrak{f}} \cdot \mathfrak{f}. \tag{15}$$

Instead of looking for the universal function H(𝔡, 𝔥, *ρ*, 𝔳), *one can also seek the universal function* Φ(𝔢, 𝔡, *φ*, 𝔣).

I will often refer to Φ more briefly as the *world function.*

Φ *must be invariant under Lorentz transformations, just as H is.*

Just like *H*, Φ may be broken into two pieces:

$$\Phi = \tfrac{1}{2}(\mathfrak{d}^2 - \mathfrak{e}^2) + \Phi_1,$$

in which the first term is dominant in pure ether and the second one is dominant in the interior of atoms.

9. We may build a 4×4 matrix [1]) with the help of the world function that includes the energy current and the Maxwellian ether stresses for our general ether dynamics:

$$S = \begin{bmatrix} \Phi - \mathfrak{b}\mathfrak{h} + \mathfrak{e}_x\mathfrak{d}_x + \mathfrak{h}_x\mathfrak{b}_x + \mathfrak{f}_x\mathfrak{v}_x & \mathfrak{e}_x\mathfrak{d}_y + \mathfrak{h}_x\mathfrak{b}_y + \mathfrak{f}_x\mathfrak{v}_y & \mathfrak{e}_x\mathfrak{d}_z + \mathfrak{h}_x\mathfrak{b}_z + \mathfrak{f}_x\mathfrak{v}_z & -i(\mathfrak{d}_y\mathfrak{b}_z - \mathfrak{d}_y\mathfrak{b}_z - \}\mathfrak{f}_x) \\ \mathfrak{e}_y\mathfrak{d}_x + \mathfrak{h}_y\mathfrak{b}_x + \mathfrak{f}_y\mathfrak{v}_x & \Phi - \mathfrak{b}\mathfrak{h} + \mathfrak{e}_y\mathfrak{d}_y + \mathfrak{h}_y\mathfrak{b}_y + \mathfrak{f}_y\mathfrak{v}_y & \mathfrak{e}_y\mathfrak{d}_z + \mathfrak{h}_y\mathfrak{b}_z + \mathfrak{f}_y\mathfrak{v}_z & -i(\mathfrak{d}_z\mathfrak{b}_x - \mathfrak{d}_x\mathfrak{b}_z - \}\mathfrak{f}_y) \\ \mathfrak{e}_z\mathfrak{d}_x + \mathfrak{h}_z\mathfrak{b}_x + \mathfrak{f}_z\mathfrak{v}_x & \mathfrak{e}_z\mathfrak{d}_y + \mathfrak{h}_z\mathfrak{b}_y + \mathfrak{f}_z\mathfrak{v}_y & \Phi - \mathfrak{b}\mathfrak{h} + \mathfrak{e}_z\mathfrak{d}_z + \mathfrak{h}_z\mathfrak{b}_z + \mathfrak{f}_z\mathfrak{v}_z & -i(\mathfrak{d}_x\mathfrak{b}_y - \mathfrak{d}_y\mathfrak{b}_x - \}\mathfrak{f}_z) \\ -i(\mathfrak{e}_y\mathfrak{h}_z - \mathfrak{e}_z\mathfrak{h}_y - \varphi \cdot \mathfrak{v}_x) & -i(\mathfrak{e}_z\mathfrak{h}_x - \mathfrak{e}_x\mathfrak{h}_z - \varphi \cdot \mathfrak{v}_y) & -i(\mathfrak{e}_x\mathfrak{h}_y - \mathfrak{e}_y\mathfrak{h}_x - \varphi \cdot \mathfrak{v}_z) & \Phi + \mathfrak{e}\mathfrak{d} - \varphi \cdot \} \end{bmatrix} \tag{16}$$

If one performs the following operation:

$$\Delta \cdot v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + \frac{\partial}{i \cdot \partial t}$$

on the last row of the matrix then one obtains the energy equation when one sets the expression so obtained to zero:

$$\operatorname{div}([\mathfrak{e} \cdot \mathfrak{h}] - \varphi \cdot \mathfrak{v}) + \frac{\partial}{\partial t}(\Phi + \mathfrak{e} \cdot \mathfrak{d} - \varphi \cdot \rho) = 0,$$

[1] H. Minkowski, Zwei Abhandlungen, B. G. Teubner, 1910, pp. 36.

since (14) gives $\Phi + \mathfrak{e} \cdot \mathfrak{d} - \varphi \cdot \rho = W$. From the relativity principle it then follows further that:

(17) $$\Delta \cdot vS = 0.$$

What remains are three equations that correspond to the first three rows of S, and also with little difficulty we obtain the field equations, (1) through (4), directly.

As to the question of whether the matrix (16) is symmetric across the diagonal or not, we will return to it later (pp. ?).

Hamilton's Principle

10. When we were in the foregoing purely theoretical section, it was stated that the form of the field equations that was given was the possible one; let us discuss this further now. It seems to me that there is some value in showing that one can obtain the field equations through quite simple mathematical operations when one assumes the validity of Hamilton's principle.

I shall therefore make the following two assumptions: First: that the state of the ether is completely characterized by the quantities $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$; moreover, the last two are defined by the equations:

$$\rho = \operatorname{div} \mathfrak{d}, \quad \mathfrak{v} = \mathfrak{rot}\ \mathfrak{h} - \dot{\mathfrak{d}};$$

Second: that the time evolution of the ether satisfies Hamilton's Principle, which we shall now formulate:

Hamilton's Principle. There a function $H(\mathfrak{d}, \mathfrak{h}, \rho, \mathfrak{v})$, *whose integral over any given spacetime with boundary is an extremum for all real motions, when one varies the state variables at all points inside a region, but not on the boundary.*

(18) $$\int_G \delta H\ (\mathfrak{d}, \mathfrak{h}, \rho, \mathfrak{v}) \cdot dx \cdot dy \cdot dz \cdot dt = 0.$$

On the boundary of the region *G*, one has:

$$\delta\mathfrak{d} = \delta\mathfrak{h} = \delta\rho = \delta\mathfrak{v} = 0.$$

One can show that the principle of relativity is valid when *H* is invariant under Lorentz transformations. We assume that this is the case and replace the quantities $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$ with the known expressions $\mathfrak{d}', \mathfrak{h}', \rho, \mathfrak{v}'$ that are obtained by a transformation from a coordinate system (*x*, *y*, *z*, *t*) into another (x', y', z', t'). Hence, we must obtain a function $H'(\mathfrak{d}', \mathfrak{h}', \rho, \mathfrak{v}')$ that involves the new variables $(\mathfrak{d}', \mathfrak{h}', \rho, \mathfrak{v}')$ in precisely the same way that H involves the variables ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$). We conclude from this that we must set $H' = H$. Now, let G' be the region of the new coordinate system (x', y', z', t') that *H* goes over to under the transformation. One then has:

$$\int_{G'} \delta H(\mathfrak{d}', \mathfrak{h}', \rho', \mathfrak{v}') \cdot dx' \cdot dy' \cdot dz' \cdot dt' = \int_G \delta H\ (\mathfrak{d},\ \mathfrak{h},\ \rho,\ \mathfrak{v}) \cdot dx \cdot dy \cdot dz \cdot dt.$$

If Hamilton's Principle is valid for the coordinate system (x, y, z, t) then it follows from this equation that is it also valid for any other system (x', y', z', t'). Indeed, the Hamiltonian function is the same function H in every coordinate system.

The laws of nature, i.e., the differential equations that one derives from Hamilton's principle, are the same in any coordinate system that one obtains by means of a Lorentz transformation; this is the principle of relativity.

We would now like to derive the field equations from Hamilton's principle. To this end, we define the variation:

$$\delta H = \frac{\partial H}{\partial \mathfrak{d}} \cdot \delta \mathfrak{d} + \frac{\partial H}{\partial \mathfrak{h}} \cdot \delta \mathfrak{h} + \frac{\partial H}{\partial \rho} \cdot \delta \rho + \frac{\partial H}{\partial \mathfrak{v}} \cdot \delta \mathfrak{v}.$$

We would now like to introduce the following abbreviations:

$$(19) \qquad \frac{\partial H}{\partial \mathfrak{d}} = \mathfrak{e}, \qquad \frac{\partial H}{\partial \mathfrak{h}} = -\mathfrak{h}, \qquad \frac{\partial H}{\partial \rho} = -\varphi, \qquad \frac{\partial H}{\partial \mathfrak{v}} = \mathfrak{f}.$$

The variation of H is then:

$$(20) \qquad \delta H = \mathfrak{e} \cdot \delta \mathfrak{d} - \mathfrak{b} \cdot \delta \mathfrak{h} - \varphi \cdot \delta \rho + \mathfrak{f} \cdot \delta \mathfrak{v}.$$

In order to elaborate upon this expression further, we must employ a formula from four-dimensional calculus of variations, whose derivation we shall briefly recall: We use the following notation for the four-vector [1]) that is the product of the four-vector $P = (\mathfrak{f}, i\varphi)$ and the six-vector $\mathfrak{F} = (\mathfrak{h}, -i\mathfrak{d})$:

$$[P \cdot \mathfrak{F}] = ([\mathfrak{f} \cdot \mathfrak{h}] + \varphi \cdot \mathfrak{d},\ i \cdot (\mathfrak{f} \cdot \mathfrak{d})).$$

We define the Div of this vector by:

$$\mathrm{Div}[P \cdot \mathfrak{F}] = \mathrm{div}\{[\mathfrak{f} \cdot \mathfrak{h}] + \varphi \cdot \mathfrak{d}\} + \frac{\partial(\mathfrak{f} \cdot \mathfrak{d})}{\partial t}.$$

Now we have:

$$\mathrm{div}[\mathfrak{f} \cdot \mathfrak{h}] = \mathfrak{h} \cdot \mathfrak{rot}\ \mathfrak{f} - \mathfrak{f} \cdot \mathfrak{rot}\ \mathfrak{h},$$
$$\mathrm{div}(\varphi \cdot \mathfrak{d}) = \mathfrak{d} \cdot \nabla\varphi + \varphi \cdot \mathrm{div}\,\mathfrak{d},$$
$$\frac{\partial(\mathfrak{f} \cdot \mathfrak{d})}{\partial t} = \mathfrak{d} \cdot \frac{\partial \mathfrak{f}}{\partial t} + \mathfrak{f} \cdot \frac{\partial \mathfrak{d}}{\partial t}.$$

From this, we get:

[1] M. Laue, Das Relativitätsprinzip, pp. 67.

$$(21)\qquad \operatorname{div}\{[\mathfrak{f}\cdot\mathfrak{h}]+\varphi\cdot\mathfrak{d}\}+\frac{\partial(\mathfrak{f}\cdot\mathfrak{d})}{\partial t}=\mathfrak{h}\cdot\mathfrak{rot}\,\mathfrak{f}-\mathfrak{d}\cdot\left(\nabla\varphi+\frac{\partial\mathfrak{f}}{\partial t}\right)-\mathfrak{f}\cdot\left(\mathfrak{rot}\;\mathfrak{h}-\frac{\partial\mathfrak{d}}{\partial t}\right)+\varphi\cdot\operatorname{div}\mathfrak{d}.$$

In four-dimensional symbols these formulas look like:

$$(22)\qquad \operatorname{Div}[P\cdot\mathfrak{F}]=-(\mathfrak{F}\cdot\mathfrak{Rot}\,P)-(P\cdot\Delta\cdot\mathfrak{F}).$$

We would now like to use these formulas in our problem. Therefore, we remark that:

$$\mathfrak{rot}\;\mathfrak{h}-\frac{\partial\mathfrak{d}}{\partial t}=\mathfrak{v},\qquad \operatorname{div}\mathfrak{d}=\rho.$$

One thus has:

$$\operatorname{Div}[P\cdot\mathfrak{F}]=\mathfrak{h}\cdot\mathfrak{rot}\,\mathfrak{f}-\mathfrak{d}\cdot\left(\nabla\varphi+\frac{\partial\mathfrak{f}}{\partial t}\right)-\mathfrak{f}\cdot\mathfrak{v}+\varphi\cdot\rho.$$

If we put the variations $\delta\mathfrak{d}$ and $\delta\mathfrak{h}$ in place of $\mathfrak{d}$ and $\mathfrak{h}$ then we obtain:

$$\mathfrak{f}\cdot\delta\mathfrak{v}+\varphi\cdot\delta\rho=\mathfrak{rot}\,\mathfrak{f}\cdot\delta\mathfrak{h}-\left(\nabla\varphi+\frac{\partial\mathfrak{f}}{\partial t}\right)\cdot\delta\mathfrak{d}-\operatorname{Div}[P\cdot\delta\mathfrak{F}].$$

Now, we see that the integral:

$$\int_G \operatorname{Div}[P\cdot\delta\mathfrak{F}]\cdot dx\cdot dy\cdot dz\cdot dt\,,$$

which is precisely the space integral of a three-dimensional divergence, turns into an integral over the boundary of *G*. However, since Hamilton's principle assumes that the variations of all of the state variables, as well as $\delta\mathfrak{F}$, are null on the boundary then one has:

$$\int_G \operatorname{Div}[P\cdot\delta\mathfrak{F}]\cdot dx\cdot dy\cdot dz\cdot dt=0.$$

As a result, when one uses formula (20) for δH one obtains:

$$\int_G \delta H\cdot dx\cdot dy\cdot dz\cdot dt$$
$$=\int_G\left(\left(\mathfrak{e}+\nabla\varphi+\frac{\partial\mathfrak{f}}{\partial t}\right)\cdot\delta\mathfrak{d}+(\mathfrak{rot}\,\mathfrak{f}-\mathfrak{d})\cdot\delta\mathfrak{h}\right)\cdot dx\,dy\,dz\,dt.$$

Since there are no further relations between $\mathfrak{d}$ and $\mathfrak{h}$, and consequently $\delta\mathfrak{h}$ and $\delta\mathfrak{d}$ are completely independent of each other, one can satisfy Hamilton's principle only if the following differential equation is satisfied:

$$\mathfrak{e} + \nabla\varphi + \frac{\partial \mathfrak{f}}{\partial t} = 0,$$
$$\mathfrak{rot}\ \mathfrak{f} - \mathfrak{d} = 0.$$

From these two equations, it follows that:

$$\frac{\partial \mathfrak{b}}{\partial t} + \mathfrak{rot}\ \mathfrak{e} = 0.$$

Since the differential equations (19) for $\mathfrak{e}$, $\mathfrak{b}$, φ, ρ are in complete agreement with equation (9), these equations are identical with the field equations (2) and (4), since we started with equations (1) and (3) as defining equations to begin with.

From this, it is proved that the form of the field equations that I chose is the only one that is consistent with Hamilton's principle.

In conclusion, let us remark that one can give equation (21) an interesting form when one takes:

$$\mathfrak{rot}\ \mathfrak{f} = \mathfrak{b}, \quad \nabla\varphi + \frac{\partial \mathfrak{f}}{\partial t} = -\mathfrak{e}, \qquad \mathfrak{rot}\ \mathfrak{h} - \frac{\partial \mathfrak{d}}{\partial t} = \mathfrak{v}, \qquad \operatorname{div} \mathfrak{d} = \rho.$$

If we recall equation (11) then we obtain:

$$\frac{\partial(\mathfrak{f} \cdot \mathfrak{d})}{\partial t} + \operatorname{div}\{[\mathfrak{f} \cdot \mathfrak{h}] + \varphi\ \cdot \mathfrak{d}\} = \Phi - H. \tag{23}$$

The invariants.

11. Should the function $H(\mathfrak{d}, \mathfrak{h}, \rho, \mathfrak{v})$ be invariant under Lorentz transformations, i.e., should it be a four-dimensional scalar, then it must be a function of other four-dimensional scalars that one can construct out of $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$. There are four [†]) such quantities that are independent of each other:

1. The absolute value of the four-vector $P = (\mathfrak{v}, i\rho)$. It is:

$$\sigma = \sqrt{\rho^2 - \mathfrak{v}^2} = \rho \cdot \sqrt{1 - \beta^2}\ , \qquad \beta = \frac{\mathfrak{v}}{\rho}.$$

2. The absolute value of the six-vector $\mathfrak{F} = (\mathfrak{h}, -i\mathfrak{d})$. For this, we will use the quadratic expression:

$$p = \mathfrak{d}^2 - \mathfrak{h}^2.$$

[†] Translators note: This list is incomplete; it lacks the invariant that would correspond to $F \wedge F$ in modern electrodynamics. Other researchers had pointed out this fact at the time, such as Pauli (*The Theory of Relativity*), Weyl (*Space, Time, and Matter*), and Born (??).

3. The scalar product of the six-vector $\mathfrak{F} = (\mathfrak{h}, -i\mathfrak{d})$ with its dual $\mathfrak{F}^* = (-i\mathfrak{d}, \mathfrak{h})$. If we multiply this product by $i/2$ then we obtain the quantities:

$$q = (\mathfrak{h} \cdot \mathfrak{d}).$$

4. If we multiply the four-vector P by the six-vector $\mathfrak{F}$ and its dual $\mathfrak{F}^*$ then one obtains two new four-vectors:

$$\begin{aligned} A = P \cdot \mathfrak{F} \quad &= ((\rho \cdot \mathfrak{d} + [\mathfrak{v} \cdot \mathfrak{h}]), -i \cdot (\mathfrak{v} \cdot \mathfrak{d})), \\ B = P \cdot \mathfrak{F}^* \quad &= (i(\rho \cdot \mathfrak{h} - [\mathfrak{v} \cdot \mathfrak{b}]), (\mathfrak{v} \cdot \mathfrak{h})). \end{aligned}$$

The squares of their absolute values are:

$$\begin{aligned} A^2 = \quad & (\rho \cdot \mathfrak{d} + [\mathfrak{v} \cdot \mathfrak{h}])^2 - (\mathfrak{v} \cdot \mathfrak{d})^2, \\ B^2 = - & (\rho \cdot \mathfrak{h} + [\mathfrak{v} \cdot \mathfrak{d}])^2 + (\mathfrak{v} \cdot \mathfrak{h})^2. \end{aligned}$$

These two quantities are no longer independent of each other, as one easily sees, but:

$$A^2 + B^2 = (\mathfrak{h}^2 - \mathfrak{d}^2) \cdot (\mathfrak{v}^2 - \rho^2) = \sigma^2 \cdot p.$$

Likewise, the scalar product of the two also gives us nothing new:

$$\begin{aligned} (A \cdot B) &= i(\rho \cdot \mathfrak{d} + [\mathfrak{v} \cdot \mathfrak{h}]) \cdot (\rho \cdot \mathfrak{h} - [\mathfrak{v} \cdot \mathfrak{d}]) - (\mathfrak{v} \cdot \mathfrak{d}) \cdot (\mathfrak{v} \cdot \mathfrak{h}) \\ &= - i \cdot (\mathfrak{h} \cdot \mathfrak{d}) \cdot (\mathfrak{v}^2 - \rho^2) = i \cdot \sigma^2 \cdot q. \end{aligned}$$

We thus obtain only *one* fourth scalar, and indeed for this we will choose the quantity $s = -B^2$:

$$s = (\rho \cdot \mathfrak{h} - [\mathfrak{v} \cdot \mathfrak{d}])^2 - (\mathfrak{v} \cdot \mathfrak{h})^2.$$

From the theory of four-dimensional vectors, one may prove that no more independent scalars can be given; however, I shall skip the proof here.

We have thus found the following four possible independent variables:

(24)
$$\begin{cases} \sigma = \sqrt{\rho^2 - \mathfrak{v}^2} = \rho \cdot \sqrt{1 - \dfrac{v^2}{c^2}} \\ p = \mathfrak{d}^2 - \mathfrak{h}^2 \\ q = (\mathfrak{d} \cdot \mathfrak{h}), \\ s = (\rho \cdot \mathfrak{h} - [\mathfrak{v} \cdot \mathfrak{d}])^2 - (\mathfrak{v} \cdot \mathfrak{h})^2. \end{cases}$$

12. The intensity variables $\mathfrak{e}$, φ, $\mathfrak{b}$, $\mathfrak{f}$ can now be computed in the following way:

$$(25)\qquad \begin{cases} \mathfrak{e} = 2\cdot\dfrac{\partial H}{\partial t}\cdot\mathfrak{d} + \dfrac{\partial H}{\partial q}\cdot\mathfrak{h} + 2\cdot\dfrac{\partial H}{\partial s}\cdot\left[\mathfrak{v}\cdot(\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}])\right], \\ \varphi = -\dfrac{\partial H}{\partial \sigma}\cdot\dfrac{\rho}{\sigma} - 2\cdot\dfrac{\partial H}{\partial s}\cdot(\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}])\cdot\mathfrak{h}, \\ \mathfrak{b} = 2\cdot\dfrac{\partial H}{\partial p}\cdot\mathfrak{h} - \dfrac{\partial H}{\partial q}\cdot\mathfrak{d} - 2\cdot\dfrac{\partial H}{\partial s}\cdot(\rho\cdot(\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}]) - \mathfrak{v}\cdot(\mathfrak{v}\cdot\mathfrak{h})), \\ \mathfrak{f} = -\dfrac{\partial H}{\partial \sigma}\cdot\dfrac{\mathfrak{v}}{\sigma} - 2\cdot\dfrac{\partial H}{\partial s}\cdot\left(\left[\mathfrak{d}\cdot(\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}\right] + \mathfrak{h}\cdot(\mathfrak{v}\cdot\mathfrak{h})\right). \end{cases}$$

We observe that:

$$(\mathfrak{v}\cdot\mathfrak{h}) = \frac{1}{\rho}\cdot(\mathfrak{v}\cdot(\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}])),$$

and one immediately recognizes that the factor $\partial H/\partial s$ vanishes in expression (25) when:

$$\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}] = 0.$$

We make the further assumption that an electron is at rest in the field $\mathfrak{b} = 0$, so that $\partial H/\partial q$ must have either the factor q or the factor s, because it does not, on the other hand, vanish for $\mathfrak{v} = 0$, $\mathfrak{h} = 0$; however, we now have:

$$q = (\mathfrak{d}\cdot\mathfrak{h}) = \frac{1}{\rho}\cdot(\mathfrak{d}\cdot(\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}])).$$

Thus, under the same restriction, $\partial H/\partial q$ must be null, like the factor $\partial H/\partial s$, namely, when:

$$\rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}] = 0.$$

However, one obtains the quantity: $\rho'\cdot\mathfrak{h}' = \rho\cdot\mathfrak{h} - [\mathfrak{v}\cdot\mathfrak{d}]$ when one subjects the ether state to a Lorentz transformation that takes it from its coordinate system into another one that moves with the velocity $\mathfrak{q} = \mathfrak{v}/\rho$. If q is constant in space and time then one can transform to a rest system, in which $\mathfrak{h}' = 0$; i.e., the condition that we just described is satisfied for a stationary motion.

When we make the assumption that not only $\mathfrak{v}$ *and* $\mathfrak{h}$, *but also* $\mathfrak{b}$ *and* $\mathfrak{f}$, *are null in the field of an electron at rest, all terms in the intensity variables that do not* (?) *contain the invariants q and s drop out for a stationary motion.*

Now, since certainly all of the applications of this to electrons and matter correspond to only quasi-stationary motions, and there is no point in burdening ourselves with the

search for quantities that obviously have no influence on the results, in what follows, we will make the simplifying assumption that *q* and *s* do not enter into *H* at all.

