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On the transformation of the equations of dynamics

By **PAUL PAINLEVÉ**

Translated D. H. Delphenich

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INTRODUCTION

1. – If one is given a system of Lagrange equations:

$$(A) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i\,, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k)\,,$$

in which the Q_i depend upon neither velocity nor time, and in which T, which is a quadratic form in the q_i', is also independent of t :

$$2T \equiv \sum A_{ij}\, q_i'\, q_j' \equiv \frac{ds^2}{dt^2} \qquad (A_{ij} \equiv A_{ji}),$$

then one can demand to know if there exist other systems that are analogous (A_i) that define *the same motion* as (A). The question thus-posed is highly restricted, but it will acquire a different significance if one subjects the system (A_i) to only the condition that the *trajectories* of (A) and (A_i) *must coincide* then the motion along those trajectories will differ from one system to another, in general. In other words, the problem consists of defining the systems:

$$(A_1) \qquad \frac{d}{dt_1}\left(\frac{\partial T_1}{\partial q_i'}\right) - \frac{\partial T_1}{\partial q_i} = Q_i'(q_1, \ldots, q_k), \qquad \frac{dq_i}{dt_1} = q_i' \qquad (i = 1, 2, \ldots, k),$$

in which:

$$2T_1 \equiv \sum A_{ij}'\, q_i'\, q_j' \equiv \frac{ds_1^2}{dt_1^2},$$

that define *the same relations between the* q_i as (A). Two such systems (A) and (A_1) will be called *correspondents*.

2. – That problem is attached to a problem that appears to be more general and demands some explanations if it is to be posed clearly. The change of variables:

$$(1) \qquad q_1 = \varphi_1(r_1, r_2, \ldots, r_k), \qquad \ldots, \qquad q_k = \varphi_k(r_1, r_2, \ldots, r_k),$$

from which one infers, inversely, that:

$$(2) \qquad r_1 = \psi_1(q_1, q_2, \ldots, q_k), \qquad \ldots, \qquad r_k = \psi_k(q_1, q_2, \ldots, q_k),$$

transforms ds^2 into an expression of the same nature $d\sigma^2$:

$$d\sigma^2 \equiv \sum A_{ij}(\varphi_1, \varphi_2, \ldots, \varphi_k)\, d\varphi_i\, d\varphi_j \equiv \sum B_{ij}(r_1, r_2, \ldots, r_k)\, dr_i\, dr_j,$$

and the system (A) into a system:

$$(B) \qquad \frac{d}{dt}\left(\frac{\partial \tau}{\partial r_i'}\right) - \frac{\partial \tau}{\partial r_i} = R_i(r_1, r_1, \ldots, r_k), \qquad \frac{dr_i}{dt} = r_i' \qquad (i = 1, 2, \ldots, k),$$

in which:

$$2\tau \equiv \frac{d\sigma^2}{dt^2}, \qquad R_i \equiv Q_1(\varphi_1, \varphi_2, \ldots, \varphi_k)\frac{\partial \varphi_1}{\partial r_i} + \ldots + Q_k(\varphi_1, \varphi_2, \ldots, \varphi_k)\frac{\partial \varphi_k}{\partial r_i}.$$

We say that the expressions ds^2 and $d\sigma^2$, and similarly the systems (A) and (B), are *homologous* ([1]) and admit the transformation (1) as a *transformation of passage*. In particular, if ds^2 and $d\sigma^2$ [or (A) and (B)] coincide when one sets $q_i = r_i$ ($i = 1, 2, \ldots, k$) then the transformation (1) will be a transformation of ds^2 [or the system (A)] *into itself*.

A transformation (1) will make one *and only one* homologue correspond to a given ds^2 [or to a system (A)]. Conversely, there exists only one transformation of passage between two homologous expressions ds^2 and $d\sigma^2$ [or between two homologous systems (A) and (B)], unless ds^2 [or (A)] admits transformations into itself. Indeed, upon combining *one* transformation of passage with an *arbitrary* transformation of ds^2 [or (A)] into itself, one will get a new transformation of passage, and one will get all of them in that way. Those transformations of ds^2 [or (A)] into itself always define a *group*, which will be continuous if it depends upon arbitrary constants and discontinuous otherwise. (One easily shows that it cannot depend upon arbitrary functions.) Therefore, it is never difficult to recognize when two given expressions ds^2 and $d\sigma^2$ [or two given systems (A) and (B)] are homologous or to determine the transformations of passage in the case where the group of transformations of ds^2 into itself is discontinuous, and especially when it reduces to the identity transformation. However, in the case where that group is continuous, the transformations of passage depend upon differential equations. From Lie's theories, the whole problem comes down to determining the transformations of ds^2 [or (A)] into itself, and that study will come down to the *integration of a complete linear system*.

Finally, observe that if (A) and (B) are homologous then the same thing will be true *a fortiori* for ds^2 and $d\sigma^2$, but the converse is obviously not true. In particular, a transformation $q_i = \varphi_i$ of ds^2 into itself will preserve (A) only if one has:

$$\sum_i Q_j(\varphi_1, \ldots, \varphi_k)\frac{\partial \varphi_j}{\partial r_i} \equiv Q_i(r_1, \ldots, r_k) \qquad \text{for} \qquad i = 1, 2, \ldots, k.$$

More generally, let (A) and (B) be two homologous systems: If ds^2 and $d\sigma^2$ admit several transformations of passage then those transformations $q_i = \varphi_i$ will be of two types according to whether they do or do not satisfy the conditions:

$$R_i \equiv Q_1(\varphi_1, \varphi_2, \ldots, \varphi_k)\frac{\partial \varphi_1}{\partial r_i} + \ldots + Q_k(\varphi_1, \varphi_2, \ldots, \varphi_k)\frac{\partial \varphi_k}{\partial r_i} \qquad (i = 1, 2, \ldots, k).$$

([1]) If the two ds^2 that one compares include the same letters, such as ds^2 and ds_1^2, then ds_1^2 will be called *homologous* to ds^2 if it coincides with one of the homologues to ds^2, such as $d\sigma^2$, in which one has set $r_i = q_i$ ($i = 1, 2, \ldots, k$). Similarly, (A) and (A_1) will be called *homologous* if (A) coincides with one of the systems (B) when one sets $r_i = q_i$ and $t = t_1$.

Only the former ones transform (A) into (B).

3. – Having said that, we look for all of the systems (B_1):

$$(B_1) \qquad \frac{d}{dt_1}\left(\frac{\partial \tau_1}{\partial r_i'}\right) - \frac{\partial \tau_1}{\partial r_i} = R_i'(r_1, r_1, \dots, r_k) , \quad \frac{dr_i}{dt} = r_i' \qquad (i = 1, 2, \dots, k) ,$$

in which:

$$2\tau_1 \equiv \sum B_{ij}'(r_1, r_2, \dots, r_k)\, r_i'\, r_j' \equiv \frac{d\sigma_1^2}{dt_1^2} ,$$

such that the trajectories of (B_1) *are deduced from those of* (A) *by a change of variables* (1), $q_i = \varphi_i$. The inverse change of variables (2) transforms (B_1) into one (A_1) that corresponds to (A), so the systems (B_1) in question will be composed of *homologues of* (A) *and homologues of all of its correspondents*. The only difficulty then consists of determining the *correspondents* (A_1) and (A).

Among those systems (B_1), it is remarkable that there are two of them for which ds_1^2 will agree with ds^2 when one sets $q_i = r_i$ ($i = 1, 2, \dots, k$). If such a system (B_1) does exist then the motion defined by (A) will enjoy an important property: *One can replace the forces Q_i in* (A) *with some other forces,* namely, with forces $R_i'(r_1, r_1, \dots, r_k)$ *such that new trajectories are deduced from the former ones by changing the q_i into φ_i* ($q_1, \dots, q_k$). In the particular case where the Q_i and the R_i' are *identical* [i.e., where (A) and (B_1) coincide when one sets $q_i = r_i$, $t = t_1$], *the transformation $q_i = \varphi_i$ will transform the set of trajectories of* (A) *into itself.* On the other hand, it is clear that the inverse transformation (2) will make (B_1) become a *correspondent* (A_1) to (A) whose ds_1^2 is *homologous* to ds^2 . With that, we pose the following two problems:

I. *Determine the substitutions* (1) $q_i = \varphi_i$ *that transform the set of trajectories of A into itself.*

II. *Determine the systems of forces* $R_i'(q_1, q_1, \dots, q_k)$ *such that when one substitutes them for Q_i in* (A)*, the new trajectories will be deduced from the former ones by changing the q_i into φ_i* ($q_1, \dots, q_k$).

In order to solve the first problem, one must calculate *all of the correspondents* (A_1) *to* (A) *that are, at the same time, its homologues*. The desired transformations are composed of all transformation that take (A) to each system (A_1). In particular, they include the transformation of (A) into itself.

In order to solve the second problem, one must calculate *all of the correspondents* (A_1) *to* (A) *for which the ds_1^2 is homologous to ds^2*. All of the transformations of passage that exist between ds^2 and each ds_1^2, namely, $q_i = \varphi_i$, define the desired systems of forces R_i' , namely:

$$R_i' = Q_1\,(\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_k)\,\frac{\partial \varphi_1}{\partial r_i} + \ldots + Q_k\,(\varphi_1,\ \varphi_2,\ \ldots,\ \varphi_k)\,\frac{\partial \varphi_k}{\partial r_i} \qquad (i = 1, 2, \ldots, k)\,.$$

In particular, they include the transformations of ds^2 into itself.

4. – The foregoing will suffice to show the interest that is attached to the study of *corresponding* systems. The present treatise is devoted to proving some general properties of those systems. In another work, I will develop the main applications of those properties, and especially the solutions to problems I and II in the case of two or three parameters.

If one agrees to represent a system (A) by the symbol $\left(\frac{ds^2}{dt^2}, Q_i\right)$, or rather $\left(\frac{ds^2}{dt^2}, U\right)$, when the Q_i are derived from a potential U, then the main results that I have obtained can be summarized as follows:

In the first place, *an arbitrary system* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *will always admit an infinitude of correspondents*, namely, the systems $\left(C\,\frac{ds^2}{dt_1^2}, c\,Q_i\right)$, where C and c are two constants. One can pass from the system (A) to one of its correspondents (A_1) by the transformation: $\frac{dt_1}{dt} = \sqrt{\frac{C}{c}}$ ([1]). When all of the forces Q_i are zero, one passes from (A) to (A_1) by setting $dt_1 / dt = c$, where c denotes an *arbitrary* constant. In what follows, I will often say that ds^2 and $C\,ds^2$ are two *similar* ds^2, and likewise that the systems of forces Q_i and $c\,Q_i$ are two *similar* systems of forces, or rather that ds^2 and $C\,ds^2$ (and likewise the systems Q_i and $C\,Q_i$) are not *distinct*.

An arbitrary system (A) *does not admit other correspondents, in general*. If it does admit one, say $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$, then it will admit an infinitude of them, namely $\left(C\,\frac{ds_1^2}{dt_1^2}, c\,Q_i'\right)$. We say that those correspondents are not *distinct* from the former.

In the second place, assume that *the Q_i are derived from a potential*. As Darboux pointed out, *the system* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *will admit an infinitude of correspondents* $\left[(\alpha U + \beta)\,\frac{ds_1^2}{dt_1^2}, \frac{\gamma U + \delta}{\alpha U + \beta}\right]$, in which α, β, γ, δ are constants that are subject to the single condition that $\alpha\delta - \beta\gamma \neq 0$. The correspondence between (A) and one such system (A_1) enjoys a remarkable property: Group the trajectories of (A) into a *natural congruence*, by which I mean a congruence that satisfies the condition that $T - U = h$, where h is a well-defined constant, and compare the natural congruences of (A) and (A_1). One will find that *any natural congruence of* (A) *will coincide with a natural*

([1]) These are well-known properties that were pointed out a long time ago by Bertrand in his work on *similitude* in mechanics and from which Appell inferred an interpretation of *imaginary time* by setting $C / c = -1$.

congruence of (A_1), for which the values of h and h_1 will correspond to each other by the relation $h = \dfrac{\beta h_1 + \delta}{\alpha h_1 + \gamma}$. That property *is characteristic of the Darboux transformation*. One passes from (A) to (A_1) by the transformation:

$$(\alpha\delta - \beta\gamma)\,dt_1^2 = (\alpha U + \beta)^2[\alpha\, ds^2 - dt^2(\alpha U + \beta)] .$$

The systems (A_1) coincide with the ones that I have indicated to begin with for $\alpha = 0$. *An arbitrary system* $\left(\dfrac{ds^2}{dt^2}, U\right)$ *does not admit other correspondents, in general.* We shall give the name of *ordinary correspondents* to (A) to all of those systems (A_1).

5. – I have now arrived at the systems (A) that possess correspondents that are distinct from those *ordinary* correspondents. Here, we agree to study the case in which there are forces and the case where all of the Q_i are zero separately.

FIRST CASE. – *All of the coefficient* Q_i *are zero in* (A). The same thing will necessarily be true for any corresponding system (A_1) then. One finds that one then comes down to the study of *pairs of corresponding* ds^2 when one calls two ds^2 *correspondents* when their geodesics coincide. For $k = 2$, that is Dini's problem, and the theorem that was proved by that geometer proves to be a special case of the following one:

Let ds^2 *and* ds_1^2 *be two corresponding* ds^2 *(i.e., non-similar ones), and let* Δ *and* Δ_1 *be their discriminants (relative to the* dq_i). *The expression:*

$$\left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}} \frac{ds_1^2}{ds^2}$$

is a first integral of the geodesics. The expressions:

$$\left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}} \frac{ds_1^2}{dt^2}, \qquad \left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}} \frac{ds^2}{dt_1^2}$$

are then the quadratic integrals of the two systems:

$$\left(\frac{ds^2}{dt^2}, Q_i = 0\right) \qquad \text{and} \qquad \left(\frac{ds_1^2}{dt_1^2}, Q_i' = 0\right) ,$$

respectively. Moreover, one passes from one system to the other by the transformation:

$$\text{(1)} \qquad \frac{dt}{\Delta^{\frac{1}{1+k}}} = C\,\frac{dt_1}{\Delta_1^{\frac{1}{1+k}}},$$

in which C denotes an arbitrarily-chosen number (or even, if one prefers, an arbitrary first integral of the geodesics). *A* ds^2 *cannot admit the (non-similar) correspondent* ds_1^2 *without the system* $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ *admitting at least one quadratic integral that is distinct from that of vis viva* ([1]).

The study of the particular case in which the forces are zero implies some important consequences for the general case, notably, these: IF ds^2 AND ds_1^2 ARE CORRESPONDENTS THEN:

1. *For any system of forces* Q_i *, one can find forces* Q_i' *such that the two systems* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ *are correspondents, and one can then pass from one system to the other by a transformation of the form* (1), *in which C is a well-defined number*.

2. *Two arbitrary correspondents* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ *are included in the preceding ones, i.e., one can pass from one to the other by a transformation* (1).

I. *If one can pass from one system* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *to a system* $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ *for which the* Q_i *are* GIVEN *by a change of variables such that:*

([1]) That integral will agree with that of *vis viva* only when $ds_1^2 \equiv C\,ds^2$. Moreover, it can happen that ds^2 admits a correspondent and that the system $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ possesses *only one quadratic integral* besides the *vis viva* integral, as is shown by the example of the pair of correspondents:

$$ds^2 \equiv \varphi(q_1, q_2)(dq_1^2 + dq_2^2) + dq_3^2$$

and

$$ds_1^2 = \varphi(q_1, q_2)(dq_1^2 + dq_2^2) + c\,dq_3^2,$$

in which c is an arbitrary number.

$$\frac{dt_1}{dt} = \lambda\,(q_1, q_2, \ldots, q_k)$$

then ds^2 *and* ds_1^2 *will be correspondents, and the preceding results will apply.*

II. *If two systems* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ *correspond for* TWO *distinct systems of associated forces, say,* Q_i *and* Q_i'*, on the one hand, and* (Q_i) *and* (Q_i') *, on the other, then* ds^2 *and* ds_1^2 *will also correspond, and as a result, the system* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *will admit correspondents of the form* $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ FOR ANY Q_i .

