"Die Grundgleichungen der elastischen Stabilität in allgemeinen Koordinaten und ihre Integration," Zeit. angew. Math. Mech. **23** (1943), 321-330.

The basic equations of elastic stability in general coordinates and their integration ([1]).

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Abstract. – Starting from force equilibrium, the author exhibits the basic equations of the theory of stability in arbitrary curvilinear coordinates with the use of the absolute differential calculus. The advantages of that derivation are that no assumptions need to be made initially about the laws of deformation and that the quantities, which also acquire tensor properties under finite displacements, emerge clearly as tensors. The integration of the basic equations of the anisotropic continuum for stationary preloading will be performed with a vectorial stress function that satisfies a sixth-order differential equation. In the case of isotropy, the process arrives at second-order differential equations.

1. – Introduction.

In the presentations of stability theory up to now [**1-8**] ([2]), Cartesian coordinates were employed for exhibiting the basic equations. In the present article, that theory will be derived by applying the absolute differential calculus in arbitrary curvilinear coordinates. The representation offers the advantage that those quantities, which also acquire tensor properties for finite displacements, will emerge clearly as tensors. Above all, an essential difference from the previous representations consists of the fact that the equilibrium conditions will be derived from force equilibrium, not from the energy principle, i.e., with no assumptions on the law of deformation (the elastic potential, resp.). After eliminating the additional stress tensor that is required by elastic deformation with the help of the stress-extension equations, that will yield a new system of equations for the displacement vector in anisotropic continua that corresponds to a modification of the law of anisotropic elasticity in such a way that a new tensor will appeal in place of the original elasticity tensor that will depend upon the preloading. The general integral of the basic equations for stationary preloads will be performed with a vectorial stress function that satisfies a sixth-order differential equation. In the case of an isotropic continuum, it can be shown that the

([1]) Talk given at the meeting of the Society of Mathematicians in Würzburg in 1943.

([2]) The numbers in square brackets refer to the list of references that is given at the end of the article. During its printing, I was made aware of a paper by **E. A. Denker** [Deutsche Math. **5** (1940), 546-563], in which general coordinates were already employed. The problem was represented as a variational problem there by introducing an auxiliary parameter. However, the variation was not performed in general, such that the basic equations did not appear explicitly. The integration was not treated there, either.

same Ansatz that the author had successfully applied in the usual spatial theory of elasticity will lead to a simple path to solution. Only second-order differential equations will appear along it.

2. – Notations and basic geometric relations.

The notations $x = \bar{x}^1$, $y = \bar{x}^2$, $z = \bar{x}^3$, and in general $\bar{x}^k$, might characterize orthogonal Cartesian coordinates, and Roman indices will always refer to Cartesian coordinates. The quantities $\bar{x}^k$ are assumed to be differentiable functions of the new curvilinear coordinates x^1, x^2, x^3, and in general x^λ, and Greek indices will always refer to curvilinear coordinates. Each of the components of a quantity $A^k = A_k$ that represents a vector in Cartesian coordinates is assigned the quantity:

$$A_\lambda = \sum_k A_k \frac{\partial \bar{x}^k}{\partial x^\lambda} \tag{1}$$

in the new coordinates, which can also be referred to as a vector or a first-rank tensor. That way of distinguishing the Cartesian starting coordinates from the general coordinate indeed deviates from the usual notation, but it has certain advantages under differentiation insofar as Cartesian vectors will be treated as invariants for the new coordinates. Moreover, the known notations (cf., e.g., **Levi-Civita** [**9**]). With the abbreviation:

$$\frac{\partial \bar{x}^k}{\partial x^\lambda} = c_\lambda^k , \tag{2}$$

the quantities $B_{\lambda\mu}$, which are correspondingly obtained from the quantities B_{kl} by the relation:

$$B_{\lambda\mu} = \sum_{k,l} B_{kl}\, c_\lambda^k\, c_\mu^l , \tag{3}$$

generally define a second-rank "tensor." Corresponding statements are true for higher-rank tensors. The metric tensor, which is crucial for the determination of length, will be:

$$g_{\lambda\mu} = \sum_k c_\lambda^k\, c_\mu^k , \tag{4}$$

with the Cartesian components $g_{kl} = \delta_l^k$, in which:

$$\delta_l^k = \begin{cases} 1 & \text{when} \quad k = l \\ 0 & \text{when} \quad k \neq l \end{cases} \tag{5}$$

is the **Kronecker** symbol. If one further defines the adjoint of the determinant c_λ^k and divides by the determinant $|c_\lambda^k|$ then that will yield the quantities:

$$c_k^\lambda = \frac{1}{2|c_\lambda^k|}\sum_{l,m,\alpha,\beta} e_{klm}\, e^{\lambda\alpha\beta} c_\alpha^l\, c_\beta^m \,. \tag{6}$$

For the auxiliary quantities e_{klm} that appear in it, one has:

$$e_{klm} = \begin{cases} 1 & \text{when } k, l, m \quad \text{define a cyclic form} \\ -1 & \qquad\qquad\quad \text{for odd permutations of the upper indices,} \\ 0 & \text{when} \qquad\quad\; \text{at least two indices coincide.} \end{cases} \tag{7}$$

Corresponding statements are true for the quantities $e^{\lambda\alpha\beta}$. It follows for the determinant that:

$$|c_\lambda^k| = \sum_{k,l,m,\alpha,\beta,\gamma} \tfrac{1}{6}\, e_{klm}\, e^{\lambda\alpha\beta}\, c_\alpha^k\, c_\beta^l\, c_\gamma^m \,. \tag{8}$$

