"Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie. Der Hohlkegel unter Einzellast als Beispiel," Zeit. angew. Math. Mech. **14** (1934), 203-212.

# A new Ansatz for the solution of spatial problems in the theory of elasticity. The hollow cone under an isolated load as an example ([1]).

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Translated by D. H. Delphenich

**1. Introduction.** – As is known, stress functions are available for the solution of planar and axially-symmetric problems in the theory of elasticity from which displacements stresses can be obtained by differentiation from the outset that will satisfy all elastic equations. By contrast, such a method for the solution of general spatial problems is obviously still lacking. The basis for that is in the types of Ansätze that have been explored up to now that would all lead to integrals. **Maxwell** ([2]) already related stresses and displacements to three functions and exhibited the differential equation that exists between the functions by substituting them in the basic spatial equation, but without solving them in general form without the help of a further Ansatz. With later Ansätze, on the one hand, the same path was taken and in that way, one will succeed in satisfying the differential equation that appears in each case in integral form. Hence, e.g., one of those types of solutions ([3]) contains an integration over $x$ (Cartesian coordinates), and another, an integration over $r$ (polar coordinates). On the other hand, one is anxious to bring the problem back to a boundary-value problem in potential theory ([4]), which has already been done. The integral equations that appear in that way are solved by either the **Fredholm** theory or the process of successive approximations. Although each of those Ansätze represents a mathematically-unimpeachable solution of the problem, the search for more rigorous solutions in arbitrary coordinates is linked with a relatively-large expenditure of calculation. It would then be interesting to learn about a very simple Ansatz in what follows, with whose help, one will succeed in obtaining displacements and stresses by differentiation alone from general spatial stress functions that are composed of three harmonic functions. The associated differentiation schema will be completely symmetric, and for that reason can be easily adapted to arbitrary coordinates.

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([1]) The present treatise is a theoretical result of the research that the author carried out at the Mech. Techn. Labor. der Techn. Hochsch. in Munich on the initiative of Prof. Dr. **L. Föppl**. Therefore, I would like to express my most sincere thanks to the Notgemeinschaft der Deutschen Wissenschaft for their kind support.

([2]) **J. C. Maxwell**, *Scientific Papers of J. C. Maxwell*, Paris, 1927, v. 2, pp. 198, *et seq.*

([3]) **E. Trefftz**, *Mathematische Elastizitätstheorie*, Handbuch. d. Phys., Bd. VI, pp. 92.

([4]) **L. Lichtenstein**, "Über die erste Randwertaufgaben der Elastizitätstheorie," Math. Zeit. **20** (1924), pp. 21; furthermore, **A. Korn**., "Über die Lösung des Grundproblems der Elastizitätstheorie," Math. Ann. **75** (1914), pp. 497.

As an example, the problem of the hollow cone under an isolated load will be treated. The precise expressions for the stresses will be given for three different types of loads.

**2. The solution of the basic equations of elasticity with the help of the new Ansatz**. – In Cartesian coordinates $x$, $y$, $z$, with the normal stresses $\sigma_x$ , $\sigma_y$ , $\sigma_z$ , shear stresses $\tau_{xy}$ , $\tau_{yz}$ , $\tau_{zx}$ , and the displacements $\xi$, $\eta$, $\zeta$, the equilibrium conditions will read ([5]):

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0, \tag{1}$$

etc., with cyclic permutations. Moreover, the following known relations exist between stresses and displacements on the basis of **Hooke**'s law and the assumption of small deformations ([1]):

$$\sigma_x = 2G\left(\frac{\partial\xi}{\partial x}+\frac{e}{m-2}\right), \ldots, \tag{2}$$

in which one sets:

$$\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}=e, \tag{3}$$

and

$$\tau_{xy} = G\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right), \ldots \tag{4}$$

By combining these with eq. (1), one will get the so-called "basic equations of elasticity":

$$\Delta\xi+\frac{m}{m-2}\frac{\partial e}{\partial x} = 0, \ldots, \tag{5}$$

in which one defines:

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\,. \tag{6}$$

Now, it so happens that the displacements can be derived from harmonic functions with the help of a new Ansatz in such a way that eqs. (3) and (5) can be fulfilled without any integrals appearing, as in the methods up to now. *One succeeds in doing that with the following Ansatz:*

([5]) **A. and L. Föppl**, *Drang und Zwang*, 1st ed., Bd. I, pp. 16 *et seq.*

$$\left.\begin{aligned} 2G\xi &= -\frac{\partial F}{\partial x} + C\,\Phi_1, \\ 2G\eta &= -\frac{\partial F}{\partial y} + C\,\Phi_1, \\ 2G\zeta &= -\frac{\partial F}{\partial z} + C\,\Phi_3. \end{aligned}\right\} \tag{7}$$