13. *Hypothesis: The Hamiltonian function H depends only upon the invariants* σ *and p.*

We then have the following simple expressions for the intensity variables:

$$\text{(26)} \qquad \begin{cases} \mathfrak{e} = 2 \cdot \dfrac{\partial H}{\partial p} \cdot \mathfrak{d}, \quad \mathfrak{b} = 2 \cdot \dfrac{\partial H}{\partial p} \cdot \mathfrak{h}, \\ \varphi = -\dfrac{\partial H}{\partial \sigma} \dfrac{\rho}{\sigma}, \quad \mathfrak{f} = -\dfrac{\partial H}{\partial \sigma} \cdot \dfrac{\mathfrak{v}}{\sigma}. \end{cases}$$

Each of the intensity vectors $\mathfrak{e}$, $\mathfrak{b}$, $\mathfrak{f}$ *has the same direction as the corresponding magnitude vectors* $\mathfrak{d}$, $\mathfrak{h}$, $\mathfrak{v}$, *and furthermore, one has the two proportions:*

$$\mathfrak{f} : \mathfrak{v} = \varphi : \rho, \qquad\qquad \mathfrak{b} : \mathfrak{h} = \mathfrak{e} : \mathfrak{d}.$$

From this, one deduces the theorem: *The world matrix* (16) *is symmetric about the diagonal.*

Just like *H*, Φ also naturally depends on two variables, and in order to exhibit this, we shall take the following two quantities:

$$\text{(27)} \qquad \begin{cases} \chi = \sqrt{\varphi^2 - \mathfrak{f}^2} \\ \eta = \sqrt{\mathfrak{e}^2 - \mathfrak{b}^2}. \end{cases}$$

If we set:

$$\frac{\mathfrak{v}}{\rho} = \frac{\mathfrak{f}}{\varphi} = \mathfrak{q},$$

then we can also write:

$$\text{(27a)} \qquad \chi = \varphi \cdot \sqrt{1 - \mathfrak{q}^2}.$$

In conclusion, one must remark that one can find an interesting meaning for the quantity:

$$\mathrm{Div}(\mathfrak{f}, i\varphi) = \mathrm{div}\, \mathfrak{f} + \frac{\partial \varphi}{\partial t}.$$

To abbreviate, I will set:

$$-\frac{1}{\sigma} \cdot \frac{\partial H}{\partial \sigma} = \psi.$$

One then has:

$$\varphi = \psi \cdot \rho, \qquad \mathfrak{f} = \psi \cdot \mathfrak{v};$$

hence:

$$\operatorname{div} \mathfrak{f} + \frac{\partial \varphi}{\partial t} = \psi \cdot \left(\operatorname{div} \mathfrak{v} + \frac{\partial \rho}{\partial t} \right) + (\mathfrak{v} \cdot \nabla \psi) + \rho \cdot \frac{\partial \psi}{\partial t}.$$

Now, we have:

$$\operatorname{div} \mathfrak{v} + \frac{\partial \rho}{\partial t} = 0,$$

and we can further set $\mathfrak{v} = \rho \cdot \mathfrak{q}$, in which $\mathfrak{q}$ may be understood to means the *velocity* with which the charge is moving at the point in question. One then has:

$$(\mathfrak{v} \cdot \nabla \psi) + \rho \cdot \frac{\partial \psi}{\partial t} = \rho \cdot \left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} \cdot \mathfrak{q}_x + \frac{\partial \psi}{\partial y} \cdot \mathfrak{q}_y + \frac{\partial \psi}{\partial z} \cdot \mathfrak{q}_z \right).$$

If we now consider a single individual volume element that contains the charge, as we are wont to do with material volume elements, and regard ψ as a characteristic property of the moving charge element then the time variation of ψ is:

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} \cdot \mathfrak{q}_x + \frac{\partial \psi}{\partial y} \cdot \mathfrak{q}_y + \frac{\partial \psi}{\partial z} \cdot \mathfrak{q}_z .$$

We thus arrive at the equation:

$$\operatorname{div} \mathfrak{v} + \frac{\partial \rho}{\partial t} = \rho \cdot \frac{\mathrm{D}\psi}{\mathrm{D}t}. \tag{28}$$

This last equation has particular interest in regard to Abraham's recently proposed theory of gravitation [1]). In a region where the electric field is null the same equations follow for the quantities that Abraham called $\mathfrak{F}_x$, $\mathfrak{F}_y$, $\mathfrak{F}_z$, $\mathfrak{F}_u$ as the ones that follow for the quantities that I denoted by $\mathfrak{f}_x$, $\mathfrak{f}_y$, $\mathfrak{f}_z$, $i\varphi$, except with one difference, that Abraham set:

$$\operatorname{Div} \mathfrak{F} = -4\pi\gamma \cdot \nu,$$

in which γ means the gravitational constant and ν means the mass density, whereas the equation that we just derived follows for my vector:

$$\operatorname{Div}(\mathfrak{f}, i\varphi) = \rho \cdot \frac{\mathrm{D}\psi}{\mathrm{D}t}.$$

One will therefore go from my Ansätze to Abraham's gravitational field theory when one makes the assumption that wherever there is a material mass one finds a constant increase in the quantity ψ. The current $\mathfrak{f}$ that converges upon the mass distribution is then the gravitational field. However, since such an assumption is physically absurd, it is therefore out of the question that such a simple path exists from my Ansätze to a

[1] M. Abraham, Physik. Zeitschr., **13**, pp. 1, 1912.

gravitational theory. I have suggested how this probably might happen in the Introduction (pp. ? and ?).

In the following chapter I will next examine whether the existence of irreducible electrons is consistent with my Ansätze.

Greifswald, Physikalisches Institut, 6 January 1912.

(Submitted 9 January 1912.)

Ann. d. Phys. **39** (1912), 1-40.

(Second Part. [1])

Chapter Two

The Problem of the Electron.

Knot Singularities in the Field.

11. It is well known that the Nature of the electromagnetic field originates in a six-vector that is capable of introducing transversal waves into the ether. A superficial examination of the fundamental equations for ether dynamics, (1) through (4), that I gave previously can also lead to the conclusion that, analogously, the four-vector ($\mathfrak{v}$, $i\rho$) can give rise to longitudinal waves; however, that was a mistake. In the interior of atoms, where the term H_I in the Hamiltonian function (I, pp. 524) is appreciable, transversal and longitudinal momenta might possibly obey similar laws. However, these cannot be the correct laws of wave motion because, by assumption, the differential equations are linear, which is not the case in the interior of atoms. At great distances from the atom, where H_I is negligible compared to $(\mathfrak{d}^2 - \mathfrak{h}^2)/2$, the *transverse* momentum converges to a spherical shell that expands with constant velocity 1 (velocity of light). However, the *longitudinal* momentum exhibits no such motion since the equations for ρ and $\mathfrak{v}$ are never linear. Often, the longitudinal momentum essentially stays inside of a small volume, namely the volume of an electron, which neither contracts nor expands as a spherical surface, and its velocity can have all possible values, but always less than 1. In other words: the momentum of the four-vector ($\mathfrak{v}$, $i\rho$) never has the character of a longitudinal wave, but that of "quantum radiation;" it is the electronic radiation. If longitudinal waves were possible in the ether then the existence of unchanging accumulation points for the state ρ, and thus, the existence of electronic radiation, could not be compatible with the fundamental equations. Conversely, the fact of electronic radiation thus excludes the appearance of longitudinal waves.

Before we do anything else, we must find the conditions on the character of the Hamiltonian function H (or the world function F) in order for it to lead to the existence of unchanging knot singularities in the field. These conditions then apply just as well for the existence of quantum radiation. Namely, let ρ be the charge density and let $\mathfrak{d}$ be the electrical displacement in the neighborhood of a knot singularity, as functions of the distance r from the center, such that the equilibrium conditions $=\varphi + \mathfrak{e} = 0$ is satisfied. If we now apply a Lorentz transformation to the four-vector (0, $i\rho$) and the six-vector (0, $-i\mathfrak{d}$) along with the coordinate system (x, y, z, t) then we obtain a coordinate system (x', y', z', t'), a four-vector $(\mathfrak{v}', i\rho')$, and a six-vector $(\mathfrak{h}', -i\mathfrak{d}')$, as known functions of (x', y', z', t'), and indeed both vectors satisfy the fundamental equations of ether dynamics that we stated. From the transformation formula, one immediately sees that $\mathfrak{h}' =$

[1] Continuation of the article in Ann. d. Phys. **37**, pp. 511; referred to as I.

$[\mathfrak{q} \cdot \mathfrak{d}']$, $\mathfrak{v}' = \rho' \cdot \mathfrak{q}$, in which $\mathfrak{q}$ is a three-dimensional vector that is less than 1 and is constant over all time and space. The new solution that one obtains by the use of this transformation thus represents a knot singularity that moves forwards with the constant velocity $\mathfrak{q}$.

When equilibrium is possible for a knot singularity at rest then there are also knot singularities that move with any arbitrary velocity less than 1, hence, quantum radiation without longitudinal waves.

12. One arrives at an interesting conclusion concerning the motion of electric charge in general when one makes the following two assumptions: *first, that the function H is not an even function of* σ, and *second, that both H and* Φ *have no jumps or breaks for all physically meaningful values of the variables* ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$) *and* ($\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$). The first of these two assumptions is identical with the assumption that positive and negative charges exhibit a fundamentally different behavior that they do in reality, where electrons have only *one* sign. If *H* were an even function of σ then one could change the sign of σ, i.e., of ρ, without changing the equations in any way; therefore, positive and negative charges must behave the same way. The second assumption arises from the fact that the state variables of one type (e.g., the intensity variables) can only be infinite or vary in a discontinuous way when one allows the corresponding quantities of the other type (the magnitude variables) to become infinite or vary discontinuously.

In many respects, the quantity ρ plays a role in the equations of ether dynamics that is analogous to the role that is played by the deviations of the density from their normal state in the equations of aerodynamics (cf. I, pp. 520). In aerodynamics, one can consider positive and negative density variations – for example, interference – and add them to zero or at least a small density variation. Similarly, the charge density ρ in the ether can be chosen in such a way that positive and negative charges cancel each other upon superposition, and, conversely, nothing new can come of separating them. In fact, these assumptions rest on the foundations of modern atomic theory if the atom is considered to be a large volume of positive electrical charge that is completely permeated by negatively charged electrons. If, when an electron enters into the positive electrical charge volume, one superimposes its charge upon the latter charge then one will obtain a smaller charge than that of a free electron. If an electron leaves the atom then during the separation it restores the large negative charge of the electron and the large positive charge of what is left of the atom.

When one accepts both of the previous assumptions, our theory now implies that ideas of this sort or to be ruled out. From the equations of ether dynamics, one obtains that when a state variable ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$), or, what amounts to the same thing, according to the second assumption, ($\mathfrak{e}$, $\mathfrak{b}$, φ, $\mathfrak{f}$) has a jump as a function of (x, y, z) then one must have that another state variable is infinite. Thus, as long as we exclude singular points at which the state variables, in whole or in part, become infinite, hence, singularities that cannot enter into any integrals of the equations that involve *everywhere real* motions, one must make all of the state variables *constant* functions of x, y, z, t. Moreover, it follows from the first of our assumptions that one must always have $\mathfrak{v}^2 < \rho^2$, or $\mathfrak{v}^2 / \rho^2 < 1$. If we had $\mathfrak{v}^2 > \rho^2$ then s would have an imaginary value, and therefore since H is not an even

function of σ it would be complex. Since that cannot be the case for all *real* motion one must always have $\mathfrak{v}^2 / \rho^2 < 1$. However, we can think of charges in the real world that are subdivided into spacelike elements of the aforementioned sort, and any variation in the charge density that the existence of the vector $\mathfrak{v}$ induces can be thought of as translating each of the elementary components with the (variable) velocity $\mathfrak{q} = \mathfrak{v} / \rho$. From this, as we have always assumed, $\mathfrak{q}$ is an *everywhere finite and constant* function of the coordinate. However, if such is not the case then one can never interchange any of the elementary components that we introduced. Every single element then remains distinct, and it cannot be the case that the positive and negative charges can cancel each other or that they can be created from nothing.

The laws regarding charges are valid not only for the sum of all charges, but they also apply equally to positive and negative charges.

Thus, if an electron enters the positive spatial region of an atom then, according to our theory, the positive charge must first evade it, and then enter it, like a liquid into which a solid body has penetrated.

The equilibrium conditions.

13. The energy in the entire space in which the time evolution is defined may be computed from equation (7) (I, pp. 523):

$$E = \int (H + \mathfrak{b} \cdot \mathfrak{h} - \mathfrak{f} \cdot \mathfrak{v}) \cdot dV.$$

When everything is at rest, one has:

$$E_0 = \int H \cdot dV.$$

The condition for equilibrium is that for any small virtual variation $\delta\mathfrak{b}$, $\delta\rho$ (in which $\delta\rho =$ div $\delta\mathfrak{d}$) no energy can be converted into the energy of motion. One must therefore have $\delta E = 0$ when one varies ρ and $\mathfrak{d}$:

$$\begin{aligned} \delta E &= \int \left(\frac{\partial H}{\partial \rho} \cdot \delta\rho + \frac{\partial H}{\partial \mathfrak{d}} \cdot \delta\mathfrak{d} \right) \cdot dV \\ &= \int (-\varphi \cdot \delta\rho + \mathfrak{e} \cdot \delta\mathfrak{d}) \cdot dV. \end{aligned}$$

Now, one has:

$$\varphi \cdot \delta\rho = \operatorname{div}(\varphi \cdot \delta\mathfrak{d}) - \nabla\varphi \cdot \delta\mathfrak{d},$$

and further:

$$\int_V \operatorname{div}(\varphi \cdot \delta\mathfrak{d}) \cdot dV = \int_S \varphi \cdot \delta\mathfrak{d}_N \cdot dS,$$

in which S represents the bounding surface of the space V and N is the direction of the surface normal. If one chooses V sufficiently large then the bounding surface integral is vanishingly small, and it follows that:

$$\delta E = = \int (\nabla \varphi + \mathfrak{e}) \; \delta \mathfrak{d} \cdot d\mathrm{V}.$$

The following equation, which is already known to us, then results as the *equilibrium condition* that we obtain from $\delta \mathrm{E} = 0$:

$$\nabla \varphi + \mathfrak{e} = 0. \tag{29}$$

14. It is very difficult to determine whether the equilibrium is stable or unstable. Let the quantities $\mathfrak{e}_0$, $\mathfrak{d}_0$, φ_0, ρ_0 be computed in such a way that the equilibrium conditions are satisfied. The system then moves infinitely slowly in an infinitesimal neighborhood of the equilibrium point, in such a way that:

$$\mathfrak{d} = \mathfrak{d}_0 + \delta \mathfrak{d}, \qquad \rho = \rho_0 + \delta \rho, \quad \mathfrak{h} = \delta \mathfrak{h}, \qquad \mathfrak{v} = \delta \mathfrak{v},$$

just as:

$$\mathfrak{e} = \mathfrak{e}_0 + \delta \mathfrak{e}, \qquad \varphi = \varphi_0 + \delta \varphi.$$

The states $\delta \mathfrak{h}$ and $\delta \mathfrak{v}$ bring with them the consequence that throughout the variation of $\mathfrak{d}$ and ρ over an infinitesimal time dt, which we shall call $d\mathfrak{d}$ and $d\rho$ (cf. eq. (1) and (3)), we have:

$$d\mathfrak{d} = -(\delta \mathfrak{v} - \mathfrak{rot}\, \delta \mathfrak{h}) \cdot dt,$$
$$d\rho = -\operatorname{div} \delta \mathfrak{v} \cdot dt.$$

On the other hand, if we have the small deviation from equilibrium that is due to a variation of the motion state variables $\mathfrak{h}$ and $\mathfrak{v}$, or, what amounts to the same thing, $\mathfrak{b}$ and $\mathfrak{f}$, over the time interval dt (I, eq. (2) and (4)), then:

$$d\mathfrak{b} = -\mathfrak{rot}\, \delta \mathfrak{e} \cdot dt,$$
$$d\mathfrak{f} = -(\delta \mathfrak{e} + \nabla \delta \varphi) \cdot dt.$$

In order to clearly understand the energy balance, we divide $d\mathfrak{d}$ into two pieces $d\mathfrak{d} = d\mathfrak{d}' + d\mathfrak{d}''$, where:

$$\begin{cases} d\mathfrak{d}' = -\delta \mathfrak{v} \cdot dt \\ d\mathfrak{d}'' = \mathfrak{rot}\, \delta \mathfrak{h} \cdot dt. \end{cases} \tag{30}$$

One immediately sees that:

$$\delta \mathfrak{e} \cdot d\mathfrak{d}' - \delta \varphi \cdot d\rho \quad = \quad \delta \mathfrak{v} \cdot d\mathfrak{f} + \operatorname{div}(\delta \varphi \cdot \delta \mathfrak{v}) \cdot dt,$$

$$\delta \mathfrak{e} \cdot d\mathfrak{d}'' \qquad = -\delta \mathfrak{h} \cdot d\mathfrak{b} - \operatorname{div}[\delta \mathfrak{e} \cdot \delta \mathfrak{h}] \cdot dt.$$

If we integrate over a region of space V in which only internal energy balance is found and on whose boundary the energy current $-\delta\varphi \cdot \delta \mathfrak{v}$ and $[\delta \mathfrak{e} \cdot \delta \mathfrak{h}]$ are null then one obtains:

$$(31) \qquad \begin{cases} \int_V (\delta \mathfrak{e} \cdot d\mathfrak{d}' - \delta\varphi \cdot d\rho) \cdot dV & = \int_V \delta \mathfrak{v} \cdot d\mathfrak{f} \cdot dV \\ \int_V \delta \mathfrak{e} \cdot d\mathfrak{d}'' & = -\int_V \delta \mathfrak{h} \cdot d\mathfrak{b} \cdot dV. \end{cases}$$

We shall assume that in the moment considered $\delta \mathfrak{h}$ and $\delta \mathfrak{v}$ are directed in such a way that $d\mathfrak{d}'$ and $d\mathfrak{d}''$ have the same direction as $\delta \mathfrak{d}$, and $d\rho$ has the same sign as $d\rho$, so that the deviation of the equilibrium state increases. If equilibrium is to be stable then one must remove the motion state variables; one must therefore give $\delta \mathfrak{f}$ the opposite direction to $d\mathfrak{f}$, and $\delta \mathfrak{v}$ must be given the opposite direction to $\delta \mathfrak{h}$. Conversely, if the motion state variables $\delta \mathfrak{f}$ and $\delta \mathfrak{v}$ increase for the given directions of $\delta \mathfrak{v}$ and $\delta \mathfrak{h}$ then the equilibrium is unstable. If the equilibrium is to be stable then:

$$\int (\delta\varphi \cdot \delta\rho - \delta \mathfrak{e} \cdot \delta \mathfrak{d}') \cdot dV,$$

must have the same sign as:

$$\int \delta \mathfrak{v} \cdot \delta \mathfrak{f} \cdot dV,$$

and:

$$\int \delta \mathfrak{e} \cdot \delta \mathfrak{d}'' \cdot dV$$

must have the same sign as:

$$\int \delta \mathfrak{h} \cdot \delta \mathfrak{b} \cdot dV,$$

where $\delta \mathfrak{d}' + \delta \mathfrak{d}'' = \delta \mathfrak{d}$ and indeed $\operatorname{div} \delta \mathfrak{d}' = \delta\rho$, $\operatorname{div} \delta \mathfrak{d}'' = 0$, and in which we have further chosen the variations $\delta \mathfrak{v}$ and $\delta \mathfrak{h}$ in such a way that $\delta \mathfrak{v}$ is proportional to $\delta \mathfrak{d}'$ and $\operatorname{rot} \delta \mathfrak{h}$ is proportional to $\delta \mathfrak{d}''$.

We consider the two conditions separately by first assuming that $\delta \mathfrak{d}' = \delta \mathfrak{d}$, and then that $\delta \mathfrak{d}'' = \delta \mathfrak{d}$. This leads to a very important case in which absolutely no magnetic effects interfere; thus, $\delta \mathfrak{d}'' = 0$, and, as a result, $\delta \mathfrak{d}' = \delta \mathfrak{d}$. We then introduce infinitesimal variations into a centrally symmetric field of a spherical electron in such a way that everything remains centrally symmetric, and thus, that the concentration and dilution of the charges happens only on concentric spherical shells; we also assume that $\delta\rho$, and thus $\delta \mathfrak{b}$, are radially directed. Furthermore, $\delta \mathfrak{v}$ and $\delta\rho$ must be functions of only r, the distance from the center. For any such variation $\delta \mathfrak{e}$ must always have a potential; no magnetic fields ever appear. In the sequel, $\delta \mathfrak{v} = \partial \delta \mathfrak{d} / \partial t$ is always radially directed and a function of only r; the central symmetry is never lost during the further evolution of the variations.

Now, if the electron, together with its field, is to be in stable equilibrium, in such a way that the charge in its interior and atmosphere will not explode under any sort of displacement, then the following condition must be valid for the centrally symmetric variations that we chose, that the expression:

$$\int (\delta\varphi \cdot \delta\rho - \delta\mathfrak{e} \cdot \delta\mathfrak{D}) \cdot dV,$$

must have the same sign as:

$$\int \delta\mathfrak{v} \cdot \delta\mathfrak{f} \cdot dV.$$

We would now like to examine a variation of ρ that is only found on two infinitesimally thin spherical shells of thickness ε_1 and ε_2. Let the distance between the shells be a, where a is small compared to the mean radius r of the two shells. Furthermore, let us denote the variations of ρ by $\delta\rho_1$ and $-\rho_2$, in such a way that:

$$\varepsilon_1 \cdot \delta\rho_1 = \varepsilon_2 \cdot \delta\rho_2$$

(if we neglect the quantities of order a/r larger than 1). We then have a variation of the electric field in the space between the two layers:

$$\delta\mathfrak{D} = \delta\rho_1 \cdot \varepsilon_1 = \delta\rho_2 \cdot \varepsilon_2,$$

where $\delta\mathfrak{D}$ converges continuously to zero with the layers ε_1 and ε_2.

With the hypothesis that was introduced in section **13** (I, pp. ?), I further compute that H depends only upon the two variables:

$$\sigma = \rho \cdot \sqrt{1 - \mathfrak{v}^2/\}^2} \quad \text{and} \quad p = \mathfrak{d}^2 - \mathfrak{h}^2.$$

One then has:

$$\delta\varphi = \frac{\partial\varphi}{\partial\sigma} \cdot \delta\rho + 2 \cdot \frac{\partial\varphi}{\partial p} \cdot \mathfrak{d} \cdot \delta\mathfrak{D},$$

$$\delta\mathfrak{e} = \frac{\partial\mathfrak{e}}{\partial\sigma} \cdot \delta\rho + 2 \cdot \frac{\partial\mathfrak{e}}{\partial p} \cdot \mathfrak{d} \cdot \delta\mathfrak{D},$$

in which $\delta\mathfrak{v}/\rho$ is ignored as infinitesimal when compared to 1, so one can set $\sigma = \rho$. Now, since eq. (26) of part I, pp. ?? gives:

$$\mathfrak{e} = 2 \cdot \frac{\partial H}{\partial p} \cdot \mathfrak{d}, \qquad \varphi = -\frac{\partial H}{\partial\sigma},$$

so one has:

$$2 \cdot \frac{\partial\varphi}{\partial p} \cdot \mathfrak{d} \cdot \delta\mathfrak{D} \cdot \delta\rho = -2 \cdot \frac{\partial\mathfrak{e}}{\partial\sigma} \cdot \delta\rho \cdot \delta\mathfrak{D},$$

thus:

$$\delta\varphi \cdot \delta\rho - \delta\mathfrak{e} \cdot \delta\mathfrak{d} = \frac{\partial\varphi}{\partial p} \cdot \delta\rho^2 + 4 \cdot \frac{\partial\varphi}{\partial p} \cdot \mathfrak{d} \cdot \delta\mathfrak{d} \cdot \delta\rho - 2 \cdot \frac{\partial\mathfrak{e}}{\partial p} \cdot \mathfrak{d} \cdot \delta\mathfrak{d}^2.$$

If we now integrate over the entire shell in which the variation has been introduced then we obtain:

$$\int (\delta\varphi \cdot \delta\rho - \delta\mathfrak{e} \cdot \delta\mathfrak{d}) \cdot dV =$$
$$4\pi \cdot r^2 \cdot \left(\frac{\partial\varphi}{\partial\sigma} \cdot (\delta\}_1^2 \cdot \varepsilon_1 + \delta\}_2^2 \cdot \varepsilon_2) + 2 \cdot \frac{\partial\varphi}{\partial p} \cdot \mathfrak{d} \cdot (\delta\}_1^2 \cdot \varepsilon_1^2 + \delta\}_2^2 \cdot \varepsilon_2^2) - 2 \cdot \frac{\partial\mathfrak{e}}{\partial p} \cdot \mathfrak{d} \cdot \delta\}_1^2 \cdot \varepsilon_1^2 \cdot a \right).$$

If we choose ε_1 and ε_2 to be sufficiently small that the only term in the brackets above that determines the sign is the first term, and the sign of the overall expression is then the same as that of $\partial\varphi / \partial\sigma$, or, as we can also say in the rest case ($\sigma = \rho$), that of $\partial\varphi / \partial\rho$.