However, the last proposition supposes that $k > 2$. For $k = 2$, one knows only that the number ν of (distinct) associated systems of forces Q_i , Q_i' *cannot exceed* 3 (ds^2 and ds_1^2 being given) without the geodesics of ds^2 and ds_1^2 coinciding (so ν will then be infinite). If $n = 3$ then ds^2 will be the ds^2 of a surface of constant curvature (and similarly for ds_1^2).

SECOND CASE. – *The forces* Q_i *of* (A) *are not all zero.* One proves that one can pass from the system (A) to a corresponding system (A_1) by a well-defined change of variables of the form:

$$\frac{dt_1^2}{dt^2} = \lambda^2(q_1, q_2, \ldots, q_k)\left(\frac{d\sigma^2}{dt^2} - V\right) = \lambda^2(\tau - V)\,,$$

if the equality $\tau - V$ = const. *is verified for any motion of* (A), which demands that $\tau - V$ *is either a quadratic integral of* (A) *or an absolute constant.* One will then be led to distinguish several possible hypotheses:

I. ($\tau - V$) *reduces to an absolute constant:* $dt_1 / dt = \lambda$. That is the case that was treated before in which ds^2 and ds_1^2 are CORRESPONDENTS. *The system* $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ *will admit a quadratic integral.*

II. *There exists a force function U, and* $\tau - V$ *coincides with* $T - (U + a)$. *The two systems* $\left[(U + a)\frac{ds^2}{dt'^2}, \frac{1}{U + a}\right]$ *and* $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$, *the first of which is an ordinary correspondent of* (A), *are,* *at the same time, correspondents to* $(U + a)\, ds^2$ *and* ds_1^2. It will then enjoy the properties that were indicated above: The system:

$$\left[(U+a)\frac{ds^2}{dt'^2}, Q_i = 0\right]$$

admits a quadratic integral. As for the system:

$$\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right),$$

it will admit a quadratic integral not only when one annuls the Q_i', but for the given Q_i'.

III. (General hypothesis). – *The equality* $\tau - V = const.$ *defines an integral of* (A) *that is distinct from that of vis viva. The systems* (A) *and* (A_1) *then admit a quadratic integral.* It is convenient to point out two particular cases under that hypothesis: *The case in which* U_1 *exists and the geodesics of* ds^2 *coincide with natural congruence* $T_1 - U_1 = a_1$ of (A_1) [this is the hypothesis II when one permutes (A) and (A_1)], and *the case in which* U *and* U_1 *exist and the two natural congruences* $T - U = a$ *and* $T_1 - U_1 = a_1$ *of* (A) *and* (A_1), *resp., coincide.* In one case and the other, the Darboux transformation will permit one to return to the hypothesis I *in which the geodesics of* ds^2 *and* ds_1^2 *coincide,* and as a result, to apply the conclusions that were stated in regard to the first case.

6. – The properties that I just enumerated are *necessary, but not sufficient,* conditions for a system (A) to admit ordinary correspondents: They are sufficient for only $k = 2$. However, those properties permit one to effortlessly form sufficient conditions upon singularly simplifying them, and among those conditions, they represent the most important ones, since they are the ones that exhibit the essential character of the systems (A) under study. Among the consequences that they imply, I shall cite these:

Let $\left[\frac{ds^2}{dt^2}, Q_i\right]$ and $\left[\frac{ds_1^2}{dt_1^2}, Q_i'\right]$ be two non-ordinary corresponding systems:

1. *One never has* $ds_1^2 = \mu(q_1, \ldots, q_k)\, ds^2$.

2. *If* Q_i *and* Q_i' *are derived from potentials* U *and* U_1, *resp., then there will not generally exist a natural congruence* $T - U = a$ *for* (A) *that coincides with a natural congruence* $T_1 - U_1 = a$ *for* (A_1), *and there will NEVER exist more than one.*

[Among the natural congruences, we include the congruence of geodesics that correspond to a (or a_1) $= \infty$.]

However, here is another consequence that is even more important:

The search for correspondents (A_1) *to a given system* (A), *and in particular the search for groups of transformations for the trajectories of* (A), *never imply the integration of complete linear systems.*

Finally, any integral of (A) *that is algebraic and entire (or rational) in* $q_1', \ldots, q_k'$ *corresponds to an analogous integral of the same degree of* (A_1) ([1]).

That applies to the *linear integrals*, in particular: Therefore, from a theorem of Lie, it will result that two corresponding ds^2 will possess the *same number of infinitesimal transformations into themselves*. That remark and the theorems that were established above on the correspondences that preserve geodesics immediately imply all of the propositions that were known already in regard to the correspondence between planar motions and motions on a surface of constant curvature, and analogous propositions are thus found to be established for an *arbitrary* number of parameters.

7. – We now return to the problems that I posed at the beginning of this introduction:

First of all, the necessary and sufficient conditions for the motion that is defined by (A) *to be defined by another system* (A_1) *are obviously the following ones:*

1. (A) *and* (A_1) *must be correspondents at the same time as* ds^2 *and* ds_1^2.
2. Δ *and* Δ_1 *must be identical (up to a constant factor).*

As for the systems B_1 (see pp. 4) whose trajectories are deduced from those of (A) by a transformation $q_i = \varphi_i (r_1, r_2, \ldots, r_k)$, their properties result immediately from the properties of the system (A_1). I shall confine myself to explicitly pointing out this now-obvious theorem:

In every case, one can pass from (A) *to* (B_1) *by a change of variables:*

$$q_i = \varphi_i (r_1, r_2, \ldots, r_k), \qquad \frac{dt_1}{dt} = \lambda (q_1, q_2, \ldots, q_k) [\tau - V] \qquad (i = 1, 2, \ldots, k),$$

in which the expression $\tau - V$ *defines a quadratic integral of* (A) *unless it reduces to a constant.*

In the last case, the substitution $q_i = \varphi_i$ will transform *the two geodesic congruences into each other. Conversely, if the geodesics of* (A) *and* (B_1) *correspond under the transformation* $q_i = \varphi_i$ *then one will have:*

([1]) This theorem is hardly obvious but results from the particular form of the relation that exist between dt and dt_1.

$$\frac{dt_1}{dt} = \lambda\,(q_1, q_2, \ldots, q_k)\,,$$

and there will exist systems (B_1) *whose vis viva is* $d\sigma_1^2 / dt_1^2$ *FOR ANY* Q_i *in* (A).

In particular, if one knows *a transformation* $q_i = \varphi_i$ *of the geodesics of* ds^2 *into themselves* then for any system of forces Q_i of (A), one can calculate the forces R_i' such that the trajectories of the system $\left[\frac{ds^2}{dt_1^2}, R_i'\right]$ reduce to the trajectories of (A) by changing q_i into $\varphi_i\,(q_1, q_2, \ldots, q_k)$. For example, take the most general *homographic transformation* that preserves the geodesics of $ds^2 \equiv dq_1^2 + dq_2^2 + dq_3^2$. For any system of forces Q_i, one can associate forces R_i' such that the trajectories of $\left[\frac{ds^2}{dt^2}, Q_i\right]$ and $\left[\frac{ds^2}{dt_1^2}, R_i'\right]$ can be deduced from each other by a given homographic transformation. One will recover Appell's well-known results upon applying the general correspondence formulas that were established in this article to that particular case.

Finally, I shall say a few words about a problem that is quite analogous to the search for correspondents and is concerned with the systems (A) for which the forces are derived from a potential U. One knows that each *natural congruence* of trajectories $T - U = a$ coincides with the geodesics of $(U + a)\,ds^2$. One can investigate whether $ds'^2 \equiv (U + a)\,ds^2$ *admits a (non-similar) correspondent* ds^2 *for any* a, namely, $ds_1'^2$. It is clear that this investigation will revert completely to the study of pairs of corresponding ds^2. However, what analogy might exist between the $ds_1'^2$ and the correspondents (A_1) and (A)? First of all, one effortlessly sees that if ds'^2 possesses a correspondent $ds_1'^2$ (for any a) then the system (A) will always possess an infinitude of distinct correspondents that depend upon an arbitrary constant: Moreover, the converse is not true. However, the precise question that is of interest to us is the following one: Can one of the systems $\left[\frac{ds_1'^2}{dt_1'^2}, Q_i' = 0\right]$ (where $ds_1'^2$ depends upon a) be attached to a certain system $\left[\frac{ds_1^2}{dt_1^2}, U_1\right]$ that is independent of a in the same way that [ds'^2, $Q_i = 0$] is attached to (A)? That amounts to demanding to know whether (A) can admit non-ordinary correspondents $\left[\frac{ds_1^2}{dt_1^2}, U_1\right]$ such that any natural congruence of (A) will be a natural congruence of (A_1). We have said that this is never true. *The search for corresponding systems to* (A) *and that of the* ds^2 *that correspond to* $(U + a)\,ds^2$ *always constitute two distinct problems then.*

8. – I shall conclude this introduction with a brief historical overview of the prior research. It was the work of Appell on homographies in mechanics that led me to study the general questions that are treated in this article. In two publications in the *American Journal* (1889-1890), Appell showed that any planar motion (or in ordinary space) can be made to correspond to another planar (or spatial) motion that is produced by some other forces (those forces always being independent of velocity) with the aid of an arbitrary homographic transformation, and he gave some remarkable applications of that principle to the theory of central forces. At the end of the first paper, Appell, following Goursat, posed the more general problem: *If one is given two* ds^2*, namely,* ds^2 *and* ds_1^2*, then for every system of forces* Q_i*, do there exist forces* R_i' *such that one can pass from the system* $\left[\frac{ds^2}{dt^2}, Q_i\right]$ *to the system* $\left[\frac{ds_1^2}{dt_1^2}, R_i'\right]$ *by changing the* q_i *into* φ_i $(q_1, q_2, \ldots, q_k)$ *and dt into* λ $(q_1, q_2, \ldots, q_k)$? He indicated, in that regard, that the following proposition seemed reasonable (which he proved in the case of homography): *If the substitution* $q_i = \varphi_i$, $dt_1 = \lambda\, dt$ *transforms the system* $\left[\frac{ds^2}{dt^2}, Q_i\right]$ *into a system* $\left[\frac{ds_1^2}{dt_1^2}, R_i'\right]$ *for arbitrary forces* Q_i (ds^2 *and* ds_1^2 *being given*) *then it will make the geodesics of* ds^2 *correspond to those of* ds_1^2. That proposition, which was verified by Dautheville for $k = 2$, was proved, along with its converse, by Appell himself in a note in the Bulletin de la Société mathématique (15 March 1892). In a note that appeared almost simultaneously in the Comptes rendus de l'Académie des Sciences (12 April 1892) ([1]), I have summarized the main results that were contained in that paper, which are results that refer to the preceding proposition, in particular, but completed them, as one saw above (no. **5**, pp. 6-8). One of the most important complements consists of the fact that *if the two geodesic congruences of* ds^2 *and* ds_1^2 *are transformed into each other by a change of variables* q_i *then one can always pass from the system* $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$ *to the system* $\left[\frac{ds_1^2}{dt_1^2}, R_i' = 0\right]$ *by changing the* q_i *into* φ_i $(q_1, q_2, \ldots, q_k)$ ***and*** *dt* ***into*** $\lambda\, dt_1$. For example, from that, it will suffice to know that any surface of constant curvature can be represented geodesically on the plane in order for one to be assured that any planar motion [where the forces Q_1 (q_1, q_2), Q_2 (q_1, q_2) are arbitrary] can be made to correspond to a motion on a surface of constant curvature.

The question that I have posed naturally led me to generalize Dini's problem, which coincides with the search for correspondents in the particular case where $k = 2$ and the forces are zero. Liouville had previously published two notes on that problem: In the first one (Comptes rendus, 6 April 1891), he determined *all* ds^2 *with two or three parameters such that the motion that is defined by the system* $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$ *could also be defined by another system* $\left[\frac{ds_1^2}{dt^2}, Q_i' = 0\right]$ *and that, moreover, the discriminants* Δ *and* Δ_1 *of* ds^2 *and* ds_1^2*, resp., would be identical* ([2]). In the second one (Comptes rendus, 16 December 1891), which was devoted to quadratic integrals, Liouville

([1]) *See* also the Comptes rendus of 16 May, 13 June, 10 October, 7 November, 21 November 1892 and 2 January 1893.

([2]) From the foregoing, this second condition is pointless, since it is always a consequence of the first one (no. **7**, pp. 10). The ds^2 that Liouville calculated are therefore the only ds^2 with three parameters such that the motion along their geodesics coincides with another analogous motion.

observed that if, for $k = 2$, the cases in which the system $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$ admits a quadratic integral are also the ones for which Dini's problem has solutions then for $k > 2$, the same thing will no longer be true, and he announced some later work on the question. After my publication on 11 April 1892, that author made known (*loc. cit.*, 25 April 1892) ([1]) the results that he had obtained by a very different method from my own. That method, which is based upon the sufficient conditions for two ds^2 to be *corresponding*, exhibits the very remarkable fact that *a ds^2 cannot possess one correspondent without possessing an infinitude of them of the form:*

$$ds_1^2 \equiv \frac{C^{k-1}\,d\sigma_{k-1}^2 + C^{k-2}\,d\sigma_{k-2}^2 + \cdots + C\,d\sigma_1^2 + d\sigma^2}{\delta^2},$$

in which C is an arbitrary constant that depends upon δ. It results from this that there will exist (k – 1) quadratic integrals (in addition to the *vis viva* integral) for the system $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$. Nonetheless, it remains to be seen whether those integrals are *distinct*. An example that was cited above (*see* the note on page 7) shows that they can reduce *to just one*.

Liouville's method obviously applies to the study of the case in which $ds'^2 \equiv (U + h)\,ds^2$ admits correspondents for any h. However, as I have said, that study is *always distinct* from the study of the correspondents of $\left[\frac{ds^2}{dt^2}, U\right]$, and one cannot deduce any property of the latter systems from it. Therefore, Liouville's work and my own *meet up only in the case where all of the forces are zero.* It would nonetheless be legitimate to appeal to Liouville's results that concern the corresponding ds^2 in order to study the case in which the systems $\left[\frac{ds^2}{dt^2}, Q_i\right]$ and $\left[\frac{ds_1^2}{dt_1^2}, Q'_i\right]$ correspond *with preservation of the geodesics*, as well as the case that reduces to it under the Darboux transformation. However, even when I was treating those particular cases, I exclusively appealed to the method that I presented at the time of my first communication in that work and the applications that followed it.

Before passing on to the proof of the theorems that were enumerated above, I shall immediately indicate a notation that has been useful for me: I must frequently take the derivatives of the same variables $q_1, q_2, \ldots, q_k$ with respect to the two different variables t and t_1, or with respect to one of them, say, q_1. I shall invariably represent the derivative dq_i / dt by q'_i, the derivative dq_i / dt_1 by $(q')_i$, and the derivative dq_i / dq_1 by $q'_{(i)}$; from that, $q'_{(1)}$ will be equal to unity.

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([1]) *See* also the Comptes rendus on 23 May, 12 September, 31 October and 14 November 1892.

CHAPTER I

General properties of trajectory equations.

I. - NUMBER OF CONSTANTS UPON WHICH THE TRAJECTORIES DEPEND.

1. – I shall first establish some very simple properties of the differential equations that the trajectories depend upon.