From the theory of determinants, when one sums over an upper index and a lower one, the summation sign will be omitted, so:

$$c_\alpha^k\, c_k^\beta = \delta_\alpha^\beta\,, \qquad c_\alpha^k\, c_l^\alpha = \delta_l^k\,, \qquad \text{etc.} \tag{9}$$

If one further sets:

$$|g_{\alpha\beta}| = \tfrac{1}{6} e^{\alpha\beta\gamma}\, e^{\lambda\mu\nu}\, g_{\alpha\lambda}\, g_{\beta\mu}\, g_{\gamma\nu} = g \tag{10}$$

and

$$g^{\alpha\beta} = \frac{1}{2g} e^{\alpha\gamma\delta}\, e^{\beta\mu\nu}\, g_{\gamma\mu}\, g_{\delta\nu} \tag{11}$$

then it will follow that:

$$g_{\alpha\beta}\, g^{\alpha\beta} = \delta_\beta^\gamma\,, \qquad \text{etc.,} \tag{12}$$

i.e., the $g^{\alpha\beta}$ will also define a tensor. Furthermore, one will have:

$$|c_\lambda^k| = \sqrt{g}\,, \quad |c_k^\lambda| = \frac{1}{\sqrt{g}}\,, \tag{13}$$

and the quantities c_k^λ, δ_β^α, as well as:

$$\frac{1}{\sqrt{g}} e^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma} \qquad \text{and} \qquad \sqrt{g}\, e_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}\,, \tag{14}$$

will represent all tensors. (In so doing, one must generally agree that the new coordinate system must rotate in the same sense as the Cartesian one.)

According to eq. (3), the quantities $B_{\lambda\mu}$ are covariant tensor components, while the associated quantities:

$$B^{\alpha\beta} = B^{kl} c_k^\alpha c_l^\beta \tag{15}$$

are contravariant components. It then follows that:

$$B^{\alpha\beta} = g^{\alpha\lambda} g^{\beta\mu} B_{\lambda\mu} . \tag{16}$$

Differentiating an invariant C will produce a vector:

$$\frac{\partial C}{\partial \bar{x}^k} = C_{,k} = c_k^\alpha C_{,\alpha} . \tag{17}$$

Differentiating a vector A_k will produce a tensor:

$$A_{k,l} = \frac{\partial A_k}{\partial \bar{x}^l} = c_l^\beta A_{k,\beta} = c_l^\beta \frac{\partial}{\partial x^\beta}(c_k^\alpha A_\alpha) . \tag{18}$$

The corresponding curvilinear components of that tensor are then:

$$A_{\alpha,\beta} = c_\alpha^k c_\beta^l A_{k,l} = \frac{\partial A_\alpha}{\partial x^\beta} - \Gamma_{\alpha\beta}^\gamma A_\gamma . \tag{19}$$

In that, one has set:

$$c_\alpha^k \frac{\partial c_k^\gamma}{\partial x^\beta} = - c_k^\gamma \frac{\partial c_\alpha^k}{\partial x^\beta} = - \Gamma_{\alpha\beta}^\gamma \tag{20}$$

(which correspond to the **Christoffel** symbols of the second type).

In general, it follows from the rules of covariant differentiation that the operations:

$$B_{\alpha\beta,\gamma} = \frac{\partial B_{\alpha\beta}}{\partial x} - \Gamma_{\alpha\gamma}^\lambda B_{\lambda\beta} - \Gamma_{\beta\gamma}^\lambda B_{\alpha\lambda} , \tag{21}$$

$$B_{\cdots\gamma}^{\alpha\beta} = \frac{\partial B^{\alpha\beta}}{\partial x^\gamma} + \Gamma_{\gamma\lambda}^\alpha B^{\lambda\beta} + \Gamma_{\gamma\lambda}^\beta B^{\alpha\lambda} , \tag{22}$$

etc., always produce new tensors. Their contravariant components are:

$$B_{\alpha\beta}|^\gamma = g^{\gamma\lambda} B_{\alpha\beta,\lambda} , \qquad B^{\alpha\beta}|^\gamma = B_{\cdots,\lambda}^{\alpha\beta} g^{\gamma\lambda} , \qquad \text{etc.} \tag{23}$$

Furthermore, the **Ricci** relations follow from this:

$$c^k_{\alpha,\beta} = c^\alpha_{k,\beta} = g_{\alpha\beta,\gamma} = g^{\alpha\beta}_{\cdots,\gamma} = g^{\alpha\beta}|^\gamma = g_{\alpha\beta}|^\gamma = 0 \,. \tag{24}$$

3. – Stress tensor and equilibrium conditions.

If the normal stresses σ_x, σ_y, σ_z are characterized by t_{11}, t_{22}, t_{33} and the shear stresses τ_{yz} , τ_{zx} , τ_{xy} are characterized by t_{23}, t_{31}, t_{12} then t_{kl} will generally represent the components of the stress tensor in Cartesian coordinates. The associated covariant components in curvilinear systems are then:

$$t_{\lambda\mu} = t_{kl}\, c^k_\lambda\, c^l_\mu \,, \tag{25}$$

and the respective contravariant components are:

$$t^{\lambda\mu} = t^{kl}\, c^\lambda_k\, c^\mu_l \,. \tag{26}$$

The equilibrium conditions under a displacement in the x-direction:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + P_x = 0 \,, \quad \text{etc.,} \tag{27}$$

will go to the form:

$$t^{kl}_{\cdots,k} + P^l = 0 \,. \tag{28}$$

The P^l in that is the body force vector. Correspondingly, one has:

$$t^{\lambda\mu}_{\cdots,\lambda} + P^\mu = 0 \tag{29}$$

as the equilibrium conditions in the new coordinates, in which P^μ is determined from:

$$P^\mu = P^k\, c^\mu_k \,. \tag{30}$$

When eq. (29) is written out in detail, it takes the form:

$$\frac{\partial t^{\lambda\mu}}{\partial x^\lambda} + \Gamma^\lambda_{\lambda\nu}\, t^{\nu\mu} + \Gamma^\mu_{\lambda\nu}\, t^{\lambda\nu} + P^\mu = 0 \,. \tag{31}$$