The sign of:

$$\int \delta\mathfrak{d} \cdot \delta\mathfrak{f} \cdot dV$$

is easy to deduce from this when we observe that, from eq. (26), part I, pp. ?:

$$\delta\mathfrak{f} = \frac{\varphi}{\rho} \cdot \delta\mathfrak{f},$$

that sign is identical with the sign of φ / ρ. From this, one obtains the following theorem:

A necessary condition for the stability of the equilibrium is that the differential quotient $\partial\varphi / \partial\}$ *must have the same sign everywhere as the quotient* φ / ρ.

This condition is always satisfied, for example, when φ always has the same sign as ρ, and when φ always increases along with an increase in ρ. This is the case when H is an even function of ρ that can be represented by means of series development with only negative coefficients.

If H also includes odd powers of ρ, as is true in any case (cf. section **12**), then the sign of $\partial\varphi / \partial\rho$, as well as φ / ρ, can change. When H_I (cf. I, pp. ?) depends only upon σ and likewise on φ, in turn, then $\partial\varphi / \partial\rho$ can be null only wherever φ attains a maximum or minimum value. It will then be impossible for the signs of $\partial\varphi / \partial\rho$ and φ / ρ to change simultaneously. However, if φ (and thus also H_I) contains both of the variables σ and p then one cannot exclude the existence of fields that satisfy this condition; naturally, the field of the electron must belong to this category. This consideration is therefore important because the principle of superposition for the electromagnetic vectors, upon which the Maxwell equations rest, in no longer valid in the interior of the atom when H_I depends not only upon σ, but also p. It is therefore not for us to choose whether Maxwell's equations shall or shall not be abandoned in the interiors of atoms, and we are

forced to abandon then as long as we make the fundamental equations asymmetric in the positive and negative charges. In a later section, we will also examine further whether this asymmetry is necessary.

In order to discuss the second stability condition, we consider a perturbation of the equilibrium that gives an increment $\delta\mathfrak{d}$ to the field in such a way that $\delta\rho = 0$ around a closed curve. The curve may have the form of a rectangle with two sides parallel to the field lines $\mathfrak{d}$ and two sides perpendicular to them at every point. We focus our attention on a moment at which $\mathfrak{v} = 0$, so that only the second of equations (31) is of interest. Along the curves that are at right angles to the lines of $\mathfrak{d}$, we construct the integral:

$$\int \delta\mathfrak{e} \cdot \delta\mathfrak{d} \cdot dV.$$

Along the lines that are perpendicular to the lines of $\mathfrak{d}$, we have:

$$\delta\mathfrak{e} = 2 \cdot \frac{\partial H}{\partial p} \cdot \delta\mathfrak{d},$$

since p remains unaffected by the quantities of order $\delta\mathfrak{d}^2$. On the other hand, along the curves that are parallel to $\mathfrak{d}$, we have:

$$\delta\mathfrak{e} = 4 \cdot \frac{\partial^2 H}{\partial p^2} \cdot \mathfrak{d}^2 \cdot \delta\mathfrak{d} + 2 \cdot \frac{\partial H}{\partial p} \cdot \delta\mathfrak{d}.$$

To abbreviate, we shall now set:

$$4 \cdot \frac{\partial^2 H}{\partial p^2} \cdot \mathfrak{d}^2 + 2 \cdot \frac{\partial H}{\partial p} = \frac{\partial \mathfrak{e}}{\partial \mathfrak{d}}.$$

This therefore means the differential quotient of $\mathfrak{e}$ that one obtains when one varies only $\mathfrak{d}$ and, moreover, in such a way that it increases *for any direction* about $\delta\mathfrak{d}$. We thus have:

$$\int \delta\mathfrak{e} \cdot \delta\mathfrak{d} \cdot dV = \int_{\mathrm{I}} \frac{\partial \mathfrak{e}}{\partial \mathfrak{d}} \cdot \delta\mathfrak{d}^2 \cdot dV + \int_{\mathrm{II}} 2 \cdot \frac{\partial H}{\partial p} \cdot \delta\mathfrak{d}^2 \cdot dV,$$

in which I means the region of space in which the lines $\delta\mathfrak{d}$ run parallel to $\mathfrak{d}$, and II is then one in which they are perpendicular to them. On the other hand, we have:

$$\int \delta\mathfrak{h} \cdot \delta\mathfrak{b} \cdot dV = \int 2 \cdot \frac{\partial H}{\partial p} \cdot \delta\mathfrak{h}^2 \cdot dV.$$

If both integrals should have the sign in all circumstances, as the stability conditions demand, and also if the curves $\delta\mathfrak{d}$ are so directed that integral I greatly dominates integral II, then $\partial\mathfrak{e}/\partial\mathfrak{d}$ must have the same sign as $2\partial H/\partial p$ or $\mathfrak{e}/\mathfrak{d}$.

A necessary condition for the stability of the equilibrium is that the partial differential quotient $\partial\mathfrak{e}/\partial\mathfrak{d}$ *that one obtains when one varies the magnitude of* $\mathfrak{d}$ *without varying the direction has the same sign as* $\mathfrak{e}/\mathfrak{d}$ *everywhere.*

This condition is always satisfied when $\mathfrak{e}$ always has the same sign as $\mathfrak{d}$ and when, moreover, $\mathfrak{e}$ always grows with increasing $\mathfrak{d}$. As is well known in conventional electrostatics the equilibrium of the field is thus always stabile.

In our general theory the situation is no longer quite so simple. We will soon see that if the sign of $\mathfrak{e}/\mathfrak{d}$ must necessarily change in the interior of an atom then the sign of $\partial\mathfrak{e}/\partial\mathfrak{d}$ must also change in that same place.

Whether both conditions that we know to be necessary for stability of the equilibrium are also sufficient cannot be stated, for certain. This much is clear in any case: that unstable equilibria can also enter into the general theory. In fact, it would be impossible for the theory to make any pretense of being a general theory of matter if it did not also embrace the cases of unstable equilibrium that actually occur.

15. As we already saw before, it is often more convenient to compute with the world function Φ instead of the Hamiltonian function H. One must then (cf. I, pp ?) take the intensity variables ($\mathfrak{e}$, $\mathfrak{h}$, φ, $\mathfrak{f}$) to be the independent state variables. In the special case $\mathfrak{h} = 0$, $\mathfrak{f} = 0$, so that η is simply the absolute value of $\mathfrak{e}$ and $\chi = \varphi$ in equation (27), part I, pp. ?, one then has:

$$d\rho = \frac{\partial\rho}{\partial\varphi}\cdot d\varphi + \frac{\partial\rho}{\partial\eta}\cdot\frac{\mathfrak{e}\cdot d\mathfrak{e}}{\eta},$$

$$d\mathfrak{d} = \frac{\partial\mathfrak{d}}{\partial\varphi}\cdot d\varphi + \frac{\partial\mathfrak{d}}{\partial\eta}\cdot\frac{\mathfrak{e}\cdot d\mathfrak{e}}{\eta}.$$

If we set $d\mathfrak{d} = 0$ then we have:

$$\frac{\mathfrak{e}\cdot d\mathfrak{e}}{\eta} = -\frac{\dfrac{\partial\mathfrak{d}}{\partial\varphi}}{\dfrac{\partial\mathfrak{d}}{\partial\eta}}\cdot d\varphi,$$

in which $\partial\mathfrak{d}$ means that we vary the magnitude of $\mathfrak{d}$ without varying its direction. One thus has:

$$d\rho = \frac{\dfrac{\partial\rho}{\partial\varphi}\cdot\dfrac{\partial\mathfrak{d}}{\partial\eta} - \dfrac{\partial\rho}{\partial\eta}\cdot\dfrac{\partial\mathfrak{d}}{\partial\varphi}}{\dfrac{\partial\mathfrak{d}}{\partial\eta}}\cdot d\varphi,$$

$$\frac{\partial \rho}{\partial \varphi} = \frac{\dfrac{\partial \mathfrak{d}}{\partial \eta}}{\dfrac{\partial \rho}{\partial \varphi} \cdot \dfrac{\partial \mathfrak{d}}{\partial \eta} - \dfrac{\partial \rho}{\partial \eta} \cdot \dfrac{\partial \mathfrak{d}}{\partial \varphi}}.$$

However, one then has:

$$\frac{\partial \mathfrak{d}}{\partial \varphi} = -\frac{\partial^2 \Phi}{\partial \varphi \cdot \partial \eta} \cdot \frac{\mathfrak{e}}{\eta},$$

or, since $\partial \mathfrak{d}$ shall mean the variation of the absolute value of $\mathfrak{d}$:

$$\frac{\partial \mathfrak{d}}{\partial \varphi} = -\frac{\partial^2 \Phi}{\partial \varphi \cdot \partial \eta} = -\frac{\partial \rho}{\partial \eta}.$$

With that, we ultimately come to:

(32)
$$\frac{\partial \varphi}{\partial \rho} = \frac{\dfrac{\partial \mathfrak{d}}{\partial \eta}}{\left(\dfrac{\partial^2 \Phi}{\partial \varphi \cdot \partial \eta}\right)^2 + \dfrac{\partial \mathfrak{d}}{\partial \eta} \cdot \dfrac{\partial \rho}{\partial \varphi}}.$$

Likewise, when one sets $d\rho = 0$ one obtains:

(33)
$$\frac{\partial \mathfrak{e}}{\partial \mathfrak{d}} = \frac{\dfrac{\partial \rho}{\partial \varphi}}{\left(\dfrac{\partial^2 \Phi}{\partial \varphi \cdot \partial \eta}\right)^2 + \dfrac{\partial \mathfrak{d}}{\partial \eta} \cdot \dfrac{\partial \rho}{\partial \varphi}}.$$

The expressions (32) and (33) must therefore always have the same signs as φ / ρ ($\mathfrak{e}$/$\mathfrak{d}$, resp.) if the equilibrium is to be stable.

16. All that remains is for us to address the question of what sort of conditions that the world function must satisfy in order for it to be possible that a spherical knot singularity with an extremely thin electrical atmosphere is an equilibrium point in the ether state. We think of a radius r pointing outward from the center of a knot singularity that is so long that its end already satisfies the principle of superposition $\mathfrak{e} = \mathfrak{d}$. Let m be the total charge in the interior of a ball of this large radius r, so the potential of the electric field at the endpoint of r is $\psi = m / 4\pi r$. In equilibrium, the quantity ψ must be identical with φ: $\varphi = m / 4\pi r$. Now, let the charge density of the electrical atmosphere at the endpoint of r equal ρ. Hence, the charge that is contained in a spherical shell of thickness dr is $dm = 4\pi r^2 \cdot \rho \cdot dr$, which makes the variation of φ along the increment dr:

$$d\varphi = -\frac{m}{4\pi \cdot r^2} \cdot dr + \frac{dm}{4\pi \cdot r}.$$

The second term must become vanishingly small compared to the first one. I.e, since:

$$\frac{dm}{4\pi \cdot r} = r \cdot \rho \cdot dr,$$

$r \cdot \rho$ must be vanishingly small compared to $m / 4\pi \cdot r^2$, and therefore $r \cdot \rho$ must converge to zero faster than r^{-2}.

The charge density ρ of the atmosphere of an electrical knot singularity must converge to zero faster than r^{-3}.

If the world function Φ is to be represented by a power series for small values of:

$$\chi = \sqrt{\varphi^2 - \mathfrak{f}^2} \qquad \text{and} \qquad \eta = \sqrt{\mathfrak{e}^2 - \mathfrak{d}^2}$$

then this sum must look like the following:

$$\Phi = -\tfrac{1}{2}\eta^2 + \sum \alpha_\mu \cdot \chi^\mu + \sum \beta_\nu \cdot \eta^\nu + \sum \gamma_{hk} \cdot \chi^h \cdot \eta^k,$$

$$m > 4,\ n > 2,\ \ h \geq 1,\ \ \eta \geq 2.$$

In the event that the exponents μ, h involve fractional numbers their denominators must always be odd, or else Φ would become imaginary for negative values of φ; this rule is not valid for ν, k, since η is always positive.

When and only when the series for Φ satisfies these conditions, the following are true:

First: $\mathfrak{d}$ and ρ cannot become infinite for $\eta = 0$ ($\chi = 0$, resp.).

Second: $\partial\mathfrak{d} / \partial\mathfrak{e}$ goes to 1 for small values of η and χ.

Third: ρ converges to zero faster than r^{-3} for $\chi = m / 4\pi r$ and $\eta = m / 4\pi r^2$ when one lets r go to infinity.

For small values of η and χ – i.e., in a vacuum – the stability conditions are admittedly not satisfied for just any sort of function Φ, but still in very many cases; for example, this is always the case when both of the smallest exponents μ and h are even and the coefficients of the term with these smallest powers of χ is positive. One then has that for small values of the variables $\partial\varphi / \partial\}$ is always positive, just like $\partial\mathfrak{d} / \partial\mathfrak{e}$; from (32) and (33), we therefore also have $\partial\varphi / \partial\rho > 0$, $\partial\mathfrak{d} / \partial\mathfrak{e} > 0$. Furthermore, since $\mathfrak{e}/\mathfrak{d}$ and φ / ρ are likewise always positive in weak fields, the vacuum stability conditions are always satisfied in these cases.

The differential equation of the electrostatic field for the case of spherical symmetry.

17. For a static field one has $\varphi = \chi$ and, up to direction, $\mathfrak{e} = \eta$, hence:

$$|\mathfrak{d}| = -\frac{\partial \Phi}{\partial \eta}, \qquad \rho = \frac{\partial \Phi}{\partial \varphi}.$$

Furthermore, when the field is spherically symmetric, and we abbreviate the magnitude of $\mathfrak{d}$ by writing $\mathfrak{d}$, instead of $|\mathfrak{d}|$, we have:

$$\frac{1}{r^2} \cdot \frac{d}{dr}(r^2 \cdot \mathfrak{d}) = \frac{d\mathfrak{d}}{dr} + \frac{2\mathfrak{d}}{r} = \rho.$$

From this, it follows that:

$$r \cdot \frac{\partial^2 \Phi}{\partial \eta^2} \cdot \frac{d\eta}{dr} + r \cdot \frac{\partial^2 \Phi}{\partial \eta \partial \varphi} \cdot \frac{d\varphi}{dr} + 2 \cdot \frac{\partial \Phi}{\partial \eta} + r \cdot \frac{\partial \Phi}{\partial \varphi} = 0.$$

However, at equilibrium, one has:

$$\eta = -\frac{d\varphi}{dr} = -\varphi', \qquad \frac{d\eta}{dr} = -\frac{\partial^2 \varphi}{\partial r^2} = -\varphi''.$$

Φ is a given function of φ and φ': $\Phi(\varphi, \varphi')$. We then obtain the following differential equation for φ as the *equilibrium condition with spherical symmetry:*

$$r \cdot \frac{\partial^2 \Phi}{\partial \varphi'^2} \cdot \varphi'' + 2 \cdot \frac{\partial \Phi}{\partial \varphi'} + r \cdot \frac{\partial}{\partial \varphi}\left(\frac{\partial \Phi}{\partial \varphi'} \cdot \varphi' - \Phi\right) = 0. \tag{34}$$

This is a second order differential equation whose general integral therefore has two arbitrary constants. Since the equation in φ'' is of first degree, it has no singular integrals. Both arbitrary constants are determined when the potential φ and field strength $-\varphi'$ are given for a definite r, or when the potentials φ_1 and φ_2 are given on both boundary components of a spherical condensor. One can thus obtain any arbitrary field between the shells of a spherical condensor as an integral of equation (34). It is self-explanatory that nothing should be noticeably different in the experimentally achieved cases of ordinary electrostatics; the exact integral gives only the imperceptible electrical atmosphere that lies on both boundary components beyond the usual situation. Hence, we will next focus our attention on the region between the spherical surfaces. If one traverses this region then, as we will see, the integral has singular points for certain values of r, which makes it physically impossible, so it is valid only in a neighborhood of the boundary.

In general, the integral has *one* singularity, for the value $r = 0$. One thus sees from this that terms of order higher than φ'' must be multiplied by r. Indeed, the singularity in φ at $r = 0$ is, in general, an *essential* one that only transcendental functions (e.g., elliptic functions, exponential functions, etc.) can have. A peculiar property of these functions is that these essential singularities can be made to vanish for a certain choice of arbitrary constants in the integral. One can see this easily when one develops the world function Φ in a power series in $(\varphi - a)$ and φ' about the system of values $\varphi = a$, $\varphi' = 0$. From the Ansatz on pp ?, the lowest power of φ' is the second. Therefore, let:

$$\begin{aligned}\Phi = \Phi_0 &+ a_{20} \cdot \varphi'^2 + a_{30} \cdot \varphi'^3 + \dots \\ &+ (\varphi - a) \cdot \ (a_{10} + a_{21}\varphi'^2 + a_{31}\varphi'^3 + \dots \\ &+ (\varphi - a)^2 \cdot \ (a_{02} + a_{22}\varphi'^2 + a_{32}\varphi'^3 + \dots\end{aligned}$$

in which the coefficients of all of the given quantities are to be determined. We now set $\rho = \partial\Phi / \partial\varphi$, $\mathfrak{d} = \partial\Phi / \partial\varphi'$:

$$\begin{aligned}(\varphi - a) &= \alpha_2 r^2 + \alpha_3 r^3 + \dots, \\ \varphi' &= 2\alpha_2 r + 3\alpha_3 r^2 + \dots,\end{aligned}$$

in which α_2, α_3, … are unknown, and equation (34), which can also be written:

$$\frac{d\left(r^2 \cdot \dfrac{\partial\Phi}{\partial\varphi'}\right)}{dr} = r^2 \cdot \frac{\partial\Phi}{\partial\varphi},$$

then gives recursion formulas by which one can sequentially express α_2, α_3, etc., in terms of a and the coefficients of the world function. We have thus obtained an integral that has no singularity at the point $r = 0$ and that also involves only *a single* arbitrary constant, namely a. It remains for us to prove that other integrals can be given in terms of a power series about $r = 0$, possibly one with fractional exponents or logarithmic terms. From that, we can prove that all other integrals must have an *essential* singularity at $r = 0$.

$$\frac{d\left(u^{-2} \cdot \dfrac{\partial\Phi}{\partial\varphi'}\right)}{du} = -u^4 \cdot \frac{\partial\Phi}{\partial\varphi}.$$

If one develops Φ in a power series in φ and φ' about $\varphi = 0$ and $\varphi' = 0$ (cf. pp. ?) and then sets:

$$\begin{aligned}\varphi &= a \cdot u \cdot (1 + \alpha_1 u + \alpha_2 u^2 + \dots), \\ \varphi' &= -a \cdot u^2 \cdot (1 + 2\alpha_1 u + 3\alpha_2 u^2 + \dots),\end{aligned}$$

then one obtains a recursion formula by which one can sequentially obtain α_2, α_3, etc., in terms of a and the coefficients of the world function. We thus have an integral in a neighborhood of $u = 0$ that has no singularities at that point. Moreover, this integral involves *only one* arbitrary constant, namely a. All other solutions of the differential equation (34) have an *essential* singularity at $u = 0$.

Whereas, as we saw on pp ?, the general integral of the field between two boundary components of a spherical condensor, on which we are given arbitrary potentials φ_1 and φ_2, can be computed, the particular integral that has no singularity at the point $r = 0$ gives us the field in the interior of a charged hollow sphere. In conventional electrostatics, one has φ = const., $\mathfrak{e} = 0$ in that region, and in all practically realizable situations the integral that we spoke of will not differ appreciably. The power series expansion on pp. ?:

$$(\varphi - a) = \alpha_2 \cdot r^2 + \alpha_3 \cdot r^3 + \ldots$$

must therefore have vanishingly small coefficients in all practically realizable situations, and the quantity a gives an almost precisely constant potential inside and outside the hollow sphere. Meanwhile, from our theory a field must exist in the interior, although it has an extremely weak electrical atmosphere, and it corresponds to the electrical field that was computed by the power series expansion on pp. ?. In any event, the state inside the hollow sphere is completely determined when the potential on it (and therefore a) is given. From this, one recognizes the manner by which integral that we obtain inside the hollow sphere may involve only one arbitrary constant.

The particular integral for which the point $r = \infty$ or $r^{-1} = u = 0$ is not a singularity gives us a representation of the field outside of a charged ball, from which, everything else may be infinitely extended. In conventional electrodynamics, one would have $\varphi = a / r$, where $a = m / 4\pi$, and m is the charge of the ball. Therefore, in all practically realizable situations the coefficients α_1, α_2, etc., in the series on pp. ?:

$$\varphi = a \cdot u \cdot (1 + \alpha_1 u + \alpha_2 u^2 + \ldots)$$

are vanishingly small. One carries out the computations for extremely weak fields, which corresponds to a thin atmosphere around the charged ball. In any case, one recognizes that the external field is completely determined when one knows the charge m of the ball, and thus, the constant a. One then understands the manner in which the integral, which has no singularity at $u = 0$, involves only one arbitrary constant.

We thus see how the integrals of equation (34) exhaust all of the possibilities for spherically symmetric fields, and, with a slight generalization, we may deduce the following conclusion from this:

There are infinitely many forms for the world function Φ that do not bring one into conflict with conventional electrodynamics.

Discussion of an example: $\Phi = -\frac{1}{2}\eta^2 + \frac{1}{6}a\chi^6$.

18. The function:

(35) $$\Phi = -\tfrac{1}{2}\,\eta^2 + \tfrac{1}{6}\,a \cdot \chi^6$$

satisfies all of the conditions that were specified for the world function in **16**. In a static field, it gives:

(36) $$\mathfrak{d} = \mathfrak{e}, \quad \rho = a \cdot \varphi^5.$$

Since everything is symmetric in positive and negative charges, the superposition principle d = e is also valid in the interior of the know singularity, or else the equilibrium would be unstable.

The differential equation (34) may be simplified considerably in this case; it becomes:

(37) $$r \cdot \varphi'' + 2 \cdot \varphi' + a \cdot r \cdot \varphi^5 = 0.$$

This equation may be immediately integrated once. When one introduces the abbreviation $r \cdot \varphi^2 = v$ then if we multiply (37) by $4r \cdot (2r\varphi' + \varphi)$ may be written in the following way:

(38) $$\begin{cases} r^2 \cdot \dfrac{d}{dr}\left(\dfrac{v'^2}{v}\right) + 2r \cdot \dfrac{v'^2}{v} - v' + 4a \cdot v^2 \cdot v' = 0, \\ v = r \cdot \varphi^2. \end{cases}$$

By integration, we obtain:

$$\frac{r^2 \cdot v'^2}{v} - v + \frac{4a}{3} \cdot v^3 = \mathrm{C},$$

in which C is an integration constant. This equation may be solved by a quadrature when one introduces the independent variable $\xi = \ln r / r_0$ in place of r, in which r_0 is the second arbitrary constant.