A system of Lagrange equations:

$$(A) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i\,(q_1, q_2, \ldots, q_k)\,, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k)\,,$$

in which:

$$2T \equiv \sum A_{ij}(q_1, q_1, \ldots, q_k)\, q_i' q_j' \equiv \frac{ds^2}{dt^2} \qquad (A_{ij} \equiv A_{ji})$$

defines $(2k - 1)$ of the variables $q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'$ as functions of one of them and $(2k - 1)$ arbitrary constants. For example, those constants permit one to give arbitrary values to $q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'$ for $q_1 = q_1^0$. The functions $q_2, q_3, \ldots, q_k$ of q_1 that are defined by (A) then satisfy a differential system whose order n can neither exceed $2k - 1$ nor, on the other hand, become less than $2k - 2$, because the functions $q_2, q_3, \ldots, q_k$, $\frac{dq_2}{dq_1} = \frac{q_2'}{q_1'}, \ldots, \frac{dq_k}{dq_1} = \frac{q_k'}{q_1'}$ can take on arbitrary values for q_1^0 (1).

There exist systems (A) for which n effectively reduces to $2k - 2$: They are the ones *in which all of the coefficients Q_i are zero*. The trajectories of (A) are then the *geodesics of ds^2* of T, and those geodesics will depend upon $(2k - 2)$ arbitrary constants. Moreover, it is easy to form the differential equations of the geodesics in this case. Indeed, suppose that the system (A) is solved for the q_i'', which is always possible since the discriminant Δ of T is non-zero. We obtain the five equations:

$$\frac{d^2 q_i}{dt^2} = P_i\,(q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k') \qquad (i = 1, 2, \ldots, k),$$

in which P_i is *a quadratic form* with respect to the q_i'. Upon supposing that the differentials are taken with respect to an auxiliary variable $\theta = g\,(t)$, those equations can be further written:

(1) It is well-known that it follows from this that (A) cannot admit a first integral of the form $\varphi\,(q_1, q_2, \ldots, q_k)$ = const. It is nonetheless implicit that the discriminant Δ of T is not identically zero.

$$d^2q_i\,\theta_t'^2 + dq_i\,d\theta\,\theta_{t^2}'' = P_i\,(q_1, q_2, \ldots, q_k,\ q_1',\ q_2',\ \ldots,\ q_k') \times \theta_t'^2 = \Pi_i\,\theta_t'^2\,,$$

so, upon eliminating $d\theta\,\theta_{t^2}''\dfrac{1}{\theta_t'}$ from those two relations:

$$(1) \qquad d^2q_i\,dq_j - dq_i\,d^2q_j = \Pi_i\,dq_j - \Pi_j\,dq_i\,.$$

If one sets $\theta = q_1$, for example, then one will have $(k-1)$ second-order equation to solve for $\dfrac{d^2q_2}{dq_1^2}$, , …, $\dfrac{d^2q_k}{dq_1^2}$. Moreover, those equations are given explicitly by the least-action principle.

2. – I would now like to show that, *when that case is overlooked, the trajectories will depend upon* $(2k-1)$ *arbitrary constants* ([1]). Indeed, one infers from equations (A), as above, that:

$$\frac{d^2q_i}{dt^2} = P_i\,(q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k') + \frac{\alpha_i}{\Delta}\,.$$

α_i denotes what Δ will become when one replaces the terms in the i^{th} column with $Q_1, Q_2, \ldots, Q_k$, and as a result:

$$d^2q_i\,\theta_t'^2 + dq_i\,d\theta\,\theta_{t^2}'' = \Pi_i\,\theta_{t^2}' + \frac{\alpha_i}{\Delta}\,d\theta^2 = \theta_t'^2\left[\Pi_i + \frac{\alpha_i}{\Delta}\,dt^2\right],$$

so finally:

$$(2) \qquad \begin{cases} \dfrac{dt^2}{\Delta} = \dfrac{d^2q_2\,dq_1 - d^2q_1\,dq_2 - (\Pi_2\,dq_1 - \Pi_1\,dq_2)}{\alpha_2\,dq_1 - \alpha_1\,dq_2} \\[2ex] \qquad = \dfrac{d^2q_j\,dq_i - d^2q_i\,dq_j - (\Pi_j\,dq_i - \Pi_j\,dq_i)}{\alpha_j\,dq_i - \alpha_j\,dq_i}\,. \end{cases}$$

If one takes q_1 to be the independent variable, in particular, then one will have:

$$(3) \qquad \frac{d^2q_i}{dq_1^2} + \left(\Phi_1\frac{dq_i}{dq_1} - \Phi_i\right) = \left(\frac{\alpha_i}{\Delta} - \frac{\alpha_1}{\Delta}\frac{dq_i}{dq_1}\right)\frac{1}{\left(\dfrac{dq_1}{dt}\right)^2}\,,$$

in which:

([1]) This supposes that $k > 1$. For $k = 1$, one can no longer speak of relations between the q_i.

$$\Phi_i \equiv P_i\left(q_1, q_2, \ldots, q_k, 1, \frac{dq_2}{dq_1}, \ldots, \frac{dq_k}{dq_1}\right).$$

From the equality (3), q_2, q_3, …, q_k, and $\frac{dq_2}{dq_1}$, …, $\frac{dq_k}{dq_1}$ have received arbitrary values for $q_1 = q_1^0$, so one can once more choose $q_1'^0$ in such a fashion as to give $\frac{d^2q_i}{dq_1^2}$ an arbitrary value, at least as long as the binomial $\alpha_i - \alpha_1 \frac{dq_2}{dq_1}$ is not zero. In order for the functions q_2, …, q_k of q_1 to depend upon only $2k - 2$ constants, it is only necessary that the conditions:

$$\frac{\alpha_1}{q_1'} = \frac{\alpha_2}{q_2'} = \ldots = \frac{\alpha_k}{q_k'} \tag{4}$$

should be verified identically. The α_i do not contain the velocities, which can be true only if all of the α_i, and as a result, *all of the* Q_i, *are zero* ([1]).

3. – In the case where the Q_i are not all zero, here is how one can define the differential equations of the trajectories. Let $\alpha_1 \neq 0$. One first writes down the $(2k - 2)$ equations:

$$\frac{\dfrac{d^2q_2}{dq_1^2} + \Phi_1 \dfrac{dq_2}{dq_1} - \Phi_2}{\alpha_2 - \alpha_1 \dfrac{dq_2}{dq_1}} = \frac{\dfrac{d^2q_i}{dq_1^2} + \Phi_1 \dfrac{dq_i}{dq_1} - \Phi_i}{\alpha_i - \alpha_1 \dfrac{dq_i}{dq_1}} \qquad (i = 3, \ldots, k). \tag{5}$$

On the other hand, if one sets:

$$\chi_i \equiv \frac{d^2q_i}{dq_1^2} + \Phi_1 \frac{dq_i}{dq_1} - \Phi_i, \qquad \psi_i \equiv \frac{1}{\Delta}\left(\alpha_i - \alpha_1 \frac{dq_i}{dq_1}\right) \qquad (i = 2, 3, \ldots, k)$$

then one can infer from the equality:

$$\left(\frac{dq_1}{dt}\right)^2 = \frac{\alpha_2 - \alpha_1 \dfrac{dq_2}{dq_1}}{\left(\dfrac{d^2q_2}{dq_1^2}\right) + \Phi_1 \dfrac{dq_2}{dq_1} - \Phi_2} = \frac{\psi_2}{\chi_2}$$

that:

([1]) When the forces Q_i depend upon velocities, it will suffice (in order that $\nu = 2k - 2$) that the α_i should satisfy the conditions (4).

$$2\,\frac{d^2 q_1}{dt^2} = \frac{d}{dq_1}\frac{\psi_2}{\chi_2}\,,$$

and upon replacing $\frac{d^2 q_1}{dt^2}$ with its value $\Phi_1\left(\frac{dq_1}{dt}\right)^2 + \frac{\alpha_1}{\Delta} \equiv \Phi_1\frac{\psi_2}{\chi_2} + \frac{\alpha_1}{\Delta}$, one will get:

(6) $$\frac{d}{dq_1}\log\chi_2 + 2\,\Phi_1 = \frac{d}{dq_1}\log\psi_2 - 2\frac{\alpha_1}{\Delta}\frac{\chi_2}{\psi_2}\,,$$

which is an equation of the form:

(6′) $$\frac{d}{dq_1}\chi_2 + \chi_2\frac{M}{\psi_2} = 0\,,$$

in which M is a polynomial in the derivatives.

By definition, one then defines a system of the form:

$$q'''_{(2)} = f_2(q_1, q_2, \ldots, q_k, q'_{(2)}, q'_{(3)}, \ldots, q'_{(k)}, q''_{(2)})\,,$$
$$q''_{(i)} = f_i(q_1, q_2, \ldots, q_k, q'_{(2)}, q'_{(3)}, \ldots, q'_{(k)}, q''_{(2)}) \qquad (i = 3, 4, \ldots, k),$$

upon setting:

$$q'_{(i)} = \frac{dq_i}{dq_1}\,, \qquad q''_{(i)} = \frac{d^2 q_i}{dq_1^2}\,, \qquad q'''_{(i)} = \frac{d^3 q_i}{dq_1^3}\,,$$

which can be made more symmetric, but that is irrelevant to our purposes.

I immediately point out that *the geodesics of* ds^2 *belong to the trajectories, no matter what the forces* Q_i *are.* Indeed, the equations:

$$\chi_2 \equiv \frac{d^2 q_2}{dq_1^2} + \Phi_1\frac{dq_2}{dq_1} - \Phi_2 = 0\,, \qquad \ldots, \qquad \chi_k \equiv \frac{d^2 q_k}{dq_1^2} + \Phi_1\frac{dq_k}{dq_1} - \Phi_k = 0\,,$$

which define the geodesics, imply the relations (5), (6). The equality (3) shows us, moreover, that q'_1 *is infinite* at an arbitrary point of those trajectories: In other words, the geodesics form a ($2k$ – 2)-parameter congruence of trajectories, namely, the congruence that is obtained by imposing the condition that $1/q'_1 = 0$ (or $1/T_0 = 0$, if T_0 denotes the initial semi-*vis viva*) on the initial constants. That condition will be realized all along the trajectory. Furthermore, that is a proposition that we will have to establish in a very different manner.

I must now insist upon some characteristic differences that separate the case in which the forces are zero from the general case.

II. – SYSTEMS IN WHICH ALL OF THE COEFFICIENTS Q_i ARE ZERO.

4. – We have said that if all of the forces are zero then the trajectories will depend upon $(2k - 2)$ constants, and from the principle of least action, it can be defined by the system:

$$(\alpha) \qquad \frac{d}{dq_1}\left(\frac{\partial f}{\partial q'_{(2)}}\right) - \frac{\partial f}{\partial q_2} = 0, \qquad \ldots, \qquad \frac{d}{dq_1}\left(\frac{\partial f}{\partial q'_{(2)}}\right) - \frac{\partial f}{\partial q_2} = 0,$$

upon setting:

$$q'_{(2)} = \frac{dq_2}{dq_1}, \quad \ldots, \quad q'_{(k)} = \frac{dq_k}{dq_1},$$

and

$$f = \sqrt{T(q_1, q_2, \ldots, q_k, 1, q'_{(2)}, \ldots, q'_{(k)})} .$$

Now assume that one has integrated those equations and that, as a result, one knows $q_2, q_3, \ldots, q_k$ as a function of q_1 and $(2k - 2)$ arbitrary constants $a_1, a_2, \ldots, a_{2k-2}$. How does one determine t ? From the *vis viva* theorem:

$$dt = h\, dt = h \times (f) \times dq_1 ,$$

in which h is a new constant, and (f) is the function of q_1 that is obtained from f by replacing $q_2, q_3, \ldots, q_k$ and $q'_{(2)}, \ldots, q'_{(k)}$ as functions of q_1 and the constants. It is legitimate to write:

$$h = g\,(a_1, a_2, \ldots, a_{2k-2}, h_0) ,$$

and since, on the one hand:

$$\begin{aligned} a_i &= F_i[q_1, q_2, \ldots, q_k, q'_{(2)}, \ldots, q'_{(k)}] \\ &= F_i[q_1^0, q_2^0, \ldots, q_k^0, q'^0_{(2)}, \ldots, q'^0_{(k)}] \end{aligned}$$

is a first integral of the geodesics, one will see that *dt verifies the equation:*

$$(\beta) \qquad dt = G[q_1, q_2, \ldots, q_k, q'_{(2)}, \ldots, q'_{(k)}, h_0]\, f\, dq_1 ,$$

in which G represents an arbitrary first integral of the geodesics that depends upon an arbitrary parameter h_0 .

Conversely, assume that a relation:

$$(\gamma) \qquad dt = H[q_1, q_2, \ldots, q_k, q'_{(2)}, \ldots, q'_{(k)}, h_0]\, dq_1 ,$$

are compatible with (α). I intend that to mean that the function $t\,(q_1)$ that is defined by (γ) when one replaces the q_i and $q'_{(i)}$ in H as functions of q_1 verifies the equations of motion. One must have (from that substitution):

$$H = h\,f\,,$$

in which *h* is constant for the same geodesic (and that will be true for any geodesic one considers). Hence, H / f is *a first integral of the geodesic*.

If one is given a system (*A*) without forces Q_i then one will see that the system (*A*) will not be altered when one replaces *dt* with *G dt*, where *G* is either a constant or an arbitrary first integral of the geodesic.

5. – From a remark by Darboux, the systems (*A*) *in which the forces are derived from a potential U* reduce to systems (*A*) without forces. That results from the principle of least action: The equations:

$$(a) \qquad \frac{d}{dq_1}\left(\frac{\partial f}{\partial q'_{(i)}}\right) - \frac{\partial f}{\partial q_{(i)}} = 0\,, \qquad \frac{dq_i}{dq_1} = q'_{(i)} \qquad (i = 2, 3, \ldots, k)\,,$$

in which:

$$f = \sqrt{(U+h)\,T\,(q_1, q_2, \ldots, q_k, 1, q'_{(2)}, \ldots, q'_{(k)})}\,,$$

define both the geodesics of $ds_1^2 = (U+h)\,ds^2$ and the trajectories of (*A*) that correspond to the value *h* of the *vis viva* constant. However, one must indeed observe that the motion along the trajectories that are defined by (*A*):

$$(A) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q'_i}\right) - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i}\,, \qquad \frac{dq_i}{dt} = q'_{(i)} \qquad (i = 1, 2, \ldots, k)$$

differs from the motion that is defined by (A_1):

$$(A_1) \qquad \frac{d}{dt}\left[\frac{\partial T_1}{\partial (q'_i)}\right] - \frac{\partial T_1}{\partial q_i} = 0\,, \qquad \frac{dq_i}{dt} = (q'_i) \qquad (i = 1, 2, \ldots, k)\,,$$

in which:

$$T_1 \equiv (U+h)\frac{ds^2}{dt_1^2} = \frac{ds_1^2}{dt_1^2}\,.$$

Indeed, from (*A*), one has:

$$dt^2 = \frac{ds^2}{U+h}\,,$$

and from (A_1):

$$dt_1^2 = \alpha\,(U+h)\,ds^2\,,$$

in which α denotes a new arbitrary constant [or an arbitrary first integral of (a)]. One will then go from the first motion to the second one by changing dt^2 into $\frac{dt_1^2}{\alpha(U+h)^2}$, where α is an arbitrary constant.

With that, introduce canonical variables into (A) and (A_1): Let $p_i = \frac{\partial T}{\partial q_i'}$ and $p_i' = \frac{\partial T_1}{\partial (q_i')} = (U+h)\, p_i \frac{(q_i')}{q_i'}$. Along each trajectory, one will have:

$$p_i = \sqrt{\alpha}\, p_i',$$

in which α is a constant.

Upon letting T' and T_1' denote what T and T_1 will become when one replaces the q_i' and (q_i') as functions of p_i and p_i', respectively, from (A) one will have:

$$T' = U + h,$$

and from (A_1):

$$T_1' = \alpha\,(U + h).$$

Any first integral of (A_1), which one can always suppose to be homogeneous in $p_1', \ldots, p_i'$, namely:

$$F_1(q_1, q_2, \ldots, q_k, p_1', p_2', \ldots, p_k', , h) = C,$$

corresponds to an integral of (A):

$$F_1(q_1, q_1, \ldots, q_k, p_1, p_1, \ldots, p_k, h) = F_1[q_1, q_1, \ldots, q_k, p_1, p_1, \ldots, p_k, (T' - U)] = C.$$

Conversely, any first integral of (A):

$$F(q_1, q_1, \ldots, q_k, p_1, p_1, \ldots, p_k) = C$$

can be made homogeneous by the substitution of $p_i \sqrt{\frac{U+h}{T'}}$ for p_i, and the expression $F_1(q_1, q_1, \ldots, q_k, p_1, p_1, \ldots, p_k, h)$ that one will obtain, in which one replaces the p_i with p_i', will be a first integral of (A_1).

