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General considerations on the moments of forces

By OSTRAGRADSKY

Translated by D. H. Delphenich

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The principle of virtual velocities, which one attributes to Jean Bernoulli, remained sterile in its consequences until Lagrange made it the basis for his *Mécanique analytique*, which is a remarkable work that changed the face of the science of equilibrium and motion. Lagrange was not merely content to infer the consequences of the principle of Jean Bernoulli, but extended and generalized that principle, and appealed to it in order to solve the most difficult problems of the equilibrium and motion of systems. One might believe that the matter had been exhausted and that there was nothing to add to the theories that Lagrange had just posed.

Meanwhile, since the publication of *Mécanique analytique* (*), some geometers glimpsed that the principle of virtual velocities could be extended even more than Lagrange himself had supposed. Along with Jean Bernoulli, that great geometer had thought that in order to get equilibrium in the system, he could make the total moment (i.e., the sum of the moments of all forces) equal to zero for all displacements that the system could receive. Now, some geometers have remarked that Lagrange was demanding too much, and that in order to get equilibrium, it would suffice that the total moment should not take on a positive value for any possible displacement, in such a way that if the value of the moment were negative or zero for all of those displacements then equilibrium would be assured. The principle of virtual velocities acquires much greater generality when presented in that manner, and becomes susceptible to a greater number of applications, and truly encompasses all of the questions that one can propose in regard to the equilibrium of forces.

It is quite surprising to see that in the new edition of *Mécanique analytique*, which was an edition that was published during the era in which one already knew the full extent of the principle of virtual velocities, Lagrange not only made no use of the observation that in force equilibrium, the total moment could acquire a negative value, but in a sense he discarded it when it presented itself in the proof that he gave to the principle of virtual velocities (**). Meanwhile, without having noticed that fact, that great geometer incompletely enumerated the possible displacements in most of the questions in the first part of *Mécanique analytique*, and it is easy to recognize that the displacements that he neglected to consider were not impeded by any condition, in such a way that

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(*) 1788.

(**) *Mécanique analytique*, 1811 edition, pages 25 and 26.

if all of the equations that he established for equilibrium were satisfied then equilibrium would nonetheless not exist.

In this paper, we propose to present the analysis that relates to the use of the principle of virtual velocities when it is considered in full generality and to complete the solution of several questions that were treated in the first part of *Mécanique analytique*.

1. – Let P, Q, R, … denote the forces that are applied to a system, and let $P\,dp + Q\,dq + R\,dr + \ldots$ denote the total moment of those forces under an arbitrary displacement. In order to have equilibrium in the system, it is necessary and sufficient that the differential $P\,dp + Q\,dq + R\,dr + \ldots$ must become positive for any possible displacement.

It is obvious that there can be no question of finding the equilibrium conditions for a system that is not defined completely. The definition of the system must encompass the complete enumeration of all the displacements that it can experience, and in order to distinguish those displacements from the ones that the system can never experience, due to the obstacles that oppose them, one must have some conditions that only the possible displacements will satisfy and the impossible displacements will never verify.

The conditions that one deals with are most often expressed by linear functions of the quantities that fix the displacements of the system, which are functions that cannot change sign when one considers only the possible displacements, in such a way that if one lets dL, dM, … denote those functions then the quantities dL, dM, … will be zero for any of the possible displacements, and that will not be true for the others. However, some of them can change sign when one passes from possible displacements to ones that are not.

Having said that, it is clear that in place of the infinitely small variations that are contained in a linear form in dp, dq, dr, …, one can introduce some other variations $d\xi$, $d\eta$, $d\zeta$, …, which are the same in number and coupled to the former ones by first-degree equations. Each differential dp, dq, dr, … will become a linear function of $d\xi$, $d\eta$, $d\zeta$, …, and therefore the total moment $P\,dp + Q\,dq + R\,dr + \ldots$ will take the following form $A\,d\xi + B\,d\eta + C\,d\zeta + \ldots$ Now, since all of the differentials dL, dM, … are coupled by first-degree equations with the ones that are found in dp, dq, dr, …, one can find dL, dM, … among the quantities $d\xi$, $d\eta$, $d\zeta$, …, which will give an expression of the form:

$$\lambda\,dL + \mu\,dM + \ldots + A\,d\xi + B\,d\eta + C\,d\zeta + \ldots$$

for the total moment.

Having replaced the total moment $P\,dp + Q\,dq + R\,dr + \ldots$ with $\lambda\,dL + \mu\,dM + \ldots + A\,d\xi + B\,d\eta + \ldots$, we see that it is necessary for $\lambda\,dL + \mu\,dM + \ldots + A\,d\xi + B\,d\eta + \ldots$ to be zero or negative for the possible displacements and to not become positive for any of those displacements.

In order to do that, we pass from all imaginable displacements to only the possible displacements. The quantities dL, dM, … cannot change signs, but they can become zero. However, the differentials $d\xi$, $d\eta$, $d\zeta$, … will also remain arbitrary if one considers all imaginable displacements, and one can arrange those differentials in such a manner as to give the sign that one desires to the function $\lambda\,dL + \mu\,dM + \ldots + A\,d\xi + B\,d\eta + \ldots$: Hence, the total moment must keep the same sign for all possible motions, at least one does not have $A\,d\xi + B\,d\eta + C\,d\zeta + \ldots =$

0 for any $d\xi$, $d\eta$, … that will give $A = 0$, $B = 0$, …, separately. Those conditions often include all of the equilibrium conditions for the system, so they will always include some of them. However, since obviously everything comes down to the equality $P\,dp + Q\,dq + R\,dr + \ldots = \lambda\,dL + \mu\,dM + \ldots$, which must be true for all imaginable displacements, one can consider only that equality.