(39) $$\begin{cases} \left(\dfrac{dv}{d\xi}\right)^2 = C \cdot v + v^2 - \dfrac{4a}{3} \cdot v^4, \\ \xi = \ln \dfrac{r}{r_0}, \quad \varphi = \sqrt{\dfrac{v}{r}}. \end{cases}$$

In general, the integral of this equation is an elliptic function of ξ. As we might expect, it therefore has an essential singularity for $\xi = \infty$, i.e., for $r = 0$ and $r = \infty$.

Since φ must naturally be real, I will now discuss the solutions of (39) that is real and positive for real arguments. We will distinguish three cases, namely: C positive, C negative, C precisely zero.

19. I. Case: $C > 0$.

I will always denote the (single) *positive* solution of the cubic equation:

(40) $$C + \gamma - \frac{4a}{3} \cdot \gamma^3 = 0$$

by γ:

(40a) $$\gamma > 0,$$

and use the following quantity in place of C as the integration constant:

(41) $$h = \sqrt{\frac{C}{3C + 2\gamma}}.$$

We compute the following three quantities from the integration constant h:

(42) $$\begin{cases} b = h \cdot \sqrt{\dfrac{1-h^2}{1-3h^2}} \cdot \sqrt{\dfrac{3}{4a}}, \\ k^2 = \dfrac{(1-h)\cdot(1-3h)}{(1+h)\cdot(1+3h)}, \\ p = \frac{1}{2} \cdot \sqrt{\dfrac{(1-h)\cdot(1+3h)}{1-3h^2}}. \end{cases}$$

When C increases from 0 to ∞, h steadily increases from 0 to $1/\sqrt{3}$, b and p then always remain real, and indeed they both steadily increase, b from 0 to ∞, and p from +0.5 to ∞. On the other hand, k^2, which takes the value 1 for $h = 0$, decreases steadily to zero at $h = 1/3$, changes its sign at that point, and attains the value $-(2-\sqrt{3})^2 = -0.0718$ for $h = 1/\sqrt{3}$. One summarizes the behavior of the three quantities with the help of the following table:

C	h	b	p	k^2
0	0	0	$+\frac{1}{2}$	+1
$+\dfrac{1}{3\cdot\sqrt{a}}$	$+\frac{1}{3}$	$+\dfrac{1}{3\cdot\sqrt{a}}$	+1	0
$+\infty$	$+\dfrac{1}{\sqrt{3}}$	$+\infty$	$+\infty$	$-(2-\sqrt{3})^2$

When one substitutes the function u for v, where u is defined by following equation:

$$v = b \cdot \frac{1+u}{(1-u)+h(1+u)},$$

then an elementary calculation gives that u satisfies the well-known differential equation for the Jacobi function of modulus k, and we obtain:

$$(43)\qquad \begin{cases} \varphi = \sqrt{\dfrac{b}{r}} \cdot \sqrt{\dfrac{1+u}{(1-u)+h(1+u)}}, \\ \left(\dfrac{du}{dx}\right)^2 = (1-u^2)\cdot(1-k^2u^2), \\ x = p \cdot \ln\dfrac{r}{r_0}. \end{cases}$$

If $k^2 > 0$ then when we set $u = -\,cn\,x / dn\,x$ we obtain:

$$(44)\qquad \begin{cases} \varphi = \sqrt{\dfrac{b}{r}} \cdot \sqrt{\dfrac{dn\,x - cn\,x}{(dn\,x + cn\,x) + h\cdot(dn\,x - cn\,x)}}, \\ x = p \cdot \ln\dfrac{r}{r_0}, \quad k^2 > 0, \end{cases}$$

in which $dn\,x$ and $cn\,x$ are the well-known Jacobi functions. In this formulation, the arbitrary constant r_0 means the value of the integral u that corresponds to the r at which φ becomes null, $r = r_0$; hence $x = 0$ becomes $dn\,x = cn\,x = 1$. In the vicinity of $x = 0$ we have the power series expansion:

$$dn\,x = 1 - \frac{k^2x^2}{2} + \frac{k^2\cdot(4+k^2)}{24}\cdot x^4 - \frac{k^2\cdot(16+44k^2+k^4)}{720}\cdot x^6 + \ldots$$

$$cn\,x = 1 - \frac{x^2}{2} + \frac{1+4k^2}{24}\cdot x^4 - \frac{1+44k^2+16k^4}{720}\cdot x^6 + \ldots$$

If one substitutes these series then one obtains the following development:

$$\frac{dn\,x - cn\,x}{(dn\,x + cn\,x) + h\cdot(dn\,x - cn\,x)} = \frac{x^2\cdot(1-k^2)}{4}\cdot\left(1 + \frac{x^2}{12p^2} + \frac{x^4}{360p^4} + \cdots\right).$$

The first few terms of the infinite series that we just wrote also define the beginning of the series expansion for the following function:

$$(1-k^2)\cdot p^2\cdot\left(\frac{e^{\frac{x}{2p}} - e^{-\frac{x}{2p}}}{2}\right) = \frac{x^2\cdot(1-k^2)}{4}\cdot\left(1 + \frac{x^2}{12p^2} + \frac{x^4}{360p^4} + \cdots\right).$$

From the term in x^6 onward the series begin to deviate. If one substitutes the latter function instead of the former in (44) then one obtains an approximation for φ that

deviates only very slightly from the precise value of φ for small values of x. This approximation is:

$$\varphi = \frac{p}{2} \cdot \sqrt{b \cdot (1-k^2)} \cdot \frac{1}{\sqrt{r}} \cdot \left(\sqrt{\frac{r}{r_0}} - \sqrt{\frac{r_0}{r}} \right),$$

which one may also write:

(45)
$$\begin{cases} \varphi = A \cdot \left(\dfrac{1}{r_0} - \dfrac{1}{r} \right), \\ A = \dfrac{p}{2} \cdot \sqrt{b \cdot (1-k^2) \cdot r_0}. \end{cases}$$

In the vicinity of $r = r_0$ the value of φ that one computes in our theory deviates only very unnoticeably from the value of the potential that conventional electrostatics gives for a spherical condensor for which the potential is null at $r = r_0$ and the field strength A/r_0^2 predominates.

When one is given the point r_0 and the field strength at r_0 then one can also compute the value of h from the aforementioned formula (45) for A; this makes both arbitrary constants in the integrals of (37) completely determined.

One sees from the theorem that we just proved that the electrical atmosphere in the neighborhood of the null point is extraordinarily thin. However, the further that one goes away from the null point, the stronger the electrical atmosphere becomes. Conventional electrostatics is then only valid for a spherical condensor whose shells are not to far away from each other. The larger one takes the space between them, the more one feels the influence of the electrical atmosphere that both shells present; the positive electrical atmosphere comes from the shell with the positive potential, and the negative electrical atmosphere comes from the other shell. Ultimately, the atmosphere becomes so thick when the shell is very from the point r_0 that the value x that corresponds to r becomes closely equal to the half-period !2K of the elliptic functions *dn x* and *cn x*. For the given r_0 this has the effect that the smaller one makes 2K the smaller that k becomes and the larger that h becomes. However, from formula (45) it is easy to see that the quantity A, i.e., the field strength at r_0, decreases with increasing h. The largest value of h for which formula (44) is still valid is $h = 1/3$, which corresponds to the value:

$$A = 0.289 \cdot \frac{\sqrt{r_0}}{\sqrt[4]{a}} \qquad \text{and} \qquad 2K = \pi.$$

In this special case, we have *dn x* = 1, *cn x* = cos *x*. If one wants the field strength at r_0 to be larger then one must compute with a somewhat modified formula. Since $k^2 < 0$ for $h > 1/3$, one introduces the following quantity κ as the modulus, which is positive and less than 1: $\kappa^2 = -k^2 / (1 - k^2)$, and one further replaces the quantity p with a quantity $q = p \cdot \sqrt{1-k^2}$, whereas one leaves b unchanged. One must therefore replace formula (42) with the following one:

$$(46)\qquad \begin{cases} b = h\cdot\sqrt{\dfrac{1-h^2}{1-3h^2}}\sqrt{\dfrac{3}{4a}}, \\ \kappa^2 = \dfrac{(3h-1)\cdot(1-h)}{8h} = \dfrac{-k^2}{1-k^2} \\ q = \sqrt{\dfrac{2h}{1-3h^2}} = p\cdot(1-k^2). \end{cases}$$

If h increases from 1/3 to $1/\sqrt{3}$ then all three quantities constantly decrease. The boundary values give the following table:

C	h	b	q	κ^2
$+\dfrac{1}{3\cdot\sqrt{a}}$	$+\dfrac{1}{3}$	$+\dfrac{1}{3\cdot\sqrt{a}}$	+1	0
$+\infty$	$+\dfrac{1}{\sqrt{3}}$	$+\infty$	$+\infty$	$\dfrac{2-\sqrt{3}}{4}$

If one introduces the following variable in place of x:

$$y = x/\sqrt{1-\kappa^2} = q\cdot\ln r/r_0$$

then one obtains the following equation instead of (43):

$$(47)\qquad \begin{cases} q = \sqrt{\dfrac{b}{r}}\cdot\sqrt{\dfrac{1+u}{(1-u)+h(1+u)}}, \\ \left(\dfrac{du}{dy}\right)^2 = (1-u^2)\cdot((1-\kappa^2)+\kappa^2\cdot u^2), \\ y = q\cdot\ln\dfrac{r}{r_0}. \end{cases}$$

If one replaces the solution u of the differential equation with $u = -cn\ y$, then one obtains:

$$(48)\qquad \begin{cases} \varphi = \sqrt{\dfrac{b}{r}}\cdot\sqrt{\dfrac{1+u}{(1-u)+h(1+u)}}, \\ y = q\cdot\ln\dfrac{r}{r_0}. \end{cases}$$

If one lets h increase from 1/3 to $1/\sqrt{3}$ then A in formula (45), and thus the field strengths in r_0, increase without bound. If we let r_1 denote the value r for which y equals the half-period $2K$ of the elliptic function $cn\ y$ then we have:

$$r_1 = r_0 \cdot e^{\frac{2K}{q}}.$$

If one lets the field strengths in r_0 increase without bound then q will likewise become infinitely large, whereas $2K$ remains finite, and r_1 returns ever closer to r_0; therefore, for large field strengths the context in which conventional electrostatics is valid will ultimately become infinitesimal.

Conversely, if we let the field strengths get very small then we can calculate the half-period, which corresponds to the modulus k that differs only very slightly from 1, almost precisely from the formula:

$$2K = 2 \cdot \ln \frac{4}{\sqrt{1-k^2}}.$$

In this case, since $p = \frac{1}{2}$, the value r_1, for which $x = p \cdot \ln r/r_0$ equals $2K$ is:

$$r_1 = r_0 \cdot \left(\frac{16}{1-k^2}\right)^{\frac{1}{p}} = r_0 \cdot \left(\frac{256}{(1-k^2)^2}\right).$$

As k gets closer to the value 1, r_1 increases without bound; the regime in which the electrostatic laws are valid for very small field strengths then extends to infinity.

We can summarize the results that we just obtained in the following theorem:

In a spherical condensor, the electrical field follows the laws of conventional electrostatics to a very high degree of precision in the neighborhood of the null surface r_0, on which the potential is zero. However, the distance that one may separate the two spherical condensor surfaces before the electrical atmosphere that surrounds the charged surfaces produces noticeable deviations from the usual laws of electrostatics depends on the field strengths. For very weak fields, the associated distance becomes unbounded. With increasing field strengths it becomes smaller and smaller, and at indefinitely large field strengths it ultimately grows infinitesimal.

Since elliptic functions are periodic, the integral (43) takes on the value of zero not only for $r = r_0$, but also at infinitely many other points. The null surfaces of the potential are the spherical shells of the following radii:

$$\ldots,\ r_0 \cdot e^{-\frac{4\nu K}{p}},\ r_0 \cdot e^{-\frac{4(\nu-1)K}{p}},\ \ldots,\ r_0 \cdot e^{-\frac{8K}{p}},\ r_0 \cdot e^{-\frac{4K}{p}},\ r_0,\ r_0 \cdot e^{+\frac{4K}{p}},\ r_0 \cdot e^{+\frac{8K}{p}},\ \ldots$$

The direction of the field alternates; if it points in the direction of positive r at one null point, it points in the negative direction at the next one, back to the positive direction at

the next one after that, etc. Between any two null points we have alternated between a spherical shell with a positive electrical atmosphere and one with a negative electrical atmosphere. The positive and negative maxima of the charge density are found at roughly the following places:

$$\ldots, r_0 \cdot e^{-(4\nu+2)\frac{K}{p}}, r_0 \cdot e^{-(4\nu-2)\frac{K}{p}}, \ldots, r_0 \cdot e^{-\frac{6K}{p}}, r_0 \cdot e^{-\frac{2K}{p}}, r_0 \cdot e^{+\frac{2K}{p}}, r_0 \cdot e^{+\frac{6K}{p}}, \ldots$$

The potentials at these places are:

$$\varphi_M = \pm \sqrt{\frac{b}{h}} \cdot \frac{1}{\sqrt{r_M}},$$

in which one assumes that the positive and negative signs alternate. Both of the curves:

$$\varphi = \pm \sqrt{\frac{b}{h}} \cdot \frac{1}{\sqrt{r}},$$

touch the undulating curve of the potential on both sides. The closer one comes to the null point the more the potential, and with it, the charge density, increases. In the neighborhood of the null point, they become unbounded.

The integral (44) may be described in the following manner:

The center is enveloped by concentric spherical shells of electrical charge, like an onion. Moreover, the positive and negative electrical shells regularly alternate. Between any two shells in the neighborhood of the null surface $\varphi = 0$, *there is a more or less wide region in which the electrical atmosphere is extremely thin, and in which an electrical field exists, as it should in the usual laws of electrostatics for a spherical condensor. In the neighborhood of the center, the onion shells grow denser and denser, and likewise, the charge density, potential, and field between any two shells grows without bound.*

The integral (44) therefore has an *essential* singularity at the point $r = 0$, in the sense that for $r = 0$, φ is not constrained to any finite or infinite value, but fluctuates between unbounded positive and negative values in the neighborhood of $r = 0$ over very short intervals.

The point $r = \infty$ is also an essential singularity of the function, whether or not φ converges to zero for $r = \infty$. There are always infinitely many positive and negative maxima of φ between any two arbitrary finite value of r and the value $r = \infty$, and this is what characterizes the singularity at $r = \infty$.

If one wishes to make a precise definition of the range of the integral (44) then the case of a very small value of h is well suited. In this case, one can obtain the function over its entire range by elementary computational operations.

In the sequel, I will set:

$$\ldots, r_0 \cdot e^{\frac{(2\nu-1)\cdot 2K}{p}} = r_{2\nu-1}, \qquad r_0 \cdot e^{\frac{2\nu\cdot 2K}{p}} = r_{2\nu},$$

$$r_0 \cdot e^{\frac{(2\nu+1)\cdot 2K}{p}} = r_{2\nu+1}, \ldots,$$

The series of quantities r_n ($n = 2\nu - 1, 2\nu, 2\nu + 1, \ldots$) defines a geometric progression, for which we always have:

$$r_n : r_{n-1} = r_{n+1} : r_n = \ldots = e^{\frac{2K}{p}}.$$

n may take on odd and even, positive and negative values. One then has, in general:

$$r_n^2 = r_{n-1} \cdot r_{n+1}.$$

If we now introduce the assumption that h is very small compared to 1 then we can neglect the higher powers of h in formula (42), and calculate with the following approximation:

$$b = \sqrt{\frac{3}{a}} \cdot \frac{h}{2}; \qquad 1 - h^2 = 8h; \qquad p = \tfrac{1}{2}.$$

Furthermore, we can compute the half-period $2K$ of the elliptic function as:

$$2K = \ln \frac{16}{1-k^2} = \ln \frac{2}{h}.$$

The quotient of the geometric progression r_n is thus $e^{\frac{2K}{p}} = 4 / h^2$, which is a very large number; i.e., any two successive values in the progression:

$$\ldots, r_{n-1}, \ r_n, \ r_{n+1}, \ldots$$

are of very different orders of magnitude. Compared to r_n, r_{n-1} is infinitesimal and r_{n-1} is infinite. In general, we compute them as:

$$r_n = r_0 \cdot \left(\frac{4}{h^2}\right)^n,$$

in which n may be odd or even and positive or negative.

By making use of formula (45), one obtains the value of φ in the neighborhood of the point $r_{2\nu}$:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \frac{h}{2} \cdot \sqrt{r_{2\nu}} \cdot \left(\frac{1}{r_{2\nu}} - \frac{1}{r}\right) = \sqrt[4]{\frac{3}{a}} \cdot \sqrt{r_{2\nu-1}} \cdot \left(\frac{1}{r_{2\nu}} - \frac{1}{r}\right).$$

For a value of r that is very large compared to $r_{2\nu}$, but very small compared to $r_{2\nu+1}$, one can simply compute:

$$\varphi = -\sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_{2\nu-1}}}{r_{2\nu}};$$

for a value of r that is very large compared to $r_{2\nu}$, but small compared to $r_{2\nu+1}$, one has:

$$\varphi = +\sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_{2\nu-1}}}{r_{2\nu}} = \sqrt[4]{\frac{3}{a}} \cdot \frac{1}{\sqrt{r_{2\nu+1}}}.$$

In order to compute φ in the neighborhood of $r_{2\nu+1}$ (and thus, $r_{2\nu-1}$), I set:

$$x' = p \cdot \ln \frac{r}{r_{2\nu+1}} = p \cdot \ln \frac{r}{r_0} - (2\nu + 1) \cdot 2K = x - (2\nu + 1) \cdot 2K.$$

Now:

$$cn\, x = -\, cn\, x', \qquad dn\, x = +\, dn\, x',$$

hence:

$$\varphi = \frac{\sqrt{b}}{\sqrt{r}} \cdot \sqrt{\frac{dn\, x' + cn\, x'}{(dn\, x' - cn\, x') + h \cdot (dn\, x' + cn\, x')}}.$$

Since we would like to neglect terms of higher order in h, we develop $cn\, x'$ and $dn\, x'$ in a power series in $(1 - k^2)$ about $k^2 = 1$. This gives the following expressions:

$$(dn\, x')_{k^2} = (dn\, x')_{k^2=1} + \frac{1-k^2}{4} \cdot \left[\frac{sn^2 x'}{cn\, x'} + \frac{1}{2} \cdot sn\, x' \cdot cn\, x' \cdot \ln \frac{1 + sn\, x'}{1 - sn\, x'} \right]_{k^2=1} + \ldots$$

$$(cn\, x')_{k^2} = (cn\, x')_{k^2=1} - \frac{1-k^2}{4} \cdot \left[\frac{sn^2 x'}{cn\, x'} - \frac{1}{2} \cdot sn\, x' \cdot cn\, x' \cdot \ln \frac{1 + sn\, x'}{1 - sn\, x'} \right]_{k^2=1} + \ldots$$

We have set:

$$(cn\, x')_{k^2=1} = (dn\, x')_{k^2=1} = \frac{2}{e^{x'} + e^{-x'}}$$

$$(sn\, x')_{k^2=1} = \frac{e^{x'} - e^{-x'}}{e^{x'} + e^{-x'}},$$

$$1 - h^2 = 8h^2.$$

The use of this formula gives, by a very simple computation:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \sqrt{\frac{r_{2\nu+1}}{r^2 + r_{2\nu+1}^2}}.$$

For small values of r:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \frac{1}{\sqrt{r_{2\nu+1}}}.$$

For large values of r:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_{2\nu+1}}}{r}.$$

In the neighborhood of $r_{2\nu-1}$ one has, since the opposite sign must dominate there:

$$\varphi = -\sqrt[4]{\frac{3}{a}} \cdot \sqrt{\frac{r_{2\nu-1}}{r^2 + r_{2\nu-1}^2}}.$$

Thus, for increasing values of r, φ goes through the following values of the series:

$$-\sqrt[4]{\frac{3}{a}} \cdot \frac{1}{\sqrt{r_{2\nu-1}}}; \qquad -\sqrt[4]{\frac{3}{a}} \cdot \sqrt{\frac{r_{2\nu-1}}{r_{2\nu-1}^2 + r^2}}; \qquad -\sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_{2\nu-1}}}{r};$$

$$\sqrt[4]{\frac{3}{a}} \cdot \sqrt{r_{2\nu-1}} \cdot \left(\frac{1}{r_{2\nu}} - \frac{1}{r}\right); \qquad +\sqrt[4]{\frac{3}{a}} \cdot \frac{1}{\sqrt{r_{2\nu-1}}};$$

$$+\sqrt[4]{\frac{3}{a}} \cdot \sqrt{\frac{r_{2\nu+1}}{r_{2\nu+1}^2 + r^2}}; \qquad +\sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_{2\nu+1}}}{r};$$

$$\sqrt[4]{\frac{3}{a}} \cdot \sqrt{r_{2\nu+1}} \cdot \left(\frac{1}{r} - \frac{1}{r_{2\nu+2}}\right);$$

…

20. II. Case: $C < 0$. Again, I will let γ denote the unique positive solution of the third degree equation:

$$-C + \gamma - \frac{4a}{3} \cdot \gamma^3 = 0, \qquad \gamma > 0,$$

and introduce the following quantity h in place of the C as the integration constant:

$$h = +\sqrt{\frac{-C}{-3C + 2\gamma}}.$$

Moreover, I will again compute the three quantities b, k^2, p from (42):

$$b = h \cdot \sqrt{\frac{1-h^2}{1-3h^2}} \cdot \sqrt{\frac{3}{4a}},$$

$$k^2 = \frac{(1-h)(1-3h)}{(1+h)(1+3h)},$$

$$p = \frac{1}{2} \cdot \sqrt{\frac{(1+h)(1+3h)}{1-3h^2}}.$$

Thus, if I replace v with the following quantity u:

$$v = b \cdot \frac{1 + k \cdot u}{(1 - ku) - h(1 + k \cdot u)},$$

then in the case $C < 0$, one obtains the following equation in place of equation (39):

$$(49) \qquad \begin{cases} \left(\frac{du}{dx}\right)^2 = (1 - u^2)(1 - k^2 u^2), \\ x = p \cdot \ln \frac{r}{r_0}, \\ \varphi = \sqrt{\frac{b}{r}} \cdot \sqrt{\frac{1 + k \cdot u}{(1 - ku) - h \cdot (1 + ku)}}. \end{cases}$$

The integral (49) is then real only when:

$$k^2 \geq 0.$$

We have therefore have to merely restrict ourselves to the regime $0 < h < 1/3$, in which this condition is satisfied. If one then introduces the integral $-cn\,x / dn\,x$ for u, then one obtains:

$$(50) \qquad \begin{cases} \varphi = \sqrt{\frac{b}{r}} \cdot \frac{\sqrt{dn\,x - k \cdot cn\,x}}{\sqrt{(dn\,x + k \cdot cn\,x) - h(dn\,x - k \cdot cn\,x)}}, \\ x = p \cdot \ln \frac{r}{r_0}, \quad 0 < h < +1. \end{cases}$$

The characteristic of this integral is that, by contrast with (44), it never goes to zero, so one always has:

$$dn\,x > k \cdot cn\,x,$$

and the expression under the square root sign therefore always stays greater than zero. However, when φ never goes through zero, it must therefore always retain the same sign, and it goes continuously from the very large values at small r to the very small values at large r. Meanwhile, the periodic function that appears under the square root sign regularly fluctuates back and forth between a maximum and a minimum. The maximum is attained for the value $x = (2v + 1) \cdot 2K$; hence:

$$r = \ldots,\ r_0 \cdot e^{-\frac{(2v+1)\cdot 2K}{p}},\ r_0 \cdot e^{-\frac{(2v-1)\cdot 2K}{p}},\ \ldots,\ r_0 \cdot e^{-\frac{6K}{p}},\ r_0 \cdot e^{-\frac{2K}{p}},\ r_0 \cdot e^{+\frac{2K}{p}},\ r_0 \cdot e^{+\frac{6K}{p}},\ \ldots$$

and the minimum is attained for $x = 4vK$; hence:

$$r = \ldots, r_0 \cdot e^{-\frac{4\nu \cdot K}{p}}, r_0 \cdot e^{-\frac{4(\nu-1)\cdot K}{p}}, \ldots, r_0 \cdot e^{-\frac{4K}{p}}, r_0, r_0 \cdot e^{+\frac{4K}{p}}, r_0 \cdot e^{+\frac{8K}{p}}, \ldots$$