A further condition is the equilibrium of the forces under a rotation. In Cartesian coordinates, the known symmetry property $t^{kl} = t^{lk}$ of the stress tensor will follow from that, which will also be true for $t^{\lambda\mu}$, as a result. In order to represent the physical stress components $\tau^{\lambda\mu}$, which are not identical to the quantities $t^{\lambda\mu}$, one must determine the stress vector that acts on a surface x^λ = const. If one imagines that surface as the fourth face of an elementary tetrahedron whose remaining faces are defined by surfaces $\bar{x}^k$ = const. then one will have the equilibrium condition for the resultant force in the $\bar{x}^l$-direction:

$$\sum_\mu \tau^{\lambda\mu} \frac{c_\mu^l}{\sqrt{g_{\mu\mu}}} = \sum_k t^{kl} \frac{c_k^\lambda}{\sqrt{g_{\lambda\lambda}}}, \tag{32}$$

The $\tau^{\lambda\mu}$ in that represent the μ-components of the physical stress vector that acts upon the surface x^λ = const. $c_\mu^l / \sqrt{g_{\mu\mu}}$ is the direction cosine of the μ-direction with respect to the $\bar{x}^l$-direction, while $c_k^\lambda / \sqrt{g^{\lambda\lambda}}$ represents the direction cosine of the normal to the surface x^λ = const. with respect to the $\bar{x}^k$-direction. Upon multiplying by c_l^ν and summing over l, it will follow that:

$$\tau^{\lambda\nu} = \sqrt{\frac{g_{\nu\nu}}{g^{\lambda\lambda}}}\, t^{\lambda\nu}. \tag{33}$$

The quantities $\tau^{\lambda\nu}$ are not a tensor then. The normal stress will be:

$$\sigma_\lambda = \frac{t^{\lambda\lambda}}{g^{\lambda\lambda}}, \tag{34}$$

while the total component of the shear stress that falls in the μ-direction will assume the quantities:

$$\overset{*}{\tau}_{\lambda\mu} = \sum_\nu \frac{g_{\mu\nu}}{\sqrt{g^{\lambda\lambda} g_{\nu\nu}}} t^{\lambda\nu}. \tag{35}$$

If the forces per unit area f^l are given on the surface of the body then then boundary conditions will read:

$$t^{\lambda\nu} c_\mu^l \frac{1}{\sqrt{g^{\lambda\lambda}}} = f^l \tag{36}$$

when the surface is represented by x^λ = const.

4. – Geometry of the deformed body.

After the deformation, the coordinates $\bar{x}^k$ will go to:

$$\bar{x}'^k = \bar{x}^k + V^k. \tag{37}$$

The quantities V^k represent the Cartesian components of the elastic displacement vector. If all new quantities are characterized by a prime accordingly then the relations will follow:

$$c'^k_\lambda = c^k_\lambda + V^k_\lambda = c^k_\lambda + c^k_\nu V^\nu_{\cdot,\lambda}, \tag{38}$$

$$g'_{\lambda\mu} = g_{\lambda\mu} + V_{\lambda,\mu} + V_{\mu,\lambda} + V^\nu_{\cdot,\mu} V_{\nu,\lambda}, \tag{39}$$

$$\sqrt{g'} = \sqrt{g}\,(1 + V^\nu_{\cdot,\nu} + \text{higher-order order}). \tag{40}$$

All of the other quantities can be likewise reduced to the corresponding quantities of the initial system and the derivatives of the displacement vectors. In the context of the problem statement that will be stated below for the elastic continuum, the displacement vector V_λ and its derivatives already represent second-order quantities, since the components of the preloading tensor $T^{\lambda\mu}$ are regarded as first-order quantities. Upon restricting oneself to second-order terms, that will make:

$$g'_{\lambda\mu} = g_{\lambda\mu} + 2\,d_{\lambda\mu}, \tag{41}$$

in which:

$$d_{\lambda\mu} = \tfrac{1}{2}(V_{\lambda,\mu} + V_{\mu,\lambda}) \tag{42}$$

is the deformation tensor. One further obtains:

$$c'^k_\lambda = c^k_\lambda - V^\lambda_{\cdot,\nu}\, c^\nu_k, \qquad g'^{\lambda\mu} = g^{\lambda\mu} - 2\,d^{\lambda\mu}, \tag{43}$$

$$\Gamma'^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + V^\lambda_{\cdot,\mu\nu}. \tag{44}$$

The second derivatives of the displacement vector can correspondingly be always reduced to the first derivatives of the deformation tensor by the relation:

$$V^\lambda_{\cdot,\mu\nu} = d^\lambda_{\mu,\nu} + d^\lambda_{\nu,\mu} - d_{\mu\nu}\,|^\lambda, \tag{45}$$

which emerges from eq. (42). If the covariant derivative in the deformed system is characterized by an * then it will follow that:

$$c_{*\alpha} = c_{,\alpha}, \qquad A_{\alpha*\beta} = A_{\alpha,\beta} - V^\gamma_{\cdot,\alpha\beta} A_\gamma, \tag{46}$$

$$A^\alpha_{\cdot*\beta} = A^\alpha_{\cdot,\beta} + V^\alpha_{\gamma\beta} A^\gamma, \qquad \text{etc.} \tag{47}$$

5. – Equilibrium of deformed bodies.

