"Soluzione generale delle equazioni indefinite dell'equilibrio di un corpo continuo," Rend. R. Accad. dei Lincei, Classe sci. fis., mat. e nat. (5) **1** (1892), 137-141.

General solution of the indefinite equations of equilibrium in a continuous body

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In order to find the most general solution to the system of partial differential equations:

$$\left.\begin{aligned}\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}&=X,\\ \frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}&=Y,\\ \frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}&=Z\end{aligned}\right\}\tag{1}$$

$$Y_z = Z_y, \qquad Z_x = X_z, \qquad X_y = Y_x, \tag{2}$$

in which X, Y, Z denote three given functions of x, y, z, it is obviously enough to find the general solution of those equations when one sets $X = Y = Z = 0$.

The most general solution of the partial differential equation:

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$$

is

$$P=\frac{\partial V}{\partial x}-\frac{\partial W}{\partial y}, \qquad Q=\frac{\partial W}{\partial x}-\frac{\partial U}{\partial z}, \qquad R=\frac{\partial U}{\partial y}-\frac{\partial V}{\partial x},$$

in which U, V, W denote three arbitrary functions.

The equations that one deduces from (1) by setting $X = Y = Z = 0$ can then be satisfied in a more general way by setting:

$$\left.\begin{aligned}
X_x &= \frac{\partial v_1}{\partial z} - \frac{\partial w_1}{\partial y}, \quad & X_y &= \frac{\partial w_1}{\partial x} - \frac{\partial u_1}{\partial z}, \quad & X_z &= \frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x}, \\
Y_x &= \frac{\partial v_2}{\partial z} - \frac{\partial w_2}{\partial y}, \quad & Y_y &= \frac{\partial w_2}{\partial x} - \frac{\partial u_2}{\partial z}, \quad & Y_z &= \frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x}, \\
Z_x &= \frac{\partial v_3}{\partial z} - \frac{\partial w_3}{\partial y}, \quad & Z_y &= \frac{\partial w_3}{\partial x} - \frac{\partial u_3}{\partial z}, \quad & Z_z &= \frac{\partial u_3}{\partial y} - \frac{\partial v_3}{\partial x},
\end{aligned}\right\} \qquad (1')$$

in which u_i, v_i, w_i (i = 1, 2, 3) denote nine arbitrary functions.

However, due to (2), those nine functions must satisfy the following three partial differential equations:

$$\left.\begin{aligned}
\frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x} &= \frac{\partial w_3}{\partial x} - \frac{\partial u_2}{\partial z}, \\
\frac{\partial v_3}{\partial z} - \frac{\partial w_3}{\partial y} &= \frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x}, \\
\frac{\partial w_1}{\partial x} - \frac{\partial u_1}{\partial z} &= \frac{\partial v_2}{\partial z} - \frac{\partial w_2}{\partial y},
\end{aligned}\right\}$$

which can be written:

$$\left.\begin{aligned}
\frac{\partial (v_2 + w_3)}{\partial x} &= \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}, \\
\frac{\partial (w_3 + u_1)}{\partial y} &= \frac{\partial v_3}{\partial z} + \frac{\partial v_1}{\partial x}, \\
\frac{\partial (u_1 + v_3)}{\partial z} &= \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}.
\end{aligned}\right\} \qquad (3)$$

Now, since u_2 , u_3 ; v_3 , v_1 ; w_1 , w_2 are arbitrary, one can always set:

$$\left.\begin{aligned}
u_2 &= \frac{\partial F_{12}}{\partial x}, \quad & u_2 &= \frac{\partial F_{13}}{\partial x}, \\
v_3 &= \frac{\partial F_{23}}{\partial y}, \quad & v_1 &= \frac{\partial F_{21}}{\partial y}, \\
w_1 &= \frac{\partial F_{31}}{\partial z}, \quad & w_2 &= \frac{\partial F_{32}}{\partial z},
\end{aligned}\right\} \qquad (4)$$

