"Ueber eine einfache Methode zur Begründung des Princips der virtuellen Verrückungen," Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig **38** (1886), 70-74.

On a simple method for establishing the principle of virtual velocities ([1])

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n mass-points m_1 (x_1, y_1, z_1), m_2 (x_2, y_2, z_2), …, m_n (x_n, y_n, z_n) that are subject to the condition:

$$f(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) = 0 \tag{1}$$

might be in *equilibrium* under the influence of any sort of *unknown* forces $P_1, P_2, \ldots, P_n$. The properties and formulas that this assumption of equilibrium implies for those unknown forces shall be developed.

The equilibrium that exists will obviously *not be perturbed* when one adds any sort of *new* restrictions to the condition (1) [rather, by contrast, it will only be further *strengthened* in the event that such an expression is allowed]. Thus, e.g., that equilibrium *will persist* when one *fixes* the (n – 1) points $m_2, m_3, \ldots, m_n$, so one considers their coordinates to be *unvarying*, by which the motion of the point m_1 will be restricted to that well-defined surface Ω that is represented by equation (1) [when those coordinates are thought of as *unvarying*]. However, the original state of affairs will then be reduced by the fact that the point m_1 that displaces on Ω is found to be in equilibrium under the influence of the force P_1. It follows from this that the force P_1 must be *normal* to the surface Ω, so its components X_1, Y_1, Z_1 must possess the following form:

$$X_1 = \lambda_1 \frac{\partial f}{\partial x_1},$$

$$Y_1 = \lambda_1 \frac{\partial f}{\partial y_1},$$

$$Z_1 = \lambda_1 \frac{\partial f}{\partial z_1},$$

in which λ_1 represents a still-unknown factor.

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Obviously, analogous statements will be true for the remaining forces P_2, P_3, …, P_n (their components X_2, Y_2, Z_2, X_3, Y_3, Z_3, …, X_n, Y_n, Z_n, respectively), such that one will arrive at the following $3n$ formulas:

$$X_h = \lambda_h \frac{\partial f}{\partial x_h}, \quad Y_h = \lambda_h \frac{\partial f}{\partial y_h}, \quad Z_h = \lambda_h \frac{\partial f}{\partial z_h}, \qquad h = 1, 2, 3, \ldots, n, \tag{2}$$

in which λ_1, λ_2, …, λ_n represent still-*unknown* factors.

Once the *directions* of the unknown forces P_1, P_2, …, P_n have been determined from formulas (2), one must further deal with investigating their *intensities*, or (what amounts to the same thing) with investigating the unknown factors λ_1, λ_2, …, λ_n . In order to go deeper into that, we revert to the *original* situation, which mean that the points m_1, m_2, …, m_n that are subject to the condition (1) are in *equilibrium* under the influence of the forces P_1, P_2, …, P_n. That equilibrium will persist when one *spatially fixes* the $(n - 2)$ points m_3, m_4, …, m_n, but restricts the motion of the two points m_1 and m_2 to two *fixed rectilinear tracks*, moreover, which might be *parallel*, and whose common direction cosines might be denoted by α, β, γ. Therefore, as a consequence of the condition (1), the points m_1 and m_2 can move along those tracks only in such a way that their simultaneously-traversed path-elements δs_1 and δs_2 will relate to each other as follows:

$$\left(\frac{\partial f}{\partial x_1}\alpha + \frac{\partial f}{\partial y_1}\beta + \frac{\partial f}{\partial z_1}\gamma\right)\delta s_1 + \left(\frac{\partial f}{\partial x_2}\alpha + \frac{\partial f}{\partial y_2}\beta + \frac{\partial f}{\partial z_2}\gamma\right)\delta s_2 = 0 ,$$

and those simultaneous path-elements δs_1, δs_2 will then be *equal* to each other as soon as one subjects the α, β, γ to the relation:

$$\left(\frac{\partial f}{\partial x_1}\alpha + \frac{\partial f}{\partial y_1}\beta + \frac{\partial f}{\partial z_1}\gamma\right) + \left(\frac{\partial f}{\partial x_2}\alpha + \frac{\partial f}{\partial y_2}\beta + \frac{\partial f}{\partial z_2}\gamma\right) = 0 . \tag{3}$$

Having established that, the points m_1 and m_2 can then be displaced along their *parallel* tracks G_1 and G_2 by only *equal amounts*, so they can move only in such a way that their mutual distance remains *constant*. Consequently, that mobility of the two points will suffer no obstruction or restriction in the event that one couples the two points with each other by a *rigid line L*.