Having found that:

$$P\,dp + Q\,dq + R\,dr + \ldots = \lambda\,dL + \mu\,dM + \ldots$$

for all imaginable displacements, one again considers only the possible displacements. Since dL, dM, … do not change signs, but can become zero, it is clear that the quantity $\lambda\,dL + \mu\,dM + \ldots$ will be negative or zero when one gives signs to the factors λ, μ, … that are opposite to those of the differentials dL, dM, …, respectively, and moreover it is clear that $\lambda\,dL + \mu\,dM + \ldots$ will remain negative only under that hypothesis. One can then consider the second and last condition of equilibrium to be that the signs of λ, μ, … must be opposite to those of dL, dM, …, respectively. Therefore, equilibrium will exist in the system only if one has:

$$P\,dp + Q\,dq + R\,dr + \ldots = \lambda\,dL + \mu\,dM + \ldots$$

for all imaginable displacements, and that λ, μ, … take the opposite signs to the differentials dL, dM, …, resp., that refer to the possible displacements. Upon transporting all of the terms to the same side, the equilibrium conditions of an arbitrary system will be expressed by:

1. The equation:

$$0 = P\,dp + Q\,dq + R\,dr + \ldots + \lambda\,dL + \mu\,dM + \ldots,$$

which must be true for all imaginable displacements.

2. The condition that the quantities λ, μ, … have the same signs as the differentials dL, dM, … for the possible displacements. It is obvious that if one or more of the quantities dL, dM, … can be zero for all possible displacements then the signs of the factors that refer to those quantities will be arbitrary.

II. – In order to apply the preceding considerations to some special examples, refer the system to rectangular coordinates and let X, Y, Z denote the forces that are applied to a point of the system that are parallel to the axes. If one lets dx, dy, dz represent the projections of a displacement onto the coordinates axes then the trinomial $X\,dx + Y\,dy + Z\,dz$ will express the moment that one takes when considering only one point of the system, and the sum:

$$\sum (X\,dx + Y\,dy + Z\,dz),$$

when extended over all points, will represent the value of the total moment.

Having said that, let dL, dM, … denote linear functions of dx, dy, dz, …, which, by the nature of the system, can change sign only when passing from possible displacement to the ones that are not. In order to have equilibrium, we will have the equation:

$$0 = \sum (X\,dx + Y\,dy + Z\,dz) + \lambda\,dL + \mu\,dM + \ldots,$$

which must be true for any imaginable values of dx, dy, dz, …, and in which λ, μ, … have the same signs as the functions dL, dM, …, respectively, when referred to the possible displacements.

III. – As a first example, consider the equilibrium of a point m that is located on a surface. Let $L = 0$ denote the equation of the surface. The coordinates of m must satisfy it when it is on that surface, and one lets dL denote the variation of L that is due to an arbitrary displacement of m. In order to have equilibrium for that point, one will have:

$$0 = X\,dx + Y\,dy + Z\,dz + \lambda\,dL\,.$$

In order to determine the sign of dL, and consequently, that of λ, observe that the surface $L = 0$ divides the space into two parts that are easy to distinguish, because in the one the quantity L will be greater than zero, and in the other it will be less than zero. Suppose that the point m can displace only in the space where L is greater than zero and on the surface itself. It will then follow that for the possible displacements, the function L will keep its value or even increase, in such a way that dL (always for possible displacements) will be zero or positive, and can become negative only for the impossible displacements; hence, the quantity λ must be positive.

Upon letting x, y, z denote the coordinates of the point m and considering L to be a function of x, y, z, one will have:

$$dL = \frac{dL}{dx}dx + \frac{dL}{dy}dy + \frac{dL}{dz}dz\,,$$

and as a result, the equation of equilibrium will give:

$$X + \lambda\frac{dL}{dx} = 0,$$
$$Y + \lambda\frac{dL}{dy} = 0,$$
$$Z + \lambda\frac{dL}{dz} = 0,$$

or rather:

$$-\lambda = \frac{X}{\frac{dL}{dx}} = \frac{Y}{\frac{dL}{dy}} = \frac{Z}{\frac{dL}{dz}} = \frac{-\sqrt{X^2+Y^2+Z^2}}{\sqrt{\left(\frac{dL}{dx}\right)^2+\left(\frac{dL}{dy}\right)^2+\left(\frac{dL}{dz}\right)^2}}.$$

One gives the – sign to the latter fraction because each of the first three is negative, since λ is positive. If the point is constrained to stay on the surface then the first three functions (and consequently, the fourth, as well) can have an arbitrary sign. Indeed, that is what the difference between the equilibrium conditions for a point that is located over the surface and the ones for a point that is constrained to remain on it consists of.

IV. – As a second application, we shall present the equilibrium in the system that is known by the name of the *funicular polygon*. Let n denote the number of angles, which we distinguish from each other by the numerals 1, 2, 3, …, n, and for the angle that pertains to the numeral i, let x_i, y_i, z_i denote the coordinates, let X_i, Y_i, Z_i denote the forces parallel to the axes, and let r_i denote the length of the part of the cord that is found between the consecutive angles i and $i + 1$.