If $K^{\lambda\mu}$ is the stress tensor of a deformed body, i.e.:

$$K^{\lambda\mu} = K^{lm} c_l'^{\lambda} c_m'^{\mu} , \tag{48}$$

and if $P^{\mu} + p^{\mu}$ is the vector of body forces after the deformation then it will follow from the foregoing that in addition to the symmetry property $K^{\lambda\mu} \neq K^{\mu\lambda}$, the equilibrium conditions are:

$$K^{\lambda\mu}_{\ldots,*\lambda} + P^{\mu} + p^{\mu} = 0 \tag{49}$$

or

$$K^{\lambda\mu}_{\ldots,*\lambda} + V^{\lambda\mu}_{\ldots,\lambda} K^{\nu\mu} + V^{\mu}_{\ldots,\lambda\nu} K^{\lambda\nu} + P^{\mu} + p^{\mu} = 0 , \tag{50}$$

resp. The equations thus-obtained define the foundation for the treatment of stability problems. If preloads $T^{\lambda\mu}$ were already present before the displacements V^{λ} occurred then the stress tensor $K^{\lambda\mu}$ would be combined with a tensor $\bar{T}^{\lambda\mu}$ that is due to only the preloads (first-order quantities) and a tensor $t^{\lambda\mu}$ that is required by the deformation (second-order quantities):

$$K^{\lambda\mu} = t^{\lambda\mu} + \bar{T}^{\lambda\mu} . \tag{51}$$

In that expression, $\bar{T}^{\lambda\mu}$, as well as $t^{\lambda\mu}$, represents a symmetric tensor. In order to establish the connection between $\bar{T}^{\lambda\mu}$ and the preload tensor $T^{\lambda\mu}$, one must appeal to the physical process ([3]). The force that actually acts on a surface x^1 = const. in the k-direction is:

$$\sum_{\mu} T^{1\mu} \sqrt{g}\, c^k_{\mu}\, dx^2\, dx^3 \qquad \text{(surface area } \sqrt{g\, g^{11}}\, dx^2\, dx^3\text{)}$$

before the deformation, and:

$$\sum_{\mu} \bar{T}^{1\mu} \sqrt{g'}\, c'^k_{\mu}\, dx^2\, dx^3$$

after the deformation. The projective operation, which is given by multiplication by c^k_{μ}, indeed means forming the exact k-component. In reality, however, in the immediate neighborhood of the

([3]) The exact calculation of the tensor $\bar{T}^{\lambda\mu}$ would require one to refer to the initial deformation that is coupled with the preloading by the law of deformation. Such a calculation would require an extension of the initial equations to third-order quantities, since the metric tensor in the initial state represents a first-order quantity, and each of the initial deformations already include second-order quantities. At the same time, an extension of the elasticity law to second-order quantities would be required, for which corresponding statements can hardly be guaranteed by experimental findings. In order to get around that difficulty, a special relationship must be imposed upon the behavior of the preloading tensor under deformation for which I shall give two formulations below that are based upon the behavior that one would expect physically.

point in question, the material has performed an overall rotation (i.e., also as a rigid body), i.e., apart from the deformation, that is given by the skew-symmetric tensor:

$$\omega_{\lambda\mu} = \tfrac{1}{2}(V_{\lambda,\,\mu} - V_{\mu,\,\lambda}) \ . \tag{52}$$

The derivatives of that tensor can all be reduced to the derivatives of the deformation tensor, moreover. Namely, when one recalls eq. (45), it will follow that:

$$\omega_{\lambda\mu,\,\nu} = d_{\lambda\nu,\,\mu} - d_{\mu\nu,\,\lambda} \ . \tag{53}$$

If the force quantities that were characterized above are not regarded as directly equal to each other, but only after eliminating the rotation that is included in c'^k_μ, then $c'^k_\mu = c^k_\mu + c^k_\nu (d^\nu_\mu + \omega^\nu_{\cdot\mu})$ must be replaced with the tensor:

$$c'^k_\mu - c^k_\nu\, \omega^\nu_{\cdot\mu} = c^k_\mu + c^k_\nu\, d^\nu_\mu , \tag{54}$$

which includes only a pure deformation. When one next introduces a tensor $\overset{+}{T}{}^{\lambda\mu}$ in place of $\bar{T}^{\lambda\mu}$, that line of reasoning will lead to a relation of the form:

$$\left.\begin{aligned} T^{\lambda\mu}\sqrt{g}\, c^k_\mu &= \overset{+}{T}{}^{\lambda\mu}\sqrt{g'}\,(c^k_\mu + c^k_\nu\, d^\nu_\mu), \\ \text{or} \qquad T^{\lambda\mu} &= \overset{+}{T}{}^{\lambda\mu}(1 + d^\nu_\nu) + \overset{+}{T}{}^{\lambda\nu} d^\mu_\nu , \end{aligned}\right\} \tag{55}$$

resp. Solving that for $\overset{+}{T}{}^{\lambda\mu}$ will yield:

$$\overset{+}{T}{}^{\lambda\mu} = T^{\lambda\mu}(1 - d^\nu_\nu) - T^{\lambda\nu}\, d^\mu_\nu \tag{56}$$

in the context of the second-order theory. One sees that the tensor $\overset{+}{T}{}^{\lambda\mu}$ is asymmetric, so it cannot be identified with $\bar{T}^{\lambda\mu}$ directly. If one sets its purely-symmetric part equal to $\bar{T}^{\lambda\mu}$ then that will give:

$$\bar{T}^{\lambda\mu} = T^{\lambda\mu}(1 - d^\nu_\nu) - \tfrac{1}{2}(T^{\lambda\nu}\, d^\mu_\nu + T^{\nu\mu}\, d^\lambda_\nu) \ . \tag{57}$$