However, if one imagines implementing such a thing then that will now reduce the original situation by the fact that the *rigid line L*, whose endpoints m_1 and m_2 can displace along the *fixed parallel tracks* G_1 and G_2, will be in *equilibrium* under the influence of forces P_1 and P_2 that are applied to their endpoints. It follows immediately from this that the sum of those components of P_1 and P_2 that correspond to the tracks must be equal to *zero*. One will then get the formula:

$$P_1 \cos (P_1, G_1) + P_2 \cos (P_2, G_2) = 0 .$$

However, that formula can also be written:

$$(X_1 \alpha + Y_1 \beta + Z_1 \gamma) + (X_2 \alpha + Y_2 \beta + Z_2 \gamma) = 0 ,$$

or when one recalls (2), also as:

$$\lambda_1 \left(\frac{\partial f}{\partial x_1} \alpha + \frac{\partial f}{\partial y_1} \beta + \frac{\partial f}{\partial z_1} \gamma \right) + \lambda_2 \left(\frac{\partial f}{\partial x_2} \alpha + \frac{\partial f}{\partial y_2} \beta + \frac{\partial f}{\partial z_2} \gamma \right) = 0 ,$$

or, since α, β, γ are subject to the relation (3), also as:

$$\lambda_1 = \lambda_2 .$$

Analogously, one will obviously get $\lambda_1 = \lambda_3$, and furthermore, $\lambda_1 = \lambda_3$, etc., so in general:

(4) $$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_n .$$

However, if one denotes the common value of those n quantities briefly by λ (with no index) then one will arrive at the following result, based upon formulas (2):

Theorem:

If n mass-points m_h (x_h, y_h, z_h), which are subject to a given condition:

$$f (x_1, y_1, z_1, x_2, y_2, z_2 , \ldots, x_h, y_h, z_h) = 0 ,$$

are found to be in ***equilibrium*** *under the influence of any forces F_h (X_h, Y_h, Z_h) then those forces must necessarily possess values of the following form:*

(5) $$X_h = \lambda \frac{\partial f}{\partial x_h} , \quad Y_h = \lambda \frac{\partial f}{\partial y_h} , \quad Z_h = \lambda \frac{\partial f}{\partial z_h} , \qquad h = 1, 2, 3, \ldots, n,$$

in which λ represents an unknown factor.

In so doing, it should be added that this unknown factor λ does not need to have a well-defined value, but rather, it can possess ***any arbitrary*** *value.*

In order to show the validity of the last assertion, we will obviously need to show only that *any* of the forces P_1, P_2, …, P_n (e.g., P_1) can have *arbitrary strength*.

To that end, we would like to *fix* the $(n - 1)$ points m_2, m_3, …, m_n once more, such that only the point m_1 can still be displaced along the (previously-discussed) surface Ω. Obviously, that point m_1 will then be in *equilibrium* under the influence of an *arbitrarily-strong* force P_1 in the event that it is *normal* to the surface Ω. However, if we now imagine one such normal force P_1 of *arbitrary* strength actually acts upon m_1, and we denote the forces that are required to fix the $(n - 1)$ points m_2, m_3, …, m_n by P_2, P_3, …, P_n, resp., then we will have n forces in total that will keep the point-system in equilibrium when taken together, and the strength of the first of them (namely, P_1) can be chosen *arbitrarily*. Q.E.D.

However, we shall not go further into the question of how we must ultimately proceed in order to arrive the *principle of virtual displacements* from theorem (5).

Remark. – Should one have reservations about the fact that when one introduces the fixed parallel tracks G_1, G_2, the mutual distance between the two points m_1, m_2 remains constant under the motion that takes place only *in the first instant*, then one can easily avoid that aspect of the situation by replacing those tracks G_1, G_2 with two *fixed curves* C_1, C_2. Those curves C_1, C_2 can then be easily established in such a way that the points m_1, m_2 that displace along them and are subject to the condition (1), moreover, will possess a mutual distance that remains *perpetually* constant.

First theorem:

If three points m_1, m_2, m_3 are subject to the condition that the area of the triangle that they define should remain ***constant****, and if those points are found to be in* ***equilibrium*** *under the influence of any forces P_1, P_2, P_3 then P_1, P_2, P_3 will lie in the plane of the aforementioned triangle, as well as being perpendicular to the opposite sides of the triangle and being proportional to the lengths of those sides.*

Second theorem:

If four points m_1, m_2, m_3, m_4 are subject to the condition that the volume of the tetrahedron that they define shall remain ***constant****, and if those points are found to be in* ***equilibrium*** *under the influence of any forces P_1, P_2, P_3, P_4 then P_1, P_2, P_3, P_4 will be perpendicular to the opposite faces of the tetrahedron and be proportional to the areas of those sides.*

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