Having said that, the total moment can be expressed by the sum:

$$\sum_{i=1}^{n+1}(X_i\,dx_i + Y_i\,dy_i + Z_i\,dz_i),$$

and at the same time, due to the inextensibility of the cord, it is necessary that the differential dr_i can acquire a positive value only for the displacements that the system can never exhibit, and that must be true for any numeral i. Therefore, if we let λ_i denote a negative quantity then the condition that relates to the angle i will give a term $\lambda_i\,dr_i$ in the general equation for equilibrium. Since any other angle will provide similar terms, one can express all of that by saying that the *condition equations* are introduced into the general formula for equilibrium by way of $\sum_{i=1}^{n}\lambda_i\,dr_i$, or rather, by way of $\sum_{i=1}^{n+1}\lambda_i\,dr_i$, provided that one makes $\lambda_n = 0$.

From the preceding, the equation of equilibrium of the system that we consider will become:

$$\text{(A)} \qquad 0 = \sum_{i=1}^{n+1}(X_i\,dx_i + Y_i\,dy_i + Z_i\,dz_i + \lambda_i\,dr_i),$$

and the quantity λ_i must be negative for any numeral i, moreover.

Equation (A) must be true for any differential dx_1, dy_1, dz_1, dx_2, dy_2, dz_2, …, so one must separately equate the coefficients of all the differentials to zero, which will give:

$$0 = X_i + \lambda_i \frac{dr_i}{dx_i} + \lambda_{i-1} \frac{dr_{i-1}}{dx_i},$$

$$0 = Y_i + \lambda_i \frac{dr_i}{dy_i} + \lambda_{i-1} \frac{dr_{i-1}}{dy_i},$$

$$0 = Z_i + \lambda_i \frac{dr_i}{dz_i} + \lambda_{i-1} \frac{dr_{i-1}}{dz_i}$$

for any i, provided that one nevertheless makes $\lambda_0 = 0$.

Now, it is easy to see that $\frac{dr_{i-1}}{dx_i} = -\frac{dr_{i-1}}{dx_{i-1}}$, $\frac{dr_{i-1}}{dy_i} = -\frac{dr_{i-1}}{dy_{i-1}}$, $\frac{dr_{i-1}}{dz_i} = -\frac{dr_{i-1}}{dz_{i-1}}$; hence, the preceding equations will become:

$$0 = X_i + \lambda_i \frac{dr_i}{dx_i} - \lambda_{i-1} \frac{dr_{i-1}}{dx_{i-1}},$$

$$0 = Y_i + \lambda_i \frac{dr_i}{dy_i} - \lambda_{i-1} \frac{dr_{i-1}}{dy_{i-1}},$$

$$0 = Z_i + \lambda_i \frac{dr_i}{dz_i} - \lambda_{i-1} \frac{dr_{i-1}}{dz_{i-1}}$$

or rather, upon using the notation of finite differences:

$$0 = X_i + \Delta\left(\lambda_{i-1} \frac{dr_{i-1}}{dx_{i-1}}\right),$$

$$0 = Y_i + \Delta\left(\lambda_{i-1} \frac{dr_{i-1}}{dy_{i-1}}\right),$$

$$0 = Z_i + \Delta\left(\lambda_{i-1} \frac{dr_{i-1}}{dz_{i-1}}\right),$$

so upon integrating with the sign $\sum$ and paying attention to the fact that $\lambda_0 = 0$:

$$\text{(B)} \qquad \sum_{i=1}^{n+1} X_i = -\lambda_s \frac{dr_s}{dx_s}, \quad \sum_{i=1}^{n+1} Y_i = -\lambda_s \frac{dr_s}{dy_s}, \quad \sum_{i=1}^{n+1} Z_i = -\lambda_s \frac{dr_s}{dz_s}.$$

Upon supposing that $s = n$, one will find that:

$$0 = \sum_{i=1}^{n+1} X_i , \qquad 0 = \sum_{i=1}^{n+1} Y_i , \qquad 0 = \sum_{i=1}^{n+1} Z_i .$$

Upon setting:

$$\sum_{i=1}^{n+1} X_i = A_s, \qquad \sum_{i=1}^{n+1} Y_i = B_s , \qquad \sum_{i=1}^{n+1} Z_i = C_s , \qquad \sqrt{A_s^2 + B_s^2 + C_s^2} = R_s ,$$

to abbreviate, and observing that $-\frac{dr_s}{dx_s}, -\frac{dr_s}{dy_s}, -\frac{dr_s}{dz_s}$ represent the cosines of the angles λ_s , μ_s , ν_s that the side r_s of the polygon makes with the coordinate axes, equations (B) will reduce to:

$$\lambda_s = \frac{A_s}{\cos \lambda_s} = \frac{B_s}{\cos \mu_s} = \frac{C_s}{\cos \nu_s} = - R_s .$$

One can give the minus sign to the resultant R_s , because the quantity λ_s , and as a result, the fractions $\frac{A_s}{\cos \lambda_s}$, $\frac{B_s}{\cos \mu_s}$, $\frac{C_s}{\cos \nu_s}$ are negative. The sign of λ_s , and consequently, that of R_s , will be arbitrary if the polygon is composed of straight rods, in such a way that the equations of equilibrium of a similar polygon will be:

$$\frac{A_s}{\cos \lambda_s} = \frac{B_s}{\cos \mu_s} = \frac{C_s}{\cos \nu_s} = \pm R_s .$$

Therefore, the principle of virtual velocities can indeed distinguish the case of straight rods from that of flexible cords.