in which F_{12} , F_{13} , are well-defined functions, up to arbitrary functions of y and z, …, resp. If one integrates (3) then one will have:

$$\left.\begin{aligned} v_2 + w_3 &= \frac{\partial F_{12}}{\partial y} + \frac{\partial F_{13}}{\partial z}, \\ w_3 + u_1 &= \frac{\partial F_{23}}{\partial z} + \frac{\partial F_{21}}{\partial x}, \\ u_1 + v_3 &= \frac{\partial F_{31}}{\partial x} + \frac{\partial F_{32}}{\partial y}, \end{aligned}\right\}$$

and in the last two groups of relations, the F_{ik} denote functions that are, in the final analysis, entirely arbitrary.

From the last three equations, one will obviously have:

$$\left.\begin{aligned} u_1 &= \frac{1}{2}\left\{\frac{\partial}{\partial x}(F_{21} + F_{31}) + \frac{\partial}{\partial y}(F_{32} - F_{12}) + \frac{\partial}{\partial z}(F_{23} - F_{13})\right\}, \\ v_2 &= \frac{1}{2}\left\{\frac{\partial}{\partial y}(F_{23} + F_{12}) + \frac{\partial}{\partial z}(F_{13} - F_{23}) + \frac{\partial}{\partial x}(F_{31} - F_{21})\right\}, \\ w_3 &= \frac{1}{2}\left\{\frac{\partial}{\partial z}(F_{13} + F_{23}) + \frac{\partial}{\partial x}(F_{21} - F_{31}) + \frac{\partial}{\partial y}(F_{12} - F_{32})\right\}, \end{aligned}\right\} \tag{5}$$

and therefore (4) and (5), in which one keeps the functions F_{ik} arbitrary, will answer the question in the most general way.

From (1′), one will then immediately have:

$$X_x = \frac{\partial^2}{\partial y\,\partial z}(F_{21} - F_{31}), \ \ldots,$$

$$Y_x = \frac{1}{2}\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(F_{21} - F_{31}) + \frac{\partial}{\partial y}(F_{12} - F_{32}) + \frac{\partial}{\partial z}(F_{23} - F_{13})\right\}, \ \ldots$$

If one then sets:

$$F_{21} - F_{31} = 0, \qquad F_{32} - F_{12} = V, \qquad F_{13} - F_{23} = W$$

then one will get the most general solution of the system (1), (2) for $X = Y = Z = 0$:

$$\left.\begin{aligned} X_x &= \frac{\partial^2 U}{\partial y\,\partial z}, & Y_z &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial W}{\partial z}\right), \\ Y_y &= \frac{\partial^2 V}{\partial z\,\partial x}, & Z_x &= \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y} - \frac{\partial W}{\partial z} - \frac{\partial U}{\partial x}\right), \\ Z_z &= \frac{\partial^2 W}{\partial x\,\partial y}, & X_y &= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial z} - \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right), \end{aligned}\right\}$$

in which U, V, W indicate three arbitrary functions.

If the three functions X, Y, Z are arbitrary then a particular solution of (1) and (2) is obviously:

$$\left.\begin{aligned} X_x &= \int X\,dx, & Y_y &= 0, \\ Y_y &= \int Y\,dy, & Z_z &= 0, \\ Z_z &= \int Z\,dz, & X_y &= 0, \end{aligned}\right\}$$

and therefore the most general solution of (1) and (2) is:

$$\left.\begin{aligned} X_x &= \int X\,dx + \frac{\partial^2 U}{\partial y\,\partial z}, & Y_y &= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial W}{\partial z}\right), \\ Y_y &= \int Y\,dy + \frac{\partial^2 V}{\partial z\,\partial x}, & Z_x &= \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y} - \frac{\partial W}{\partial z} - \frac{\partial U}{\partial x}\right), \\ Z_z &= \int Z\,dz + \frac{\partial^2 W}{\partial x\,\partial y}, & X_y &= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial W}{\partial z} - \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right). \end{aligned}\right\}$$

Those formulas show that such a force field can generate the stresses in a continuous medium in an infinitude of different ways.