Moreover, eq. (51) implies that:

$$K^{\lambda\mu} = t^{\lambda\mu} + T^{\lambda\mu}(1 - d^\nu_\nu) - \tfrac{1}{2}(T^{\lambda\nu}\, d^\mu_\nu + T^{\nu\mu}\, d^\lambda_\nu) , \tag{58}$$

and the equilibrium conditions (50) go to:

$$t^{\lambda\mu}_{\ldots,\,\lambda} + T^{\lambda\mu}_{\ldots,\,\lambda} + \tfrac{1}{2}(T^{\lambda\nu}\, d^\mu_\nu + T^{\nu\mu}\, d^\lambda_\nu)_{,\lambda} - d^\nu_\nu\, T^{\lambda\mu}_{\ldots,\,\lambda} + P^\mu + p^\mu = 0 \ . \tag{59}$$

In that expression, use was made of the fact that $t^{\lambda\mu}$ is small of the same order as $V^\lambda|^\mu$. If one further observes that the preloading tensor $T^{\lambda\mu}$ is in equilibrium with the original body force P^μ, i.e.:

$$T^{\lambda\mu}_{\dots,\lambda} + P^{\mu} = 0 \tag{60}$$

is fulfilled then eq. (60) can be put into the form:

$$t^{\lambda\mu}_{\dots,\lambda} + T^{\lambda\mu}_{\dots,\lambda} - \tfrac{1}{2}(T^{\lambda\nu}\, d^{\mu}_{\nu} + T^{\nu\mu}\, d^{\lambda}_{\nu})_{,\lambda} + V^{\mu}_{\dots,\lambda\nu}\, T^{\lambda\nu} - d^{\nu}_{\nu}\, T^{\lambda\mu}_{\dots,\lambda} + P^{\mu} + p^{\mu} = 0\,. \tag{61}$$

If all deformation quantities are expressed in terms of the derivatives of the displacement vector then that will give the following equilibrium condition:

$$t^{\lambda\mu}_{\dots,\lambda} + T^{\lambda\nu}\, V^{\mu}_{\dots,\lambda\nu} + p^{\mu} - \tfrac{1}{4}(V^{\mu}_{\dots,\lambda\nu} + V_{\nu}\,|^{\mu})T^{\lambda\nu} + (V^{\lambda}_{\dots,\nu} + V_{\nu}\,|^{\lambda})T^{\mu\nu}]_{,\lambda} + P^{\mu}\, d^{\nu}_{\nu} = 0\,. \tag{62}$$

On the other hand, if the derivatives of the displacement vector are expressed in terms of the derivatives of the deformation tensor [eq. (45)] then that will give:

$$t^{\lambda\mu}_{\dots,\lambda} + T^{\lambda\nu}\,(2\, d^{\mu}_{\lambda,\nu} - d_{\lambda\nu}\,|^{\mu}) - \tfrac{1}{2}(T^{\lambda\nu} d^{\mu}_{\nu} + T^{\mu\nu} d^{\lambda}_{\nu})_{,\lambda} + p^{\mu} + P^{\mu}\, d^{\nu}_{\nu} = 0\,. \tag{63}$$

The relations thus-obtained are based upon the relation (58), which has a heuristic character, strictly speaking. Another way of looking at things will be discussed below that is likewise closely related.

If one considers the components of the stress vector before and after the deformation then a certain deformation of the preloading tensor can also be assumed along with the deformation of the material. The simplest way of envisioning that would be the one in which the components of the stress vector that acts upon the surface x^{λ} = const. will experience the same deformation as the components of the line element, i.e.:

$$T^{\lambda\mu}\sqrt{g}\, c^{k}_{\mu} : d\overline{x}^{k} = \overset{+}{T}{}^{\lambda\mu}\sqrt{g'}\, c'^{k}_{\mu} : d\overline{x}'^{k}\,. \tag{64}$$

Due to the facts that $d\overline{x}^{k} = c^{k}_{\mu}\, dx^{\mu}$, $d\overline{x}'^{k} = c'^{k}_{\mu}\, dx^{\mu}$, those relations demand that:

$$T^{\lambda\mu}\sqrt{g} = \overset{+}{T}{}^{\lambda\mu}\sqrt{g'} \qquad \text{or} \qquad \overset{+}{T}{}^{\lambda\mu} = T^{\lambda\mu}(1 - d^{\nu}_{\nu})\,. \tag{65}$$