If the points of application of the forces are not fixed, but can slide along the cord then the preceding conditions will no longer be sufficient to maintain equilibrium. Indeed, one can then perturb the system in such a manner that some of the lengths r_1, r_2, … can increase, while it is only their sum $r_1 + r_2 + \ldots + r_n$ that cannot increase for any possible displacement, in such a way that one will have only one condition:

$$r_1 + r_2 + \ldots + r_n < 0$$

that the possible displacements must satisfy. Thus, it is easy to conclude that this new problem can be regarded as a special case of the preceding one, in regard to its solution, and that one, in fact, deduces that:

$$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} .$$

V. – We shall further speak of the equilibrium of a flexible string, each element of which is subject to given forces. Like the preceding one, that question is included in that of the funicular polygon, and it can be deduced from the latter by supposing that each side of the polygon becomes infinitely small and the number n becomes infinitely large. We first remark that the question depends upon a consideration of the two types of differentials, the first of which (like all of the ones that we have considered up to now) relates to the infinitely small displacements that one can imagine in the system, and the second of which refers to the passage from a point on the string to the infinitely close point. We denote the latter by the letter d, and we denote the differentials of the first type by the letter δ, as in analytical mechanics.

Having said that, let $X\,dm$, $Y\,dm$, $Z\,dm$ be the forces parallel to the axes that are applied to the element dm of the string, which is an element that pertains to the coordinates x, y, z. Upon considering only that element, the moment will be $(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm$, and the sum $\mathrm{S}\,(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm$, when extended over all elements of the string, will express the total moment.

Due to the inextensibility of the string, the element ds of its length can only diminish or remain the same under any virtual displacement. Now, since $ds^2 = dx^2 + dy^2 + dz^2$, the variation δds that is due to an arbitrary displacement will be expressed by $\dfrac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{ds}$. Therefore, the latter quantity can be only zero or negative for the possible displacements. Consequently, upon taking a negative quantity λ, the element ds will contribute the term $\lambda\left(\dfrac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{ds}\right)$ to the equations of equilibrium. Any other element of the string will contribute such a term, so one will have the following equation for equilibrium:

$$0 = \mathrm{S}\left[(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm + \lambda\left(\frac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{ds}\right)\right],$$

which must be true for all imaginable displacements. Upon pursuing the calculations as in *Mécanique analytique* (*), One will arrive at the result of that great work, except that our analysis will give us one more condition, namely, that the function λ must necessarily be negative, since otherwise there would not be equilibrium, even when all of the other conditions are satisfied.

We have tacitly supposed that the string is entirely free. However, if there are special conditions that must be fulfilled relative to its extremities then one must modify the equation of equilibrium:

$$0 = \mathrm{S}\left[(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm + \lambda\left(\frac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{ds}\right)\right]$$

to conform to those conditions.

For example, if the extremities are fixed then upon denoting the coordinates of the former point on the cord by x', y', z' and the latter point by x'', y'', z'', one will have $\delta x' = 0$, $\delta y' = 0$, $\delta z' = 0$, $\delta x'' = 0$, $\delta y'' = 0$, $\delta z'' = 0$, for all virtual displacements, which will add the function:

$$a'\,\delta x' + b'\,\delta y' + c'\,\delta z' + a''\,\delta x'' + b''\,\delta y'' + c''\,\delta z''$$

(*) Pages 137, 138, 139, and 140.

to the general equation of equilibrium, in which the quantities a', b', c', a'', b'', c'' are indeterminate in value and sign, and we will have:

$$0 = \mathrm{S}\left[(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm + \lambda\left(\frac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{ds}\right)\right]$$
$$+ a'\,\delta x' + b'\,\delta y' + c'\,\delta z' + a''\,\delta x'' + b''\,\delta y'' + c''\,\delta z''$$

for all imaginable displacements.

In that equation, all of the differential that are denoted by δ, including the ones that refer to the limits, are absolutely arbitrary. By means of an integration by parts, the preceding formula will reduce to:

$$\mathrm{S}\left[\left(X\,dm - d\cdot\lambda\frac{dx}{ds}\right)\delta x + \left(Y\,dm - d\cdot\lambda\frac{dy}{ds}\right)\delta y + \left(Z\,dm - d\cdot\lambda\frac{dz}{ds}\right)\delta z\right]$$
$$+\left(a'' + \lambda''\frac{dx''}{ds}\right)\delta x'' + \left(b'' + \lambda''\frac{dy''}{ds}\right)\delta y'' + \left(c'' + \lambda''\frac{dz''}{ds}\right)\delta z''$$
$$+\left(a' + \lambda'\frac{dx'}{ds}\right)\delta x' + \left(b' + \lambda'\frac{dy'}{ds}\right)\delta y' + \left(c' + \lambda'\frac{dz'}{ds}\right)\delta z'\,.$$

The quantities denoted with one prime refer to the beginning of the cord, and the ones that are denoted with two primes refer to its end. Upon equating the coefficients of all δ to zero separately, one will then find these three equations, which refer to all points of the string:

$$0 = X\,dm - d\cdot\lambda\frac{dx}{ds},$$
$$0 = Y\,dm - d\cdot\lambda\frac{dy}{ds},$$
$$0 = Z\,dm - d\cdot\lambda\frac{dz}{ds},$$

and one will then have:

$$0 = a' - \lambda'\frac{dx'}{ds}, \qquad 0 = b' - \lambda'\frac{dy'}{ds}, \qquad 0 = c' - \lambda'\frac{dz'}{ds},$$
$$0 = a'' + \lambda''\frac{dx''}{ds}, \qquad 0 = b'' + \lambda''\frac{dy''}{ds}, \qquad 0 = c'' + \lambda''\frac{dz''}{ds}$$

for the conditions at the extremities, which must always be satisfied by means of the indeterminates a', b', c', a'', b'', c''.