$\Phi_1$, $\Phi_2$, $\Phi_3$ are harmonic functions in this, so they all satisfy the equation:

$$\Delta\Phi = 0. \tag{8}$$

In order to explain the connection between these functions and the stress function $F$, we substitute the Ansatz (7), which corresponds to the displacements, in eq. (3) and (5). With consideration given to eq. (8), we will obtain:

$$-\Delta F + C\left(\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y} + \frac{\partial\Phi_3}{\partial z}\right) = 2Ge \tag{9}$$

and

$$\frac{\partial}{\partial x}\left(-\Delta F + \frac{m}{m-2}2Ge\right) = 0, \ldots \tag{10}$$

Eq. (10) demands that:

$$-\Delta F + \frac{m}{m-2}2Ge = \text{const.},$$

or, since the constant on the right-hand side is inessential:

$$2Ge = \left(1 - \frac{2}{m}\right)\Delta F. \tag{11}$$

If one substitutes this in eq. (9) then one will get:

$$C\left(\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y} + \frac{\partial\Phi_3}{\partial z}\right) = 2\left(1 - \frac{1}{m}\right)\Delta F. \tag{12}$$

*Whereas the corresponding equation is soluble only in integral form in the methods up to now, here, it can be satisfied in an entirely simple way.* Namely, set:

$$F = \Phi_0 + x\,\Phi_1 + y\,\Phi_2 + z\,\Phi_3, \tag{13}$$

in which $\Phi_0$ also satisfies eq. (8), so one will have:

$$\Delta F = 2\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_3}{\partial z}\right) \tag{14}$$

and

$$\Delta\Delta F = 0. \tag{15}$$

Eq. (12) will ultimately be satisfied completely when we set:

$$C = 4\left(1 - \frac{1}{m}\right). \tag{16}$$

That constant already appears in an axially-symmetric stress state ([6]), and indeed, one sets:

$$2\left(1 - \frac{1}{m}\right) = a. \tag{17}$$

One finally obtains the displacements from:

$$2G\,\xi = \frac{\partial F}{\partial x} + 2a\,\Phi_1\,,\ \ldots \tag{18}$$

From eqs. (2) and (4), by a brief computation, in which one makes use of eq. (11), (14), and (17), the stresses will take on the following expressions:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + a\left(\frac{\partial \Phi_1}{\partial x} - \frac{\partial \Phi_2}{\partial y} - \frac{\partial \Phi_3}{\partial z}\right),\ \ldots \tag{19}$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y} + a\left(\frac{\partial \Phi_1}{\partial y} + \frac{\partial \Phi_2}{\partial x}\right),\ \ldots \tag{20}$$

One easily convinces oneself that the equilibrium conditions are, in fact, fulfilled. Likewise, the six compatibility conditions are fulfilled, which can be exhibited as combinations of eqs. (2), (3), and (4).

**3. Interpretation of the Ansatz.** – The Ansatz shall now be discussed in more detail in the context of a special theory of elasticity.

*The theory of torsion of prismatic rods* corresponds to the starting equations:

$$\Phi_0 = -x\,\Phi_1\,, \qquad \Phi_1 = \Phi_1\,(y, z), \qquad \Phi_2 = \frac{G\,\vartheta}{a}\,x\,z, \qquad \Phi_3 = -\frac{G\,\vartheta}{a}\,x\,y. \tag{20.a}$$

([6]) **H. Neuber**, “Beiträge für den achssymmetrischen Spannungszustand,” Diss., Munich, 1932, pp. 3.

One will then have:

$$F = 0, \qquad \xi = \frac{a}{G}\Phi_1, \qquad \eta = \vartheta xz, \qquad \zeta = -\vartheta xy. \tag{20.b}$$

*The theory of torsion of round rods of varying cross-sections* corresponds to the equations:

$$\Phi_0 = 0, \qquad \Phi_1 = 0, \qquad \Phi_2 = \frac{G}{a} z \cdot \varphi(x, r), \qquad \Phi_3 = \frac{G}{a} y \cdot \varphi(x, r), \tag{20.c}$$

in which $x$, $r$ are based in cylindrical coordinates ($r = \sqrt{y^2 + z^2}$). One will have:

$$F = 0, \qquad \xi = 0, \qquad \eta = z \cdot \varphi, \qquad \zeta = -y \cdot \varphi. \tag{20.d}$$

The known differential equation for $\varphi$ follows from $\Delta\Phi_2 = 0$:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{3}{r}\frac{\partial \varphi}{\partial r} = 0. \tag{20.e}$$

*The case of* $F = 0$ *then corresponds to the general torsion.*

One will get the *planar theory of elasticity* with:

$$\Phi_0 = \Phi_0' + a\,\Phi_1', \qquad \Phi_1 = \frac{\partial \Phi_1'}{\partial x}, \qquad \Phi_2 = 0, \qquad \Phi_3 = 0, \qquad F = F' + a\,\Phi_1', \tag{20.f}$$

in which $\Phi_0'$ and $\Phi_1'$ are two new harmonic functions that depend upon only $x$ and $y$. $F'$ now corresponds to the **Airy** stress function.