A continuous body can be in equilibrium without the internal stresses being zero and without the intervention of internal forces; for example, that can happen in an ordinary elastic solid that is heated unequally.

Indeed, if $X = Y = Z = 0$ then the three arbitrary functions U, V, W can always be imagined to be determined such a way that over the entire surface of the body one will have:

$$\left.\begin{aligned} X_x \frac{\partial x}{\partial n} + X_y \frac{\partial y}{\partial n} + X_z \frac{\partial z}{\partial n} &= 0, \\ Y_x \frac{\partial x}{\partial n} + Y_y \frac{\partial y}{\partial n} + Y_z \frac{\partial z}{\partial n} &= 0, \\ Z_x \frac{\partial x}{\partial n} + Z_y \frac{\partial y}{\partial n} + Z_z \frac{\partial z}{\partial n} &= 0, \end{aligned}\right\}$$

in which the symbol n denotes the direction of the internal normal to that surface. In conformity with the preceding conditions, one can also determine the three arbitrary functions in an infinitude of different ways, as a rule.

Examine the case in which U, V, W are functions of only the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the point (x, y, z) to the origin of the coordinate.

Then set:

$$R_1 = \frac{1}{r}\frac{dU}{dr}, \quad R_2 = \frac{1}{r}\frac{dV}{dr}, \quad R_3 = \frac{1}{r}\frac{dW}{dr},$$

and one will easily get:

$$\frac{\partial^2 U}{\partial x^2} = R_1 + \frac{x^2}{r}\frac{dR_1}{dr}, \qquad \frac{\partial^2 U}{\partial x\,\partial y} = \frac{xy}{r}\frac{dR_1}{dr}, \qquad \ldots,$$

and therefore:

$$\left.\begin{aligned}
X_x &= \frac{yz}{r}\frac{dR_1}{dr}, & Y_z &= \frac{1}{2}R_1 + \frac{x}{2r}\left(x\frac{dR_1}{dr} - y\frac{dR_2}{dr} - z\frac{dR_3}{dr}\right),\\
Y_y &= \frac{zx}{r}\frac{dR_2}{dr}, & Z_x &= \frac{1}{2}R_2 + \frac{y}{2r}\left(y\frac{dR_2}{dr} - z\frac{dR_3}{dr} - x\frac{dR_1}{dr}\right),\\
Z_z &= \frac{xy}{r}\frac{dR_3}{dr}, & X_y &= \frac{1}{2}R_3 + \frac{z}{2r}\left(z\frac{dR_2}{dr} - x\frac{dR_1}{dr} - y\frac{dR_2}{dr}\right).
\end{aligned}\right\}$$

On the basis of some known general formulas, the components X_n , Y_n , Z_n of the unit pressure that are exerted upon the element of the spherical surface of radius r that surround the point (x, y, z) will be:

$$X_n = -\left(X_x\frac{x}{r} + X_y\frac{y}{r} + X_z\frac{z}{r}\right) = -\frac{y\,R_3 + z\,R_2}{2r},$$

$$Y_n = -\frac{z\,R_1 + x\,R_3}{2r},$$

$$Z_n = -\frac{x\,R_2 + y\,R_1}{2r}.$$

With the use of those formulas, one will soon see that if the body is bounded by two concentric spheres of radii r_1, r_2 then it will be enough to assume that R_1, R_2, R_3 are three functions of the distance from the center that vanish for $r = r_1$, as well as $r = r_2$, in order for one to solve the problem of equilibrium for the shell without the intervention of external surface pressures and forces that are diffused over its mass. To that end, it is then enough to assume that:

$$R_i = \{\varphi_i(r_2) - \varphi_i(r)\}\{\psi_i(r) - \psi_i(r_1)\} \qquad (i = 1, 2, 3),$$

in which the $\varphi_i(r)$, $\psi_i(r)$ are arbitrary functions that are still finite and continuous in the interval (r_1, r_2).

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