"Über die Zusammensetzung gerader Linien und eine daraus entrspringende neue Begründungsweise des barycentrischesn Calculs," Jour.. f. reine und ang. Math.. **28** (1844), 1-9.

On the composition of line segments and a new way of founding the barycentric calculus that arises from it.

(By Professor A. F. Möbius in Leipzig.)

Translated by D. H. Delphenich

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1. The only theorems that that I will assume to have been proved are these two:

Two lines that are each parallel to a third one are also parallel to each other.
Parallel line segments between parallel lines are equal to each other.

In the following, by setting an equal sign between the expressions for two line segments – e.g., $AB = CD$ – this will always imply that the two line segments do not merely have the same length, but also have the same direction, such that when the one line segment CD is advanced parallel to itself until C coincides with A then D also coincides with B. With this notation, one may thus briefly express the latter two theorems as:

I. If $AB = CD$ and $CD = EF$ then one also has $AB = EF$.
II. If $AB = CD$ then one also has $AC = BD$.

From this, one may further deduce:

III. If 1) $AB = A'B'$ and 2) $BC = B'C'$ then one also has $AC = A'C'$.

Then, from II, it follows from 1) that $AA' = BB'$, and from 2) that $BB' = CC'$. From I, this makes: $AA' = \mathrm{CC}'$, and from II: $AC = A'C'$.

IV. If
1) $AB = A'B'$,
2) $BC = B'C'$,
3) $CD = C'D'$,
4) $DE = D'E'$, etc.,

then one also has: $AD = A'D'$, $AE = A'E'$, etc. Then, from 1) and 2), it follows from III that $AC = A'C'$, from this and 3), one has $AD = A'D'$, etc.

2. If AB, CD, EF are some line segments whose magnitudes and directions are given and one sets, starting from an arbitrary point P, these line segments parallel to each other in direction, which then makes $PQ = AB$, $QR = CD$, $RS = EF$, and then forms the broken line segment *PQRS* then this operation shall be called the *composition* or *geometric addition* of the given line segments; this is to be distinguished from the *arithmetic* addition, as when one merely considers the magnitudes of the lines, but not their directions. One calls the line segment from the initial point P up to the and point S of the broken line segment *PQRS* the *geometric sum* of the line segments AB, … and expresses this by means of:

$$AB + CD + EF = PS.$$

If the end point S agrees with the starting point P then the geometric sum is null, and one then writes:

$$AB + CD + EF = GH + IK.$$

Moreover, it is in itself clear that just as the points A, B, C, D, … might lie in space, one will always have:

$$\begin{array}{rlr} AB + BA = 0, & AB + BC = AC, & \\ AB + BC + CA = 0, & AB + BC + CD = AD, & \text{etc.} \end{array}$$

3. When one, in order to compose AB, …, chooses a point P' for the starting point instead of P, and thus makes $P'Q' = AB$, $Q'R' = CD$, $R'S' = EF$, then one has, from I: $PQ = P'Q'$, $QR = Q'R'$, $RS = R'S'$, and therefore, from IV: $PS = P'S'$; i.e., the geometric sum remains the same regardless of which point one starts the addition with.

The geometric sum of several line segments is, however, not merely independent of the position of the starting point, but also of the order in which one composes them. In order to add AB and CD the first time, one then begins with AB, so $PQ = AB$, $QR = CD$, and the next time, one begins with CD, so $PQ' = QR$. Due to 1), it follows that $PQ' = QR'$, and from 2) $PQ' = QR$. Thus, R is identical with R', such that in either case, the sum is PR. In just this way, when one is adding several line segments one may permute any two consecutive one with each other, and since for any other sequence of consecutive elements one can arrive at another sequence of elements by successive transpositions of any two consecutive ones, so can the order in which one connects several line segments one after the other when that are to be added geometrically be arbitrary, exactly like in arithmetic addition.

4. If the magnitudes and directions of the line segments AB, CD, EF, GH, IK are given, and starting from P, one makes $PQ = AB$, $QR = CD$, $RS = EF$, $ST = GH$, $TU = IK$ then:

$$AB + CD + EF = PS, \; GH + IK = SU$$

and

$$AB + CD + \ldots + IK = PU = PS + SU,$$

from 2 to the end.

If one thus sets:

$$(a) \qquad AB + CD + EF = LM$$

and

$$(b) \qquad GH + IK = NO,$$

then:

$$LM = PS, \qquad NO = SU,$$

and

$$AB + CD + \ldots + IK = PS + SU = LM + NO;$$

i.e., one can add formulas (*a*) and (*b*) to each other, as well as three or more such formulas.