The values of the maximum and minimum are:

$$+\sqrt{\frac{1+k}{(1-k)-h(1+k)}} \qquad \text{and} \qquad +\sqrt{\frac{1-k}{(1+k)-h(1-k)}}.$$

The values of the potential at the points with $r_{2\nu+1}$ (maximum) and $r_{2\nu}$ (minimum) then become:

$$\varphi_{2\nu\text{-}1} = \sqrt{\frac{b\cdot(1+k)}{(1-k)-h(1+k)}} \cdot \frac{1}{\sqrt{r_{2\nu-1}}},$$

$$\varphi_{2\nu} = \sqrt{\frac{b\cdot(1-k)}{(1+k)-h(1-k)}} \cdot \frac{1}{\sqrt{r_{2\nu}}}.$$

As a result of the periodicity of the expression under the radical one has the drawback that φ is not regular with increasing r, but stepwise. The two smooth curves:

$$\varphi = \sqrt{\frac{b\cdot(1+k)}{(1-k)-h(1+k)}} \cdot \frac{1}{\sqrt{r}}$$

$$\varphi = \sqrt{\frac{b\cdot(1-k)}{(1+k)-h(1-k)}} \cdot \frac{1}{\sqrt{r}}$$

touch the step-wise curve of the potential and, in that way, they mutually bound it. One can best recognize the form of the potential curve in the case of a small k, where one can easily carry out an approximate calculation in exactly the same way that we did at the end of section **19**. in the case of a positive C. We again set:

$$r_0 \cdot e^{+2\nu\cdot\frac{2K}{p}} = r_{2\nu}, \qquad r_0 \cdot e^{+(2\nu+1)\cdot\frac{2K}{p}} = r_{2\nu+1},$$

and we obtain, for small h in the neighborhood of the value $r_{2\nu}$:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \sqrt{r_{2\nu-1}} \cdot \left(\frac{1}{r_{2\nu}} + \frac{1}{r}\right),$$

and, in the neighborhood of $r_{2\nu+1}$:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \sqrt{\frac{r_{2\nu+1}}{r^2 + r_{2\nu+1}^2}}.$$

With increasing r, φ then goes through the following sequence of values:

$$+\sqrt[4]{\frac{3}{a}}\cdot\frac{1}{\sqrt{r_{2\nu-1}}};\qquad +\sqrt[4]{\frac{3}{a}}\cdot\sqrt{\frac{r_{2\nu-1}}{r_{2\nu-1}^2+r^2}};\qquad +\sqrt[4]{\frac{3}{a}}\cdot\frac{\sqrt{r_{2\nu-1}}}{r};$$

$$+\sqrt[4]{\frac{3}{a}}\cdot\sqrt{r_{2\nu-1}}\cdot\left(\frac{1}{r}+\frac{1}{r_{2\nu}}\right);\qquad +\sqrt[4]{\frac{3}{a}}\cdot\frac{1}{\sqrt{r_{2\nu+1}}};$$

$$+\sqrt[4]{\frac{3}{a}}\cdot\sqrt{\frac{r_{2\nu+1}}{r_{2\nu+1}^2+r^2}};\qquad +\sqrt[4]{\frac{3}{a}}\cdot\frac{\sqrt{r_{2\nu+1}}}{r};$$

$$+\sqrt[4]{\frac{3}{a}}\cdot\sqrt{r_{2\nu+1}}\cdot\left(\frac{1}{r}+\frac{1}{r_{2\nu+2}}\right);$$

…

One then sees that in the range between $r_{2\nu\text{-}2}$ and $r_{2\nu-1}$, φ will remain at an almost constant value for a rather long time. At $r_{2\nu-1}$, the φ-curve drops down, and between $r_{2\nu-1}$ and $r_{2\nu}$ it almost exactly equals a one-sided hyperbola for a long stretch, but once it reaches $r_{2\nu}$ the curve becomes flat again. Between $r_{2\nu}$ and $r_{2\nu+1}$, it again takes on the form of an almost completely horizontal step; between $r_{2\nu+1}$ and $r_{2\nu+2}$, it goes upwards in the form of a forward-pointing hyperbola, and so on.

The closer that one comes to $r = 0$, the more frequent the steps become – ultimately, they are infinitely frequent – and in this way one also has in this case that, whether or not φ becomes infinitely large for $r = 0$ uniquely, the point $r = 0$ is an *essential* singularity of the function. Likewise, $r = \infty$ is an essential singularity, because between any finite r and $r = \infty$ there will still be an infinitude of steps.

For large values of h the behavior of the function is essentially the same as the case that we just described of infinitesimal h, only the steps become finer and finer, in such a way that in the case of infinitesimal h one can not distinguish distinctly differing regions very well. Ultimately, when one chooses $h = 1/3$, $k = 0$ the steps vanish completely, and both of the limiting curves that were computed on pp. ?, between which the step-wise curve of the potential goes back and forth, now merge together into the curve:

$$\varphi = \sqrt{\frac{3b}{2}}\cdot\frac{1}{\sqrt{r}},$$

which agrees with the potential curve. From the definition of b (on pp. ?), we have:

$$3b = \frac{1}{\sqrt{a}},$$

and, when one sets the arbitrary constant h equal to 1/3, the integral φ looks like:

$$\varphi = \frac{1}{\sqrt[4]{4a}}\cdot\frac{1}{\sqrt{r}}.$$

It is therefore algebraic, and the second arbitrary constant r_0 falls out of it.

In practically realizable cases, the integral for $C < 0$ gives us the field in a spherical condensor whose two reference potentials are charge with the same sign. If the potentials are low then one can take h to be small, and the field in the interior of the condensor is given by the formula:

$$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \sqrt{r_{2\nu-1}} \cdot \left(\frac{1}{r_{2\nu}} + \frac{1}{r} \right) = A + \frac{B}{r}.$$

This is the usual formula from electrostatics. When one is given the potential and field strength at a point of the condensor, hence, the quantities A and B, then one can immediately compute $r_{2\nu}$ and $r_{2\nu-1}$, and from $r_{2\nu}$ and $r_{2\nu-1}$, one can compute the period of the function, hence, the modulus k and r_0, as well. With the field, both integration constants are likewise known.

The formula that was just used is, however, no longer valid when the potential is sufficiently small. Large values of φ are associated such strong electrical atmospheres that the formulas of conventional electrostatics are no longer useful. There is no point in discussing these unrealizable cases any further.

21. III. Case: $C = 0$. When one divides equation (39):

$$\left(\frac{dr}{d\xi} \right)^2 = v^2 - \frac{4a}{3} \cdot v^4$$

by $4a / 3 \cdot v^4$ and sets:

$$\sqrt{\frac{3}{a}} \cdot \frac{1}{2v} = w,$$

then one obtains:

$$\left(\frac{dw}{d\xi} \right)^2 = w^2 - 1.$$

The solution of this equation is:

$$w = \tfrac{1}{2} (e^{\xi} + e^{-\xi}).$$

If one now sets:

$$w = \frac{1}{2} \sqrt{\frac{3}{a}} \cdot \frac{1}{r \cdot \varphi^2}, \qquad \xi = \ln \frac{r}{r_0}$$

then one obtains:

(52) $$\varphi = \sqrt[4]{\frac{3 \cdot r_0^2}{a}} \cdot \frac{1}{\sqrt{r^2 + r_0^2}}.$$

If one then chooses the value 0 for the integration constant C then the essential singularities at the point $r = 0$, as well as $r = \infty$, disappear. The integral now has only algebraic singularities, and for real values of r it has no singularities at all.

One goes from the integral that was discussed in **19** to the function (52) when one allows the integration constant h to sink without bound. The modulus k then rises to 1,

and the period $4K$ becomes infinitely large. The electric spherical shells that contain the shell r_0 then expand without bound on both sides. On the one had, they push the infinitude of onion shells that surround the null point down to the null point itself. On the other hand, null surface of the potential is shifted out to infinity completely, in such a way that no more electrically charged shells can exist between $r = r_0$ and $r = \infty$.

If the value of the potential φ on a sphere of radius R is given as $\varphi = A$ then we have the following equation for the determination of the integration constant r_0:

$$A^2 = \sqrt{\frac{3}{a}} \cdot \frac{r_0}{R^2 + r_0^2},$$

$$r_0{}^2 - \frac{1}{A^2} \cdot \sqrt{\frac{3}{a}} \cdot r_0 + R^2 = 0.$$

$$\frac{3}{4a \cdot A^2} - R^2 > 0.$$

The field that was represented by (52) is therefore possible only when the potential A does not exceed a well-defined large value:

$$A \gg \sqrt[4]{\frac{3}{4a}} \cdot \frac{1}{\sqrt{\mathrm{R}}}$$

on a sphere of radius R.

The usual laws of electrostatics thus remain valid when the potential A is sufficiently small in comparison to this largest possible value. This is the case when:

$$\sqrt{\frac{a}{3}} \cdot A^2 \cdot R = \varepsilon,$$

in which ε means a very small number. One can easily compute approximate values for r_0, in which ε^2 is ignored when compared to 1. The quadratic equation gives two solutions:

1. $$r_0 = \frac{1}{A^2} \cdot \sqrt{\frac{3}{a}} = \frac{R}{\varepsilon},$$

2. $$r_0 = \sqrt{\frac{a}{3}} \cdot A^2 \cdot R^2 = \varepsilon R.$$

The first value of r_0 is infinitely large compared to R; thus, in the interior of a ball of radius R (hence $r < R$), when one deletes ε^2 by comparison to 1 the potential is:

1. $$\varphi = \sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_0}}{\sqrt{r^2 + r_0^2}} = \sqrt[4]{\frac{3}{a}} \cdot \frac{1}{\sqrt{r_0^2}} = A.$$

By contrast, the second value of r_0 is infinitesimal compared to R. If one again deletes ε^2 by comparison to 1 then one has, for all values of $r > R$:

$$2. \qquad \varphi = \sqrt[4]{\frac{3}{a}} \cdot \frac{\sqrt{r_0}}{\sqrt{r^2 + r_0^2}} = \sqrt[4]{\frac{3}{a}} \cdot \sqrt{r_0} \cdot \frac{1}{r} = A \cdot \frac{r_0}{r}.$$

By comparison with the usual laws of electrostatics, the first solution then in the interior of a hollow ball that is charges with the potential A when no more charge exists in an infinitely large region around it. If one uses the precise formula without neglecting ε^2, then one can compute the weak electrical atmosphere, which, in our theory, must exist inside, as well as outside the ball.

22. We shall now consider several concentric spheres of radius R_1, R_2, R_3, R_4 that are charged with potentials of A_1, A_2, A_3, A_4 . Two integrals with $C = 0$ produce the field in the interior of the smallest sphere and the field in the exterior of the largest sphere. In the shells between these two shells one must take the integral with $C > 0$ or $C < 0$. As long as the potentials A_1, A_2, A_3, A_4 are all sufficiently small one obtains a very good approximation for the usual formulas of electrostatics, and the deviation from them is unnoticeably small. In this special case, we then see that the theorem that we discussed in general on pp. ? is valid, that there are functions Φ *that do not bring one into conflict with conventional electrostatics.*

However, we must now address the question of how one can imagine that charged spherical surfaces R_1, R_2, R_3, R_4 can occur in such a way that they separate any two spaces from each other as an instability surface, in which φ can take on various integrals. Naturally, the charge on such a spherical surface cannot be evenly distributed. We would then have solved the differential equation (37) for the case of spherical symmetry in complete generality, and there is nothing about its integrals, which can have discontinuities for the values R_1, R_2, R_3, R_4, that would correspond to the surface charge. The integrals, which must be valid on both sides of such a discontinuity surface, must therefore both converge very quickly, but still continuously, to one and the same third solution of the general equilibrium condition, which is now no longer spherically symmetric, inside a very thin layer on the surface. It is quite clear that these solutions must correspond to an atomistic distribution of charge in the charged surfaces. *It is interesting to remark that the theory that I sought did not subsequently give the basis for us to choose between a continuous versus an atomistic structure for the charge at all, but only that this is completely feasible if it leads to any sort of atoms of electricity.*

The problem of the electron.

23. On first glance, the example that was discussed in the previous chapter seems to suggest a world function that must be very carefully chosen, because, in fact, it leads to isolated knot singularities of the electrical charge. Namely, if, in formula (52):

$$\varphi = \sqrt[4]{\frac{3 \cdot r_0^2}{a}} \cdot \frac{1}{\sqrt{r^2 + r_0^2}}$$

we choose the integration constant r_0 to be infinitesimal when compared to all measurable lengths, perhaps the order of magnitude that one generally ascribes to the electron radius, then φ represents the potential in the neighborhood of a tiny electrical knot singularity whose atmosphere is practically equal to zero at any measurable distance from the center. Since the charge density ρ is given by equation (36) $\rho = a \cdot \varphi^5$, we then have:

$$\rho = \sqrt[4]{\frac{3 \cdot r_0^2}{a}} \cdot \frac{3 \cdot r_0^2}{\sqrt{(r^2 + r_0^2)^5}}.$$

The total charge e of the knot singularity is then given by integration:

(53) $$e = 4\pi \cdot \sqrt[4]{\frac{3 \cdot r_0^2}{a}}.$$

A simple computation shows that the total charge of the atmosphere of the knot singularity outside of a sphere of radius r_1 has the magnitude:

$$e \cdot \left(1 - \left(\frac{r_1}{\sqrt{r_1^2 + r_0^2}} \right)^3 \right).$$

When r_1 is large compared to r_0, this is only a vanishingly small fraction of e; in fact, the charge is then almost completely confined to a small ball.

In general, the theory provides no elementary quantum of charge. If one varies the arbitrary constant r_0 in (53) then one can obtain all possible magnitudes for e, and indeed, e can be just as likely have a positive or negative sign. The "electrons" that one obtains from the chosen world function are therefore not irreducible. Many knot singularities can be merged into a single larger one, and a single knot singularity can be subdivided into smaller ones, since knot singularities with all possible charges can exist.

Whether or not this peculiarity corresponds to the actually observed facts, one can perhaps believe that such a theory of matter that is constructed out of the world function of the example is completely feasible, since the electrical charge of a large body can be thought of as due to discrete knot singularities when this does not also lead to the property of irreducibility; however, this is a mistake. From formula (52) and (53) the potential at a sufficiently large distance from the knot singularity is always give as $\varphi = e/4\pi r$. Equilibrium then dominates in the field of the knot singularity only when there is a space that has zero potential. Among all of the integrals of equation (37), none of them represents a knot singularity in a space of a potential φ_0 that is arbitrarily different from zero, in such a way that we have, at a large distance from the knot singularity $\varphi = e / 4\pi r$

$+ \varphi_0$, in which φ_0 means a non-zero constant. Thus, there cannot be several knot singularities near other in equilibrium, as must be the case in a material body. If there are many of them then there must be an immediate reorganization of the charges. If two neighboring knot singularities have the same sign then they must seek to merge into a single larger knot singularity for which equilibrium is attained. On the contrary, two knot singularities with differing charges cannot remain close to each other at all. They must flee further and further away from each other in order to come to space with null potential. A world that is governed by the world function:

$$\Phi = -\tfrac{1}{2}\eta^2 + \tfrac{1}{6}a \cdot \chi^6,$$

must therefore ultimately coagulate into two large clumps of electrical charge – one positive and one negative – and these two clumps must always move further and further away from each other.

24. In the general discussion of the equilibrium condition (pp. ? and ?) we pointed out that there is an integral with *one* arbitrary constant that has no singularity at the point $r = 0$, also one with *one* arbitrary constant that is regular at the point $r = \infty$. In general, both of these integrals correspond to the two *different* values of the second arbitrary constant that the general integral still contains; they are completely different from each other and include the self-explanatory solution $\varphi = 0$ as the single mutual special case.

However, the example that was discussed by us exhibited the peculiarity that both of the integrals that we just spoke of were identical with each other. Namely, when one gave the value of zero to the arbitrary constant C (pp. ? and ?) then both of the essential singularities at $r = 0$ and $r = \infty$ *simultaneously* drop out of the general integral. This is the basis for the fact that the example gives knot singularities, but not elementary quantum, and likewise, the fact that the knot singularities can only exist in a space in which the potential $\varphi = 0$ in equilibrium. Apparently, the amalgamation of the omissions of both singularities is connected with the question of whether the differential equation (37) on pp. ? is reducible or not, i.e., whether it can be transformed into a first order *algebraic* differential equation by a single equation. World functions that lead to such integrals are therefore not needed.

If we would seek a world function of the form:

$$\Phi = -\tfrac{1}{2}\eta^2 + \tfrac{1}{\nu}a \cdot \chi^\nu,$$

in which ν shall mean an arbitrary even number that is different from 6 and greater than four, then we would find no solution without essential singularities(except for $\varphi = 0$). The field in the infinite external space around a charged ball would then be represented by a solution that has an essential singularity at $r = 0$, and the solution that we take for the interior of a charged hollow sphere is not regular at $r = \infty$. For this choice of world function we find no solution at all that represents an isolated spherical knot singularity with a finite charge. We will prove this rigorously in one of the following sections. World functions of this form are thus not to be used in any way, shape, or form.

If the theory is to be practicable at all then the world function must have a complicated form in any case. The electron will obviously be represented by a solution to the equilibrium condition (34) that has a singularity at the point $r = 0$ that is to be indicated function-theoretically as an essential singularity, which, however, gives no physically sensible value for φ. An example of such a singularity is the function:

$$\varphi = a + b \cdot e^{-\frac{1}{r}} \qquad \text{for } r = 0.$$

The differential equation (34) must then provide, amongst all the integrals that have an essential singularity at $r = 0$, one of them whose singularity is not associated with physical senselessness. This integral must include *one* arbitrary constant, which can be chosen in such a way that φ diminishes to an arbitrary, but chosen constant at a large distance from the center: the potential of the space in which the electron is found. For $r = \infty$ this integral will generally have an essential singularity that must either drop out or have no physical meaning only in the case where the potential of the neighboring space is null.

Whether or not it is amusing to find world function that really leads to an electron, nevertheless one must not deny the possibility of that such a world function exists. Therefore, in the sequel, I will next simply make the assumptions that bring about a world function F that leads to knot singularities of electrical charge in the same way that one finds in the electron. In an eventual continuation of this research, I will compute the dynamics of such knot singularities, their inertial mass, and the forces that they experience.

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(Third Part [1]), Conclusion.)

Chapter Three

Force and Inertial Mass.

Computation of the force that acts on a massive particle.

25. In order to compute the force we use the world matrix (16) that was described in I, pp. ? However, we will make no sort of restricting assumptions on the invariants that appear in the world function, but we will assume in full generality that all four of the variables that were listed in I, pp. ? enter into *H*. A simple calculation gives that the theorem that was next discussed on pp. ? under restricted assumptions is true in complete generality:

The world matrix is symmetric about the diagonal.

If one applies the multiplication rule:

$$[\mathfrak{a} \cdot \mathfrak{b}] \cdot \mathfrak{c} = (\mathfrak{a} \cdot \mathfrak{c}) \cdot \mathfrak{b} - (\mathfrak{b} \cdot \mathfrak{c}) \cdot \mathfrak{a}$$

and the formula that one obtains from it:

$$[[\mathfrak{a} \cdot \mathfrak{b}] \cdot \mathfrak{c}] + [[\mathfrak{b} \cdot \mathfrak{c}] \cdot \mathfrak{a}] + [[\mathfrak{c} \cdot \mathfrak{a}] \cdot \mathfrak{b}] = 0$$

then one easily finds the following two equations from the general formula (25) in I, pp. ?:

$$[\mathfrak{e} \cdot \mathfrak{d}] + [\mathfrak{h} \cdot \mathfrak{b}] + [\mathfrak{f} \cdot \mathfrak{v}] = 0, \tag{54}$$

$$[\mathfrak{e} \cdot \mathfrak{h}] + [\mathfrak{b} \cdot \mathfrak{d}] + (\rho \cdot \mathfrak{f} - \varphi \cdot \mathfrak{v}) = 0, \tag{55}$$

therefore, when one writes out the components of the expressions explicitly, one has:

$$\mathfrak{e}_x \cdot \mathfrak{d}_y + \mathfrak{h}_x \cdot \mathfrak{b}_y + \mathfrak{f}_x \cdot \mathfrak{v}_y = \mathfrak{d}_x \cdot \mathfrak{e}_y + \mathfrak{b}_x \cdot \mathfrak{h}_y + \mathfrak{v}_x \cdot \mathfrak{f}_y, \qquad \text{etc.}$$

$$\mathfrak{d}_y \cdot \mathfrak{b}_z + \mathfrak{d}_z \cdot \mathfrak{b}_y + \rho \cdot \mathfrak{f}_x = \mathfrak{e}_y \cdot \mathfrak{h}_z + \mathfrak{e}_z \cdot \mathfrak{h}_y + \varphi \cdot \mathfrak{v}_x, \qquad \text{etc.}$$

The theorem is therefore proved.