*The axially-symmetric theory of elasticity* will emerge from the same substitution when we assume that $\Phi_0'$ and $\Phi_1'$ depend upon only $x$ and $r$ (cylindrical coordinates; cf., *supra*). $F'$ is the axially-symmetric stress function ([7]).

*On the basis of that connection, we would like to refer to F as the* ***spatial stress function***.

As for the general stress state, it is remarkable that one can always set one of the four harmonic functions to zero without compromising the completeness. That will emerge from the substitution:

$$\left.\begin{aligned} \Phi_0 &= \Phi_0' + 2a\Phi_3' - \left( x\frac{\partial \Phi_3'}{\partial x} + y\frac{\partial \Phi_3'}{\partial y} + z\frac{\partial \Phi_3'}{\partial z} \right), \quad \Phi_1 = \Phi_1' + \frac{\partial \Phi_3'}{\partial x}, \\ \Phi_2 &= \Phi_2' + \frac{\partial \Phi_3'}{\partial y}, \quad \Phi_3 = \frac{\partial \Phi_3'}{\partial z}, \quad F = F' + 2a\,\Phi_3', \end{aligned}\right\} \tag{20.g}$$

([7]) **A.** and **L. Föppl**, *Drang und Zwang*, 2nd ed., Bd. II, Munich and Berlin, 1928, pp. 208.

which does not change the generality of the four functions $\Phi_0$, $\Phi_1$, $\Phi_2$, $\Phi_3$ and leads to the new system of equations:

$$\left.\begin{aligned} F' &= \Phi_0' + x\,\Phi_1' + y\,\Phi_2', \quad & 2G\xi &= -\frac{\partial F'}{\partial x} + 2a\,\Phi_1', \\ 2G\eta &= -\frac{\partial F'}{\partial y} + 2a\,\Phi_2', \quad & 2G\zeta &= -\frac{\partial F'}{\partial z}, \end{aligned}\right\} \tag{20.h}$$

which contains only the three harmonic functions $\Phi_0'$, $\Phi_1'$, $\Phi_3'$. In reality, only three harmonic functions will be required. *A well-defined harmonic function then corresponds to precisely three distinct stress states. The manifold of general elastic states is equal to the three-fold manifold of harmonic functions.*

For the search for solutions, it is preferable in each case to use a coordinate system in which the boundary surface of the body is included. On that basis, we would like to go to curvilinear coordinates.

**4. Transition to curvilinear coordinates.** – If the stress components $\sigma_u$, $\tau_{uv}$, …, and the displacement components $U$, $V$, $W$ belong to the orthogonal coordinate system whose axes define the direction cosines cos ($x$, $u$), …, with respect to the original axes then since the latter transform as vector components, one will have:

$$U = \cos(x, u) \cdot \xi + \cos(y, u) \cdot \eta + \cos(z, u) \cdot \zeta, \ \ldots \tag{21}$$

As is known, one obtains the direction cosines from:

$$\cos(x, u) = \frac{1}{h_u}\frac{\partial x}{\partial u}, \ \ldots, \tag{22}$$

in which $h_u$, $h_v$, $h_w$ carry the curvilinear deformation calculation and are determined from the equations:

$$h_u^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \ \ldots \tag{23}$$

(In tensor notation, one will have $h_u = \sqrt{g_{11}}$ .)

If one introduces the expressions for $\xi$, $\eta$, $\varphi$ that are in eq. (18) and observes that the first derivatives of a function also transform as vector components then one will ultimately get:

$$U = \frac{1}{2G\,h_u}\left(-\frac{\partial F}{\partial u} + 2a\,\Phi_1\frac{\partial x}{\partial u} + 2a\,\Phi_2\frac{\partial y}{\partial u} + 2a\,\Phi_2\frac{\partial z}{\partial u}\right), \ \ldots \tag{24}$$

In order to ascertain the stresses, it is simplest for one to employ the equations that exist between stresses and displacements in curvilinear coordinates that were probably first given by **Borchardt** ([8]). With the notation that has been chosen here, one will have:

$$\sigma_u = \frac{2G}{h_u}\left(\frac{\partial U}{\partial u} + \frac{V}{h_v}\frac{\partial h_u}{\partial v} + \frac{W}{h_w}\frac{\partial h_u}{\partial w} + \frac{2-a}{2(a-1)} h_u \cdot e\right), \ldots \tag{25}$$

$$\tau_{uv} = G\left[\frac{h_v}{h_u}\frac{\partial}{\partial u}\left(\frac{V}{h_v}\right) + \frac{h_u}{h_v}\frac{\partial}{\partial v}\left(\frac{U}{h_u}\right)\right], \ldots \tag{26}$$

The **Laplace** operator goes to:

$$\Delta = \frac{1}{h_u\,h_v\,h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v\,h_w}{h_u}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w\,h_u}{h_v}\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u\,h_v}{h_w}\frac{\partial}{\partial w}\right)\right]. \tag{26.a}$$

Moreover, once all mathematical prerequisites have been given, the process shall find an application to the hollow cone with an arbitrarily-directed isolated load.