That new tensor is manifestly symmetric and can therefore be identified with $\overline{T}^{\lambda\mu}$ directly. Eq. (51) then implies that:

$$K^{\lambda\mu} = t^{\lambda\mu} + T^{\lambda\mu}(1 - d^{\nu}_{\nu})\,, \tag{66}$$

and when one recalls eq. (60), the equilibrium conditions (50) will read:

$$t^{\lambda\mu}_{\dots,\lambda} + V^{\mu}_{\dots,\lambda\nu}\, T^{\lambda\nu} + p^{\mu} + P^{\mu}\, d^{\nu}_{\nu} = 0 \tag{67}$$

or

$$t^{\mu\lambda}_{\dots,\lambda} + (2d^{\mu}_{\lambda,\nu} - d_{\lambda\nu}|^{\mu})T^{\lambda\nu} + p^{\mu} + P^{\mu}\, d^{\nu}_{\nu} = 0\,, \tag{68}$$

resp. In the case of isotropy, those equations possess certain computational advantages over eqs. (63) and (64) that will be exhibited in Section Nine. The difference between both systems of equations mostly relates to the terms that are connected with only small deformation components and are hardly essential. Eq. (63) formally corresponds to a system of equations that was exhibited by **Biot** [**6**]. However, **Biot** did not employ all of the geometric relations in order to connect the derivatives of the deformation tensor with the rotation tensor, but only some of them [namely, the ones that emerge from eq. (53) in Cartesian coordinates when two indices coincide], such that deformation tensors and rotation tensors appear together in his final equations. Eqs. (63) and (68) that were presented here then represent a new formulation for Cartesian coordinates, as well. On the other hand, the relations (67) are similar to the equations that **Trefftz** [**3**] employed without discussing the physical meaning there in the sense that was given here. For the two authors, the derivation of the equilibrium conditions also resulted from the energy principle, i.e., only indirectly from the assumption of the elastic potential, while the derivation here is based upon only force equilibrium. The existence of the elastic potential here will first be a consequence of the superposition principle and the proportionality between stress and extension tensor that will be treated in the next section.

6. – The law of elasticity and the differential equations of the elastic displacement vector.

In the context of second-order theory, $d_{\lambda\mu}$ represents the deformation tensor directly, which is coupled with the stress tensor $t^{\lambda\mu}$ by the linear law of elasticity. For general elastic behavior (anisotropy), that proportionality will be defined by the coefficients $E^{\lambda\mu\nu\rho}$, which define a fourth-rank tensor:

$$t^{\lambda\mu} = E^{\lambda\mu\nu\rho} d_{\nu\rho}\,. \tag{69}$$

Corresponding to the symmetry of the stress and extension tensors, those coefficients will also have the symmetry property:

$$E^{\lambda\mu\nu\rho} = E^{\mu\lambda\nu\rho} = E^{\lambda\mu\rho\nu}\,. \tag{70}$$

The superposition principle, and therefore the existence of the elastic potential, corresponds to the further symmetry property:

$$E^{\lambda\mu\nu\rho} = E^{\nu\rho\mu\lambda}\,. \tag{71}$$

That will imply the expression for the elastic potential or the deformation energy:

$$A = \tfrac{1}{2} t^{\lambda\mu}\, d_{\lambda\mu} = \tfrac{1}{2} E^{\lambda\mu\nu\rho}\, d_{\lambda\mu}\, d_{\nu\rho}\,. \tag{72}$$

The equilibrium conditions (63) [(68), resp.] then go to the following differential equations of the displacement vector:

$$L^{\lambda\mu}[V_\lambda] + P^\mu = 0\ . \tag{73}$$

The $L^{\lambda\mu}[\ ]$ in that is a tensorial operator, and when one recalls eq. (62), it will have the following form:

$$\left.\begin{aligned} L^{\lambda\mu}[\ ] \;&= \left(E^{\lambda\nu\mu\rho} + \tfrac{3}{4} g^{\lambda\mu} T^{\nu\rho} - \tfrac{1}{4} T^{\lambda\nu} g^{\mu\rho} - \tfrac{1}{4} g^{\lambda\rho} T^{\nu\mu} - \tfrac{1}{4} g^{\nu\rho} T^{\lambda\mu}\right)[\ ]_{,\nu\rho} \\ &\quad + \tfrac{1}{4}\left(g^{\lambda\mu} P^\nu + g^{\nu\mu} P^\lambda - T^{\mu\nu}_{\dots,\rho}\, g^{\lambda\rho} - T^{\lambda\mu}_{\dots,\rho}\, g^{\nu\rho}\right)[\ ]_{,\nu}\ , \end{aligned}\right\} \tag{74.a}$$

or

$$L^{\lambda\mu}[\ ] = (E^{\lambda\nu\mu\rho} + g^{\lambda\mu} T^{\nu\rho})[\ ]_{,\nu\rho} + g^{\lambda\nu} T^\mu[\ ]_{,\nu}\,, \tag{74.b}$$

resp., when it is based upon eq. (68). Therefore, with the introduction of that operator, the system of equations is already formally reduced to the one for an anisotropic continuum without preloading, and one will then be dealing with the integration of partial differential equations whose coefficients are variable or constant according to whether the preloading is variable or stationary, resp.

7. – Integrating the basic equations for an anisotropic continuum with stationary preloading.

In the case of stationary preloading, one will have:

$$T^{\lambda\mu}_{\dots,\nu} = 0\ . \tag{75}$$

If it were further assumed that no body forces were present initially and that the supplementary body force vector p^μ that is required by the deformation represents a pure inertial force, i.e.:

$$p^\mu = -\ \rho\,\frac{\partial^2}{\partial t^2}(V^\mu) \tag{76}$$

(ρ is the specific mass, t is the time coordinate), then eq. (73) would go to:

$$\Pi^{\lambda\mu}[V_\mu] = 0\ , \tag{77}$$

in which:

$$\Pi^{\lambda\mu} = \bar{E}^{\lambda\nu\mu\rho}[\ ]_{,\nu\rho} - \rho\, g^{\lambda\mu}\,\frac{\partial^2}{\partial t^2}[\ ]\ . \tag{78}$$

The new tensor $\bar{E}^{\lambda\nu\mu\rho}$ that is introduced in that is calculated to be:

$$\bar{E}^{\lambda\nu\mu\rho} = E^{\lambda\nu\mu\rho} + \tfrac{3}{4}\, g^{\lambda\mu}\, T^{\nu\rho} - \tfrac{1}{4}\, g^{\mu\rho}\, T^{\lambda\nu} - \tfrac{1}{4}\, g^{\nu\rho}\, T^{\lambda\mu} \tag{79.a}$$

or

$$\bar{E}^{\lambda\nu\mu\rho} = E^{\lambda\nu\mu\rho} + g^{\lambda\mu}\, T^{\nu\rho}, \tag{79.b}$$

resp., according to whether one uses eq. (74.a) or (74.b), resp., as a basis ([4]). That implies the theorem: The elastic behavior of an anisotropic continuum under preloading corresponds to a change in the anisotropy that makes the tensor $\bar{E}^{\lambda\nu\mu\rho}$ enter in place of the tensor $E^{\lambda\nu\mu\rho}$.