26. We would now like to represent a material particle, hence, either an electrical knot singularity or a more complicated structure that is comprised of similar singularities, that moves in a widely extended electromagnetic field. At an point, let the energy current that is associated with the forward motion of the ether state be denoted by $\mathfrak{s}$, as in I. (5), pp. 522. We then have:

[1] Annalen der Physik **37** (1912), 511-534, **39** (1912), 1-40.

$$(56)\qquad \begin{cases} \mathfrak{s}_x = \mathfrak{e}_y \cdot \mathfrak{h}_z - \mathfrak{e}_z \cdot \mathfrak{h}_y - \varphi \cdot \mathfrak{v}_x = \mathfrak{d}_y \cdot \mathfrak{b}_z - \mathfrak{d}_z \cdot \mathfrak{b}_y - \} \cdot \mathfrak{f}_x, \\ \mathfrak{s}_y = \mathfrak{e}_z \cdot \mathfrak{h}_x - \mathfrak{e}_x \cdot \mathfrak{h}_z - \varphi \cdot \mathfrak{v}_y = \mathfrak{d}_z \cdot \mathfrak{b}_x - \mathfrak{d}_z \cdot \mathfrak{b}_x - \} \cdot \mathfrak{f}_y, \\ \mathfrak{s}_z = \mathfrak{e}_x \cdot \mathfrak{h}_y - \mathfrak{e}_y \cdot \mathfrak{h}_x - \varphi \cdot \mathfrak{v}_z = \mathfrak{d}_x \cdot \mathfrak{b}_y - \mathfrak{d}_y \cdot \mathfrak{b}_x - \} \cdot \mathfrak{f}_z. \end{cases}$$

Furthermore, we would also like to define the three three-dimensional vectors $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ by the following equations:

$$(57)\qquad \begin{cases} \Phi - \mathfrak{b} \cdot \mathfrak{h} + \mathfrak{e}_x \cdot \mathfrak{d}_x + \mathfrak{b}_x \cdot \mathfrak{h}_x + \mathfrak{f}_x \cdot \mathfrak{v}_x = \mathfrak{p}_{1x}, \\ \mathfrak{e}_y \cdot \mathfrak{d}_x + \mathfrak{b}_y \cdot \mathfrak{h}_x + \mathfrak{f}_y \cdot \mathfrak{v}_x = \mathfrak{p}_{1y}, \\ \mathfrak{e}_z \cdot \mathfrak{d}_x + \mathfrak{b}_z \cdot \mathfrak{h}_x + \mathfrak{f}_z \cdot \mathfrak{v}_x = \mathfrak{p}_{1z}, \\ \mathfrak{e}_x \cdot \mathfrak{d}_y + \mathfrak{b}_x \cdot \mathfrak{h}_y + \mathfrak{f}_x \cdot \mathfrak{v}_y = \mathfrak{p}_{2x}, \\ \Phi - \mathfrak{b} \cdot \mathfrak{h} + \mathfrak{e}_y \cdot \mathfrak{d}_y + \mathfrak{b}_y \cdot \mathfrak{h}_y + \mathfrak{f}_y \cdot \mathfrak{v}_y = \mathfrak{p}_{2y}, \\ \mathfrak{e}_z \cdot \mathfrak{d}_y + \mathfrak{b}_z \cdot \mathfrak{h}_z + \mathfrak{f}_z \cdot \mathfrak{v}_y = \mathfrak{p}_{2z}, \\ \mathfrak{e}_x \cdot \mathfrak{d}_z + \mathfrak{b}_x \cdot \mathfrak{h}_z + \mathfrak{f}_x \cdot \mathfrak{v}_z = \mathfrak{p}_{3x}, \\ \mathfrak{e}_y \cdot \mathfrak{d}_z + \mathfrak{b}_y \cdot \mathfrak{h}_z + \mathfrak{f}_y \cdot \mathfrak{v}_z = \mathfrak{p}_{3y}, \\ \Phi - \mathfrak{b} \cdot \mathfrak{h} + \mathfrak{e}_z \cdot \mathfrak{d}_z + \mathfrak{b}_z \cdot \mathfrak{h}_z + \mathfrak{f}_z \cdot \mathfrak{v}_z = \mathfrak{p}_{3z}. \end{cases}$$

As we saw in I on pp. ?, namely, eq. (17), the first three rows world matrix give us three differential equations that, if we keep (56) and (57) in mind, may be written in the following way:

$$(58)\qquad \begin{cases} \dfrac{\partial \mathfrak{s}_x}{\partial t} = \dfrac{\partial \mathfrak{p}_{1x}}{\partial x} + \dfrac{\partial \mathfrak{p}_{2x}}{\partial y} + \dfrac{\partial \mathfrak{p}_{3x}}{\partial z}, \\ \dfrac{\partial \mathfrak{s}_y}{\partial t} = \dfrac{\partial \mathfrak{p}_{1y}}{\partial x} + \dfrac{\partial \mathfrak{p}_{2y}}{\partial y} + \dfrac{\partial \mathfrak{p}_{3y}}{\partial z}, \\ \dfrac{\partial \mathfrak{s}_z}{\partial t} = \dfrac{\partial \mathfrak{p}_{1z}}{\partial x} + \dfrac{\partial \mathfrak{p}_{2z}}{\partial y} + \dfrac{\partial \mathfrak{p}_{3z}}{\partial z}. \end{cases}$$

We would now like to express the energy of a fluid that flows with a certain velocity $\mathfrak{q}$. If $\mathcal{W}$ is the energy density then, by definition, $\mathfrak{q}$ is determined from:

$$(59)\qquad \mathfrak{s} = \mathcal{W} \cdot \mathfrak{q}.$$

Furthermore, if we let dM denote the instantaneous total energy of the volume element $dx \cdot dy \cdot dz = dV$ then $dM = \mathcal{W} \cdot dV$, and we can also write equations (58) in the following way:

$$\frac{\partial}{\partial t}(dM \cdot \mathfrak{q}_x) = \left(\frac{\partial \mathfrak{p}_{1x}}{\partial x} + \frac{\partial \mathfrak{p}_{2x}}{\partial y} + \frac{\partial \mathfrak{p}_{3x}}{\partial z} \right) \cdot dV,$$

$$\frac{\partial}{\partial t}(dM \cdot \mathfrak{q}_y) = \left(\frac{\partial \mathfrak{p}_{1y}}{\partial x} + \frac{\partial \mathfrak{p}_{2y}}{\partial y} + \frac{\partial \mathfrak{p}_{3y}}{\partial z} \right) \cdot dV,$$

$$\frac{\partial}{\partial t}(dM \cdot \mathfrak{q}_z) = \left(\frac{\partial \mathfrak{p}_{1z}}{\partial x} + \frac{\partial \mathfrak{p}_{2z}}{\partial y} + \frac{\partial \mathfrak{p}_{3z}}{\partial z} \right) \cdot dV.$$

We would like to integrate these equations over a volume *V*. Let:

$$M = \int_V dM$$

be the total energy that is contained in *V* at the moment in question. Let $\overline{\mathfrak{q}}$ be the velocity of the "mean mass point" in *V*, which is defined by way of the equation:

(60) $$M \cdot \overline{\mathfrak{q}} = \int_V \mathfrak{q} \cdot dM \ .$$

Furthermore, let *S* denote the bounding surface of the volume *V*, let *N* be the outward-pointing normal at a point of *S*, and finally, let $\mathfrak{p}_N$ be a three-dimensional vector that is defined by the equation:

(61) $$\mathfrak{p}_N = \mathfrak{p}_1 \cdot \cos(N, x) + \mathfrak{p}_2 \cdot \cos(N, y) + \mathfrak{p}_3 \cdot \cos(N, z) \ \ldots$$

The components of $\mathfrak{p}_N$ can also be computed in the following way:

$$\mathfrak{p}_{Nx} = \mathfrak{p}_{1x} \cdot \cos(N, x) + \mathfrak{p}_{2x} \cdot \cos(N, y) + \mathfrak{p}_{3x} \cdot \cos(N, z), \text{ etc.}$$

The integration over *V* then gives the following result:

(62) $$\frac{\partial (M \cdot \overline{\mathfrak{q}})}{\partial t} = \int_S \mathfrak{p}_N \cdot dS \ .$$

The volume *V* is chosen in such a way that it is infinitesimal in comparison to the widely extended field, but infinitely large in comparison to the material particle that it encloses. The second condition shall express: first, that the energy of the singularity that the material particle possesses is as good as contained in the volume *V*, hence, only a vanishingly small fraction of the total energy exists outside of the surface *S*, and, second, the vacuum laws are as good as exactly valid, hence, one can set ρ and $\mathfrak{v}$ to zero and $\mathfrak{e} = \mathfrak{d}$, $\mathfrak{b} = \mathfrak{h}$. For this choice of volume, *V*, $M \cdot \overline{\mathfrak{q}}$ is the momentum of the particle, and thus its inertial mass is identical with its energy *M*, and the right-hand side of equation (62) gives the force that acts on the particle. Since it follows that the second condition $\mathfrak{p}_N$ is identical, up to vanishingly small correction terms, with the components of the Maxwell stress tensor on the bounding surface *S* of the element in question, one obtains a value for

the force that is independent of the value of the volume *V*, assuming that both of the aforementioned conditions are satisfied, and, moreover, a value that is completely identical with the one that is provided by the electron theory for a material particle that is surrounded by precisely the same electric and magnetic field as the particle in question. Just like in the theory of the electron, the force thus does not depend upon the particular way in which the electric charge and the electric or magnetic dipoles are arranged inside of a material particle, as long as the particle produces the same external field, and, in particular, it does not depend on the laws that pertain to the effect of the surface tension on the particle or the laws that govern the electromagnetic field inside the particle instead of Maxwell's equations. Precisely the same theorem, which we will next study in the case of the translational motion $\mathfrak{q}$ of the particle, may also be proved for a rotational motion with no further assumptions. The inertial momentum may thus be computed just as in ordinary mechanics, in which one always uses the energy in place of the inertial mass. This essentially follows from the fact that (54) implies that $\mathfrak{p}_{1y} = \mathfrak{p}_{2x}$, $\mathfrak{p}_{1z} = \mathfrak{p}_{3x}$, $\mathfrak{p}_{2z} = \mathfrak{p}_{3y}$.

The ponderomotive forces that bring a material particle into translational or rotational motion may be computed from the electric and magnetic field in which one finds the particle according to the same rules as in conventional electricity theory. The existence of a particular four-vector ($\mathfrak{v}$, $i\rho$) *in the interior of the particle and the deviation of the laws of electromagnetic fields from the Maxwell equations inside the particle have no noticeable influence on the external ponderomotice forces.*

According to our theory, by way of example, the force:

$$\mathfrak{P} = e \cdot (\mathfrak{e} + [\mathfrak{q} \cdot \mathfrak{b}]) \tag{63}$$

acts on an electron of total charge *e* that moves in an electromagnetic field with the velocity $\mathfrak{q}$.

This expression corresponds precisely with the one that the theory of the electron is founded upon.

On the contrary, the internal forces that act inside of an elementary particle of matter, which might perhaps contribute fine structure effects to this particle itself, are completely different from the ponderomotive forces of the usual theory of electricity. However, they may not be computed without further knowledge of the world function.

Among the external forces that affect the material particle, one also finds gravitation. From the theorem that we just proved, it follows *that the fundamental equations of ether dynamics,* I. (1) *through* (4), *which we have founded our theory upon, do not, however, clarify the issue of gravitation.* The hope that I spoke of at the beginning of my work (I, pp. ?, et seq.) is therefore not yet satisfied. In a later chapter, we will examine how the fundamental equations must be extended in order to encompass gravitation, as well.

The inertial mass of a material particle.

27. By the term "material particle" we mean, quite generally, a small region of the ether where the state variables take on enormously large values. In the sequel, we will frequently have to evaluate integrals of state variables over the entire volume of the particle. We understand that to mean a volume whose outer boundary is sufficiently far from the center of the particle that the state variables can be assumed to be infinitesimal on it. Thus, when one chooses the outer boundary of the volume to be completely arbitrary, but also such that this definition of particle is "completely" invalid on it, then this choice may have no noticeable influence on the value of the integral.

When we say that a particle is at rest and unchanging, we understand this to mean that either all of the state variables are constant in the volume that fills the particle or that that the average value of each state variable is constant at each point of the volume over a time interval that is infinitesimal compared to that of the experiment itself.

Let K be, e.g., the value of a state variable at a point (x, y, z) of the particle. Furthermore, let τ be a time interval that is infinitesimal compared to that of the experiment. We then have that the average value of which we spoke is:

$$\bar{K} = \frac{1}{\tau} \cdot \int_0^\tau K \cdot dt \, .$$

As is well know, the following equations are valid [1]):

$$\frac{\partial \bar{K}}{\partial t} = \frac{\partial \bar{K}}{\partial t}, \qquad \frac{\partial \bar{K}}{\partial x} = \frac{\partial \bar{K}}{\partial x}, \text{ etc.}$$

The conditions for the particle to be at rest are then:

$$\frac{\partial \bar{\mathfrak{d}}}{\partial t} = 0, \qquad \frac{\partial \bar{\mathfrak{h}}}{\partial t} = 0, \qquad \frac{\partial \bar{\rho}}{\partial t} = 0, \qquad \frac{\partial \bar{\mathfrak{v}}}{\partial t} = 0, \text{ etc.}$$

The following two relations ensue from the fundamental equations (1) through (4):

$$\mathfrak{e} \cdot \mathfrak{d} - \varphi \cdot \rho = -\operatorname{div}(\varphi \cdot \mathfrak{d}) - \mathfrak{d} \cdot \frac{\partial \mathfrak{f}}{\partial t},$$
$$\mathfrak{b} \cdot \mathfrak{h} - \mathfrak{f} \cdot \mathfrak{v} = -\operatorname{div}[\mathfrak{h} \cdot \mathfrak{f}] - \mathfrak{f} \cdot \frac{\partial \mathfrak{d}}{\partial t}.$$

For a particle at rest, one then has:

$$\mathfrak{e} \cdot \mathfrak{d} - \varphi \cdot \rho = -\operatorname{div}(\varphi \cdot \mathfrak{d}),$$
$$\mathfrak{b} \cdot \mathfrak{h} - \mathfrak{f} \cdot \mathfrak{v} = -\operatorname{div}[\mathfrak{h} \cdot \mathfrak{f}].$$

[1] H.A. Lorentz, Versuch eine Theorie der electrischen und optischen Erscheinung in bewegten Körpern, pp. 13.

If we now integrate over the entire volume that encloses the particle and observe that one may set $\varphi \cdot \mathfrak{d}$ and $[\mathfrak{h} \cdot \mathfrak{f}]$ equal to zero on the boundary surface of the volume then we obtain *for a material particle at rest:*

(64) $$\int \mathfrak{e} \cdot \mathfrak{d} \cdot dV = \int \varphi \cdot \rho \cdot dV,$$

(65) $$\int \mathfrak{b} \cdot \mathfrak{h} \cdot dV = \int \mathfrak{f} \cdot \mathfrak{v} \cdot dV,$$

From I, eq. (7) and (14) on pp. ? and pp. ?, we compute the energy density:

$$\mathcal{W} = H + \mathfrak{b} \cdot \mathfrak{h} - \mathfrak{f} \cdot \mathfrak{v} = F + \mathfrak{e} \cdot \mathfrak{d} - \varphi \cdot \rho.$$

One then obtains *the energy of a material particle at rest* from (64) and (65):

(66) $$E_0 = \int H \cdot dV = \int \Phi \cdot dV.$$

Let S be any surface without boundary [1]) that the particle intersects transversally, and let N be the normal to the surface at any point. Now, *assuming that the particle is at rest*, since lasting variation of the energy can occur on either side of this surface one must have:

(67) $$\int_N \overline{\mathfrak{s}}_N \cdot dS = 0,$$

in which $\overline{\mathfrak{s}}_N$ is the average value of the components of the vector $\mathfrak{s}$ that is normal to S. From (56), this vector is given by:

$$\mathfrak{s} = [\mathfrak{e} \cdot \mathfrak{h}] - \varphi \cdot \mathfrak{v} = [\mathfrak{d} \cdot \mathfrak{b}] - \rho \cdot \mathfrak{f}.$$

Laue [2]) has proved that, as long as equation (67) is valid – and this is therefore the case for any arbitrary material particle – the following theorem also exists:

Laue's Theorem. *The integral of a single component of the world matrix over the volume of a material particle at rest is null, except for the component with the index 4, 4, which provides the energy of the particle.*

Thus, in general, the average value of each component over a small time interval is to be taken, as in equation (67).

As M. Laue has pointed out, one can use this theorem as a means of computing the energy of a moving particle. I would like to carry out this computation for the theory that is being discussed at the moment. Let the field quantities for a particle at rest at a point x_0, y_0, z_0 all be characterized by the index 0. From the theory of relativity, they give the corresponding values at a point x, y, z of a moving particle that has a velocity q along the

[1] I.e., either closed or extending to infinity.
[2] M. Laue, Das Relativitätsprinzip, pp. 168, et seq.

z-axis when this point (*x*, *y*, *z*) has a position at the time *t* that is given by the following equations:

$$x = x_0, \qquad y = y_0, \qquad \frac{x - q \cdot t}{\sqrt{1-q^2}} = z_0,$$

by means of the following conversion formulae:

$$\mathfrak{d}_x = \frac{\mathfrak{d}_{x0} - q \cdot \mathfrak{h}_{y0}}{\sqrt{1-q^2}}, \qquad \mathfrak{d}_y = \frac{\mathfrak{d}_{y0} - q \cdot \mathfrak{h}_{x0}}{\sqrt{1-q^2}}, \qquad \mathfrak{d}_z = \mathfrak{d}_{z0},$$

$$\mathfrak{h}_x = \frac{\mathfrak{h}_{x0} - q \cdot \mathfrak{d}_{y0}}{\sqrt{1-q^2}}, \qquad \mathfrak{h}_y = \frac{\mathfrak{h}_{y0} - q \cdot \mathfrak{d}_{x0}}{\sqrt{1-q^2}}, \qquad \mathfrak{h}_z = \mathfrak{h}_{z0},$$

$$\rho = \frac{\rho_0 - q \cdot \mathfrak{v}_{y0}}{\sqrt{1-q^2}}, \qquad \mathfrak{v}_x = \mathfrak{v}_{x0}, \quad \mathfrak{v}_y = \mathfrak{v}_{y0}, \quad \mathfrak{v}_z = \frac{\mathfrak{d}_{y0} - q \cdot \mathfrak{h}_{x0}}{\sqrt{1-q^2}}.$$

Precisely the same relations that exist between ($\mathfrak{d}$, $\mathfrak{h}$) and ($\mathfrak{d}_0$, $\mathfrak{h}_0$) are also valid between ($\mathfrak{e}$, $\mathfrak{h}$) and ($\mathfrak{e}_0$, $\mathfrak{h}_0$), and, similarly, the same relations that exist between (ρ, $\mathfrak{v}$) and (ρ_0, $\mathfrak{v}_0$) is valid between (φ, $\mathfrak{f}$) and (φ_0, $\mathfrak{f}_0$).

The use of these formulae leads, by an elementary computation, to the following equation:

$$\begin{aligned}\mathfrak{d} \cdot \mathfrak{h} - \mathfrak{f} \cdot \mathfrak{v} \; &= \mathfrak{d}_0 \cdot \mathfrak{h}_0 - \mathfrak{f}_0 \cdot \mathfrak{v}_0 - \frac{q^2}{1-q^2}(\mathfrak{e}_0 \cdot \mathfrak{d}_0 - \varphi_0 \cdot \rho_0) \\ &- \frac{q^2}{1-q^2} \cdot (\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0} - \mathfrak{b}_{x0} \cdot \mathfrak{h}_{x0} - \mathfrak{b}_{y0} \cdot \mathfrak{h}_{y0} + \mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}) \\ &- \frac{q^2}{1-q^2} \cdot ([\mathfrak{e}_0 \cdot \mathfrak{h}_0] - \varphi_0 \cdot \mathfrak{v}_{z0} + [\mathfrak{d}_0 \cdot \mathfrak{b}_0] - \rho_0 \cdot \mathfrak{f}_{z0}).\end{aligned}$$

We now construct the timelike average value and integrate it over the volume that the material particle fills. If we then use equations (64), (65), (67) and further note that it follows from the definition of the point *x*, *y*, *z* that the following relationship exists:

$$dx \cdot dy \cdot dz = \sqrt{1-q^2} \cdot dx_0 \cdot dy_0 \cdot dz_0,$$

or:

$$dV = \sqrt{1-q^2} \cdot dV_0,$$

then one obtains:

(68) $$\int (\mathfrak{b} \cdot \mathfrak{h} - \overline{\mathfrak{f} \cdot \mathfrak{v}}) \cdot dV = \frac{q^2}{\sqrt{1-q^2}} \cdot \int (\mathfrak{b}_0 \cdot \mathfrak{h}_0 - \mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0} - \mathfrak{b}_{z0} \cdot \mathfrak{h}_{z0} - \overline{\mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}}) \cdot dV$$

If we let H_0 denote the quantity H at the point x_0, y_0, z_0 of the particle at rest then we can regard:

$$H_0 - F(x_0, y_0, z_0)$$

as a function of (x_0, y_0, z_0). Furthermore, let (x, y, z) be the point of the moving particle that one obtains at the time t by subjecting (x_0, y_0, z_0) to a Lorentz transformation. Since H is an invariant for the Lorentz transformation, its value at the point (x, y, z) of the moving particle at the time t must be computed as:

$$H = F\left(x, y, \frac{z - qt}{\sqrt{1-q^2}}\right)$$

where F means precisely the same function as before. From this, it follows that:

(69) $$\int H \cdot dV = \sqrt{1-q^2} \cdot \int H_0 \cdot dV_0 = \sqrt{1-q^2} \cdot E_0 .$$

Now, the energy E of the moving particle is given by adding (68) and (69):

$$E = \int (\bar{H} + \overline{\mathfrak{b} \cdot \mathfrak{h}} - \overline{\mathfrak{f} \cdot \mathfrak{v}}) \cdot dV ,$$

$$E = \sqrt{1-q^2} \cdot \int \bar{H}_0 \cdot dV_0 + \frac{q^2}{\sqrt{1-q^2}} \cdot \int (\overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0}} - \overline{\mathfrak{b}_{z0} \cdot \mathfrak{h}_{z0}} - \overline{\mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}}) \cdot dV$$

We can simplify this result further with the help of Laue's theorem. Namely, if we apply this theorem to the term of the world matrix (16) that has the index 3, 3, then we obtain:

$$\int (\bar{\Phi}_0 - \overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0}} - \overline{\mathfrak{b}_{z0} \cdot \mathfrak{h}_{z0}} - \overline{\mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}}) \cdot dV = 0.$$

Therefore, from (66):

$$\int \bar{\Phi}_0 \cdot dV = \int \bar{H}_0 \cdot dV = E_0,$$

$$\int (\overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0}} - \overline{\mathfrak{b}_{z0} \cdot \mathfrak{h}_{z0}} - \overline{\mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}}) \cdot dV = \int \bar{\Phi}_0 \cdot dV = E_0 .$$

From this, as M. Laue has already prove (Das Relativitätsprinzip, pp. 170), one then obtains:

(70) $$E = \frac{E_0}{\sqrt{1-q^2}} .$$

28. One can deduce another interesting consequence of Laue's theorem. If one applies it to the three terms in the diagonal of the world matrix with the indices (1, 1), (2, 2), and (3, 3) then one obtains:

$$\int (\overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{b}_{x0} \cdot \mathfrak{h}_{x0}} - \overline{\mathfrak{e}_{x0} \cdot \mathfrak{d}_{x0}} - \overline{\mathfrak{f}_{x0} \cdot \mathfrak{v}_{x0}}) \cdot dV_0 = E_0,$$
$$\int (\overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{b}_{y0} \cdot \mathfrak{h}_{y0}} - \overline{\mathfrak{e}_{y0} \cdot \mathfrak{d}_{y0}} - \overline{\mathfrak{f}_{y0} \cdot \mathfrak{v}_{y0}}) \cdot dV_0 = E_0,$$
$$\int (\overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{b}_{z0} \cdot \mathfrak{h}_{z0}} - \overline{\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0}} - \overline{\mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}}) \cdot dV_0 = E_0.$$

When one adds these terms, one obtains:

$$\int (2 \cdot \overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} - \overline{\mathfrak{e}_0 \cdot \mathfrak{d}_0} - \overline{\mathfrak{f}_0 \cdot \mathfrak{v}_0}) \cdot dV_0 = 3 \cdot E_0,$$

so, keeping (64) and (65) in mind:

(71)
$$\begin{cases} E_0 = -\frac{1}{3} \cdot \int (\overline{\mathfrak{e}_0 \cdot \mathfrak{d}_0} - \overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0}) \cdot dV_0, \\ \quad = -\frac{1}{3} \cdot \int (\overline{\rho_0 \cdot \varphi_0} - \overline{\mathfrak{f}_0 \cdot \mathfrak{v}_0}) \cdot dV_0. \end{cases}$$

In addition, one immediately sees from the previous three equations that:

(72)
$$\begin{cases} \int (\overline{\mathfrak{b}_{x0} \cdot \mathfrak{h}_{x0}} + \overline{\mathfrak{e}_{x0} \cdot \mathfrak{d}_{x0}} + \overline{\mathfrak{f}_{x0} \cdot \mathfrak{v}_{x0}}) \cdot dV_0 \\ = \int (\overline{\mathfrak{b}_{y0} \cdot \mathfrak{h}_{y0}} + \overline{\mathfrak{e}_{y0} \cdot \mathfrak{d}_{y0}} + \overline{\mathfrak{f}_{y0} \cdot \mathfrak{v}_{y0}}) \cdot dV_0 \\ = \int (\overline{\mathfrak{b}_{z0} \cdot \mathfrak{h}_{z0}} + \overline{\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0}} + \overline{\mathfrak{f}_{z0} \cdot \mathfrak{v}_{z0}}) \cdot dV_0 \\ = \frac{1}{3} \int (\overline{\mathfrak{b}_0 \cdot \mathfrak{h}_0} + \overline{\mathfrak{e}_0 \cdot \mathfrak{d}_0} + \overline{\mathfrak{f}_0 \cdot \mathfrak{v}_0}) \cdot dV_0. \end{cases}$$

These equations become particularly interesting when $\mathfrak{h} = 0$, $\mathfrak{v} = 0$, which is the case for the electron.