**5. The hollow cone with arbitrarily-directed isolated load. –** It is preferable to base the discussion on polar coordinates with:

$$x = u \cos v, \qquad y = u \sin v \cos w, \qquad z = u \sin v \sin w. \tag{27}$$

One gets from eq. (25) that:

$$h_u = 1, \qquad h_v = u, \qquad h_w = u \cdot \sin v. \tag{28}$$

As is clear from Secs. **1** and **2**, the axis of the cone coincides with the $X$-axis; moreover, the vertex of the cone is, at the same time, the coordinate origin. The hollow cone will be bounded by the surfaces $v = \gamma$ and $v = \delta$. The freedom in the load on the outer surface implies the first group of boundary conditions:

$$\left.\begin{array}{l}\text{For } v = \gamma \\ \text{and } v = \delta\end{array}\right\} \sigma_v = 0, \qquad \tau_{uv} = 0, \qquad \tau_{vw} = 0, \tag{29}$$

and thus, six conditions. Further conditions will be deduced from the fact that for an arbitrary section that is made through the hollow cone, the stresses that act in the surface of the section must define an equilibrium system along with the isolated force. The $x$-component of the isolated force must be equal and opposite to the sum of all the $x$-components of the stresses that act in the surface of the section. Corresponding

([8]) **Borchardt**: J. f. Math. (Crelle) **76** (1873), 45-48.

statements must also be true for the $y$ and $z$-components. If the section is made along a spherical surface, and $P_x$ , $P_y$ , $P_z$ are the components of the isolated force then it will follow that:

$$-P_x - \int_F \sum_{\mu=u,v,w} \tau_{uu} \cos(x,\mu)\, dF\,, \ldots \tag{30}$$

In this, one sets $\sigma_u = \tau_{uu}$ for the sake of simplicity. Further conditions will come from requiring equilibrium under rotations. For the reference point, the moment of the isolated force must be equal and opposite to the moment of the stresses that act in the surface of the section. If the vertex of the cone is chosen to be the reference point and if $M_x$ , $M_y$ , $M_z$ are the moments of the isolated force around the $X$, $Y$, $Z$ axes, resp., then one will have:

$$M_x = \int_F \sum_{\mu=u,v,w} \tau_{uu} [y\cos(z,\mu) - z\cos(y,\mu)]\, dF\,, \ldots \tag{31}$$

*The general solution to the problem will first decompose into six distinct particular solutions when one only demands that only one of those six integrals should be non-zero.* Since the solutions for $P_y$ and $P_z$ , and likewise the ones for $M_y$ and $M_z$ , will go to each other when one switches the $Y$ and $Z$-axis, that will reduce their number to four. Furthermore, since the solution for pure torsion ($M_x \neq 0$) was given already by **A. Föppl** ([9]) with the help of the theory of the torsion of round rods with varying cross-section, only three more particular solutions will remain to be given.

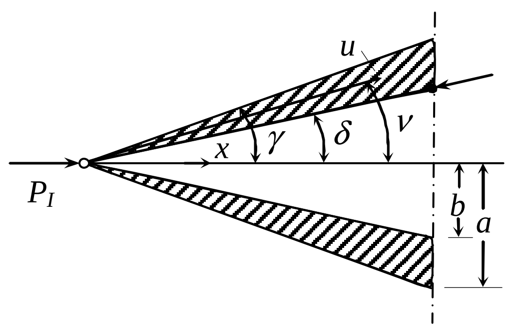


Figure 1.

**A. The isolated load acts in the direction of the *X*-axis and is applied to the vertex of the cone (see Fig. 1).**

In this case, all integrals must vanish, except for $P_x$ . In detail, the expression for $P_x$ reads:

$$-P_x = u^2 \int_0^{2\pi} \int_\delta^\gamma (\sigma_u \cos v - \tau_{uv} \sin v)\, \sin v\, dv\, dw\,. \tag{32}$$

In one's search for suitable harmonic functions that satisfy those conditions, one will arrive at the following Ansatz:

([9]) **A.** and **L. Föppl**, *Drang und Zwang*, 2nd ed., Bd. II, Munich and Berlin, 1928, pp. 108.

$$\left.\begin{aligned} &\Phi_0 = A[\ln u + \ln(1+\cos v)] + C(\ln u + \ln \sin v), \\ &\Phi_1 = B \cdot \frac{1}{u}, \quad \Phi_2 = 0, \quad \Phi_3 = 0. \end{aligned}\right\} \tag{33}$$