In particular, when (A) admits an integral that is algebraic and entire with respect to the velocities, namely, $P_m + P_{m-2} + P_{m-4} + \ldots = C$, the system ($A_1$) will admit an analogous integral of the same degree, namely, $P_m + \frac{T'}{U+h} P_{m-2} + \frac{T'^2}{(U+h)^2} P_{m-4} + \ldots = C$, in which p_i are replaced with the p_i' (1). Conversely, if (A_1) admits an integral of that form for any h then (A) will admit an entire integral of degree m. However, one poses the question here: *Does any entire algebraic integral of* (A_1) *that exists for any h necessarily have that form?* For example, when A_1 admits a quadratic integral for any h, can that integral can always be written:

$$P_2 + \frac{T'}{U+h} P_0 = C ,$$

in which P_2 and P_0 are independent of h ? The answer is affirmative, but it is hardly obvious that this must be true. I shall confine myself here to pointing out that proposition, which is indispensable for us, but not developing the proof, which is delicate.

Some analogous remarks apply to rational integrals.

III. – SYSTEMS IN WHICH THE FORCES ARE NOT ZERO.

6. – When the coefficients Q_i of a system (A) (in which k is much greater than 1) are non-zero, once the differential equations of the trajectories have been integrated, dt / dq_1 will be given as a function of q_1 by any one of the equalities (*see* pp. 15-16):

$$\frac{1}{\Delta}\frac{dt^2}{dq_1^2} = \frac{\dfrac{d^2 q_i}{dq_1^2} + \Phi_1 \dfrac{dq_i}{dq_1} - \Phi_i}{\alpha_i - \alpha_1 \dfrac{dq_i}{dq_1}} = \frac{\chi_i}{\psi_i} , \tag{2}$$

in which $q_2, q_3, \ldots, q_k$ are expressed as functions of q_1 and $(2k - 1)$ arbitrary constants.

One might remark in passing that those equalities lead one to distinguish two classes Γ and Γ' of the *real* trajectories Γ of (A) according to whether the common sign of the expressions χ_i / ψ_i (which is that of T) is positive or negative, resp., along one of those trajectories: The motion is real only along the former, while it is imaginary along the latter.

The trajectories will not be modified if one replaces the Q_i *in* (A) *with forces* $Q_i' \equiv c\, Q_i$, and one passes from the first system to the second one by changing t into $\sqrt{c}\, t + a$, i.e., changing dt into $\sqrt{c}\, t\, dt$, which is a transformation that is *unique*, from (2), at the moment when the forces Q_i are non-zero.

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(1) We remark that although this essentially supposes that one has introduced the canonical variables, if one keeps the variables q_i and their differentials then a quadratic integral of (A), namely, $d\sigma^2 - V\,dt^2 = C\,dt^2$, will correspond to the integral of (A_1):

$$(U+h)^2 \left[d\sigma^2 - \frac{V}{(U+h)} ds^2 \right] = C\,dt_1^2 .$$

If c is positive then the real motions will remain real. If c is negative then the real trajectories Γ' of the first system will become the trajectories Γ'' of the second, and *vice versa*. The particular transformations $t = i\, t_1$ and $t = -t_1$ give rise to some well-known remarks about the case in which one changes the sense of either all forces or all velocities without changing their direction or magnitude.

It is important to observe that *the forces $Q'_i = c\, Q_i$ are the only ones that will generate the same trajectories when they are substituted for the forces Q_i in* (A). Indeed, consider the differential equations of the trajectories:

(5)
$$\frac{\chi_2}{\psi_2} = \frac{\chi_3}{\psi_3} = \ldots = \frac{\chi_k}{\psi_k},$$

(6)
$$\frac{d}{dq_1}\log \chi_2 + 2\,\Phi_1 = \frac{d}{dq_1}\log \psi_2 - \frac{2\alpha_1}{\Delta}\frac{\chi_2}{\psi_2},$$

in which [1]:

$$\chi_i = \frac{d^2 q_i}{dq_1^2} + \Phi_1 \frac{dq_i}{dq_1} - \Phi_i, \qquad \psi_i = \frac{1}{\Delta}\left(\alpha_i - \alpha_1 \frac{dq_i}{dq_1}\right).$$

The $(k-2)$ equations (5) have the form:

(5′)
$$\frac{d^2 q_i}{dq_1^2} = \frac{d^2 q_2}{dq_1^2} \cdot \frac{\alpha_i - \alpha_2 \dfrac{dq_i}{dq_1}}{\alpha_2 - \alpha_1 \dfrac{dq_2}{dq_1}} + L_i \qquad (i = 3, 4, \ldots, k),$$

in which L_i contain only the first derivatives, and equation (6) can be written:

(6′)
$$\frac{d^3 q_2}{dq_1^3} = L_2 = \chi_2 \frac{d}{dq_1}\log \alpha_1 + L'_2,$$

in which L'_2 is defined with the aid of the coefficients of T and the ratios α_i / α_1.

Now suppose that one replaces the Q_i with forces Q'_i: In order for the trajectories to remain the same, it is necessary that the left-hand sides of (5′) and (6′) should not be altered. One must then have:

$$\frac{\alpha_i - \alpha_2 \dfrac{dq_i}{dq_1}}{\alpha_2 - \alpha_1 \dfrac{dq_2}{dq_1}} \equiv \frac{\alpha'_i - \alpha'_2 \dfrac{dq_i}{dq_1}}{\alpha'_2 - \alpha'_1 \dfrac{dq_2}{dq_1}} \qquad (i = 3, 4, \ldots, k),$$

(1) It is appropriate to observe that the c_i are defined with the aid of only the coefficients of T without involving the Q_i.

i.e.:

$$\frac{\alpha_1}{\alpha'_1} \equiv \frac{\alpha_2}{\alpha'_2}, \ldots, = \frac{\alpha_k}{\alpha'_k},$$

and on the other hand [from (6′)]:

$$\frac{d}{dq_1} \log \alpha_1 \equiv \frac{d}{dq_1} \log \alpha'_1,$$

or rather:

$$\frac{\partial}{\partial q_1} \log \alpha_1 \equiv \frac{\partial}{\partial q_1} \log \alpha'_1, \qquad \ldots, \qquad \frac{\partial}{\partial q_k} \log \alpha_1 \equiv \frac{\partial}{\partial q_k} \log \alpha'_1,$$

and as a result:

$$\alpha'_1 = c\,\alpha_1,$$

in which c is a constant. One will then arrive at the conditions:

$$\alpha'_1 = c\,\alpha_1, \qquad \alpha'_2 = c\,\alpha_2, \qquad \ldots, \qquad \alpha'_k = c\,\alpha_k,$$

from which, one immediately deduces that:

$$Q'_1 = c\,Q_1, \; Q'_2 = c\,Q_2, \qquad \ldots, \qquad Q'_k = c\,Q_k. \qquad\qquad \text{Q. E. D.}$$

More generally, the system (A_1):

$$(A_1) \qquad \frac{d}{dt_1}\left(\frac{\partial T_1}{\partial q'_i}\right) - \frac{\partial T_1}{\partial q_i} = Q'_i, \qquad \frac{dq_i}{dt} = q'_i \qquad (i = 1, 2, \ldots, k),$$

in which:

$$T = \frac{ds_1^2}{dt_1^2} \equiv C\,\frac{ds^2}{dt_1^2}, \qquad Q'_i = c\,Q_i,$$

defines the same trajectories as (A): From the preceding, *those systems constitute the only correspondents to* (A) *for which* ds_1^2 *differs from* ds^2 *only by a constant factor*.

I add that one passes from (A) to (A_1) by the transformation:

$$dt = \sqrt{\frac{c}{C}}\,dt_1.$$

That transformation is *determined completely*, which is the opposite of what happens in the case where the forces are zero. As one knows, in the latter case, one has $dt / dt_1 = \alpha$, where α denotes an arbitrary constant or an arbitrary first integral of the geodesics.

7. – If one is given a system (A), in which T is a well-defined *vis viva*, then the trajectories that correspond to a system of arbitrary forces Q_i ($q_1, q_2, \ldots, q_k$) include a common ($2k - 2$)-parameter congruence, namely, the geodesics of ds^2. *Do there exist other* ($2k - 2$)*-parameter congruences that belong to the trajectories for any forces* Q_i ? It is easy to see that the answer is no in the following manner: Such a congruence must verify equations (5) and (6), no matter what the Q_i are, and as a result, it must verify equation (7), which is obtained by subtracting the two equations (6) that relate to the forces Q_i and Q_i', respectively. If one observes that the Φ_i depend upon only T and only the α_i vary with the forces then one will see that equation (7) can be written:

$$\frac{d}{dq_1} L \frac{\psi_2}{\psi_2'} - 2\frac{\chi_2}{\Delta}\left(\frac{\alpha_1}{\psi_2} - \frac{\alpha_1'}{\psi_2'}\right) = 0 ,$$

when one suppresses the factor $\chi_2 = 0$ that gives the geodesics, or rather:

$$(7) \quad \left\{ \begin{aligned} 3\frac{d^2 q_2}{dq_1^2} + 2\,\Phi_1 \frac{dq_2}{dq_1} - 2\,\Phi_2 = \frac{1}{\dfrac{\alpha_2}{\alpha_1} - \dfrac{\alpha_2'}{\alpha_1'}} &\left\{ \left[\left(\frac{\alpha_2}{\alpha_1} - \frac{dq_2}{dq_1} \right) \frac{d}{dq_1} \frac{\alpha_2'}{\alpha_1'} - \left(\frac{\alpha_2'}{\alpha_1'} - \frac{dq_2}{dq_1} \right) \frac{d}{dq_1} \frac{\alpha_2}{\alpha_1} \right] \right. \\ &\left. + \left(\frac{\alpha_2}{\alpha_1} - \frac{dq_2}{dq_1} \right) \left(\frac{\alpha_2'}{\alpha_1'} - \frac{dq_2}{dq_1} \right) \frac{d}{dq_1} \frac{\alpha_1'}{\alpha_1} \right\}. \end{aligned} \right.$$

If one now replaces the Q_i' with some other forces Q_i'' then one will get a new equation (7), and upon subtracting the corresponding sides of those two equations, one will get a relation in which only *the first derivatives* appear, and which will not reduce to an identity when the Q_i, Q_i', Q_i'' are taken arbitrarily. On the other hand, the trajectories considered satisfy equations (5): Therefore, they can only depend upon at most ($2k - 3$) constants.

However, one can go further when the number k of parameters exceeds 2 and show that *if one replaces the forces Q_i in a system* (A) *with some other forces* Q_i' *then there can exist no* ($2k - 2$)*-parameter congruence of trajectories that is common to the first and second motion besides the geodesics.*

Of course, that supposes that *one does not have* $Q_i' = c\, Q_i$ ($i = 1, 2, \ldots, k$), where c is a constant, since all of the trajectories would coincide then.

In order to prove that proposition, assume that there exists one such congruence and represent its defining equations by:

$$\frac{d^2 q_i}{dq_1^2} = f_i\left(q_1, q_2, \ldots, q_k, \frac{dq_2}{dq_1}, \ldots, \frac{dq_k}{dq_1}\right) \qquad (i = 2, 3, \ldots, k).$$

From (5), one must have:

$$\frac{f_2 + \dfrac{dq_2}{dq_1}\Phi_1 - \Phi_2}{\alpha_2 - \alpha_1\dfrac{dq_2}{dq_1}} \equiv \frac{f_i + \dfrac{dq_i}{dq_1}\Phi_1 - \Phi_i}{\alpha_i - \alpha_1\dfrac{dq_i}{dq_1}} \qquad (i = 2, 3, \ldots, k),$$

in which at least one of the numerators in those ratios (say, the first one χ_2) is not identically zero, because otherwise the congruence would be that of the geodesics. One infers from this that:

$$\frac{\alpha_i - \alpha_1\dfrac{dq_i}{dq_1}}{\alpha_2 - \alpha_1\dfrac{dq_2}{dq_1}} \equiv \frac{f_i + \dfrac{dq_i}{dq_1}\Phi_1 - \Phi_i}{f_2 + \dfrac{dq_2}{dq_1}\Phi_1 - \Phi_2}.$$

One will similarly have:

$$\frac{\alpha_i' - \alpha_1'\dfrac{dq_i}{dq_1}}{\alpha_2' - \alpha_1'\dfrac{dq_2}{dq_1}} \equiv \frac{f_i + \dfrac{dq_i}{dq_1}\Phi_1 - \Phi_i}{f_2 + \dfrac{dq_2}{dq_1}\Phi_1 - \Phi_2},$$

so

$$\frac{\alpha_i' - \alpha_1'\dfrac{dq_i}{dq_1}}{\alpha_2' - \alpha_1'\dfrac{dq_2}{dq_1}} \equiv \frac{\alpha_i - \alpha_1\dfrac{dq_i}{dq_1}}{\alpha_2 - \alpha_1\dfrac{dq_2}{dq_1}} \qquad (i = 2, 3, \ldots, k),$$

which demands that:

$$\frac{\alpha_1}{\alpha_1'} = \frac{\alpha_2}{\alpha_2'} = \ldots = \frac{\alpha_k}{\alpha_k'}.$$

On the other hand, if that were true then equation (6) could be written (for the forces Q_i):

$$\frac{d^3q_2}{dq_1^3} = \chi_2\frac{d}{dq_1}\log\alpha_1 + L_2',$$

and for the forces Q_i':

$$\frac{d^3q_2}{dq_1^3} = \chi_2\frac{d}{dq_1}\log\alpha_1' + L_2',$$

in which L_2' is the same in both cases because, from a previous remark, neither T nor the ratios α_i / α_1 will change. As a result (χ_2 being non-zero), the equality:

$$\frac{d}{dq_1}\log\alpha_1 - \frac{d}{dq_1}\log\alpha_1' = 0 ,$$

which does not involve the second derivatives, must be verified *identically*, i.e., one will have:

$$\alpha_1' = c\ \alpha_1 ,$$

in which c is a constant, which will imply that:

$$\alpha_i' = c\ \alpha_i \qquad \text{and} \qquad Q_i' = c\ Q_i \qquad\qquad (i = 1, 2, \ldots, k).$$

The theorem is thus proved.

One sees that the argument supposes essentially that $k > 2$. *For* $k = 2$, *the theorem is no longer exact.* For example, the two systems of Lagrange equations:

$$(A) \qquad \begin{cases} \dfrac{d^2x}{dt^2} = 0, \\ \dfrac{d^2y}{dt^2} = g, \end{cases}$$

and

$$(A') \qquad \begin{cases} \dfrac{d^2x}{dt^2} = k\,y^{-3/2}, \\ \dfrac{d^2y}{dt^2} = k', \end{cases}$$

in which g, k, k' are constants that correspond to the same *vis viva* $T = \frac{1}{2}(x'^2 + y'^2)$ and distinct forces $Q_1 = 0$, $Q_2 = g$ on the one hand, and $Q_1' = k\,y^{-3/2}$, $Q_2' = k'$, on the other, which do not satisfy the conditions that $Q_1' = c\ Q_1$, $Q_2' = c\ Q_2$. The trajectories of (A) and (A') nonetheless comprise a common two-parameter congruence, besides the geodesics, namely, the parabolas:

$$y = (a\,x + b)^2,$$

in which a and b are two arbitrary constants. However, the preceding argument shows that *there cannot exist more than one* $(2k - 2) \equiv 2$*-parameter congruence that is common to the two systems* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(\frac{ds^2}{dt^2}, Q_i'\right)$ *(besides the geodesics).*

IV. – ORDINARY CORRESPONDENTS TO A SYSTEM (A).

8. – The preceding considerations will be of great use to us in our study of corresponding systems. Right now, we shall see that they exhibit certain *correspondents* that are attached to any system. If one is given an arbitrary system (A) then the system:

$$(A_1) \qquad \frac{d}{dt_1}\left[\frac{\partial T_1}{\partial (q_i')}\right] - \frac{\partial T_1}{\partial q_i} = Q_i', \qquad \frac{dq_i}{dt_1} = (q_i') \qquad (i = 1, 2, \ldots, k),$$

in which:

$$T_1 = C\,\frac{ds^2}{dt_1^2}, \qquad Q_i' = c\,Q_i,$$

will define the same trajectories as (A). There exist no other correspondents where ds_1^2 differs from ds^2 for T by only a constant. One passes from the first motion to the motion (A_1) by the change of variables $\frac{dt}{dt_1} = \sqrt{\frac{c}{C}}$, which is completely determined. Nonetheless, in the case where the forces are zero, the most general transformation that allows one to pass from (A) to (A_1) has the form $\frac{dt}{dt_1} = \alpha$, where α denotes, if desired, an arbitrary constant or an arbitrary first integral of the geodesics.