In the field of an electron, one has:

(73)
$$E_0 = -\frac{1}{3} \int \overline{\mathfrak{e}_0 \cdot \mathfrak{d}_0} \cdot dV_0 = -\frac{1}{3} \int \overline{\varphi_0 \cdot \rho_0} \cdot dV_0,$$

and:

(74)
$$\int \overline{\mathfrak{e}_{x0} \cdot \mathfrak{d}_{x0}} \cdot dV_0 = \int \overline{\mathfrak{e}_{y0} \cdot \mathfrak{d}_{y0}} \cdot dV_0 = \int \overline{\mathfrak{e}_{z0} \cdot \mathfrak{d}_{z0}} \cdot dV_0 = \frac{1}{3} \int \overline{\mathfrak{e}_0 \cdot \mathfrak{d}_0} \cdot dV_0.$$

29. For the special case that we discussed in II, pp. ?, the relation (73) is easily proved to be true. If we introduce the static field quantities $\eta = \mathfrak{e}_0$, $\chi = \varphi_0$ into the world function:

$$\Phi = -\frac{1}{2} \eta^2 + \frac{1}{6} a \cdot \chi^6$$

then we obtain:

$$E_0 = \int \Phi_0 \cdot dV = -\frac{1}{2} \int \mathfrak{e}_0^2 \cdot dV + \frac{1}{6} a \cdot \int \varphi_0^6 \cdot dV.$$

However, we have:

$$\mathfrak{d}_0 = -\frac{\partial \Phi}{\partial \mathfrak{e}_0} = \mathfrak{e}_0, \qquad \rho_0 = \frac{\partial \Phi}{\partial \varphi_0} = a \cdot \varphi_0,$$

so we can therefore write:

$$E_0 = -\tfrac{1}{2}\int \mathfrak{e}_0 \cdot \mathfrak{d}_0 \cdot dV + \tfrac{1}{6} a \cdot \int \varphi_0 \cdot \rho_0 \cdot dV \ .$$

If one applies (64) to this then the result is (73).

If we had given the world function in the general form:

$$\Phi = -\tfrac{1}{2}\eta^2 + \tfrac{1}{\nu} a \cdot \chi^{\nu}$$

then a completely analogous computation would give:

$$E_0 = -\tfrac{1}{2}\int \mathfrak{e}_0 \cdot \mathfrak{d}_0 \cdot dV + \tfrac{1}{\nu} a \cdot \int \varphi_0 \cdot \rho_0 \cdot dV \ ,$$

but the relation (73) would be impossible to satisfy unless $\nu = 6$. From this, it follows that:

Among all of the world functions of the form:

$$\Phi = -\tfrac{1}{2}\eta^2 + \tfrac{1}{\nu} a \cdot \chi^{\nu}$$

only the case $\nu = 0$ *leads to isolated knot singularities of electric charge.*

If one takes any value for ν then all of the integrals of equation (34) in II, pp. ? must have essential singularities, either a singularity at the null point or one at infinity, or both. There is therefore no single integral that could represent an electron.

One sees from this that can occasionally use equation (73) as a criterion for determining whether a given form of the world function is consistent with the existence of isolated knot singularities or not.

30. From formula (73), it follows that in the example that we discussed in II, the energy of a knot singularity is *negative.* It therefore follows in this case that the surface tension of the charge contributes negative energy to the positive energy of the electric field. Since $\mathfrak{d}_0$ and ρ_0 are completely distinct from each other in the Hamiltonian function:

$$H(\mathfrak{d}_0, 0, \rho_0, 0) = \Phi_0 + \mathfrak{e}_0 \cdot \mathfrak{d}_0 - \varphi_0 \cdot \rho_0 = \mathcal{W}$$
$$= \tfrac{1}{2}\mathfrak{d}_0^{\,2} - \tfrac{5}{6}\sqrt[5]{\frac{\rho_0^{\,6}}{a}} \ ,$$

one can also compute both of the total energies separately. One obtains:

$$\tfrac{1}{2}\cdot\int \mathfrak{d}_0^{\,2} \cdot dV_0 = \tfrac{1}{2}\cdot\int \mathfrak{e}_0 \cdot \mathfrak{d}_0 \cdot dV_0 \ ,$$

for the energy of the electric field, and:

$$-\tfrac{5}{6}\cdot\int \sqrt[5]{\frac{\rho_0^{\,6}}{a}}\cdot dV_0 \;=\; -\tfrac{5}{6}\cdot\int \rho_0\cdot\varphi_0\cdot dV_0 = \;-\tfrac{5}{6}\cdot\int \mathfrak{e}_0\cdot\mathfrak{d}_0\cdot dV_0\;.$$

for the energy of the surface tension.

However, if the energy of the particle is negative then the same must be true for the its inertial mass. The knot singularities, of which we spoke in II on pp. ?, thus have a negative inertial mass, and they must therefore acquire an acceleration in a force field that is in the opposite direction to the force. On first glance, this appeared to be such an absurd behavior, which we were led to in II, pp. ?, by general considerations, that knot singularities with the same sign tended to amalgamate together, whereas knot singularities with opposite signs tended to repel away from each other, whether or not the ponderomotive force of the electric field acted in opposite direction.

One can deduce a very important consequence of (73):

The necessary and sufficient condition for the inertial mass of an electron to be positive is that:

$$\int \mathfrak{e}_0\cdot\mathfrak{d}_0\cdot dV < 0$$

or:

$$\int \varphi_0\cdot\rho_0\cdot dV < 0.$$

At a great distance from the electron one has $\mathfrak{e} = \mathfrak{d}$, hence $\mathfrak{e}_0\cdot\mathfrak{d}_0$ is certainly positive. From this, it follows that:

Both of the vectors $\mathfrak{e}$ *and* $\mathfrak{h}$ *must have opposite signs in the interior of an electron.*

One sees from this that it is completely impossible for the Maxwell equations to still be valid in the interior of the electron.

Likewise, φ, since it is precisely the electric potential, also has the same sign as ρ outside of the sphere of the electron. Indeed, φ attains its maximum at the point where $\mathfrak{e}$ goes through zero, in order to take on the opposite sign in the interior of the electron. Furthermore, in the interior, φ must become so strong that it ultimately changes its sign, as well, and $\varphi_0\cdot\rho_0$ becomes quite negative, since the space integral of must be negative.

In the interior of the electron φ *must have the opposite sign to* ρ.

Chapter Four.

The problem of the quantum action.

Elementary dipole.

31. When both of the vectors $\mathfrak{d}$ and $\mathfrak{e}$ always have the same sign then there can be only one sort of elementary particle of matter that is characterized by the fact that div $\mathfrak{d}$ takes on considerable values in its interior. However, if we can, as we may, from the last statement, assume, place very strong fields $\mathfrak{d}$ and $\mathfrak{e}$ in opposition to each other then there is a second type of elementary particle that one can imagine, in which $\mathfrak{rot}\ \mathfrak{d}$ takes on very large values, but div $\mathfrak{d}$ gets very small. In these elementary particles, since div $\mathfrak{d}$ is very small the vector $\mathfrak{d}$ will therefore quickly converge to closed lines, similar to the lines of magnetic induction around a small permanent magnet. We shall call such a particle an elementary electric dipole. The behavior of the vector $\mathfrak{e}$ in equilibrium is characterized by the fact that $\mathfrak{rot}\ \mathfrak{e} = 0$, whereas div $\mathfrak{e}$! 0 is possible. The lines of the vector $\mathfrak{e}$ thus surround the dipole like the lines of magnetic force of a permanent magnet; thus, in the interior of dipole they are directed oppositely to the vector $\mathfrak{d}$. The possibility of elementary dipoles is therefore linked with the condition that the vector $\mathfrak{e}$ changes its sign for very large values of $\mathfrak{d}$.

The precise conditions for the possibility of such particles existing is extremely difficult and might also use very little of the intuition that we have gained on the nature of the world function. I will assume once and for all that there is an elementary dipole, and then deduce whatever consequences this assumption suggests.

32. From the theorem that we proved in section **26**, we can easily compute the ponderomotive force that a dipole experiences in an electric field. It is precisely the same force that an electrical double point with the same electrical moment as our dipole would experience, as it is presented in conventional electrostatics; the electrical moment may then be computed from the external field. From this, we see that an electrical field mostly exerts a torque on the dipole that makes it seeks to adjust the direction of the field, which is very small compared to the force that gives the dipole a translatory motion. Thus, external forces will powerfully drive the dipole outward from the atomic bond – like electrons – and there is very little hope of finding corpuscular rays that originate in the dipole.

By contrast, one may presume that the dipole, unlike the electron, can experience very small internal force effects, similar to ones that the electrical double points suffer in the usual conception of electrical fields when both of the charges attract each other. More precisely, one can say nothing about these internal effects, as we saw on pp ?, but presumably also look for the electrical field that will unlock the structure of our dipole. I shall now show that too large of an unlocking field will be ruinous for the dipole.

In the vacuum, where Maxwell's equations are valid, there are dipoles without electrical charge, though certainly not in the equilibrium state; such dipoles are the

spherical electrical waves. According to H. Hertz (Ges. Werke **2**, pp. 150), one can compute the field of a spherical wave with the help of a function $\Pi = f(r - t)$, where r denotes the radius vector from the center out. Let z, ρ be cylindrical coordinates, where z is the symmetry axis of the spherical wave and ρ is the direction that is perpendicular to it. One then has:

$$\mathfrak{d}_\rho = -\frac{\partial^2 \Pi}{\partial \rho \, \partial z}, \qquad \mathfrak{d}_z = \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial \Pi}{\partial \rho} \right), \quad \mathfrak{h} = \frac{\partial^2 \Pi}{\partial \rho \, \partial t},$$

where the magnetic field $\mathfrak{h}$ is parallel to the z-axis. One must observe that in these equations $r = \sqrt{\rho^2 + z^2}$. If we substitute the value that was given above for Π and denote the angle between r and z by ϑ, so that: $\cos \vartheta = z / r$, $\sin \vartheta = \rho / r$, then one finds the following formula:

$$(75)\quad \begin{cases} \mathfrak{d}_\rho = -\left(\dfrac{f''(r-t)}{r} - \dfrac{3 \cdot f'(r-t)}{r^2} + \dfrac{3 \cdot f(r-t)}{r^3} \right) \cdot \cos\vartheta \cdot \sin\vartheta, \\ \mathfrak{d}_z = +\left(\dfrac{f''(r-t)}{r} - \dfrac{3 \cdot f'(r-t)}{r^2} + \dfrac{3 \cdot f(r-t)}{r^3} \right) \cdot \sin^2\vartheta + \left(\dfrac{2 \cdot f'(r-t)}{r^2} - \dfrac{2 \cdot f(r-t)}{r^3} \right) \\ \mathfrak{h} = -\left(\dfrac{f''(r-t)}{r} - \dfrac{f'(r-t)}{r^2} \right) \cdot \sin\vartheta. \end{cases}$$

We shall now choose a function for $f(r - t)$ whose first two differential quotients take on their values in the interval between a and b, and whose first two differential quotients will be equal to zero at the boundary points a and b; let the values that f takes on at a and b be A and 0:

$$f(a) = A, \qquad f'(a) = 0, \qquad f''(a) = 0,$$
$$f(a) = A, \qquad f'(a) = 0, \qquad f''(a) = 0.$$

For all $r - t > a$, we set f equal to the constant A, and for all $r - t > b$, we set it equal to the constant zero.

With this, the field quantities in the region at a given moment are given by (75):

$$(76)\quad \begin{cases} \infty > r \geq a + t, \\ \mathfrak{d}_\rho = -\dfrac{3A}{r^3} \cdot \cos\vartheta \cdot \sin\vartheta, \\ \mathfrak{d}_z = +\dfrac{3A}{r^3} \cdot \sin^2\vartheta - \dfrac{2A}{r^3}, \\ \mathfrak{h} = 0. \end{cases}$$

This is the field of an electric dipole of moment $4\pi \cdot A$.

In the region where $a + t \geq r \geq b + t$, one finds spherical waves with electric field lines that flow back into the interior of a ball of radius $b + t$; hence, $\mathfrak{d} = \mathfrak{h} = 0$ for $r \leq b + t$.

From this, we see that a spherical momentum wave actually represents an uncharged electric dipole.

An example of a function *f*, for which one can easily show uniqueness, is:

$$(77)\qquad \begin{cases} 1. \quad \infty > r-t \geq \dfrac{\pi}{m}, & f(r-t) = 2\pi \cdot a = A, \\ 2. \quad \dfrac{\pi}{m} \geq r-t \geq -\dfrac{\pi}{m} & f(r-t) = a\sin(\sin m \cdot (r-t) + m(r-t) + ?) \\ 3. \quad -\dfrac{\pi}{m} \geq r-t \geq -\infty & f(r-t) = 0. \end{cases}$$

The dipole that we just described is not in equilibrium; rather, its hollow interior seeks to expand with light velocity and dissipate into nothing. Since a continuous transition to the spherical wave dipole must be possible for the elementary dipole that we chose – when it exists, – it then follows that there is an attainable expansion of the elementary dipole, perhaps by strong internal pulsations, for the case of unstable equilibrium, such that the dipole, when it goes over into that state, is forced to take the form of a spherical wave dipole that dissipates completely, whereas, for a weaker expansion it seeks to return to the equilibrium point. Thus, when there is an elementary dipole, it must be explosive; its explosion transforms matter into a light pulse. Since all transitions are reversible in relativity theory, it must naturally be possible for a light pulse to create a new elementary dipole. In order to mathematically represent a spherical wave that moves towards the center, which might possibly condense into an elementary dipole, one must replace $(r - t)$ with $(r + t)$ in formula (75), and likewise switch the sign of $\mathfrak{h}$.

33. From these considerations, it follows that one should not expect a new corpuscular wave, but a light pulse, to follow from the existence of a possible elementary dipole. It might be possible, e.g., for the light in the band spectrum to be emitted by exploding dipoles. The absence of a true Zeeman effect for this light draws attention to the fact that it is not excited by electronic oscillations. When one assumes that each radiating atom does not produce merely a single pulse, but a multitude, perhaps thousands of pulses in regular intervals, the fine channelization of the bands must certainly be understood as originating in the interference due to the fine structure in the bands. The emission of the pulse must be connected with the proper oscillations of the atom, and the complicated laws that these proper oscillations must be discovered if we are to clarify the notable laws of the bands. One must naturally fail when one attempts to test this theory on the most complicated aspects of band spectra. Perhaps one can, however, enlighten the simpler case of the resonance spectra as a logical consequence of pulses, once one proves their existence or nonexistence.

34. In conclusion, we must remark that it is just as possible for magnetic dipoles to exist as it is for electric dipoles. They are the elementary particles for which ρ and $\mathfrak{v}$ are taken to be very small, and for which $\mathfrak{rot}\ \mathfrak{h}$ is then as good as null, although div $\mathfrak{h} \neq 0$. As always, the vector $\mathfrak{b}$ must, by contrast, satisfy the condition div $\mathfrak{b} = 0$ precisely. Thus,

there will be closed curves in the field lines of $\mathfrak{b}$ for an elementary magnetic dipole that appear to be similar to the lines of magnetic induction for a permanent magnet. On the contrary, the lines of vector $\mathfrak{h}$ will be similar to the lines of magnetic force for a permanent magnet and inside the dipole they have the opposite direction to the lines of the vector $\mathfrak{b}$.

Planck's quantum of action.

35. As long as one thinks of the heat exchanging relationship between ether and matter in terms of resonators, one will never arrive at an interpretation of Planck's laws of radiation that is free of internal contradictions. When one thinks of resonators, as one will, one must always think of the laws of mechanical oscillators, so energy must be emitted and absorbed according to Maxwell's laws, which is in complete contradiction to the existence of quantum effects. It is self-explanatory that the existence of electrical resonators in the atom must be beyond question. The validity of the Zeeman phenomenon, particularly the simple kind that the helium lines exhibit, plainly indicates that the sine waves of the line series will be emitted by electrons in the atom that exhibit a regular pendulum oscillation. Moreover, the Zeeman phenomenon certainly shows that one can completely attribute the motion of the oscillating electrons to the laws of mechanics by means of forces that obey the usual rules of the theory of electricity. However, from this it is absolutely necessary that, from the laws of mechanics and the theory of electricity, the oscillating electrons can continually absorb energy, and from Maxwell's laws, it can also emit it back again. If one would rather avoid contradicting the accepted meaning of the Zeeman phenomenon then one must also accept that the aforementioned quantum theory is not applicable to the electronic oscillations. Therefore, there is nothing else to say about this, since it certainly leads to a large number of consequences in which quantum effects play no role whatsoever. One thinks, to take an example, of a cathode ray particle that flies between the poles of an electromagnet with a large velocity. Whenever the electron is in the space between them the magnetic field must be excited for a particular instant so that the path of the particle must assume the form of a closed circle. Now, the particle emits a wave that gets briefer as the magnetic field gets stronger. Naturally, this emitting particle suffers no quantum effects, and, as a result, one can give no upper bound on its oscillations. At the very least, one can think of long infrared waves being produced in this fashion quite well.

These considerations in no way contradict the appearance of quantum effects in the exchange of heat between matter and ether. It only leads to the idea that we must ascribe a truly essential role to the electronic oscillations in this heat exchange. Of course, resonators, when they are not coupled with any other mechanisms, certainly do not mediate the exchange of heat between ether and matter; when they do not absorb radiation they only scatter it. One imagines a hollow space that is surrounded by a reflecting wall, and in which, very many electrical resonators are distributed. If one allows any radiation to enter this hollow space then the resonators do not alter the spectral composition of the radiation at all, nor does it alter the blackbody radiation, …(?) It is therefore quite obvious that one can do without the resonators completely if one wishes to understand the exchange of heat between matter and ether, since their

contribution to the behavior is of a secondary nature and theoretically it represents only an unnecessary complication.

In section **32** we have just learned of the possibility of radiation that is not produced by resonators. During the previously described explosion of a dipole, the energy of matter will be immediately transformed into the energy of radiation. An elementary dipole, just like an electron, must then be regarded as an elementary building block of the atom. It takes part in the internal motion of the atom and might sometimes take on such an unusually large energy that it crosses its stability envelope and explodes. With that, it and all of its internal energy is transformed into an impulse wave.

The explosion of a dipole has an exact analog in the emission of an electron from an atom. This emission then takes place when an electron is in a state of motion that is so violent that it passes beyond a certain stability limit in the atom. Just as the kinetic energy of a free electron, which can assume various values, whose violent motion leads to its emission from the atomic bonds, so will the pulse wave that is produced by the exploding dipole have incommensurably many energies, depending upon the violence with which the stability limit is exceeded. The more violent the transition the higher the energy of the pulse wave; similarly, when the duration of the transition becomes quite small, so does the width of the impulse. If we assume that there is only one kind of dipole then there will be one definite degree of violence for the transition, hence, one definite impulse width, and always the same quantum of energy, which increases with decreasing impulse width.

The radiation that an exploding dipole produces must be quantized, assuming that there is only one sort of elementary dipole, and furthermore, the energy quantum must be a radiation pulse that gets bigger as the impulse width gets smaller.

Since electron emission and dipole explosion are very closely related transitions, one must expect that both of them go hand-in-hand. In fact, one can observe this in the case of fluorescent particles. On the other hand, since the two transitions are not causally linked to each other, one must expect that generally valid qualitative laws do not govern their common behavior.

Even more difficult than the problem of the emission of radiation in the theory that we are presently treating is the question of absorption. Absorption must naturally be synonymous with the creation of material dipoles by the concentration of radiant energy. Obviously, one must assume that every exploding dipole in an atom leaves behind a germ on which subsequent radiation can easily condense. One would then do well to first approach the question of absorption in earnest, when experimental exploration teaches us something of the possibility of the theory of elementary dipoles, and, in the affirmative case, to give more substance to a precise elaboration of the theory.

36. The question now arises of what the intuitive meaning of the quantum effect in our theory might be. As for that question, one must remark that this meaning will further depend on the details of the theory that we have, for the moment, left completely undetermined. Thus, in the sequel, when we seek a meaning for the quantity h, after making an additional assumption that we add to the theory, then we must not forget that

another specialization of the theory can lead to other meanings. We thus make the following:

Additional assumption: *The impulse wave of the exploding dipole has the same form for all impulse widths.*

The function that was introduced in (75) for the computation of the wave shall then have the form:

$$f(r-t) = \varphi\, m(r-t),$$

in which φ is a universal function and m is a factor that can take on all possible values. The factor m is inversely proportional to the impulse width.

With this assumption, I can prove the following theorem:

The number of closed electrical field lines that are in the hollow spherical vortex ring that is created by the explosion of a dipole is a universal constant, and the quantum of action h is nothing but the square of this constant, multiplied by a numerical factor that depends only upon the choice of units.

On the boundary surface of a very large ball of radius r, the field quantities are computed from (75), after ignoring the higher negative powers of r, as follows:

$$\mathfrak{d} = m^2 \cdot \frac{\varphi'' m(r-t)}{r} \cdot \sin\vartheta,$$

$$\mathfrak{h} = m^2 \cdot \frac{\varphi'' m(r-t)}{r} \cdot \sin\vartheta.$$

$\mathfrak{d}$ points along the longitude circles of the sphere and $\mathfrak{h}$ points along the latitude circles. The following total energy passes through a spherical area of width $d\vartheta$, whose spherical area is $2\pi r^2 \cdot \sin\vartheta \cdot d\vartheta$, and during the time interval t:

$$dE = \mathfrak{d} \cdot \mathfrak{h} \cdot 2\pi r^2 \cdot \sin\vartheta \cdot d\vartheta \cdot dt = 2\pi \cdot m^4 \cdot \varphi'' \cdot \sin\vartheta \cdot d\vartheta \cdot dt.$$

When we let a and b denote the limiting values of $r - t$, between which the spherical wave is trapped, as we did above on pp. ?, then the values $a' = m \cdot a$ and $b' = m \cdot b$ are independent of our additional assumption on m. We further set $m(r-t) = \omega$, which then makes the energy of the impulse wave:

$$E = 2\pi \cdot m^3 \cdot \int_{b'}^{a'} \varphi''(\omega)^2 \cdot d\omega \cdot \int_0^{\pi} \sin^3\vartheta \cdot d\vartheta.$$

If we set:

$$\int_{b'}^{a'} \varphi''(\omega)^2 \cdot d\omega = \alpha \tag{78}$$

then a is a quantity that is independent of m, and:

(79) $$E = \frac{8\pi}{3} \cdot \alpha \cdot m^3.$$

We now imagine that the impulse has been decomposed into a continuous sequence of sine waves, as in Fourier's theorem. In precisely the same way, a spectroscope would resolve it into a continuous spectrum. We may assume that the form of the function φ is such that the spectrum of the impulse is a rather narrow band whose mean wavelength may be called λ. The corresponding mean frequency ν is then computed from:

$$\nu = \frac{c}{\lambda},$$

and Planck's radiation theorem is identical with the theorem that:

$$E = h \cdot \nu.$$

If we now observe that the quotient:

(80) $$\frac{m}{\nu} = \beta$$

is a number that does not depend on the impulse width then, from (79), we obtain:

(81) $$h = \frac{8\pi}{3} \cdot \alpha \cdot \beta \cdot m^3.$$

$8\pi \cdot \alpha \cdot \beta / 3$ is a numerical factor that does not depend on the impulse width.

We now like to compute the number of field lines that comprise the vortex ring of the impulse. To that end, we consider the equatorial circle ($\vartheta = \pi/2$). Through it, and during the time interval *t*, a number *dn* of electrical field lines crosses it transversally, which is easily computed from:

$$dn = 2\pi r \cdot \mathfrak{d} \cdot dt = 2\pi \cdot m^2 \cdot \varphi'' \cdot dt.$$

If we now set the argument ω in the function φ'' to b', where it is null and increasing then we must ultimately come to a value c' of the argument where the sign of φ'' changes, in order to be null again at the other limit a'. c', like a' and b', is independent of the impulse width, and likewise, the integral:

(82) $$\int_{b'}^{c'} \varphi''(\omega) \cdot d\omega = -\int_{c'}^{a'} \varphi''(\omega) \cdot d\omega = \gamma.$$

The total number of lines that comprise the electrical field vortex is:

(83) $$n = 2\pi\gamma \cdot m.$$

By combining (81) and (83) one ultimately obtains:

(84) $$h = C \cdot n^2, \qquad C = \frac{2}{3\pi} \cdot \frac{\alpha \cdot \beta}{\gamma^2}.$$

C is a numerical factor that is independent of the impulse width. The theorem is thus proved.