The solution to eq. (77) represents a problem that was treated before by **Herglotz** [**10**] and **Weierstrass** [**11**]. In the notation that was given here, the differential equation for the components of the displacement vector is given by the determinant of the operator $\Pi^{\lambda\mu}[\;]$:

$$\varepsilon_{\alpha\beta\gamma}\, \varepsilon_{\lambda\mu\nu}\, \Pi^{\alpha\mu}\, \Pi^{\beta\mu}\, \Pi^{\gamma\mu}[V_\rho] = 0\,. \tag{80}$$

If a vectorial stress function were introduced then the Ansatz:

$$V_\lambda = \tfrac{1}{2}\, \varepsilon_{\lambda\mu\nu}\, \varepsilon_{\alpha\beta\gamma}\, \Pi^{\beta\mu}\, \Pi^{\gamma\mu}[X^\alpha] \tag{81}$$

would yield solutions of eq. (77) when X^α satisfies the differential equation (80), which has order six. It can be shown that one of the three Cartesian components of the stress function will generally suffice to represent the solution.

8. – Propagation of perturbations in an anisotropic continuum for stationary preloading.

An instructive example of how preloading can influence elastic properties is given by their behavior under perturbations. For example, if one is dealing with the propagation of a planar wave front then the following Ansatz will be true for the displacement vector (f_l is an arbitrary function of the argument $a_k\, \bar{x}^k + ct$):

$$V_l = f_l(a_k\, \bar{x}^k + ct)\,, \qquad V_\lambda = c^l_\lambda\, f_l\,. \tag{82}$$

a_k might be a unit vector in that. The spreading of the perturbation is then characterized by the temporal evolution of each location in the medium for which the argument $a_k\, \bar{x}^k + ct$ possesses the same value. At equal times, those locations lie on a plane whose velocity of propagation in the direction of the normal is given by c. The differential equations (77) then give:

([4]) In so doing, one should observe that the symmetry properties (70) and (71) are no longer true for the new tensor $\bar{E}^{\lambda\nu\mu\rho}$.

$$(\bar{E}^{\lambda\nu\mu\rho}\, a_\nu\, a_\rho - g^{\lambda\mu}\, \rho c^2)\, f_l'' c_\lambda^l = 0\,. \tag{83}$$

Finally, the differential equation (80) leads to a relation that represents an equation that has degree three in c^2. The known phenomenon for anisotropic media of the existence of three differential sound velocities ([5]) will then be modified by the presence of stationary preloading only insofar as other elastic numbers will be true.

9. – Integrating the basic equations for stationary preloading in the special case of isotropy.

Isotropic media represent one special case. The tensor $E^{\lambda\mu\nu\rho}$ then has the special form:

$$E^{\lambda\mu\nu\rho} = G\left[g^{\lambda\nu}\, g^{\mu\rho} + g^{\mu\nu}\, g^{\lambda\rho} + \frac{2}{m-2}\, g^{\lambda\mu}\, g^{\nu\rho}\right] \tag{84}$$

(G is the shear modulus, m is the **Poisson** constant), and when one recalls eq. (79.b), the tensor will go to:

$$\bar{E}^{\lambda\mu\nu\rho} = G\left[g^{\lambda\nu}\, g^{\mu\rho} + g^{\mu\nu}\, g^{\lambda\rho} + \frac{2}{m-2}\, g^{\lambda\mu}\, g^{\nu\rho}\right] + g^{\mu\nu}\, T^{\lambda\rho}\,. \tag{85}$$

As will be derived below, that expression admits an especially-simple integration of the basic equations. Eqs. (77) and (78) next give the differential equation of the displacement vector as:

$$V^\mu\,|^{\lambda}_{\cdot,\lambda} + \frac{m}{m-2}\, V\,|^{\mu}_{\lambda} + T^{\lambda\nu}\, V^{\mu}_{\ldots,\lambda\nu} \cdot \frac{1}{G} - \frac{\rho}{G}\frac{\partial^2}{\partial t^2}\,(V^\mu) = 0\,. \tag{86}$$

It was, in fact, with the help of that same Ansatz, which the author [**12**] applied in the usual three-dimensional theory of elasticity, that he succeeded in finding a direct general path of solution that led to differential equations of order only two. The Ansatz read:

$$2\, G\, V_\lambda = -F_{,\lambda} + \Phi_\lambda\,. \tag{87}$$

F represents a scalar and Φ_λ represents a vectorial stress function in that. After substitution in eq. (86), that will give:

$$\frac{-2m+2}{m-2}\, F\,|^{\mu\lambda}_{\cdot,\lambda} + \Phi^\mu\,|^{\lambda}_{\cdot,\lambda} + \frac{m}{m-2}\,\Phi^\lambda\,|^{\mu}_{\ldots,\lambda} + \frac{1}{G}\, T^{\lambda\nu}(-F\,|^{\mu}_{\cdot,\lambda\nu} + \Phi^{\mu}_{\cdot,\lambda\nu}) + \frac{\rho}{G}\frac{\partial^2}{\partial t^2}(F\,|^{\mu} - \Phi^\mu) = 0\,. \tag{88}$$