If one of the extremities (for example, the first one) is constrained to remain on a surface $L = 0$, where L is a function of x', y', z', and the other extremity remains free then the equation of equilibrium will be:

$$0 = \mathrm{S}\left[(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm + \lambda\left(\frac{dx\,d\delta x + dy\,d\delta y + dz\,d\delta z}{ds}\right)\right] + \mu\,\delta L,$$

in which μ is a quantity whose value and sign are indeterminate. However, if the point (x', y', z') is only located over the surface $L = 0$ then the sign of the quantity μ will be fixed (see paragraph III).

VI. – As a last application, we shall say a few words about the equilibrium of incompressible fluids. If we let $X\,dm$, $Y\,dm$, $Z\,dm$ denote the components parallel to the axes of the force that are applied to a molecule dm of the liquid, and as above, we let δx, δy, δz denote the projections of an arbitrary displacement of the element dm onto the coordinate axes then the molecule dm will contribute the term $(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm$ to the total moment. When the sum:

$$\mathrm{S}\,(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm$$

is extended over all liquid masses, it will express the value of the total moment for an arbitrary displacement. One can divide all of the displacements that one can imagine in an incompressible liquid into three classes:

1. The displacements that accompany the reduction of volume.

2. The displacements in which the volume does not change.

3. The displacements that accompany the increase in volume.

The displacements of the first type are impossible by the nature of the system, so it is pointless to deal with them. As for the other two types: For there to be equilibrium, it is necessary that the total moment that relates to those displacements must be zero or negative.

Upon denoting the volume of the molecule dm by $dx\,dy\,dz$, the variation $\delta\,(dx\,dy\,dz)$ of that volume that is due to an arbitrary displacement can be expressed, as one knows, by $\left(\frac{d\delta x}{ds} + \frac{d\delta y}{ds} + \frac{d\delta z}{ds}\right) dx\,dy\,dz$. The preceding variation must be zero or positive for the possible displacements. Hence, from the general theory, upon taking a positive quantity p that is a function of x, y, z, one will have:

$$0 = \mathrm{S}\left[\rho\,(X\,\delta x + Y\,\delta y + Z\,\delta z) + p\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right)\right] dx\,dy\,dz$$

for all imaginable displacements, in which ρ is the density of the liquid.

It will then result that:

$$\frac{dp}{dx} = \rho X, \qquad \frac{dp}{dy} = \rho Y, \qquad \frac{dp}{dz} = \rho Z$$

for any liquid mass, and one will have $p = 0$ for the entire extent of the surface. Our analysis is the same as in *Mécanique analytique* (*), except that there, it was not clear that the quantity p would necessarily have to be positive for all points of the liquid, and that if that condition is not fulfilled then the liquid will displace in such a manner as to no longer form a continuous mass.

Lagrange did not consider the displacements that accompany the increase in volume, and he did not obstruct them with any condition, which is why if everything that he assumed were fulfilled then the fluid could still break into pieces. To give an example, one needs only to consider a liquid whose molecules are each subjected to a repulsive force that emanates from a fixed center and whose surface experiences no pressure. Assume that the repulsive force is proportional to the distance. We will have $X = x$, $Y = y$, $Z = z$, and:

$$dp = \rho\,(x\,dx + y\,dy + z\,dz)\,,$$

and if we suppose that the density is constant then we will find that:

$$p = \tfrac{1}{2}\rho\,(x^2 + y^2 + z^2) + c,$$

or rather, upon setting $x^2 + y^2 + z^2 = r^2$:

$$p = \tfrac{1}{2}\rho\,r^2 + c\,.$$

One has $0 = \tfrac{1}{2}\rho\,r^2 + c$ for the free surface. Therefore, that surface is spherical; it has the focus of the repulsive force for its center. Upon denoting its radius by R, we will have $c = -\tfrac{1}{2}\rho\,R^2$, and consequently:

$$p = \tfrac{1}{2}\rho\,(r^2 - R^2)\,.$$

Therefore, from the preceding analysis, one can believe that equilibrium can persist with a spherical surface. Meanwhile, since nothing prevents the molecules of the liquid from dissipating into space, the repulsive force will necessarily dissipate. Now, from the analysis that we have established, equilibrium will be impossible, because the quantity $p = \tfrac{1}{2}\rho\,(r^2 - R^2)$ is negative.

However, if the force is attractive and the sphere is hollow then equilibrium can exist. Indeed, one will find that:

$$dp = -\rho\,r\,dr$$

in that case, so:

$$p = C - \tfrac{1}{2}\rho\,r^2$$

for the surface that is most distant from the center. One finds that:

(*) Page 194, *et seq.*

$$0 = C - \tfrac{1}{2}\rho\, r^2,$$

and upon subtracting:

$$p = \tfrac{1}{2}\rho\,(R^2 - r^2),$$

so the pressure is positive. If one denotes the radius of the surface that is closest to the center by r_0 then one will have:

$$\tfrac{1}{2}\rho\,(R^2 - r_0^2)$$

for the pressure at each point of that surface.