With this assignments the quantum action – or, more precisely, the square of the quantum action – is a quantity that is completely analogous to the quantum of electrical charge. Just as the latter gives the number of field lines that terminate in the electron in the form of a knot singularity, $\sqrt{h}$ represents the number of field lines that comprise the dipole in the form of a cluster, at least after its explosion. It is perhaps of interest to compare these two numbers. Indeed, an unknown factor C enters into (84) that will depend on the form of the function φ. In order to have, at the very least, some idea of the order of magnitude of the number n, we would like to introduce a completely determined function that was already mentioned in the example (77) on pp. ?, hence:

$$\varphi'' = -a \cdot \sin m(r - t).$$

One then computes the constant C from the equation:

$$C = \frac{\pi}{3 \cdot C},$$

where c shall mean the velocity of light. However, the number n can be computed this way only in the system of electrostatic units in which the known factor 4π has been removed ($\mathfrak{d} = \mathfrak{e}$, div $\mathfrak{d} = \rho$). If we compute the number of field lines in the practical unit of Coulomb, and we then denote it by N, then if we set:

$$N = \frac{10}{c \cdot \sqrt{4\pi}} \cdot n,$$

equation (84) then looks like:

$$h = \frac{4\pi^2 \cdot c}{800} \cdot N^2.$$

If we introduce the value of h: $h = 65.5 \cdot 10^{-28}$, then we obtain:

$$N = 129 \cdot 10^{-20} \text{ Coulomb.}$$

The elementary quantum of electrical charge is:

$$e = 15.6 \cdot 10^{-20} \text{ Coulomb.}$$

From the special assumption that $\varphi'' = a \cdot \sin m(r - t)$ one then obtains the number of electrical field lines that comprise the vortex cluster of an elementary dipole as almost exactly eight times the elementary electric quantum. In any case, both of the quantities, N and e, lie in roughly the same range of magnitude, and N is larger than e.

Chapter Five.

Gravitation.

The extended fundamental equations of ether dynamics.

37. We saw on pp. ? that the surface tension we chose for an electric charge, along with the electromagnetic field, still did not suffice to clarify all of the forces that act in the material world. We are still lacking gravitation, and we are now compelled to extend the system of fundamental quantities, in which we have considered (I, pp ?) only the fewest number of quantities possible, namely, only the six-vector ($\mathfrak{h}$, $-i\cdot\mathfrak{d}$) and the four-vector ($\mathfrak{v}$, $i\rho$).

What we must do next is to consider gravitation as an energy that lives in the surface tension. However, if we wish to agree upon the validity of the principle of relativity then we cannot introduce the energy for its own sake into the extended equations, where the energy density is the last term in the world matrix in relativity theory; it must therefore already enter into the entire matrix as such in the equations. If one seeks to link the matrix with any other four-dimensional quantities through four-dimensional operators, in order to obtain equations in this way that satisfy the causality principle (I., pp. ?), as well as the energy principle, then one comes up against insuperable difficulties. For a long time now, I have sought for such a link that would lead back to the existing system of equations, with great pains, and I am convinced that it is completely impossible to arrive at a theory of gravitation in this way that obeys both the principle of relativity and the energy principle.

On the other hand, one reaches this goal quite simply and easily when one attributes the work due to surface tension, not to the term W, but to the quantity H that was defined in I on pp. ?? by way of equation (7) as $H = W - \mathfrak{b}\cdot\mathfrak{h} + \mathfrak{v}\cdot\mathfrak{f}$. As long as the velocity of the elementary particle of matter is small compared to the velocity of light, one cannot experimentally distinguish whether W or H equals the action due to gravity. From equations (69) and (70), we have, for a moving mass particle:

$$\int H\cdot dV = \sqrt{1-q^2}\cdot E_0,$$

$$\int W\cdot dV = \frac{1}{\sqrt{1-q^2}}\cdot E_0,$$

in which the integral is to be taken over volume that the particle fills up, E_0 is the energy of a particle at rest, and q means is its velocity over the velocity of light. One thus sees that both of the integral are not noticeably distinct in practical terms.

Now, the quantity H is, however, a four-dimensional scalar, and this means that the differential operator can be applied in only one fashion; it gives a four-vector: the gradient of the scalar. Conversely, one can also link the four-vector with a scalar by applying the "divergence" operator. On the other hand, a six-vector cannot lead to a scalar by means of a four-dimensional operator of first order. From this, it follows that:

The gravitational field must necessarily be represented by a four-vector, not a six-vector.

This theorem rests completely on the assumption that the gravitational mass is to be computed by means of a four-dimensional scalar, namely, the quantity H. Otherwise, if the gravitational mass density were the fourth component of a four-vector, it would be like the electric charge density. The gravitational field would be given by a six-vector, like the electromagnetic field. As far as I see, it is impossible to find a four-vector whose fourth component is close to the energy density, and the theory of gravitation, in which, according to O. Heaviside [1]), H. A. Lorentz [2]), R. Gans [3]), the gravitational field behaves just like the electromagnetic field, can therefore either fail to be consistent with the principle of relativity or the gravitational mass can be unequal to the inertial mass.

In order to present the equations of the gravitational field, we now proceed just as we did in the presentation of the electromagnetic field equations in I, sections **2** to **5**. We assume that in order to give a complete description of the material world, along with the six-vector ($\mathfrak{h}$, $-i\cdot\mathfrak{d}$) and the four-vector ($\mathfrak{v}$, $i\rho$), it necessary to also consider a second four-vector ($\mathfrak{g}$, $i\cdot u$) and a scalar ω. This system of quantities runs parallel to a second one that is completely determined when all of the quantities of the first system are given. We already know that in the second system the six-vector ($\mathfrak{b}$, $-i\mathfrak{e}$), the four-vector ($\mathfrak{v}$, $i\rho$), which now depends not only on ($\mathfrak{h}$, $-i\mathfrak{d}$) and ($\mathfrak{v}$, $i\rho$), but also on ($\mathfrak{g}$, iu) and ω. In addition, we must introduce four-vector ($\mathfrak{t}$, iw) and a scalar H, which correspond to ($\mathfrak{h}$, $-i\mathfrak{d}$) and ($\mathfrak{v}$, $i\rho$). The scalar H shall be essentially identical to the quantities that were defined in I., pp ?. Just like the energy density W it therefore depends not only on ($\mathfrak{h}$, $-i\mathfrak{d}$) and ($\mathfrak{v}$, $i\rho$), but also on the quantities of the gravitational field, hence ($\mathfrak{g}$, iu) and ω, and the relation (7) must consequently experience a slight modification. We now apply one of the two possible four-dimensional vector operations to ($\mathfrak{g}$, iu) and ω, and the other one to ($\mathfrak{t}$, iw) and H. In this way, we obtain the one possible form for the laws of gravitation that is in harmony with the principle of relativity:

$$\text{(85)} \qquad \begin{cases} \mathfrak{g}_x = & \dfrac{\partial \omega}{\partial x}, \\ \mathfrak{g}_y = & \dfrac{\partial \omega}{\partial y}, \\ \mathfrak{g}_z = & \dfrac{\partial \omega}{\partial z}, \\ u = & -\dfrac{\partial \omega}{\partial t}, \end{cases}$$

[1] O. Heaviside, Electromagnetic Theory **1**., pp. 455, 1894.
[2] H. A. Lorentz, Vers. Kon. Ak. Wet. Amsterdam **8**., pp. 603, 1900.
[3] R. Gans, Physik, Zeitschr. **6**, pp. 803, 1905.

(86) $$\frac{\partial \mathfrak{t}_x}{\partial x} + \frac{\partial \mathfrak{t}_y}{\partial y} + \frac{\partial \mathfrak{t}_z}{\partial z} + \frac{\partial w}{\partial t} = -\gamma \cdot H.$$

Here, γ shall mean a universal constant. Equations (85) are equivalent to the following ones:

(87) $$\begin{cases} \frac{\partial \mathfrak{g}_x}{\partial t} + \frac{\partial u}{\partial x} = 0, \\ \frac{\partial \mathfrak{g}_y}{\partial t} + \frac{\partial u}{\partial y} = 0, \\ \frac{\partial \mathfrak{g}_z}{\partial t} + \frac{\partial u}{\partial z} = 0, \end{cases}$$

(88) $$\frac{\partial \omega}{\partial t} = -u.$$

Equations (86), (87), (88) collectively define a system of five independent equations, each of which includes a first differential quotient of one of the five state variables with respect to time. The causality principle is therefore satisfied.

The complete system of fundamental equations for ether dynamics including gravitation is comprised of equations (1), (2), (3), (4), (86), (87), (88).

With the nomenclature of four-dimensional vector analysis, one can also write equations (85) to (88):

$$(\mathfrak{g}, iu) = \Gamma \rho \alpha \delta \omega,$$
$$\mathrm{div}(\mathfrak{t}, iw) = -\gamma \cdot \mathrm{H},$$
$$\mathfrak{rot}(\mathfrak{g}, iu) = 0.$$

The system of equations (85) and (86) formally agrees with the ones that M. Abraham [1]) based his theory of gravitation upon when one sets the vectors ($\mathfrak{g}$, *iu*) and ($\mathfrak{t}$, *iw*) equal to each other. In his theory M. Abraham started with the assumption that the density of the gravitating mass (which he called ν) is a four-dimensional scalar, and since he employed the methods of relativity theory in the cited work then he necessarily arrived at this system of equations, which is the only one that relativity theory can deliver.

38. The first question that must be addressed is whether the principle of energy is also satisfied when we add equations (86), (87), (88). We will thus multiply equations (87) by the components of a three-dimensional vector – say $\mathfrak{a}$, – multiply equation (86) by a scalar *s*, and then we add the equations. The terms that contain differential quotients of the coordinates are then:

[1] M. Abraham, Physik. Zeitschr. **13**, pp. 1, 1912.

$$\mathfrak{a}_x \cdot \frac{\partial u}{\partial x} + \mathfrak{a}_y \cdot \frac{\partial u}{\partial y} + \mathfrak{a}_z \cdot \frac{\partial u}{\partial z} + s \cdot \left(\frac{\partial \mathfrak{t}_x}{\partial x} + \frac{\partial \mathfrak{t}_y}{\partial y} + \frac{\partial \mathfrak{t}_z}{\partial z} \right).$$

Since this expression represents a divergence, one must have: $\mathfrak{a} = \mathfrak{f}$, $s = u$. With this, we have found the last part of the energy equation:

$$\operatorname{div}(u \cdot \mathfrak{t}) + \mathfrak{t} \cdot \frac{\partial \mathfrak{g}}{\partial t} + u \cdot \frac{\partial w}{\partial t} - \gamma \cdot H \cdot \frac{\partial \omega}{\partial t} = 0,$$

in which, from equation (88), the last summand for u is replaced with the value $\partial w / \partial t$.

When we add the gravitational action we then obtain the *total energy current:*

(89) $$\mathfrak{s} = [\mathfrak{e} \cdot \mathfrak{h}] - \varphi \cdot \mathfrak{v} + u \cdot \mathfrak{t}$$

instead of I, equation (5) on pp. ?, and furthermore, the *total variation of the energy density:*

(90) $$dW = \mathfrak{e} \cdot d\mathfrak{d} + \mathfrak{h} \cdot d\mathfrak{b} - \varphi \cdot d\rho - \mathfrak{v} \cdot d\mathfrak{f} + \mathfrak{t} \cdot d\mathfrak{g} + u \cdot dw - \gamma \cdot H \cdot d\omega$$

The function H must now be defined by the following equation:

(91) $$W = H + \mathfrak{h} \cdot \mathfrak{d} - \mathfrak{v} \cdot \mathfrak{f} + u \cdot w$$

instead of equation (7) in I, pp. ? From (90), it then follows that:

(92) $$dH = \mathfrak{e} \cdot d\mathfrak{d} + \mathfrak{b} \cdot d\mathfrak{h} - \varphi \cdot d\rho - \mathfrak{f} \cdot d\mathfrak{v} + \mathfrak{t} \cdot d\mathfrak{g} - w \cdot du - \gamma \cdot H \cdot d\omega$$

Since H is a function of the following variables: ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$, $\mathfrak{g}$, u, ω), we then have:

(93) $$\begin{cases} \mathfrak{e} = \dfrac{\partial H}{\partial \mathfrak{d}}, \quad \mathfrak{b} = \dfrac{\partial H}{\partial \mathfrak{h}}, \quad \varphi = -\dfrac{\partial H}{\partial \rho} \quad \mathfrak{f} = \dfrac{\partial H}{\partial \mathfrak{v}} \\ \mathfrak{t} = \dfrac{\partial H}{\partial \mathfrak{g}} \quad w = -\dfrac{\partial H}{\partial u} \quad \dfrac{\partial H}{\partial \mathfrak{d}} = -\gamma \cdot H. \end{cases}$$

From the last equation in (93), it follows that:

(94) $$H = e^{-\gamma\omega} \cdot H'(\mathfrak{d}, \mathfrak{h}, \rho, \mathfrak{v}, \mathfrak{g}, u).$$

If we now define:

$$(95)\qquad \begin{cases} \mathfrak{e}' = \dfrac{\partial H'}{\partial \mathfrak{d}}, & \mathfrak{b}' = \dfrac{\partial H'}{\partial \mathfrak{h}}, & \varphi' = -\dfrac{\partial H'}{\partial \rho} & \mathfrak{f}' = \dfrac{\partial H'}{\partial \mathfrak{v}} \\ \mathfrak{t}' = \dfrac{\partial H'}{\partial \mathfrak{g}} & w' = -\dfrac{\partial H'}{\partial u}, \end{cases}$$

in which the primed quantities all depend on ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$, $\mathfrak{g}$, u), but not ω, then we have:

$$(96)\qquad \begin{cases} \mathfrak{e} = e^{-\gamma\omega}\cdot\mathfrak{e}', & \mathfrak{b} = e^{-\gamma\omega}\cdot\mathfrak{b}', & \varphi = e^{-\gamma\omega}\cdot\varphi', \\ \mathfrak{f} = e^{-\gamma\omega}\cdot\mathfrak{f}', & \mathfrak{t} = e^{-\gamma\omega}\cdot\mathfrak{t}', & w = e^{-\gamma\omega}\cdot w'. \end{cases}$$

If equations (93) are satisfied then the energy principle is still valid for the extended fundamental equations, and if all of the variables in H appear only in combinations that are invariant under Lorentz transformations then the relativity principle is also valid.

We have thus succeeded in presenting a theory of gravitation in which not only the energy principle, but also the relativity principle is valid.

I would like to place particular weight on this last fact, since an Ansatz in the theory of matter that we develop here that contradicts the principle of relativity must be rejected immediately. M Abraham maintained the viewpoint in his work on gravitation [1]) that gravitation and relativity theory are not compatible with each other. If this were the case, then one must reach the conclusion that gravitation is a so-called "purely external" field that is indifferent to the existence of matter. It then, as I shall assume, belongs to the forces that essentially determine the form of elementary particles of matter and the entire inner structure of the atom, and if it did not obey the principle of relativity then it would be unthinkable that the elementary particles of matter and forces that bind them together into atoms, molecules, and tangible bodies can, by a complete general motion of the matter through space, avoid the variation that leads to the contraction of matter that was shown in the Michelson experiment. On the other hand, I also believe that one will come up against very great difficulties when one tries to treat gravitation as an effect that plays no appreciable role in the internal motions of atoms, and I then believe that one must give up the viewpoint of M. Abraham, as long as the theory of gravitation is not to be treated as detached from the theory of matter. It therefore seems to me that it is very important that one can bind gravitation and relativity theory in such a simple fashion as we just did.

It must still be noted that *Hamilton's Principle*, in the form that we learned in I, section **10**, is also valid for the equations (86), (87), of extended ether dynamics. The proof of this presents no hardships.

Invariants.

39. The number of invariants will be significantly increased by the addition of the gravitational quantities. Along with the gravitational potential ω, four more quantities ($\mathfrak{t}$,

[1] M. Abraham, Ann. d. Phys. **38.**, pp. 1056, 1912.

iw) must be added to the four in I on pp. ?, and the function H' in (94) can therefore depend upon possibly *eight* independent variables. One can then take the following combinations of state variables:

$$
\text{(97)}\qquad
\begin{cases}
p = \mathfrak{d}^2 - \mathfrak{h}^2, \\
q = (\mathfrak{d} \cdot \mathfrak{h}), \\
\sigma = \sqrt{\rho^2 - \mathfrak{v}^2}, \\
s = (\rho \cdot \mathfrak{h} - [\mathfrak{v} \cdot \mathfrak{d}])^2 - (\mathfrak{v} \cdot \mathfrak{h})^2, \\
\kappa = \sqrt{\mathfrak{g}^2 - u^2}, \\
\mathfrak{t} = (u \cdot \mathfrak{h} - [\mathfrak{g} \cdot \mathfrak{d}])^2 - (\mathfrak{g} \cdot \mathfrak{h})^2, \\
h = (\mathfrak{g} \cdot \mathfrak{v}) - u \cdot \rho, \\
\mathfrak{b} = (\rho \cdot \mathfrak{h} - [\mathfrak{v} \cdot \mathfrak{d}]) \cdot (u \cdot \mathfrak{h} - [\mathfrak{g} \cdot \mathfrak{d}]) - (\mathfrak{v} \cdot \mathfrak{h}) \cdot (\mathfrak{g} \cdot \mathfrak{h}).
\end{cases}
$$

By the methods of four-dimensional vector analysis, the proof leads to the fact that all of the other invariants may be computed from these eight invariants. However, I would prefer to not give that proof here.

Likewise, I would also prefer not to write the formulas that are analogous to formulas (25) in I, pp. ?, for computing $\mathfrak{e}', \mathfrak{b}', \varphi', \mathfrak{f}', w', \mathfrak{t}'$ from the function H', since they are quite easy to deduce.

The differential equation of the electron.

40. The following quantities are naturally also invariants for the Lorentz transformation:

$$
\begin{aligned}
\mathfrak{e} \cdot \mathfrak{d} - \mathfrak{b} \cdot \mathfrak{h} &= e^{-\gamma\omega} \cdot (\mathfrak{e}' \cdot \mathfrak{d} - \mathfrak{b}' \cdot \mathfrak{h}), \\
\varphi \cdot \rho - \mathfrak{f} \cdot \mathfrak{v} &= e^{-\gamma\omega} \cdot (\varphi' \cdot \} - \mathfrak{f}' \cdot \mathfrak{v}), \\
\mathfrak{t} \cdot \mathfrak{g} - w \cdot u &= e^{-\gamma\omega} \cdot (\mathfrak{t}' \cdot \mathfrak{g} - w' \cdot u).
\end{aligned}
$$

For many purposes, it is more convenient to use a different function in place of H, one that differs from it only by a summand that we will construct out of the aforementioned invariants. Exactly as in I, pp. 524, we shall now define:

$$
\text{(98)}\qquad F = H - (\mathfrak{e} \cdot \mathfrak{d} - \mathfrak{b} \cdot \mathfrak{h}) + (\varphi \rho - \mathfrak{f} \cdot \mathfrak{v}).
$$

We can also set:

$$
\text{(99)}\qquad
\begin{cases}
\Phi = e^{-\gamma \cdot \omega} \cdot \Phi', \\
\Phi' = H' - (\mathfrak{e}' \cdot \mathfrak{d} - \mathfrak{b}' \cdot \mathfrak{h}) + (\varphi' \cdot \rho - \mathfrak{f}' \cdot \mathfrak{v}),
\end{cases}
$$

in which Φ' is a quantity that depends upon only the variables $\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$, $\mathfrak{g}$, u, but not on ω. Since $(\mathfrak{e}', \mathfrak{b}', \varphi', \mathfrak{f}', w', \mathfrak{t}')$ can be computed from the variables ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$, $\mathfrak{g}$, u), one can, conversely, compute ($\mathfrak{d}$, $\mathfrak{h}$, ρ, $\mathfrak{v}$) from $(\mathfrak{e}', \mathfrak{b}', \varphi', \mathfrak{f}', \mathfrak{g}, u)$, and one may then consider Φ' to be a function of this new system of variables:

$$\Phi = e^{-\gamma\omega} \cdot \Phi'(\mathfrak{e}', \mathfrak{b}', \varphi', \mathfrak{f}', \mathfrak{g}, u). \tag{100}$$

Now, from (99) and (95), it follows that:

$$d\Phi' = -\mathfrak{d} \cdot d\mathfrak{e}' + \mathfrak{h} \cdot d\mathfrak{b}' + \rho \cdot d\varphi' + \mathfrak{v} \cdot d\mathfrak{f}' + \mathfrak{t}' \cdot d\mathfrak{g} - w' \cdot du,$$

hence:

$$\begin{cases} \mathfrak{d} = -\dfrac{\partial\Phi'}{\partial\mathfrak{e}'}, \quad \mathfrak{h} = -\dfrac{\partial\Phi'}{\partial\mathfrak{b}'}, \quad \rho = -\dfrac{\partial\Phi'}{\partial\varphi'}, \quad \mathfrak{v} = -\dfrac{\partial\Phi'}{\partial\mathfrak{f}'}, \\ \mathfrak{t}' = -\dfrac{\partial\Phi'}{\partial\mathfrak{g}}, \quad w' = -\dfrac{\partial\Phi'}{\partial u}. \end{cases} \tag{101}$$

In the case of an electron at rest the quantities $\mathfrak{b}', \mathfrak{f}'$, u are a constant that equals zero, and the remaining three depend only upon the distance r. I set:

$$\begin{cases} X = \mathfrak{e}' = -e^{+\gamma\omega} \cdot \dfrac{d\varphi}{dr}, \\ Y = \varphi' = e^{+\gamma\omega} \cdot \varphi, \\ Z = \mathfrak{g} = \dfrac{d\omega}{dr}. \end{cases} \tag{102}$$

We thus have a function Φ' of only three variables:

$$\Phi'(X, Y, Z).$$

However, since:

$$\mathrm{X} = -\frac{d\mathrm{Y}}{dr} + \gamma \cdot Y \cdot Z,$$

we have, in reality, only two unknown variables, Y and Z, and the derivative of one of them, dY / dr, in Φ'.

We then have the following two differential equations for these two unknown variables:

$$\frac{1}{r^2} \cdot \frac{d}{dr}(r^2 \cdot \mathfrak{d}) = \rho,$$
$$\frac{1}{r^2} \cdot \frac{d}{dr}(r^2 \cdot \mathfrak{t}) = -\gamma \cdot H,$$

or:

(103)
$$\begin{cases} \frac{1}{r^2}\cdot\frac{d}{dr}(r^2\cdot\frac{d\Phi'}{dX})+\frac{d\Phi'}{dY}=0 \\ \frac{1}{r^2}\cdot\frac{d}{dr}(r^2\cdot\frac{d\Phi'}{dZ})+\gamma\cdot\left(\Phi'-X\cdot\frac{d\Phi'}{dX}-Y\cdot\frac{d\Phi'}{dY}-Z\cdot\frac{d\Phi'}{dZ}\right)=0. \end{cases}$$

If one wishes to discuss the problem of the electron with consideration to gravitation then one must replace equation (34) in II, pp?, with these two equations. By the way, one can also eliminate the unknown Z and its derivative from the two equations by the usual procedures of differential equations. One then obtains a third order equation for the unknown $Y = e^{\gamma\omega}\cdot\varphi$, whereas (34) was an equation of second order for φ.

The world matrix.

41. (…)