Since two correspondents to the same system correspond to each other, one sees that the existence of one arbitrary correspondent (A_1) to (A) implies the existence of an infinitude of other correspondents, namely, the ones that one deduces from the first (A_1) upon multiplying T_1 and the Q_i' by two constant factors C and c.

9. – In a later chapter, we will see that *a system* (A) *that is taken at random will admit no other correspondents, in general*. However, now suppose that the forces Q_i are derived from a potential U. The trajectories of (A) for the value h of the *vis viva* constant coincide with the geodesics of $ds'^2 = (U + h)\,ds^2$. With that, consider the system (A_1), in which $T_1 \equiv (\alpha\,U + \beta)\,\frac{ds^2}{dt_1^2}$, and in which the Q_i' are derived from the potential $U' = \frac{\gamma U + \delta}{\alpha U + \beta}$ (with the condition that $\alpha\,\delta - \gamma\,\beta \neq 0$). The trajectories of ($A_1$) for the value h_1 of the *vis viva* constant coincide with the geodesics of:

$$ds_1'^2 = [\gamma\,U + \delta + h_1\,(\alpha\,U + \beta)]\;ds^2.$$

The trajectories of (A) for a given value of h coincide with the trajectories of (A_1) for which the constant h_1 verifies the equality:

$$h = \frac{\delta + \beta h_1}{\gamma + \alpha h_1} \quad \text{or} \qquad h_1 = \frac{\delta - \gamma h}{\alpha h - \beta}.$$

The systems (A) and (A_1) are then *correspondents*, and each *natural* family $h = h_0$ of trajectories of (A) coincides with a natural family $h_1 = h_1^0$ of (A_1). On the other hand, one has:

$$ds^2 = (U + h)\, dt^2 = \left(U + \frac{\delta + \beta h_1}{\gamma + \alpha h_1} \right)$$

and

$$(\alpha U + \beta)\, ds^2 = \left(\frac{\gamma U + \delta}{\alpha U + \beta} + h_1 \right) dt_1^2 \,,$$

from which one infers that:

$$(a) \qquad (\alpha\delta - \beta\gamma)\; dt_1^2 = (\alpha U + \beta)^2\, [\alpha\, ds^2 - (\alpha U + \beta)\, dt^2]\ .$$

That transformation (a), which permits one to pass from (A) to (A_1), is *unique*, moreover. Indeed, in (A) and in (A_1), one can express $\frac{d^2 q_2}{dq_1^2}$ as a function of $q_1, q_2, \ldots, q_k, \frac{dq_2}{dq_1}, \ldots, \frac{dq_k}{dq_1}, \frac{dq_k}{dt}$, and upon equating those two values of $\frac{d^2 q_2}{dq_1^2}$, one will get a well-defined relation between $q_1, q_2, \ldots, q_k, dq_1, dq_2, \ldots, dq_k, dt$ and dt_1 (¹). That unique relation must then coincide with the one that we just obtained, which is quite easy to verify when we do the calculation.

Those new *correspondents* (A_1) coincide with the first ones for $\alpha = 0$.

Since it is legitimate to add a constant to a force function, for $\alpha \neq 0$, one can always suppose that U' of the form $U' = \delta / \alpha U$. The equation (a) will then become:

$$(a') \qquad \left(\frac{dt_1}{dt} \right)^2 = \frac{a^2}{\delta} U^2 \left(\frac{ds^2}{dt^2} - U \right) = \frac{a^2}{\delta} U^2\, h\,,$$

or rather:

$$\frac{dt^2}{dt_1^2} = \frac{1}{\alpha U^2} \left(\alpha U \frac{ds^2}{dt_1^2} - \frac{\delta}{\alpha U} \right) = \frac{h_1}{\alpha U^2}\ .$$

Those equalities show that the expressions $\frac{1}{U} \left(\frac{dt_1}{dt} \right)$ and $U \frac{dt}{dt_1}$ are integrals of (A) and (A_1), namely, the two *vis viva* integrals.

(¹) We shall return to this point at the beginning of Chapter Three, moreover.

Any first integral of (A) corresponds to a first integral of (A_1) that is obtained by replacing dt as a function of dt_1 using (a'). An entire (or rational) algebraic integral corresponds to an analogous integral of the same degree. For example, an integral of degree two of (A), say:

$$d\sigma^2 - V\,dt^2 = k\,dt^2,$$

will correspond to the integral of (A_1):

$$d\sigma^2 - \frac{V\,ds^2}{U} + \frac{\delta}{\alpha^2}\frac{V\,dt_1^2}{U^2} = \frac{k\,h_1}{\alpha U^2}\,dt^2,$$

i.e.:

$$U^2\left(d\sigma^2 - \frac{V\,ds^2}{U} + \frac{\delta}{\alpha^2}\frac{V\,dt_1^2}{U^2}\right) = k_1\,dt_1^2.$$

That transformation was pointed out by Darboux. It is clear that the correspondents (A_1) that are deduced from (A) by that transformation coincide with the ones that one deduces from any of the transforms (A_1).

A system (A) with a potential that is taken at random will not admit other correspondents, in general. That results from the general study of the corresponding systems (A), (A_1), in which (A_1) is not one of the *ordinary* correspondents $\left(C\frac{ds^2}{dt_1^2}, c\,Q_i\right)$ or $\left[(\alpha U+\beta)\frac{ds^2}{dt_1^2}, \frac{\gamma U+\delta}{\alpha U+\beta}\right]$ of (A).

CHAPTER II.

Corresponding systems in which all forces are zero.

I. – PROOF OF A GENERAL PROPERTY OF THOSE SYSTEMS.

1. – Let (A) and (A_1) be two corresponding systems: If all of the forces Q_i are zero in (A) *then they will also be zero in* (A_1). Indeed, the trajectories of (A) depend upon only ($2k$ – 2) parameters, so the same thing will be true for the trajectories of (A_1), and from a theorem in Chapter One, all of the forces in (A_1) must be zero.

We shall first study the correspondence between two systems (A) and (A_1) *without forces* then. The fundamental theorem that we shall prove is the following one:

If a system (A) *is without forces, namely* $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$, *possesses a correspondent* (A_1) *that is distinct from the ordinary correspondents* $\left[C\,\frac{ds^2}{dt_1^2}, Q_i' = 0\right]$ *then it will admit a quadratic integral (in addition to that of vis viva).*

That can also be stated as:

If the geodesics of the two (non-similar) ds^2 *coincide then they will admit a rational integral of degree two.*

Two such ds^2 will be called *correspondents*.
If k is equal to 2 then that theorem will coincide with that of Dini.

2. – In order to prove that proposition, I will appeal to the following:

Let a system of equations be given:

$$\text{(1)} \qquad \frac{d}{dq}\left(\frac{\partial f}{\partial q_i'}\right) - \frac{\partial f}{\partial q_i} = 0, \qquad \frac{dq_i}{dq} = q_i' \qquad (i = 1, 2, \ldots, k),$$

in which f is an arbitrary of $q, q_1, q_2, \ldots, q_k, q_1', \ldots, q_k'$ *subject to only the condition that the system* (1) *must be soluble for the* $d^2 q_i / dq^2$, *in other words, that the Hessian d of f relative to the variable* q_i', *namely:*

$$d = \begin{vmatrix} \frac{\partial^2 f}{\partial q_1'^2} & \frac{\partial^2 f}{\partial q_1' \partial q_2'} & \cdots & \frac{\partial^2 f}{\partial q_1' \partial q_k'} \\ \frac{\partial^2 f}{\partial q_1' \partial q_2'} & \frac{\partial^2 f}{\partial q_2'^2} & \cdots & \frac{\partial^2 f}{\partial q_2' \partial q_k'} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial q_1' \partial q_k'} & \cdots & \cdots & \frac{\partial^2 f}{\partial q_k'^2} \end{vmatrix},$$

is not identically zero: That Hessian is a last multiplier of (2).

Indeed, reduce the system (1) to the canonical form with the aid of the change of variables:

$$p_i = \frac{\partial f}{\partial q_i'} \qquad (i = 1, 2, \ldots, k),$$

from which, one infers, inversely, that:

$$q_i' = \frac{\partial f}{\partial p_i},$$

upon setting:

$$f_1(q, q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k) \equiv p_1 q_1' + \cdots + p_k q_k' - f .$$

The new equations admit unity as a multiplier. In other words, if one knows $(2k - 1)$ first integrals of the system (1), namely:

(2) $$\varphi_j(q, q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k) = c_j \qquad [j = 1, 2, \ldots, (2k - 1)],$$

when one infers $p_1, p_2, \ldots, p_k, q, q_1, q_2, \ldots, q_{k-2}$ as functions of q_{k-1} and q_k from those integral, for example, the expression:

$$\frac{1}{\delta}\left(\frac{\partial f_1}{\partial p_k}\,dq_{k-1} - \frac{\partial f_1}{\partial p_{k-1}}\,dq_k\right) \equiv \frac{1}{\delta}\,(q_k'\,dq_{k-1} - q_{k-1}'\,dq_k)$$

is an exact total differential. δ denotes the functional determinant $\dfrac{D(\varphi_1, \varphi_2, \ldots, \varphi_{2k-1})}{D(q_1, q_2, \ldots, q_{k-2}, p_1, p_2, \ldots, p_k)}$. On the other hand, if one supposes that the integrals φ_j are expressed with the aid of the q_i' then one will have:

$$\begin{aligned}\delta_1 &= \frac{D(\varphi_1, \varphi_2, \ldots, \varphi_{2k-1})}{D(q_1, q_2, \ldots, q_{k-2}, q_1', q_2', \ldots, q_k')} \\ &\equiv \frac{D(\varphi_1, \varphi_2, \ldots, \varphi_{2k-1})}{D(q_1, q_2, \ldots, q_{k-2}, p_1, p_2, \ldots, p_k)}\,\frac{D(q_1, q_2, \ldots, q_{k-2}, p_1, p_2, \ldots, p_k)}{D(q_1, q_2, \ldots, q_{k-2}, q_1', q_2', \ldots, q_k')} \\ &\equiv \frac{D(p_1, p_2, \ldots, p_k)}{D(q_1', q_2', \ldots, q_k')} \equiv \delta \times d .\end{aligned}$$

Therefore, the expression:

$$\frac{d}{\delta}\,(q_k'\,dq_{k-1} - q_{k-1}'\,dq_k)$$

is an exact differential [if one takes into account the $2k - 1$ relations (2)]. The Hessian d is a multiplier of (1).

In particular, if q does not enter into f then d will be multiplier of the system:

$$\frac{dq_1}{q_1'} = \frac{dq_2}{q_2'} = \ldots = \frac{dq_k}{q_k'} = \frac{d\,\dfrac{\partial f}{\partial q_1'}}{\dfrac{\partial f}{\partial q_1}} = \ldots = \frac{d\,\dfrac{\partial f}{\partial q_k'}}{\dfrac{\partial f}{\partial q_k}} .$$

Apply that lemma to a system (A) without forces:

$$(A) \qquad \frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = 0 , \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k),$$

upon taking $q = t$. One sees that the discriminant Δ of T is a multiplier of the system:

$$\frac{dq_1}{q_1'} = \frac{dq_2}{q_2'} = \ldots = \frac{dq_k}{q_k'} = \frac{d\,\dfrac{\partial T}{\partial q_1'}}{\dfrac{\partial T}{\partial q_1}} = \ldots = \frac{d\,\dfrac{\partial T}{\partial q_k'}}{\dfrac{\partial T}{\partial q_k}} .$$

Now assume that one knows $(2k - 3)$ first integral of the *geodesics*, i.e., $(2k - 3)$ integrals of (A) that are homogeneous and of degree zero with respect to the q_i', namely (upon setting $q_{(2)}' = q_2' / q_1' = dq_2 / dq_1 , \ldots, q_{(k)}' = q_k' / q_1' = dq_k / dq_1$):

$$(3) \qquad \psi_j [q_1, q_2, \ldots, q_k, q_{(2)}', q_{(3)}', \ldots, q_{(k)}'] = c_j \qquad [j = 1, 2, \ldots, (2k - 3)].$$

One combines those integrals with that of *vis viva*:

$$(4) \qquad T \equiv q_1'^{\,2}\, \tau [q_1, \ldots, q_k, q_{(2)}', \ldots, q_{(k)}'] = h .$$

If one infers $q_3, q_4, \ldots, q_k, q_{(2)}', \ldots, q_{(k)}'$ as functions of q_1, q_2 from (3) then the expression:

$$(4') \qquad \frac{\Delta q_1'}{\delta} [dq_2 - q_{(2)}'\, dq_1] ,$$

in which one replaces q_1' with its value that is inferred from (4) is an exact differential. Here, one has:

$$\delta_1 = \frac{D(T, \psi_1, \psi_2, \ldots, \psi_{2k-3})}{D(q_3, q_4, \ldots, q_k, q_1', q_2', \ldots, q_k')}$$

$$= \frac{D(q_1'^2\, \tau, \psi_1, \psi_2, \ldots, \psi_{2k-3})}{D[q_3, q_4, \ldots, q_k, q_1', q_{(2)}', \ldots, q_{(k)}']} \frac{D[q_3, q_4, \ldots, q_k, q_1', q_{(2)}', \ldots, q_{(k)}']}{D(q_3, q_4, \ldots, q_k, q_1', q_2', \ldots, q_k')} .$$

However, upon observing that $q'_{(i)} = q'_i / q'_1$, one will find immediately that:

$$\frac{D[q_3, q_4, \ldots, q_k, q'_1, q'_{(2)}, \ldots, q'_{(k)}]}{D(q_3, q_4, \ldots, q_k, q'_1, q'_2, \ldots, q'_k)} \equiv \frac{D[q'_{(2)}, q'_{(3)}, \ldots, q'_{(k)}]}{D(q'_2, q'_1, \ldots, q'_k)} = \frac{1}{q_1'^{\,k-1}} .$$

On the other hand, since τ, ψ_1, ψ_2, …, ψ_{2k-3} do not depend upon q'_1, but only on $q'_{(i)}$, one will have:

$$\frac{D(q_1'^2\,\tau, \psi_1, \psi_2, \ldots, \psi_{2k-3})}{D[q_3, q_4, \ldots, q_k, q'_1, q'_{(2)}, \ldots, q'_{(k)}]} \equiv 2\tau\, q'_1 \frac{D(\psi_1, \psi_2, \ldots, \psi_{2k-3})}{D[q_3, q_4, \ldots, q_k, q'_1, q'_{(2)}, \ldots, q'_{(k)}]}$$

$$\equiv 2\tau\, q'_1\, \delta' ,$$

and as a result:

$$\delta_1 \equiv \frac{2\tau\,\delta'}{q_1'^{\,k-2}} .$$

Replace δ_1 with that value in the expression (4) and set $q'_1 = h / \sqrt{\tau}$. By definition, one sees that *the expression:*

$$\frac{1}{\delta'} \frac{\Delta}{\tau^{\frac{1+k}{2}}} [dq_2 - q'_{(2)}\, dq_1]$$

will be an exact differential when one replaces q_2, …, q_k, $q'_{(2)}$, …, $q'_{(k)}$ *with* q_1, q_2 *using* (3). δ' denotes the functional determinant of the ψ_j with respect to the variables q_3, …, q_k, $q'_{(2)}$, …, $q'_{(k)}$.