If the formula that is to be added to (*a*) is identical with (*a*) then one has:

$$AB + CD + EF + AB + CD + EF = LM + LM,$$

or, since line segments can be added in any order (3), when one lets any terms with the same symbol follow each other in succession, and understands $m \cdot AB$ to mean a line segment that has the same direction and magnitude as AB and is to AB what m is to 1:

$$2 \cdot AB + 2 \cdot CD + 2 \cdot EF = 2 \cdot LM,$$

and similarly when one adds m formulas that are identical to (*a*), one gets:

$$(c) \qquad m \cdot AB + m \cdot CD + m \cdot EF = m \cdot LM,$$

where m can be any positive whole number.

Under the same assumption about m, one can infer from (*a*) that:

$$\frac{1}{m} \cdot AB + \frac{1}{m} \cdot CD + \frac{1}{m} \cdot EF = \frac{1}{m} \cdot LM.$$

If one then sets:

$$\frac{1}{m} \cdot AB + \frac{1}{m} \cdot CD + \frac{1}{m} \cdot EF = XY.$$

then one also has, because one can multiply by m:

$$AB + CD + EF = m \cdot XY;$$

as a result, due to (*a*):

$$m \cdot XY = LM,$$

and therefore:

$$XY = \frac{1}{m} \cdot LM.$$

One can therefore multiply or divide a formula like (*a*) by any positive whole number, and thus, also by any positive rational fraction, and therefore, from known results, also by any positive irrational number without destroying its validity [*]).

[*]) The same thing may also be easily demonstrated with the help of the study of similar figures, just as, conversely, this study can be derived from the theorems above. It follows, e.g., from the identity $AB + BC = AC$ that one also has $m \cdot AC = FG + GH = FH$; i.e., if FG, GH are two sides of a triangle and AB, BC are sides of another one that are parallel and proportional to the latter then the third side FH of the former

What we just now proved for the formula (a) must ultimately also be true for formulas of the general form:

$$(a^*) \qquad a \cdot AB + c \cdot CD + \ldots = l \cdot LM,$$

where a, c, …, and l mean arbitrary positive numbers. Then, when one sets $a \cdot AB = A'B'$, $c \cdot CD = C'D'$, etc., and $l \cdot LM = L'M'$, this formula will reduce to the previous one (a).

5. Just as $a + b = c$ and $a = c - b$ are identities, one can also consider the formulas:

1. $AB + CD = EF$,
2. $AB = EF - CD$

as identities, and call AB the *geometric difference* between EF and CD, when EF is the geometric sum of AB and CD. If one set adds DC to both sides of (1) then this gives:

$$AB = EF + DC = EF - CD,$$

from (2); to say that a line segment is *subtracted* from another line segment is the same thing as to say that the former line, when taken in the opposite direction, is geometrically *added* to the latter.

Therefore, when the general formula (a) above, along with the assumptions that were made regarding it, has one or a number of terms of negative signs, one can arrive at nothing but positive terms either by changing the minus sign on a term into a plus, although one must then switch the starting point and the endpoint of the associated line segment, or one can simply move a negative sign to the other side of the equal sign with a change of sign.

Above all, it emerges from the foregoing that one can treat formulas like (a) completely as one treats ordinary equations, as long as they consist of only linear forms of the line segments that enter into them, such that one can, in fact, move terms from one side of the equal sign to the other one with the opposite sign, multiply or divide all terms by the same number, and add two or more such formulas to each other or subtract one from the other.

6. We have thus arrived at a type of calculation with line segments whose validity needs no proof, as long as the line segments are part of one and the same line, or parallel to the same line, and whose admissibility flows from the general theorems of the theory of parallels when the line segments have different directions.

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triangle is also parallel to the third side AC of the latter one, and in the same ratio of proportionality. Or, if FG, GH, HF aare parallel to AB, BC, CA, resp., and for that reason one sets $FG = p \cdot AB + q \cdot BC + r \cdot CA$. However, one also always has $r \cdot AB + r \cdot BC + r \cdot CA = 0$, and therefore (no. 5) $(p - r)\, AB + (q - r)\, BC = 0$. However, as long as A, B, C do not lie in a line the geometric sum of two lines that have the directions AB and BC cannot be zero, and it must then follow that each of the coefficients $p - r$ and $q - r$ are individually zero, so $p = q = r$ and $FG : GH : HF = AB : BC : CA$; i.e., when the sides of a triangle FGH are parallel to those of another one ABC, they are also proportional to them.