With an application of eq. (13), one will obtain the stress function:

$$F = (A + C) \ln u + A \ln (1 + \cos v) + C \ln \sin v + B \cos v\,. \tag{34}$$

The displacements are defined from this corresponding to eq. (24). Finally, with the help of eq. (25) and (26), one will get the following stresses:

$$\left.\begin{aligned} &\sigma_u = \frac{1}{u^2}[A - (2+a)B\cos v + C], \\ &\sigma_v = \frac{1}{u^2}\left[-A\frac{\cos v}{1+\cos v} + (a-1)B\cos v + C\cot^2 v\right], \\ &\sigma_w = \frac{1}{u^2}\left[-A\frac{1}{1+\cos v} + (a-1)B\cos v - C\frac{1}{\sin^2 v}\right], \\ &\tau_{uv} = \frac{\sin v}{u^2}\left[-A\frac{1}{1+\cos v} + (a-1)B + C\frac{\cos v}{\sin^2 v}\right], \\ &\tau_{vw} = 0, \quad \tau_{uw} = 0. \end{aligned}\right\} \tag{35}$$

One sees that the following relation exists between $\sigma_v$ and $\tau_{uv}$ :

$$\sigma_v - \tau_{uv} \cdot \cot v = 0. \tag{36}$$

The six boundary conditions (29) then reduce to two. It follows from both equations that:

$$\frac{A}{C} = -\frac{1+\cos\gamma\cos\delta}{(1-\cos\gamma)(1-\cos\delta)}, \qquad \frac{B}{C} = -\frac{1}{(a-1)(1-\cos\gamma)(1-\cos\delta)}\,. \tag{37}$$

In order to determine $C$, one can appeal to eq. (32). It will yield:

$$C = \frac{P_x(a-1)(1-\cos\gamma)(1-\cos\delta)}{2\pi(\cos\delta - \cos\gamma)[\cos^2\gamma + \cos^2\delta + (2-a)\cos\gamma\cos\delta]}\,. \tag{38}$$

The *maximal stress* is found on the inner side ($v = \delta$, see Fig. 1). Here, we have $u = \frac{b}{\sin\delta} = a\frac{\cot\gamma}{\cos\delta}$. We will get:

$$\sigma_{\max} = -\frac{P_x(\cos\gamma - \cos\delta)[3\cos\delta - (a-1)\cos\gamma]}{2\pi(a^2 - b^2)\cos^2\gamma[\cos^2\gamma + \cos^2\delta + (2-a)\cos\gamma\cos\delta]}\,. \tag{39}$$

For the *complete cone*, one will get:

$$\sigma_{\max} = -\frac{P_x(1-\cos\gamma)[3-(a-1)\cos\gamma]}{2\pi a^2\cos^2\gamma[1+(2-a)\cos\gamma+\cos^2\gamma]}. \tag{40}$$

The solution for the complete cone coincides with another solution for the axially-stressed hyperboloid when the latter is considered to be at a large distance from the narrowest cross-section ([10]).

For the *conical shell* ([11]), one will get a simple formula when one passes to the limit $\gamma \to \delta$. If $h$ is the wall thickness then one will have $a^2 - b^2 \approx \frac{2ah}{\cos\gamma}$, and one will get:

$$\sigma_{\max} = -\frac{P_x}{2\pi a h\cos\gamma}. \tag{41}$$

**B. The isolated load acts in the direction of the *Y*-axis and is applied to the vertex of the cone (see Fig. 2).**

$$-P_y = u^2\int_0^{2\pi}\int_\delta^\gamma [\sigma_u \sin v\cos w + \tau_{uv}\cos v\cos w - \tau_{uv}\sin w]\sin v\,dv\,dw. \tag{42}$$

The following Ansatz is sufficient for the fulfillment of these conditions:

$$\left.\begin{aligned}
\Phi_0 &= \cos w\left[B\frac{\sin v}{1+\cos v} + D\frac{\sin v}{1-\cos v}\right],\\
\Phi_1 &= \frac{\cos w}{u}\left[C\frac{\sin v}{1+\cos v} + E\frac{\sin v}{1-\cos v}\right],\\
\Phi_2 &= \frac{A}{u}, \quad \Phi_3 = 0.
\end{aligned}\right\} \tag{43}$$

Eq. (13) correspondingly implies the stress function:

$$F = \cos w\sin v\left[A + C - E + \frac{B-C}{1+\cos v} + \frac{D+E}{1-\cos v}\right]. \tag{44}$$

We will get the stresses from this with the help of eqs. (24), (25), and (26):

([10]) **H. Neuber**, "Beiträge für den achssymmetrischen Spannungszustand," Diss., Munich, 1932, pp. 37, *et seq.*

([11]) The formulas for the conical shell are valid only for wall thicknesses that are not too small, since the assumption of small displacements would no longer be valid otherwise.