That amounts to saying that if one then writes the differential equations of the geodesics:

$$(5) \qquad dq_1 = \frac{dq_2}{q'_{(2)}} = \ldots = \frac{dq_k}{q'_{(k)}} = \frac{dq'_{(2)}}{\lambda_2} = \ldots = \frac{dq'_{(k)}}{\lambda_k}$$

then those equations will admit the expression:

$$\frac{\Delta}{\tau^{\frac{1+k}{2}}}$$

for a last multiplier.

The proof of the theorem that I have in mind is then achieved. Indeed, suppose that (A) and (A_1) are two corresponding systems (without forces), in other words, that the geodesics of (A) and ($A1$) coincide. Equations (5) will be the same for the two systems, and they will both admit the two multipliers:

$$\frac{\Delta}{\tau^{\frac{1+k}{2}}}, \qquad \frac{\Delta_1}{\tau_1^{\frac{1+k}{2}}}.$$

The quotient $\frac{\Delta}{\Delta_1}\frac{\tau_1^{\frac{1+k}{2}}}{\tau^{\frac{1+k}{2}}}$ *is then a first integral of* (5), and since that integral can be written:

$$\left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}}\frac{\tau_1}{\tau} = \text{const.},$$

one sees that the geodesics will admit a rational integral of degree two. As for the system (A) itself, if one takes the *vis viva* integral $T \equiv q_1'^2\,\tau = h$ into account then one will find that *it possesses a quadratic integral:*

$$\left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}} ds_1^2 = C\,dt^2. \tag{6}$$

Can that integral coincide with that of *vis viva*? In order for that to be true, it is necessary and sufficient that $\left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}} ds_1^2 = C\,ds^2$. That first demands ds_1^2 must be equal to $\mu\,ds^2$, and (since Δ_1 / Δ is equal to μ^k) that $\mu^{-\frac{2}{1+k}}$, moreover, and as a result μ must be a constant. Therefore, if (A_1) is not an *ordinary* correspondent of (A) then the integral (6) is always distinct from that of *vis viva*. The theorem that I stated is then proved completely. Observe that the preceding argument shows us that ds^2 *and* $\mu\,ds^2$ *cannot be correspondents unless* μ *is a constant*. Otherwise, μ' = const. would be a first integral of the geodesics.

Similarly, (A_1) possesses the integral:

$$\left(\frac{\Delta_1}{\Delta}\right)^{\frac{2}{1+k}} ds^2 = C_1\,dt_1^2.$$

3. – Before proceeding, I shall insist upon one of the results that was obtained just now. We have said that the differential equations (5) of the geodesics admit the expression $\Delta / \tau^{\frac{1+k}{2}}$ for a multiplier. Now, one knows an explicit form for those equations, namely, the following one:

$$(1') \qquad dq_1 = \frac{dq_2}{q'_{(2)}} = \dots = \frac{dq_k}{q'_{(k)}} = \frac{d\cdot\dfrac{\partial f}{\partial q'_{(2)}}}{\dfrac{\partial f}{\partial q_2}} = \dots = \frac{d\cdot\dfrac{\partial f}{\partial q'_{(k)}}}{\dfrac{\partial f}{\partial q_k}},$$

in which f is equal to $\sqrt{\tau}$. Conversely, any system (1), in which f is the square root of a second-degree polynomial τ in $q'_{(2)}, \dots, q'_{(k)}$, can be regarded as defining the trajectories of a system (A) without forces, namely, the system for which:

$$T \equiv q_1'^2 \, f\left(q_1, q_2, \dots, q_k, \frac{q'_2}{q'_1}, \dots, \frac{q'_k}{q'_1}\right).$$

We then arrive at this theorem:

Any system (1′), *in which f is the square root of a second-degree polynomial t in* $q'_{(2)}, \dots, q'_{(k)}$ *admits* $\Delta / \tau^{\frac{1+k}{2}}$ *as a last multiplier,* where Δ denotes the discriminant of $\frac{1}{2}\tau$, when it is made homogeneous.

I shall rapidly indicate another proof of that theorem that consists of generalizing the solution that Darboux gave to Dini's problem. From the lemma that I established before, the Hessian d of f relative to the variables $q'_{(2)}, q'_{(3)}, \dots, q'_{(k)}$ is a multiplier of (1′). Since $f \equiv \sqrt{\tau}$ here, one will have:

$$d \equiv \frac{1}{\tau^{\frac{3(k-1)}{2}}} \begin{vmatrix} \left[\dfrac{\tau}{2}\dfrac{\partial^2\tau}{\partial q'^2_{(2)}} - \dfrac{1}{4}\left(\dfrac{\partial\tau}{\partial q'_{(2)}}\right)^2\right] & \cdots & \left(\dfrac{\tau}{2}\dfrac{\partial^2\tau}{\partial q'_{(2)}\,\partial q'_{(k)}} - \dfrac{1}{4}\dfrac{\partial\tau}{\partial q'_{(2)}}\dfrac{\partial\tau}{\partial q'_{(k)}}\right) \\ \left(\dfrac{\tau}{2}\dfrac{\partial^2\tau}{\partial q'_{(2)}\,\partial q'_{(3)}} - \dfrac{1}{4}\dfrac{\partial\tau}{\partial q'_{(2)}}\dfrac{\partial\tau}{\partial q'_{(3)}}\right) & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ \left(\dfrac{\tau}{2}\dfrac{\partial^2\tau}{\partial q'_{(2)}\,\partial q'_{(k)}} - \dfrac{1}{4}\dfrac{\partial\tau}{\partial q'_{(2)}}\dfrac{\partial\tau}{\partial q'_{(k)}}\right) & \cdots & \left[\dfrac{\tau}{2}\dfrac{\partial^2\tau}{\partial q'^2_{(k)}} - \dfrac{1}{4}\left(\dfrac{\partial\tau}{\partial q'_{(k)}}\right)^2\right] \end{vmatrix} \equiv \frac{1}{\tau^{\frac{3(k-1)}{2}}}\, d_1,$$

in which d_1 is a polynomial of degree at most 2 (k – 1) with respect to $q'_{(i)}$. For k = 2, one will find immediately that $d_1 \equiv \tau \dfrac{\partial^2\tau}{\partial q'^2_{(2)}} - \left(\dfrac{\partial\tau}{\partial q'_{(2)}}\right)^2 \equiv \Delta$. For k = 3, $d_1 \equiv \Delta\ \tau$. More generally, a transformation with a very painful determinant will show that $d_1 \equiv \Delta\,\tau^{(k-2)}$. It follows from this that $\Delta / \tau^{\frac{1+k}{2}}$ is a multiplier of (1′).

Conversely, since we have established by our first method that $\Delta / \tau^{\frac{1+k}{2}}$ is a multiplier of (1′), we can conclude that $d_1 \equiv \Delta\,\tau^{(k-2)}$. First of all, the fraction:

$$D \equiv \frac{d \times \tau^{\frac{1+k}{2}}}{\Delta} \equiv \frac{d_1}{\Delta\ \tau^{k-2}}\ ,$$

in which the two terms are polynomials with respect to the $q'_{(i)}$ and the coefficients A_{ij} of τ, is an absolute constant C (viz., independent of the $q'_{(i)}$ and the A_{ij}): In other words, it will define a first integral of (1′), and the geodesics of an arbitrary ds^2 in k variables will admit an integral that is algebraic and rational with respect to the $q'_{(i)}$, which is obviously absurd ([1]). Therefore, $d_1 \equiv C\,\Delta\,\tau^{(k-2)}$. Upon taking a particular ds^2 – say, $ds^2 = dq_1^2 + dq_2^2 + \cdots + dq_k^2$ – one will see immediately that $C = 1$.

4. – I add that the preceding results are capable of being extended to more general equations that are provided by the *calculus of variations.* If two systems (1′), where *f is arbitrary*, define the same relations between the q_i then the ratio of the Hessians d and d' of f and f', resp., (relative to the variables $q'_{(2)}, q'_{(3)}, \ldots, q'_{(k)}$) will be a first integral of (1′).

In particular, when f and f' are rational (or algebraic) in $q'_{(2)}, \ldots, q'_{(k)}$, equations (1) will admit a first integral that is rational (or algebraic) with respect to the $q'_{(i)}$.

*If f is the n*th *root of a polynomial τ of degree n in $q'_{(2)}, q'_{(3)}, \ldots, q'_{(k)}$ then one will have:*

$$d \equiv \frac{1}{\tau^{(k-1)\left(2-\frac{1}{n}\right)}}\,d_1$$

$$= \frac{1}{\tau^{(k-1)\left(2-\frac{1}{n}\right)}}\begin{vmatrix} \frac{1}{n}\left[\tau\frac{\partial^2\tau}{\partial q'^2_{(2)}} - \left(1-\frac{1}{n}\right)\left(\frac{\partial\tau}{\partial q'_{(2)}}\right)^2\right] & \cdots & \frac{1}{n}\left[\tau\frac{\partial^2\tau}{\partial q'_{(2)}\,\partial q'_{(k)}} - \left(1-\frac{1}{n}\right)\frac{\partial\tau}{\partial q'_{(2)}}\frac{\partial\tau}{\partial q'_{(k)}}\right] \\ \vdots & \ddots & \vdots \\ \frac{1}{n}\left[\tau\frac{\partial^2\tau}{\partial q'_{(2)}\,\partial q'_{(k)}} - \left(1-\frac{1}{n}\right)\frac{\partial\tau}{\partial q'_{(2)}}\frac{\partial\tau}{\partial q'_{(k)}}\right] & \cdots & \frac{1}{n}\left[\tau\frac{\partial^2\tau}{\partial q'^2_{(k)}} - \left(1-\frac{1}{n}\right)\left(\frac{\partial\tau}{\partial q'_{(k)}}\right)^2\right] \end{vmatrix},$$

([1]) Moreover, it would be quite easy to prove that last point rigorously.

and upon letting Δ denote the Hessian of the homogeneous form:

$$T \equiv \frac{1}{n(n-1)}\, q_1'^{\,n}\, \tau\left(q_1, q_2, \ldots, q_k, \frac{q_2'}{q_1'}, \frac{q_3'}{q_1'}, \ldots, \frac{q_k'}{q_1'}\right),$$

one will find that:

$$d_1 = (n-1)^{k-1}\, \Delta'\, \tau^{k-2}\,.$$

Δ' represents what Δ will become when one sets $q_1' = 1$, $q_i' = q_{(i)}'$. It follows from this that $\Delta' / \tau^{\frac{k(n-1)+1}{n}} \equiv d$ will be a multiplier of (1′).

In order to prove the last propositions, one can follow the same path as in the case where T has degree two. Upon appealing to the equations:

$$(1'') \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = 0\,, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k)\,,$$

which define the same relations between the q_i as (1′), one will first establish, with no change in the argument, that $\Delta' / \tau^{\frac{k(n-1)+1}{n}}$ is a multiplier of (1′), and then that it must coincide with d. That will give the value of d_1 .

In particular, if two systems such as (1″), where T and T_1 have the same degree n, correspond, say, $T = \frac{ds^n}{dt^n}$, $T_1 = \frac{ds_1^n}{dt_1^n}$, then the equality:

$$\left(\frac{\Delta}{\Delta_1}\right) ds_1^{k(n-1)+1} = C\, dt^{k(n-1)+1}$$

will provide a first integral of (1″)[1].

---

[1] If T and T_1 have degrees n and n_1, resp., then the equality will have the form:

$$q_i'^{\,(n-n_1)}\, \frac{\Delta}{\Delta_1} \left(\frac{ds_1}{dt}\right)^{k(n_1-1)+1} = C\,,$$

in which i has any of the values 1, 2, …, k. One will necessarily have $q_i' = c_i\, q_1'$, i.e., $q_i = c_i\, q_1 + c_i'$, in which the c, c' are constants, and the same conclusion applies to the trajectories of the second system. Disregarding that special case, the two systems cannot correspond to each other unless n is equal to n_1 .

II. – PASSING FROM A SYSTEM (*A*) WITHOUT FORCES TO ITS CORRESPONDENT. CONSEQUENCE.

5. – When all of the coefficients Q_i are zero for a system (*A*), the equality:

$$dt = C\,ds\,,$$

in which C denotes either a number or a first integral of the geodesics, defines a motion of (*A*) on each geodesic. Conversely, any equality:

$$dt = f\left(q_1, q_2, \ldots, q_k, \frac{dq_2}{dq_1}, \ldots, \frac{dq_k}{dq_1}\right) ds$$

that defines a motion of (*A*) on an arbitrary geodesic will have the preceding form.

Let us apply that remark to the two corresponding systems (*A*) and (A_1) with no forces. We will have:

$$dt = C\,ds\,, \qquad dt_1 = C_1\,ds_1\,,$$

so

$$\text{(a)} \qquad \frac{dt}{dt_1} = c\,\frac{ds}{ds_1}\,,$$

in which $c \equiv C / C_1$ represents a number or a first integral of the geodesics. One then deduces a motion that is defined by (A_1) from an arbitrary motion that is defined by (*A*) by changing dt into dt_1 using (*a*). Moreover, any equality:

$$\frac{dt}{dt_1} = f\left(q_1, q_2, \ldots, q_k, \frac{dq_2}{dq_1}, \ldots, \frac{dq_k}{dq_1}\right)$$

that transforms the motions of (*A*) and (A_1) into each other will be a transformation (*a*).

However, we saw above that the expression:

$$\left(\frac{\Delta}{\Delta_1}\right)^{\frac{1}{k+1}} \frac{ds_1}{ds}$$

is a first integral of the geodesics. If one replaces C with that expression in (*a*) then that will give:

$$\text{(b)} \qquad \frac{dt}{\Delta^{\frac{1}{k+1}}} = \frac{dt_1}{\Delta_1^{\frac{1}{k+1}}}\,.$$

We will then arrive at this conclusion:

One can pass from the system (A) *to the system* (A_1) *by the transformation* (b). *That transformation is not the only one. The most general one is obtained by setting:*

$$\frac{dt}{\Delta^{\frac{1}{k+1}}} = C\,\frac{dt_1}{\Delta_1^{\frac{1}{k+1}}},$$

in which C represents a constant or a first integral of the geodesics.

That proposition plays a fundamental role in the theory of correspondents. We shall now deduce some immediate consequences from it.

6. – One of the more important ones is the following:

Let two systems (A) *and* (A_1) *be:*

$$(A) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i\,(q_1, q_2, \ldots, q_k)\,, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k),$$

and

$$(A_1) \qquad \frac{d}{dt_1}\left[\frac{\partial T}{\partial q_{(i)}'}\right] - \frac{\partial T_1}{\partial q_i} = Q_i'\,(q_1, q_2, \ldots, q_k)\,, \qquad \frac{dq_i}{dt_1} = (q_i') \qquad (i = 1, 2, \ldots, k).$$

If the geodesics of T and T_1 coincide then any system of forces Q_i of (A) *can be associated with a system of forces* Q_i' *such that* (A) *and* (A_1) *are correspondents* ([1]).

Indeed, suppose that the equations (A) are solved for $\frac{d^2q_i}{dt^2}$. We will have:

$$(a) \qquad \frac{d^2q_i}{dt^2} = P_i + \frac{\alpha_i}{\Delta} = P_i + \beta_i\,,$$

in which P_i denotes a quadratic form in the q_i' that depends upon only T, and the β_i depend upon the forces Q_i and coefficients A_{ij} of T. One will similarly have that:

([1]) One can also prove that theorem by appealing to the differential equations of the trajectories.

$$(a_1) \qquad \frac{d^2 q_i}{dt_1^2} = P_i' + \frac{\alpha_i'}{\Delta_1} = P_i' + \beta_i',$$

for (A_1).