For brevity of notation, it would be suitable to introduce the following operators:

([5]) See, e.g., **Green**, Cambridge Phil. Soc. Trans. **7** (1839).

$$(\,)|^{\lambda}_{\cdot,\lambda} + \frac{1}{G} T^{\lambda\nu} (\,)_{,\lambda\nu} - \frac{\rho}{G}\frac{\partial^2}{\partial t^2} (\,) = \bar{\Delta} \,, \tag{89}$$

$$(\,)|^{\lambda}_{\cdot,\lambda} + \frac{m-2}{2m-2}\left[\frac{1}{G} T^{\lambda\nu} (\,)_{,\lambda\nu} - \frac{\rho}{G}\frac{\partial^2}{\partial t^2} (\,)\right] = \bar{\bar{\Delta}} \,. \tag{90}$$

Eq. (88) will then go to:

$$\left(1-\frac{2}{m}\right) \bar{\Delta}\Phi^{\mu} - 2\left(1-\frac{2}{m}\right) \bar{\bar{\Delta}} F \,|^{\mu} + \Phi^{\lambda}\,|^{\mu}_{\cdot,\lambda} = 0 \,. \tag{91}$$

That vector equation will be integrable when Φ^{λ} satisfies the second-order differential equation:

$$\bar{\Delta}\Phi^{\lambda} = 0 \,. \tag{92}$$

The remaining equation (91) will then read:

$$2\left(1-\frac{1}{m}\right) \bar{\bar{\Delta}} F = \Phi^{\lambda}{}_{|,\lambda} + C_2 \tag{93}$$

after integration, in which C_2 is an integration constant, just like C_1 . For the sake of further integration, it would be convenient to introduce the vector function X^{λ} in place of Φ^{λ}, which corresponds to the relation:

$$\Phi^{\lambda} = 2\left(1-\frac{1}{m}\right) \bar{\bar{\Delta}} X^{\lambda} \,. \tag{94}$$

Eq. (93) is then, in turn, integrable, and leads to the equation:

$$F = \psi + X^{\lambda}_{\cdot,\lambda} + C_1\, \bar{x}_k\, \bar{x}^k \,, \tag{95}$$

in which ψ satisfies the homogeneous eq. (93):

$$\bar{\bar{\Delta}}\psi = 0 \,, \tag{96}$$

and C_1 is coupled with C_2 by the relation:

$$2\left(1-\frac{1}{m}\right) C_1 \left(6 + \frac{m-2}{(m-1)G} T^{\lambda}_{\lambda}\right) = C_2 \,. \tag{97}$$

It then follows from (92) and (94) that X^{λ} satisfies the fourth-order differential equation:

$$\bar{\Delta}\bar{\bar{\Delta}} X_{\lambda} = 0 \,, \tag{98}$$

which therefore combines solutions of eq. (92), as well as eq. (96). However, one needs will only the former in order to define the displacement components, because solutions of eq. (96) once more drop out of eq. (94) correspondingly when one forms Φ_λ . On the other hand, from eq. (95), they would appear in F only in the scalar form $X^{\lambda}_{\cdot,\lambda}$, which is already included in the function ψ, in general. Thus, one can make the simplifying assumption that X_λ includes only solutions of eq. (92), i.e., that:

$$\bar{\Delta} X_\lambda = 0 , \tag{99}$$

with no restriction of the general validity. As a result of the rules of covariant differentiation, that equation includes not just one of the components X_λ , but all of them. Therefore, it is convenient for the sake of practical considerations to first determine the Cartesian components X_k from the analogous equation:

$$\bar{\Delta} X_k = 0 , \tag{100}$$

which includes only one component, and then calculate X_λ from $X_\lambda = c_\lambda^k X_k$. By contrast, the operator $\bar{\Delta}$ can be written in the new system. When one knows the functions X_λ and ψ, the stress function F will be determined according to eq. (95), and it will satisfy the differential equation:

$$\bar{\Delta}\bar{\bar{\Delta}} F = 0 . \tag{101}$$

The displacement components are then calculated from:

$$2\, G\, V_\lambda = - F_{,\lambda} + 2\left(1 - \frac{1}{m}\right)\bar{\bar{\Delta}} X_\lambda . \tag{102}$$

The stress tensor can then either follow from the displacements by way of the equations:

$$t^{\lambda\mu} = G\left(V^\lambda |^\mu + V^\mu |^\lambda + \frac{2}{m-2} g^{\lambda\mu} V^{\nu}_{\cdot,\nu}\right) , \tag{103}$$

which follow from (69), (42), and (84), or they can be determined directly from the stress functions, for which the relations:

$$t^{\lambda\mu} = F |^{\lambda\mu} + \left(1 - \frac{1}{m}\right)\bar{\Delta}\,(X^\lambda |^\mu + X^\mu |^\lambda) + \frac{g^{\lambda\mu}}{m-2}\left(- F |^{\nu}_{\cdot,\nu} + 2\left(1 - \frac{1}{m}\right)\bar{\bar{\Delta}} X^{\nu}_{\cdot,\nu}\right) \tag{104}$$

are definitive. Eq. (104), which is obtained from (103) when one substitutes the expressions in (102), allows one to check the calculation by substitution in the equilibrium conditions (77). A brief calculation will show that they are, in fact, fulfilled.

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