$$\left.\begin{aligned}
\sigma_u &= \frac{\sin v\cos w}{u^2}\left[(2+a)(-A-C+E)+\frac{2aC}{1+\cos v}-\frac{2aE}{1-\cos v}\right],\\
\sigma_v &= \frac{\sin v\cos w}{u^2}\left[(a-1)(A+C-E)+\frac{2aC}{1+\cos v}+\frac{2aE}{1-\cos v}+\frac{-B+C}{(1+\cos v)^2}-\frac{D+E}{(1-\cos v)^2}\right],\\
\sigma_w &= \frac{\sin v\cos w}{u^2}\left[(a-1)(A+C-E)+\frac{B-C}{(1+\cos v)^2}+\frac{D+E}{(1-\cos v)^2}\right],\\
\tau_{uv} &= \frac{\cos w}{u^2}\left[(a-1)(-A-C-E)\cos v+2a(C+E)+\frac{B-(1+a)C}{1+\cos v}-\frac{D+(1+a)E}{1-\cos v}\right],\\
\tau_{vw} &= \frac{\sin v\cos w}{u^2}\left[\frac{aC}{1+\cos v}+\frac{aE}{1-\cos v}+\frac{B-C}{(1+\cos v)^2}-\frac{D+E}{(1-\cos v)^2}\right],\\
\tau_{uw} &= \frac{\sin w}{u^2}\left[(a-1)A+(1+a)(-C+E)+\frac{B+(1+a)C}{1+\cos v}-\frac{D+(1+a)E}{1-\cos v}\right].
\end{aligned}\right\} \quad (44.a)$$

That shows, in turn, that a relation exists between the stresses that are involved with the boundary conditions (29), and indeed in this case:

$$\sigma_v \cos v + \tau_{uv} \sin v - \tau_{vw} \cot w = 0. \tag{45}$$

Thus, the conditions (29) correspond to only four equations. Together with eq. (42), one then has five equations for the five still-unknown constants at one's disposal, such the latter will be determined uniquely.

We would like to set:

$$\cos \gamma = c, \qquad \cos \delta = d, \qquad N = (1 + c)\,(1 + d)\,[(d - c)2 + (c + d)(1 - cd)], \tag{46}$$

to abbreviate.

We then obtain:

$$\left.\begin{aligned}
\frac{A}{C} &= \frac{4}{(a-1)N}[(d-c)^2+(2-a)cd(1-cd)],\\
\frac{B}{C} &= \frac{(1+c)(1+d)}{N}[-(a-1)(d-c)^2-(a-1)(c+d)(1-cd)-2acd(1-cd)],\\
\frac{D}{C} &= \frac{(1-c)(1-d)}{N}[-(a-1)(d-c)^2+(a-1)(c+d)(1-cd)-2acd(1-cd)],\\
\frac{E}{C} &= \frac{(1+c)(1+d)}{N}[-(d-c)^2+(c+d)(1-cd)],\\
C &= \frac{P_y(a-1)N}{4\pi a(d-c)\{(d-c)^2[(3-cd)(1-cd)-(d-c)^2]+(4-a)cd(1-cd)^2\}}.
\end{aligned}\right\} \quad (46.a)$$

The *maximal bending stress* appears at the location $w = 180^{\circ}$, $v = \gamma$ (see Fig. 2). Here, one will have $u = a / \sin \gamma$. One will get:

$$\sigma_{max} = \frac{P_y \sin\gamma(1-c^2)\{3(d-c)^2 + d[3c-(a-1)d](1-cd)\}}{\pi a^2 (d-c)\{(d-c)^2[(3-cd)(1-cd)-(d-c)^2]+(4-a)cd(1-cd)^2\}}. \tag{47}$$

The *maximal shear stress* appears at the location $w = 90^{\circ}$, $v = \delta$ (see Fig. 1). Here, we will have $u = ac / (d \sin \gamma)$. We get:

$$\sigma_{max} = \frac{P_y (a-1)\, d^2 (1-c^2)(1-cd)}{\pi a^2 c\{(d-c)^2[(3-cd)(1-cd)-(d-c)^2]+(4-a)\,cd(1-cd)^2\}}. \tag{48}$$

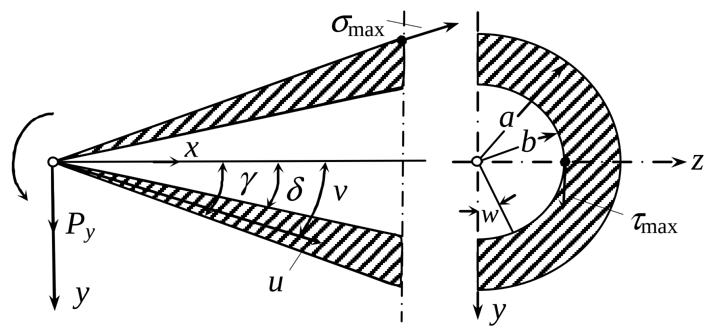


Figure 2.