We know that when all of the forces are zero, and as a result, the α_i, α_i', one can pass from (a) to (a_1) by the change of variable:

$$\frac{dt_1}{\Delta_1^{\frac{1}{k+1}}} = C\frac{dt}{\Delta^{\frac{1}{k+1}}},$$

in which C is a constant, which can be written:

$$dt_1 = \lambda\,(q_1, q_2, \ldots, q_k)\,dt\,.$$

If we perform a change of variables then that will give:

$$\frac{dq_i}{dt} = \frac{dq_i}{dt_1}\frac{dt_1}{dt} = \lambda\frac{dq_i}{dt_1}, \qquad \frac{d^2 q_i}{dt^2} = \lambda^2\frac{d^2 q_i}{dt_1^2} + \frac{dq_i}{dt_1}\lambda\frac{d\lambda}{dt_1}\,.$$

Equations (a) become:

$$(b) \qquad \frac{d^2 q_i}{dt_1^2} = (P_i) - \frac{dq_i}{dt_1}\frac{d}{dt}\log\lambda + \frac{\beta_i}{\lambda^2},$$

in which (P_i) represents P_i when one replaces $\frac{dq_i}{dt}$ with $\frac{dq_i}{dt_1}$. Since equations (b) and (a_1) coincide when the β_i, β_i' are zero, one will have:

$$(P_i) - \frac{dq_i}{dt_1}\left(\frac{dq_1}{dt_1}\frac{\partial\log\lambda}{\partial q_1} + \cdots + \frac{dq_k}{dt_1}\frac{\partial\log\lambda}{\partial q_k}\right) \equiv P_i'\,.$$

In order for them to coincide even when the β_i, β_i' are not zero, it will then be necessary and sufficient that:

$$\frac{\beta_i}{\lambda^2} = \beta_i' \qquad (i = 1, 2, \ldots, k),$$

which can also be written:

$$\beta_i\,\Delta^{\frac{2}{k+1}} = C^2\beta_i'\,\Delta_1^{\frac{2}{k+1}} \qquad (i = 1, 2, \ldots, k).$$

The theorem is thus proved.

It is easy to deduce the explicit relations that define the Q_i' as functions of the Q_i from those relations.

Let Δ^{ij} (or Δ_1^{ij}) represent the minor of Δ (or Δ_1, resp.) relative to the element A_{ij} (or A'_{ij}, resp.). One will have:

$$\beta_i = \frac{\Delta^{1i}}{\Delta} Q_1 + \frac{\Delta^{2i}}{\Delta} Q_2 + \cdots + \frac{\Delta^{ki}}{\Delta} Q_k \;,$$

and as a result (as one knows):

$$Q_i = A_{1i}\, \beta_1 + A_{2i}\, \beta_2 + \ldots + A_{ki}\, \beta_k \,.$$

We then write down the equalities:

$$Q'_i = \sum_{j=1}^{k} A'_{ij}\, \beta'_j \,, \qquad \beta'_j = C^2 \left(\frac{\Delta}{\Delta_1} \right)^{\frac{2}{k+1}} \beta_j \,, \qquad \beta_j = \frac{1}{\Delta} \sum_{l=1}^{k} \Delta^{jl}\, Q_l \;.$$

That will give:

$$(c) \qquad \Delta_1^{\frac{2}{k+1}} Q'_i = \frac{C^2}{\Delta^{\frac{k-1}{k+1}}} (\mu_{i1}\, Q_1 + \mu_{i2}\, Q_2 + \cdots + \mu_{ik}\, Q_k) \qquad (i = 1, 2, \ldots, k),$$

in which μ_{ij} denotes the determinant that is obtained by replacing the j^{th} column in Δ with the i^{th} column in Δ_1 (μ_{ij} is generally distinct from μ_{ji}, here).

7. Remarks. – That theorem can be completed by several remarks. Upon varying the constant C^2, as should be obvious from the outset, we will get an infinitude of systems Q'_i that are all deduced from each other by multiplying the Q' by a constant factor. However, it is important to observe that if the Q_i are given then those forces Q'_i will be the only ones for which (A) and (A_1) are correspondents. Indeed, if the Q_i are given then the trajectories of (A), and as a result, those of (A_1), will be well-defined. Now we saw in the previous chapter that one cannot change the forces Q'_i in a system (A_1) without changing the trajectories unless the new forces (Q'_i) differ from the first ones by only the same constant factor.

Moreover, in the present case, one passes from the system (A) to the system (A_1) [in which the Q'_i satisfy the conditions (c)] by the transformation:

$$(d) \qquad \frac{dt_1}{\Delta^{\frac{1}{k+1}}} = C \, \frac{dt}{\Delta^{\frac{1}{k+1}}} \,,$$

in which C denotes a well-defined number when the Q_i, Q'_i are given. Here again, it is important to observe that this transformation is *unique*. In other words, there exists no other change of variables:

$$(e) \qquad \frac{dt_1}{dt} = f\left(q_1, q_2, \ldots, q_k, \frac{dq_2}{dq_1}, \ldots, \frac{dq_k}{dq_1} \right)$$

that will transform one of the given systems (A) and (A_1) into each other. If one recalls the equality that was established in the first chapter (*see* pp. 21) then the equality that results from (A):

$$\frac{dt^2}{dq_1^2}\left(\frac{\alpha_2}{\Delta}-\frac{\alpha_1}{\Delta}\frac{dq_2}{dq_1}\right) = \frac{d^2q_2}{dq_1^2}+\Phi_1\frac{dq_2}{dq_1}-\Phi_2$$

will become obvious.

If one writes the analogous equality that relates to (A_1) and equates the two values of $\frac{d^2q_2}{dq_1^2}$ [which coincide since (A) and (A_1) are correspondents] then one will find that dt and dt_1 are coupled by a relation of the form (e) ([1]):

$$\frac{dt_1}{dt} = \sqrt{\varphi\left(q_2,\ldots,q_k,\frac{dq_2}{dq_1},\ldots,\frac{dq_k}{dq_1}\right)} \;.$$

That is what we would like to establish. The ratio dt_1 / dt is perfectly determined as a function of $q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'$ then, and the preceding equality must coincide with (d).

Those remarks permit us to state the following corollaries:

Let (A) *and* (A_1) *be two given corresponding systems in which the forces are not zero: If the geodesics of T and T_1 coincide then one can pass from* (A) *to* (A_1) *by the unique transformation:*

$$(d) \qquad \frac{dt_1}{\Delta_1^{\frac{1}{1+k}}} = C\,\frac{dt}{\Delta^{\frac{1}{1+k}}},$$

in which C is a well-defined number.

The forces Q_i' are then coupled to the forces Q_i by the conditions (c).

Conversely, *if one can pass from a given system* (A) *to another corresponding one* (A_1) *by a transformation:*

$$dt_1 = \lambda\,(q_1, \ldots, q_k)\; dt\;,$$

then the geodesics of T and T_1 will coincide, and one will have:

$$\lambda = C\left(\frac{\Delta_1}{\Delta}\right)^{\frac{1}{1+k}}.$$

Indeed, refer to the calculation that was developed in no. **6**. By hypothesis, equations (b) and (a_1) coincide for given Q_i, Q_i', and therefore for given β_i, β_i'. That can happen only when the terms

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([1]) On that subject, *see* the beginning of the third chapter, in the left-hand sides of (b) and (a_1) that are homogeneous of degree two in the dq_i / dq_1 and the terms that are independent of those variables are respectively identical. However, upon identifying the second-degree terms, one will define precisely the necessary and sufficient conditions for the geodesics of T and T_1 to coincide. On the other hand, since the geodesics coincide, λ will necessarily have the indicated form. Moreover, (A) will admit a correspondent with *vis viva* T_1, not only for the given forces Q_i, *but for arbitrary forces.*

8. – We finally prove this converse to the first proposition:

If ds^2 and ds_1^2 are given then one can associate arbitrary forces Q_i with forces Q_i' such that the two systems $\left(\frac{ds^2}{dt^2}, Q_i\right)$ and $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ are correspondents and the geodesics of ds^2 and ds_1^2 will coincide.

Indeed, we know that the geodesics of ds_1^2 belong to the trajectories of (A_1) for any Q_i', and therefore they will belong to the trajectories of (A) for any Q_i .

Now, for a system (A), besides the geodesics of ds^2, there exists no $(2k - 2)$-parameter congruence of trajectories that is independent of the forces Q_i . The geodesics of ds^2 then overlap with those of ds_1^2.

However, one can go further: When two systems $\left[\frac{ds^2}{dt^2}, Q_i\right]$ and $\left[\frac{ds_1^2}{dt_1^2}, Q_i'\right]$ are correspondents, the same thing will be true for the two systems $\left[\frac{ds^2}{dt^2}, c\,Q_i\right]$ and $\left[\frac{ds_1^2}{dt_1^2}, c'\,Q_i'\right]$, where c and c' denote two constants. However, assume that the systems (A) and (A_1) will again be correspondents when one replaces the Q_i with certain forces that are *distinct* from the first, namely (Q_i), and the Q_i' with (Q_i') [1]. The geodesics of ds_1^2 belong to the trajectories of the two systems $\left[\frac{ds^2}{dt^2}, Q_i\right]$ and $\left[\frac{ds^2}{dt^2}, (Q_i)\right]$. However, we have shown in the first chapter that for $k > 2$, there exists no $(2k - 2)$-parameter congruence of trajectories that is common to two such systems except for the geodesics of ds^2. We thus arrive at this conclusion:

([1]) If the systems of forces Q_i and (Q_i) are *distinct* then the systems Q_i' and (Q_i') will also be so, because otherwise the trajectories of (A_1), and as a result, those of (A), would not be modified by the change of forces, and one would have:

$$(Q_i) = c\, Q_i .$$

If two corresponding systems $\left[\frac{ds^2}{dt^2}, Q_i\right]$ *and* $\left[\frac{ds_1^2}{dt_1^2}, Q_i'\right]$ *remain correspondents when one replaces the forces* Q_i *and* Q_i' *with certain forces* (Q_i) *and* (Q_i') *that are distinct from the first then the geodesics of* ds^2 *will coincide with those of* ds_1^2. As a result, all of the preceding propositions will apply to the correspondence in question.

Of course, the last proof supposes that $k > 2$, because for $k = 2$, the lemma that it is based upon will break down.

We shall return to this point in Chapter Three, in which we shall recover all of the results that we just obtained by a different method.

III. – CONDITIONS FOR A SYSTEM (*A*) WITHOUT FORCES TO ADMIT A CORRESPONDENT. REMARK ON THE SYSTEMS (*A*) FOR WHICH THE FORCES ARE DERIVED FROM A POTENTIAL.

9. – We just saw that if a system (*A*) without forces possesses a (non-ordinary) correspondent then it will necessarily admit a quadratic integral that is distinct from that of *vis viva*. *For* $k = 2$, *that condition is sufficient*, as is well-known, and one will deduce a correspondent (A_1) from (*A*) from any quadratic integral.

For $k > 2$, it is easy to define systems that possess correspondents and admit *only one quadratic integral* besides the *vis viva* integral. For example, the ds^2 :

$$ds^2 = \varphi(q_1, q_2)(dq_1^2 + dq_2^2) + dq_3^2 ,$$

in which φ is arbitrary, is a correspondent to ds_1^2 :

$$ds_1^2 = \varphi(q_1, q_2)(dq_1^2 + dq_2^2) + C\,dq_3^2 ,$$

in which C is a number. On the other hand, the system (*A*) or $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$ admits only one quadratic integral:

$$\varphi(q_1, q_2)(dq_1^2 + dq_2^2) + C\,dq_3^2 = c\,dt^2 ,$$

which is an integral that can be written:

$$dq_3 = c\,dt ,$$

in particular.

However, in general, *the condition that there exists a quadratic integral does not suffice for* (*A*) *to admit a correspondent.* One easily assures oneself of that upon considering, for example, the ds^2 that Jacobi encountered in the theory of elliptic coordinates:

$$ds^2 = \left[\sum_{i=1}^{k} \frac{\psi_i(q_i)}{F'(q_i)}\right] \sum_{i=1}^{k} \frac{F'(q_i)}{f(q_i)} dq_i^2 \;,$$

in which one has set:

$$F = (u - q_1)\,(u - q_2)\,\ldots\,(u - q_k)\,,$$
$$f = (u - a_1)\,(u - a_2)\,\ldots\,(u - a_k)\,,$$

and in which ψ_i denotes an arbitrary function of q_i . That system $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$ admits a complete system of quadratic integrals and possesses no correspondents (besides ordinary correspondents).

On the subject of sufficient conditions for a system (A) to admit non-ordinary correspondents, I will make the following observations: Consider a system of (k – 1) second-order differential equations in $q_1, q_2, \ldots, q_k$. In order for such a system to be regarded as defining geodesics, it is necessary that:

1. There exists a function $f(q_1, q_2, \ldots, q_k, q_2', \ldots, q_k')$ such that the system:

(1) $$\frac{d}{dq_1}\frac{\partial f}{\partial q'_{(i)}} - \frac{\partial f}{\partial q_i} = 0\,, \qquad \frac{dq_i}{dq_1} = q'_{(i)} \qquad (i = 1, 2, \ldots, k)$$

agrees with the given system.

2. f is the square root of a second-degree polynomial τ in $q'_{(2)}, \ldots, q'_{(k)}$.

In order for the ds^2 that is defined by τ to admit correspondents, it is necessary that:

3. There exists at least two such functions $f = \sqrt{\tau}$ and $f_1 = \sqrt{\tau_1}$ that are distinct.

The first condition is always fulfilled for $k = 2$, but for $k > 2$, it is no longer true. Those conditions, which are sufficient moreover, imply the existence of a quadratic integral of the system $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$, but the converse is not true.

How does one define those sufficient conditions explicitly? One of the simplest means consists of appealing to the theorem that was established above:

In order for the systems $\left[\frac{ds^2}{dt^2}, Q_i = 0\right]$ *and* $\left[\frac{ds_1^2}{dt_1^2}, Q_i' = 0\right]$ *to be correspondents, it is necessary and sufficient that one can pass from one system to the other by changing dt into* $\lambda\,(q_1, \ldots, q_k)\,dt_1$.

One forms the desired sufficient conditions very elegantly upon expressing the fact that this is true, which are conditions that obviously involve the expressions $a_{ij} = \dfrac{\partial \log \Delta}{\partial A_{ij}}$, which I will study in another article ([1]).

I shall add only that it is easy to form a ds^2 that possesses correspondents: In particular, if one knows a *transformation of the geodesics of* ds^2 *into themselves* then that transformation will generate a *correspondent* ds_1^2 to ds^2 that is, at the same time, one of its *homologues*. Hence, the ds^2 of the form $\sum_{i=1}^{k} dq_i^2$ admit an infinitude of correspondents ds_1^2 that one deduces with the aid of the most general homographic transformation in k variables.

When one has such a form ds^2, any system $\left[\dfrac{ds^2}{dt^2}, Q_i\right]$ will admit correspondents of the form $\left[\dfrac{ds_1^2}{dt_1^2}, Q_i'\right]$. Observe that there exist systems (A) that possess non-ordinary correspondents and which admit *no quadratic integral*. The system $\left[\dfrac{ds^2}{dt^2}, Q_i = 0\right]$ will necessarily possess one such integral (in addition to that of *vis viva*). One sees that with the correspondents:

$$(A) \qquad ds^2 = \varphi(q_1, q_2)[dq_1^2 + dq_2^2] + dq_3^2 \qquad (Q_1, Q_2, Q_3),$$

and

$$(A_1) \qquad ds^2 = \varphi(q_1, q_2)[dq_1^2 + dq_2^2] + C\,dq_3^2 \qquad (Q_1, Q_2, C\,Q_3), \qquad (C \neq 1),$$

in which Q_1, Q_2, Q_3 are taken arbitrarily, which are correspondents that define not only the same trajectories, but also the same motion, because one has:

$$\frac{dt_1}{dt} = 1$$

here.