Thus, a spherical shell whose molecules are all subjected to an attractive force that emanates from the center of the shell will remain in equilibrium, while if the force is attractive then that shell will disperse into space. That result should not be surprising at all, because the systems whose equilibrium we have considered are the such that if the forces that mutually cancel out are reversed – i.e., they receive directions that are opposite to the ones that they first had – then equilibrium will no longer persist, and that if one system of forces is equivalent to another then the latter will not be true of the former. (We mean a "system that is equivalent to another system" to mean one whose forces would all equilibrate the forces of another system if they were reversed.)

VII. (*) – We shall conclude this memoir with some considerations that relate to the motion of systems.

If the forces P, Q, R, … (art. I) are not in equilibrium then the total moment $P\,\delta p + Q\,\delta q + R\,\delta r + \ldots$ will necessarily become positive for some possible displacements, and therefore the equation:

$$P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots$$

will not be true. However, if one lets P', Q', R', … denote the *dynamical* forces on the system and lets $P'\,\delta p' + Q'\,\delta q' + R'\,\delta r' + \ldots$ denote the moment of those forces then one will have:

$$0 = P'\,\delta p' + Q'\,\delta q' + R'\,\delta r' + \ldots + P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots$$

for all imaginable displacements, because the dynamical forces equilibrate the forces of motion.

With Ampère, we shall say "dynamical force" to mean the reaction of the matter that opposes any change in motion; i.e., the change in velocity, like the change in direction. If an arbitrary system is at rest or in uniform rectilinear motion then the dynamical forces will obviously be zero; if the motion is varied then they will not be. However, they are always capable of cancelling out the forces of motion, or rather (if one prefers), they can always cancel out the latter.

Let m, m', m'', … denote the masses that comprise the system, and let x, y, z denote the rectangular coordinates of the mass m, let x', y', z' denote those of the mass m', and so on; all of those coordinates are referred to the same axis. If we let t denote the time that has elapsed since

(*) § VII and the following ones were added during printing.

some fixed epoch, moreover, and let d characterize the differences that relate to t then the moment of the dynamical force that relates to the mass m will be expressed by $-\frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2}m$, and the sum $-\sum\frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2}m$, when extended over all masses m, m', m'', … will represent the total moment of the dynamical forces, in such a way that:

$$P'\,\delta p' + Q'\,\delta q' + R'\,\delta r' + \ldots = -\sum\frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2}m\,;$$

hence, the formula:

$$0 = P'\,\delta p' + Q'\,\delta q' + R'\,\delta r' + \ldots + P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots$$

will become:

$$\sum\frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2}m = P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots$$

That equation is the Lagrange equation, except that in *Mécanique analytique*, neither the values not the signs of the quantities λ, μ, … were known, which amounts to saying that Lagrange considered only the systems whose possible displacements satisfied some equations. Now, it can happen that the displacements that we speak of do not satisfy any equation in a state of motion or equilibrium. The signs of the quantities λ, μ, … will then be known in advance, because they must be the same as those of the functions δL, δM, …, respectively, when they are referred to the possible displacements.

Upon separately equating the coefficients of δx, δy, δz, $\delta x'$, … to zero, the formula:

$$\sum\frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2}m = P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots$$

will give as many equations for the determination of the unknowns x, y, z, x', …, λ, μ, … as there are coordinates x, y, z, x', …, and if one adds the equations $\delta L = 0$, $\delta M = 0$, …, whose number is equal to that of the quantities λ, μ, … , to those equations then one will have as many equations as unknowns, and the determination of those unknowns will become a problem in integral calculus. We let dL, dM, … denote what δL, δM, …, respectively, will become when one sets $\delta x = dx$, $\delta y = dy$, $\delta z = dz$, $\delta x' = dx'$, …

VIII. – However, there an essential remark that must be made here: It is that since most possible or virtual displacements will not make the functions δL, δM, … equal to zero, it might be that when one starts from a certain epoch, the effective displacements dx, dy, … will no longer satisfy some of the equations $dL = 0$, $dM = 0$, …, because since dx, dy, … are included among the values that δx, δy, …, respectively, can take on without ceasing to belong to the possible displacements, one imagines that the differences dx, dy, … can even become equal to those of the variations δx, δy, … that do not give $\delta L = 0$, $\delta M = 0$, … Furthermore, the number of equations that serve to determine the motion seems to be less than the number of unknowns. One must keep that fact in mind whenever one treats the motion of a system whose possible displacements cannot be expressed by equations, and it is very important to add the proper considerations to analytical mechanics that will establish all of the equations that are necessary to determine the motion of such a system.