For the *complete cone,* the constants $D$ and $E$ must be set to zero from the outset, since the associated functions possess poles along the axis. One will then get:

$$\sigma_{max} = \frac{P_y(4-a)\sin\gamma(1+c)}{\pi a^2(1-c)[2+(2-a)c]}, \tag{49}$$

$$\tau_{max} = \frac{P_y(a-1)(1+c)}{2\pi a^2 c[2+(2-a)c]}, \tag{50}$$

On the other hand, if one sets $d = 1$ in eqs. (47) and (48) then one will get the formulas for *the complete cone with a fine axial drill-hole*. $\sigma_{max}$ will not change, but one will get $\tau_{max}$ from:

$$\tau_{max} = \frac{P_y(a-1)(1+c)}{\pi a^2 c[2+(2-a)c]}; \tag{51}$$

i.e., precisely twice the value. *A fine axial drill-hole then raises the shear stress by 100%.*

Furthermore, the stresses for the *conical shell* can also be given here by passing to the limit, and indeed, one will have:

$$\sigma_{max} = \frac{P_y}{\pi a h \sin\gamma}, \qquad \tau = 0. \tag{52}$$

**C. The isolated load acts in the direction of the *Y*-axis. Its point of application lies on the *X*-axis at infinity (pure bending).**

In this case, all integrals must vanish, with the exception of the one for $M_z$ . In detail the expression for $M_z$ reads:

$$M_z = u^3 \int_0^{2\pi} \int_\delta^\gamma (\tau_{uv} \cos w - \tau_{uv} \cos v \sin w) \sin v \, dv \, dw \, . \tag{53}$$

In order to look for suitable functions, one arrives at the following Ansatz:

$$\left.\begin{aligned}
\Phi_0 &= \frac{\cos w}{u}\left[ B\frac{\sin v}{1+\cos v} + D\frac{\sin v}{1-\cos v} + E\frac{1}{\sin v}\right], \\
\Phi_1 &= \frac{\cos w}{u^2}\left[ C\sin v + E\left(\sin v \ln \tan\frac{v}{2} - \cot v\right)\right], \\
\Phi_2 &= \frac{1}{u^2}\left[ A\sin v + E\left(\cos v \ln \tan\frac{v}{2} + \cot v\right)\right], \quad \Phi_2 = 0.
\end{aligned}\right\} \tag{54}$$

With an application of eq. (13), the stress function will become:

$$F = \frac{\cos w \sin v}{u}\left[ (A+C)\cos v + 2E\left(\cos v \ln \tan\frac{v}{2} + 1\right) + \frac{B}{1+\cos v} + \frac{D}{1-\cos v}\right]. \tag{55}$$

When one calculates the stresses with the help of eqs. (24), (25), and (26), that will show that in that case, the following relation exists between $\sigma_y$ and $\tau_{vw}$ :

$$\sigma_y - \tau_{vw} \cot w \cos v = 0. \tag{56}$$

The six boundary conditions (29) then reduce to four again. Eq. (53) gets added as a further condition. The solution to that integral also poses no complications here. One will once more have five equations for five constants at one's disposal. Let us now perform the calculation. Let the *maximal stress* now be given. For the sake of brevity, we would like to set:

$$\cos \gamma = c, \qquad \cos \delta = d, \qquad \ln \tan\frac{\gamma}{2} = g, \quad \ln \tan\frac{\gamma}{2} = k, \tag{57}$$

and further:

$$\left.\begin{aligned}N &= \frac{4a(1+cd)(1-cd)^2}{3(1-c^2)(1-d^2)}[6(1-c^2d^2)+(4-a)\,cd\,(2+cd)^2] \\ &+\frac{4}{3}(1-c^2d^2)[6(1-c^2d^2)+(4-a)\,cd\,(18+5a+acd)] \\ &+\frac{g-k}{c-d}\Big[12(4-a)cd(1-cd)^2(2-a\_2cd)-\tfrac{4}{3}(4-a)(2+a)cd(c-d)^4 \\ &+8(4-a)(c-d)^2(1-cd)\{1+(1+a)cd-c^2d^2\}-12a(c-d)^2(1-c^2)(1-d^2)\Big].\end{aligned}\right\} \quad (58)$$

The *maximal bending stress*, which will once more appear at the location $w = 180^{o}$, $v = \gamma$ (cf., Fig. 2), proves to be:

$$\left.\begin{aligned}\sigma_{\max} &= \frac{M_z(1-c^2)^2}{\pi a^2(d-c)N}\Big\{4\frac{1-c^2d^2}{(1-c^2)(1-d^2)}[(12+5a+a^2)\,c\,(1-d^2) \\ &+a\,(10-a)\,c+2a\,(1-a)\,d]+4\,(48-11a-a^2)\,d\,(1+cd)-240d \\ &+12\,(1-cd)\,\frac{g-k}{c-d}[2\,(10-a)\,cd^2-(4+5a)\,c-a\,(1-a)\,d] \\ &+4\,(1-a)(8+a)\,d\,(c-d)(g-k)\Big\}.\end{aligned}\right\} \quad (59)$$

The *maximal shear stress* [once more at the location $w = 90^{o}$, $v = \delta$ (cf., Fig. 2)] will be:

$$\left.\begin{aligned}\tau_{\max} &= \frac{M_z\sin\gamma(1-c^2)\,d^2}{\pi a^2(d-c)c^3N}\Big\{4(1-c^2d^2)\Big[c\,\frac{2(1-a)+(4-a)\,cd}{1-c^2}+\frac{6d}{1-d^2}\Big] \\ &+4a[9+(11+a)\,cd]+(d-c)+4\,(4-a)\,c\,[a+6cd-(6+a)\,cd^2] \\ &+12\,(1-cd)(1-d^2)\,\frac{g-k}{c-d}[-a\,(1-a)\,c-3a\,d+2\,(4-a)c^2\,d] \\ &+4\,(1-a)(8+a)\,c\,(c-d)(g-k)\Big\}.\end{aligned}\right\} \quad (60)$$

For the *complete cone*, the constants $D$ and $E$ must be set to zero from the outset, since the associated functions will be infinite along the axis. One will have:

$$\sigma_{\max} = \frac{3M_z(1+c)^2[(10-a)c+2(1-a)]}{\pi a^3[6\,(1-c^2)+(4-a)\,c\,(2+c)^2]}, \quad (61)$$

$$\tau_{\max} = \frac{9M_z(1+c)^2\sin\gamma}{\pi a^3c^3[6\,(1-c^2)+(4-a)\,c\,(2+c)^2]}. \quad (62)$$

For the *complete cone with a fine axial drill-hole*, [from eq. (60) with $d = 1$], one will have:

$$\tau_{\max} = \frac{18M_z(1+c)^2 \sin\gamma}{\pi a^3 c^3[6(1-c^2)+(4-a)c(2+c)^2]} \qquad \text{(i.e., raised by 100\%).}$$

For the *conical shell*, when one passes to the limit $\gamma \to \delta$, one will get:

$$d - c \approx \frac{ha}{u^2}, \qquad \frac{g-k}{c-d} \approx -\frac{1}{\sin^2\gamma}, \tag{63}$$

and one will get:

$$\sigma_{\max} = \frac{M_z}{\pi a^2 h \cos\gamma}, \qquad \tau_{\max} = \frac{M_z \tan\gamma}{\pi a^2 h} \tag{64}$$

for the stresses.

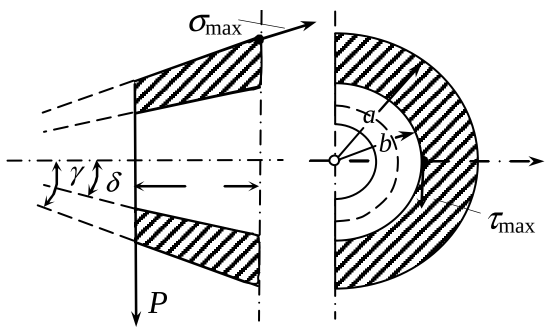


Figure 3.

By superimposing this with the stress state *B*, we can also give the formulas for the *truncated cone* (see Fig. 3). To that end, we decompose the force *P*, which should act upon the vertex perpendicular to the axis at a distance of $a \cot\gamma - l$ (let $l$ be the length of the truncated cone), into a force of equal magnitude that acts upon the vertex, which is identified with $P_y$ , and a bending moment of magnitude $-P(a\cot\gamma - l)$, which is set equal to $M_z$ .

For the *truncated conical shell*, one will have:

$$\sigma_{\max} = \frac{P\cdot l}{\pi a^2 h\cos\gamma}, \qquad \tau_{\max} = -\frac{P}{\pi a^2 h}\left(1 - \frac{l}{a}\tan\gamma\right) \tag{65}$$

Finally, let us especially give the formulas here for *weakly-damped waves* (viz., the truncated complete cone for small $\gamma$), which will be:

$$\sigma_{\max} = \frac{4P\cdot l}{\pi a^3}\left[1 + \frac{2+\frac{1}{m}}{3\left(1+\frac{1}{m}\right)}\frac{a}{l}\tan\gamma\right], \quad \tau_{\max} = -\frac{P\left(3+\frac{2}{m}-4\frac{l}{a}\tan\gamma\right)}{2\pi\left(1+\frac{1}{m}\right)a^2}, \tag{66}$$

with $c \approx 1 - \tan^2\gamma$, and when one neglects the higher powers of $\tan\gamma$.

On the other hand, we can ascertain precisely values for the *hollow cylinder* when set $c \approx 1 - \frac{a^2}{2u^2}$, $d \approx 1 - \frac{b^2}{2u^2}$, and $u = -\infty$. The maximal shear stress will become:

$$\tau_{\max} = -\frac{P\left[\left(3+\frac{2}{m}\right)a^2 + \left(1+\frac{2}{m}\right)b^2\right]}{\pi\left(1+\frac{1}{m}\right)(a^4 - b^4)} \quad . \tag{67}$$

*For the cone under combined bending, one will then get larger bending stresses, but smaller shear stresses than for the cylinder.* That can be explained simply by saying that for the cone, the bending stresses also possess transverse force components. The maximal shear stresses will be raised by 100% by means of fine axial drill-hole.

**6. Summary.** – A new Ansatz makes it possible to derive the three components of the elastic displacement vector from four harmonic functions by differentiation alone. The associated system of equations can be converted to a curvilinear coordinate system in a simple way. It will also represent the general solution to the elastic state when one of the four functions is set to zero. As an example, the stress distribution in the hollow cone with an isolated load was given.