In order to achieve that objective, one poses all of the equations:

$$\sum \frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2} m = P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots,$$

$dL = 0$, $dM = 0$, … as if one were assured that the functions dL, dM, …, would each be equal to zero in the course of their motion, and one seeks to solve those equations by the rules of integral calculus. When inferring the values of λ, μ, …, one must pay attention to the signs of those quantities. If their signs are the same as those of the functions δL, δM, … , respectively, for all values of t when they refer to the possible displacements then one will be assured that the effective displacements will satisfy the equations $dL = 0$, $dM = 0$, … during the course of motion and that the solution that one gave to the question is exact and complete. However, things will be different when one starts from an instant $t = \tau$ and one or more factors λ, μ, … take on a sign that is opposite to the one that it would need in order to have equilibrium of the forces of motion and the dynamical forces. Those forces will no longer cancel, so one must conclude that from $t = \tau$ on, the motion that produces the dynamical forces, and which one presupposes, cannot exist, because the dynamical forces must always cancel the forces of motion that arise from them. Now, since one has presupposed only the equations $dL = 0$, $dM = 0$, …, it will follow that some of them will not exist, and that they will obviously be the equations that correspond to those of the factors λ, μ, … whose signs have changed. One then suppresses those equations and at the same time deletes them from the formula:

$$\sum \frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2} m = P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots,$$

and consequently, from all of the ones that are derived from it. All of the terms contain those same factors; i.e., the factors whose signs have changed. In that way, one will find as many equations as unknowns for determining the motion when one starts from $t = \tau$, because from the preceding, each equation $dL = 0$, $dM = 0$, … that disappears will, in a sense, take one of the unknowns of the question with it.

One can point out that one will have:

$$\sum \frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2} m = P\,\delta p + Q\,\delta q + R\,\delta r + \ldots + \lambda\,\delta L + \mu\,\delta M + \ldots,$$

$$\lambda\, dL = 0, \qquad \mu\, dM = 0, \ldots$$

for all values of t, in such a way that the number of equations will always be the same as that of the unknowns. However, the equations will change at various epochs, since one must suppose, for example, that $dL = 0$ for a certain time interval, and that $\lambda = 0$ for another interval. It is clear, moreover, that λ will become zero when dL ceases to be so.

In order to clarify the preceding with an example, suppose that the integration of the equations $\sum \frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2} m = P\ \delta p + Q\ \delta q + R\ \delta r + \ldots + \lambda\ \delta L + \mu\ \delta M + \ldots$, $dL = 0$, $dM = 0$, …, and the discussion of the values of λ, μ, … will show that ever since the beginning of the motion up to $t = \tau$, those factors have had the signs that are required for the equilibrium of the forces of motion and the dynamical ones, but at the instant $t = \tau$, the factor λ will become zero, and then the same factor will change sign. The motion from $t = 0$ to $t = \tau$ will be defined by the equations:

$$\sum \frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2} m = P\ \delta p + Q\ \delta q + R\ \delta r + \ldots + \lambda\ \delta L + \mu\ \delta M + \ldots,$$

$$dL = 0, \qquad dM = 0, \ldots$$

and after $t = \tau$, it will be defined by:

$$\sum \frac{d^2x\,\delta x + d^2y\,\delta y + d^2z\,\delta z}{dt^2} m = P\ \delta p + Q\ \delta q + R\ \delta r + \ldots + \lambda\ \delta L + \mu\ \delta M + \ldots$$

One just saw that the equations of motion can change at various epochs; however, that will not happen with all of the equations: Some of them will remain unaltered during the entire course of motion; for example, the one that is known by the name of the *vis viva principle.* One obtains it by replacing the characteristic δ with d in the general formulas of dynamics. Making that change of characteristic, while paying attention to the conditions $\lambda\, dL = 0$, $\mu\, dM = 0$, …, which persist for all of the epochs of the motion, will lead to the equation:

$$d \sum \frac{dx^2 + dy^2 + dz^2}{dt^2} m = 2\,(P\ \delta p + Q\ \delta q + R\ \delta r + \ldots),$$

which will be true for any t. The same thing will be true for all of the equations that are, like the preceding ones, independent of the possible displacements of the system. Hence, whenever the principle of areas and the center of gravity principle are true, the equations that express them will persist during the entire duration of the motion.

IX. – In order to give some idea of the application of the preceding theory, consider the motion of a heavy point that is placed on a vertical circle. Upon taking the x and y coordinate axes to be in the plane of the circle, with the first one horizontal and the second one vertical upwards, and the origin to be the center of the circle, the equation of motion will be:

$$\frac{d^2x\,\delta x + d^2y\,\delta y}{dt^2} = -\,g\,\delta y + \lambda\,(x\;\delta x + y\;\delta y) = 0, \qquad \lambda\,(x\;\delta x + y\;\delta y) = 0\,.$$

λ can never become negative, because the function $x^2 + y^2$ can only remain the same or increase for the possible displacements. First suppose that $x\;\delta x + y\;\delta y = 0$. Upon setting $\delta x = dx$, $\delta y = dy$, we will have:

$$d\,\frac{dx^2 + dy^2}{dt^2} + 2\,g\,dy = 0,$$

so upon integrating that and assuming that the motion begins at rest:

$$\frac{dx^2 + dy^2}{dt^2} = 2g\,(y_0 - y),$$

in which y_0 is the ordinate of the initial position of the point. Upon setting $\delta x = dx$, $\delta y = dy$, and denoting the radius of the circle by r, one will have:

$$\frac{x\,d^2x + y\,d^2y}{dt^2} + g\,y = \lambda\,r^2.$$

Now:

$$d\,(x\;dx + y\;dy) = x\;d^2\,x + y\;d^2\,y + dx^2 + dy^2 = 0,$$

so:

$$\lambda\,r^2 = g\,y - \frac{dx^2 + dy^2}{dt^2} = g\,(3y - 2y_0)\,.$$

It is clear that for that value of λ, the motion will not take place on the circle at just one instant when y_0 is not positive. Consequently, suppose that $y_0 > 0$. It is easy to see that y will decrease with time, because if one sets $y = r\cos\theta$, $x = r\sin\theta$ then one will find form the equation $\frac{dx^2 + dy^2}{dt^2}$ $= 2g\,(y_0 - y)$ that $\frac{d\theta^2}{dt^2} = \frac{2g}{r}(\cos\theta_0 - \cos\theta)$, in which θ_0 is the initial value of θ, so $\frac{d\theta}{dt} = \sqrt{\frac{2g}{r}}$ $\sqrt{\cos\theta_0 - \cos\theta}$. One must first have $\theta_0 < \theta$, and then θ increases to the start of the motion, and then θ cannot begin to diminish before $d\theta\,/\,dt$ becomes zero; hence, θ will not diminish before it becomes equal to $2\pi - \theta_0$. One must then set $\frac{d\theta}{dt} = \sqrt{\frac{2g}{r}}\;\sqrt{\cos\theta_0 - \cos\theta}$, and then $\theta = \theta_0$ until $\theta = 2\pi - \theta_0$. It will then certainly decrease, since $\frac{dy}{dt} = -\,r\sin\theta\,\frac{d\theta}{dt}$, and become zero, and even negative. However, once it reduces to $\frac{2}{3}y_0$, the quantity λ will be equal to zero, and will then become negative much later. Hence, the point will move in the circle up to the point that $y = \frac{2}{3}y_0$, and then it will leave the circle, in such a way that when one starts from $y = \frac{2}{3}y_0$, its motion will be given by the equation $\frac{d^2x\,\delta x + d^2y\,\delta y}{dt^2} + g\,\delta y = 0$, which will decompose into two equations:

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} + g = 0.$$

One can regard the velocity $\sqrt{\frac{2g\,y_0}{3}}$ and the coordinates y_0, $\sqrt{r^2 - y_0^2}$ as belonging to the initial state of motion, which will be the case starting from $y = \frac{2}{3}y_0$. We shall not enter into the other details that relate to the special example that we chose, since we chose it for its great simplicity.

X. – In all of the preceding, we have tacitly supposed that the coefficients of δx, δy, δz, $\delta x'$, … in the functions δL, δM, … do not include time t explicitly. However, if that variable is contained in them then the preceding considerations will not suffice to establish all of the equations of motion, because the effective displacements will not be included among the possible displacements that satisfy the equations $\delta L = 0$, $\delta M = 0$, … We will not have $dL = 0$, $dM = 0$, … then, and as a result, we will not have $\lambda\, dL = 0$, $\mu\, dM = 0$, … at all epochs.

Upon assuming that time t enters explicitly into the coefficients of δL, δM, …, those functions will themselves be mobile, in a sense, and one can regard the quantities dx, dy, dz, dx', … as each being composed of two parts: One of them is due to the displacements of the functions δL, δM, …, and the other one pertains to the motions of the m, m', … with respect to those functions. Consequently, suppose that $dx = \Delta x + Dx$, $dy = \Delta y + Dy$, $dz = \Delta z + Dz$, $dx' = \Delta x' + Dx'$, …, where Δx, Δy, Δz, $\Delta x'$, … pertain to the displacements of the functions δL, δM, …, and Dx, Dy, Dz, Dx', … pertain to the motions of the points m, m', … with respect to those functions. One will first have $DL = 0$, $DM = 0$, …. DL, DM, … are what δL, δM, …, respectively, will become when one sets $\delta x = Dx$, $\delta y = Dy$, …; i.e., if one has, for example, $\delta L = A\,\delta x + B\,\delta y + C\,\delta z + A'\,\delta x' + …$ Upon setting Dx, Dy, Dz, Dx', … equal to their values $dx - \Delta x$, $dy - \Delta y$, $dz - \Delta z$, $dx' - \Delta x'$, …, resp., we will have $dL = \Delta L$, $dM = \Delta M$, … Now, the quantities ΔL, ΔM, … must be given. Upon representing them by $T\,dt$, $T'\,dt$, …, respectively, one will have $dL = T\,dt$, $dM = T'\,dt$, … Those are the equations that the formulas $dL = 0$, $dM = 0$, …, which relate to the hypothesis that δL, δM, … are independent of time, will become in the case being examined. One makes the same use of $dL = T\,dt$, $dM = T'dt$, … that one makes of $dL = 0$, $dM = 0$, … It can also happen that when one starts from a certain epoch that one determines as before, the equations $dL = T\,dt$, $dM = T'\,dt$, … will not be satisfied, but that $\lambda\,(dL - T\,dt) = 0$, $\mu\,(dM - T'\,dt) = 0$, … will be satisfied during the entire duration of the motion, and upon combining them with the general formulas of dynamics, one will always have as many equations as are necessary to determine the motion completely.

We propose to publish a treatise on the science of equilibrium and motion in which we will present in detail everything that we pointed out into this paper. In our treatise, one will see that the extension that Lagrange gave to Jean Bernoulli's principle, which is an extension that seems obscure or inexact to some of the most celebrated geometers of our time, is nonetheless legitimate and results from the principles of the thing itself. One will also see that all of the questions that one can pose about the equilibrium or motion of systems can be resolved quite easily by the principle of virtual velocities, but above all, we will develop the conditions for stability of equilibrium, which I believe is a subject that has not been treated in full generality and to the full extent that one might desire.