Furthermore, observe that *if the forces* Q_i *are derived from a potential* U *then the same thing will not be true for the* Q_i', *in general*, as the same example will show when one sets $Q_i \equiv \partial U / \partial q_i$, in which U is an *arbitrary* function of q_i . Nonetheless, the latter situation *can* present itself, as one sees when one take U to be a function of the form:

$$U = \psi(q_1, q_2) + \chi(q_3) .$$

([1]) On that subject, *see* the note that was cited before by R. Liouville (Comptes rendus, May 1892).

It is fitting to remark that in the latter example, a natural congruence of trajectories $h = a$ of (A) will *never* coincide with a natural congruence of trajectories $h_1 = a_1$ of (A_1) (h and h_1 denote the two constants of the *vis viva* integrals). Indeed, one will have both:

$$\varphi(q_1, q_2)[dq_1^2 + dq_2^2] + \quad dq_3^2 - \quad [\psi + \chi] dt^2 = a\, dt^2$$

and

$$\varphi(q_1, q_2)[dq_1^2 + dq_2^2] + C\, dq_3^2 - [\psi + C\, \chi] dt^2 = a_1\, dt^2$$

for such a congruence, and those two conditions must coincide, which is impossible, no matter how one chooses a and a_1. Later on, we shall show that this is true in general.

10. – I will conclude this study of systems in which the forces are zero by addressing the problem that was treated in which chapter, which is a problem that concerns the case in which the forces Q_i in (A) are not zero, but are derived from a potential U. One knows that for each value of the constant $h = T - U$, the trajectories of (A) coincide with the geodesics of $ds'^2 = (U + h)\, ds^2$. One can pose the following question:

Under what conditions does the system $\left[(U+h)\dfrac{ds^2}{dt^2}, Q_i = 0\right]$ *admit a non-ordinary correspondent* $\left[\dfrac{ds_1^2}{dt_1^2}, Q_i' = 0\right]$ *for any constant h?*

It is clear that this question gets back to what was treated in this chapter and that all of the properties that were proved in regard to corresponding ds^2 will apply here to the pair $(U + h)\, ds^2$ and ds_1^2, where ds_1^2 depends upon h. In particular, *the system* $\left[(U+h)\dfrac{ds^2}{dt^2}, Q_i = 0\right]$ *must admit a quadratic integral for any h*. From a theorem that I stated above without giving its proof, it will follow that the system $\left[\dfrac{ds^2}{dt^2}, Q_i\right]$ must also admit a quadratic integral.

What relations exist between that problem and the search for correspondents to the system $\left[\dfrac{ds^2}{dt^2}, U\right]$? First of all, if the geodesics of $ds'^2 \equiv (U + h)\, ds^2$ and ds_1^2 coincide for an arbitrarily-chosen h then any system $\left[(U+h)\dfrac{ds^2}{dt^2}, Q_i\right]$, in particular, the system:

$$\left[(U+h)\frac{ds^2}{dt^2}, \frac{1}{U+h}\right],$$

will admit correspondents of the form $\left[\frac{ds_1^2}{dt_1^2}, Q_i'\right]$. The system $\left[\frac{ds^2}{dt^2}, U\right]$ will then admit an infinitude of *distinct* correspondents that depend upon an arbitrary constant ([1]). *In a system* (A) *or* $\left[\frac{ds^2}{dt^2}, U\right]$, *the expression* $ds'^2 \equiv (U + h)\, ds^2$ *cannot admit a correspondent* ds_1^2 *without* (A) *admitting an infinitude of distinct correspondents. However, the converse is not true.* For example, the system $\left[\frac{ds^2}{dt^2}, U\right]$, in which:

$$ds^2 \equiv \varphi(q_1, q_2)[dq_1^2 + dq_2^2] + dq_3^2$$

and U is an *arbitrary* function of the q_i, possesses an infinitude of correspondents without $ds'^2 \equiv (U + h)\, ds^2$ admitting a correspondent ds^2 (for any value of h).

However, can it happen that *the search for a correspondent to the system* (A) *or* $\left[\frac{ds^2}{dt^2}, U\right]$ *coincides with the search for a correspondent (for arbitrary h) of the system* $\left[(U+h)\frac{ds^2}{dt'^2}, Q_i = 0\right]$? More precisely, can it happen that a correspondent $\left[\frac{ds_1'^2}{dt_1'^2}, Q_i = 0\right]$ to the latter system, in which $ds_1'^2$ depends upon h, is attached to a system $\left[\frac{ds_1^2}{dt_1^2}, U\right]$ in the same way that $\left[\frac{ds_1'^2}{dt_1'^2}, Q_i = 0\right]$ as is attached to (A)? In order for that to be true, it is necessary and sufficient that $ds_1'^2$ must have the form $ds_1'^2 \equiv (U_1 + h_1)\, ds_1^2$, in which h_1 denotes a certain function of h that U_1 and ds_1^2 no longer depend upon. If one still desires, it is necessary that there should exist a correspondent $\left[\frac{ds_1^2}{dt_1^2}, U\right]$ of (A) such that every congruence $h = a$ of (A) will coincide with a congruence $h_1 = a_1$ of (A_1). That condition is fulfilled under the Darboux transformation, but one will then have $ds_1'^2 = C\, ds'^2$. In the next chapter, I will show that it is *never* fulfilled for two correspondents that are not ordinary. In other words, the natural congruences are never preserved.

([1]) If there exists a correspondent ds_1^2 to $(U + h)\, ds^2$ *for a well-defined value of* h then the system $\left[\frac{ds^2}{dt^2}, U\right]$ will admit (*non-distinct*) correspondents of the form $\left[C\frac{ds^2}{dt^2}, cQ_t'\right]$, where ds_1^2 no longer depends upon an arbitrary constant.

Therefore, the search for a correspondent to $\left[\dfrac{ds^2}{dt^2}, U\right]$ *and the search for a correspondent to* $\left[(U+h)\dfrac{ds^2}{dt^2}, Q_i = 0\right]$ *are always two distinct problems.*

CHAPTER III

Corresponding systems in which all forces are non-zero.

I. – PROOF OF A GENERAL PROPERTY OF THOSE SYSTEMS.

1. – Let (A) and (A_1) be two corresponding systems: If the forces Q_i of (A) are not all zero then the forces Q'_i of (A_1) will not all be zero either. Having recalled that, suppose that (A) admits a correspondent (A_1) that is distinct from its ordinary correspondents ([1]). We shall show that (A) then enjoys several properties, one of the most important ones being this: *At least one of the systems* (A) *and* (A_1), *in which annuls the forces, possesses a quadratic integral.*

In order to prove that proposition, I shall appeal to the following lemma:

If the systems (A) *and* (A_1), *in which the forces are not all zero, are correspondents then one can pass from one to the other by a change of variables of the form:*

$$dt^2 = d\sigma^2 + \mu(q_1, q_2, \ldots, q_k)\, dt_1^2,$$

in which $d\sigma^2$ *represents a quadratic form in* $dq_1, \ldots, dq_k$ *whose coefficients depend upon* $q_1, \ldots, q_k$.

First of all, observe that since the Q_i are not all zero, the function t (q_1) that is defined by (A) will be determined along each trajectory (up to an additive constant) by the equality:

$$\left(\frac{dt}{dq_1}\right)^2 = \frac{q''_{(i)} + \Phi_1\, q'_{(i)} - \Phi_i}{\beta_i - \beta_1\, q'_{(i)}}, \tag{1}$$

upon setting $\beta_i = \alpha_i / \Delta$, $q'_{(i)} = dq_i / dq_1$, $q''_{(i)} = d^2 q_i / dq_1^2$ (*see* Chapter I, pp. 21) Along the same trajectory, from (A_1), one will have:

([1]) As we know, the *ordinary* correspondents of (A) are the systems $\left(C\dfrac{ds^2}{dt_1^2}, cQ_i\right)$ and $\left[(\alpha U+\beta)\dfrac{ds^2}{dt_1^2}, \dfrac{\gamma U+\delta}{\alpha U+\beta}\right]$ (if the Q_i are derived from a potential U).

$$(2)\qquad \left(\frac{dt_1}{dq_1}\right)^2 = \frac{q''_{(i)} + \Phi'_1\, q'_{(i)} - \Phi'_i}{\beta'_i - \beta'_1\, q'_{(i)}}.$$

If one eliminates d^2q_i / dq_1^2 from (1) and (2) then one will get a relation of the form:

$$\left(\frac{dt_1}{dt}\right)^2 = f\,(q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k) \qquad \left(q'_i = \frac{dq_i}{dt}\right)$$

that permits one to pass from (A_1) to (A). *Arbitrary* initial values of the q_i, q'_i correspond to a trajectory along which $\left(\frac{dt_1}{dq_1}\right)^2$, $\left(\frac{dt}{dq_1}\right)^2$, and as a result, $\left(\frac{dt_1}{dt}\right)^2$ are well-defined functions of q_1. Therefore, f and f' coincide for the initial value of the q_i, q'_i, and since those values are arbitrary, f and f' will be identical.

Having said that, form that relation explicitly from (1) and (2): It will be:

$$(3)\qquad \left(\frac{dt_1}{dt}\right)^2 = \frac{-q_1'^2\,[q'_{(i)}\,(\Phi_1 - \Phi'_1) - (\Phi_i - \Phi'_i)] + (\beta_i - \beta_1\, q'_{(i)})}{\beta'_i - \beta'_1\, q'_{(i)}} ,$$

which can also be written as:

$$(4)\qquad dt^2(\beta_i\, dq_1 - \beta_1\, dq_i) - dt_1^2(\beta'_i\, dq_1 - \beta'_1\, dq_i) = (\Pi_1 - \Pi'_1)\, dq_i - (\Pi_i - \Pi'_i)\, dq_1 ,$$

in which the Π, Π' denote quadratic forms in dq_1, dq_2, …, dq_k .

Since the relation (3) is *unique*, it must remain the same when one successively gives the values 2, 3, …, k to the index i. Now, the numerator on the right-hand side is a polynomial in q'_1, $q'_{(2)}$, … $q'_{(k)}$; the denominator includes only $q'_{(i)}$. In order for that function to not change when one sets i = 2, 3, …, k, it is then necessary that its denominator $(\beta'_i - \beta'_1\, q'_{(i)})$ should divide its numerator, and as a result, it should divide the two parts:

$$[q'_{(i)}\,(\Phi_1 - \Phi'_1) - (\Phi_i - \Phi'_i)] \qquad \text{and} \qquad \beta_i - \beta_1\, q'_{(i)}$$

separately. Consequently, one will then have:

$$\frac{\beta'_1}{\beta_1} \equiv \frac{\beta'_2}{\beta_2} \equiv \ldots \equiv \frac{\beta'_k}{\beta_k} ,$$

and on the other hand, once the division by $(\beta_i' - \beta_1' q_{(i)}')$ has been performed, the relation (3) will take the form:

$$\beta_1' dt_1^2 - \beta_1 \, dt^2 = d\sigma^2 \tag{5}$$

Q. E. D.

However, the proof supposes that $k > 2$. Here is how one can proceed for $k = 2$. Write the differential equation of the trajectories:

$$\frac{d}{dq_1} \log \chi + 2\Phi_1 = \frac{d}{dq_1} \log \psi - \frac{2\beta_1 \chi}{\psi}, \tag{6}$$

upon setting:

$$\psi \equiv \beta_2 - \beta_1 q_{(2)}', \qquad \chi \equiv q_{(2)}'' + \Phi_1 q_{(2)}' - \Phi_2 \; .$$

Since the forces are not zero, at least one of the coefficients β_1, β_2 is non-zero, and we can always assume that it is β_1 ; otherwise, we could permute q_1 and q_2 . Under those conditions, equation (6) can be written:

$$q_{(2)}'' + \frac{d}{dq_1} (\Phi_1 q_{(2)}' - \Phi_2)$$
$$= \chi \left[-2\Phi_1 + \frac{d}{dq_1} \log \beta_1 + \frac{-3 q_{(2)}'' - \Phi_1 q_{(2)}' + \Phi_2 + \dfrac{d}{dq_1}\left(\dfrac{\beta_2}{\beta_1}\right)}{\dfrac{\beta_2}{\beta_1} - q_{(2)}'} \right],$$

or rather:

$$q_{(2)}'' = \frac{-3 q_{(2)}'' - 4 q_{(2)}'' (\Phi_1 q_{(2)}' - \Phi_2) + V}{\dfrac{\beta_2}{\beta_1} - q_{(2)}'} + W = S, \tag{7}$$

in which V and W represent polynomials, the first of which is in $q_{(2)}'$, while the second is in $q_{(2)}'$ and $q_{(2)}''$, and the coefficients depend upon q_1, q_2 . The fraction that appears in the right-hand side of (7) is irreducible, moreover. In other words, $\left(\dfrac{\beta_2}{\beta_1} - q_{(2)}'\right)$ does not divide the numerator, because in order to do that, it must divide the coefficient of $q_{(2)}''^{\,2}$, which is – 3. Now express the idea that equation (7), which relates to (A_1), namely, $q_{(2)}''' = S'$, coincides with the preceding one or that $S \equiv S'$. One first finds that β_1' cannot be zero, since otherwise S' would be a polynomial with respect to the derivatives. Moreover, S' and S must become infinite for the same value of $q'_{(2)}$, so $\frac{\beta_2}{\beta_1} \equiv \frac{\beta'_2}{\beta'_1}$. Finally, the difference:

$$\frac{+4q''_{(2)}[(\Phi_1 - \Phi'_2)\,q'_{(2)} - (\Phi_1 - \Phi'_2)] - (V - V')}{\frac{\beta_2}{\beta_1} - q'_{(2)}} - (W - W')$$

must be identically zero. The fraction that appears in that difference then reduces to a polynomial (with respect to the derivatives), i.e., its numerator is divisible by its denominator $\frac{\beta_2}{\beta_1} - q'_{(2)}$, which demands that the binomial $\left(\frac{\beta_2}{\beta_1} - q'_{(2)}\right)$ must divide both $(V - V')$ and $[(\Phi_1 - \Phi'_2)\,q'_{(2)} - (\Phi_1 - \Phi'_2)]$. One will then have indeed:

$$\frac{\beta_2}{\beta_1} \equiv \frac{\beta'_2}{\beta'_1} \qquad \text{and} \qquad \beta_1\,dt^2 - \beta'_1\,dt_1^2 = d\sigma^2 ,$$

and even for $k = 2$.

2. – We shall now show that *the expression:*

$$\frac{\Delta}{\Delta_1}\,\frac{\beta'_i}{\beta_i}\left(\frac{dt_1}{dt}\right)^{k+3}$$

is a first integral of (A).

Indeed, we saw in Chapter II that Δ is a multiplier of the system (A):

$$(A) \qquad dt = \frac{dq_1}{q'_1} = \frac{dq_2}{q'_2} = \ldots$$

In other words, when one knows $(2k - 2)$ integrals of (A) that are independent of t, namely:

$$\varphi_j(q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k) = C_j \qquad [j = 1, 2, \ldots, (k - 2)],$$

if one infers $q_3, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$ as functions of q_1, q_2 then the expression:

$$\frac{\Delta}{\delta}\,q'_1\,(dq_2 - q'_{(2)}\,dq_1)$$

is an exact differential. δ represents the functional determinant $\dfrac{D(\varphi_1,\varphi_2,\dots,\varphi_{(2k-2)})}{D(q_3,q_4,\dots,q_1',q_2',\dots,q_k')}$.

Perform a first change of variables by setting $q_1' = q_1'$, $q_2' = q_{(2)}'\,q_1'$, …, $q_k' = q_{(k)}'\,q_1'$. The functions φ_j will become functions ψ_j of $q_1, \dots, q_k$, q_1', $q_{(2)}'$, …, $q_{(k)}'$, and one will have:

$$\delta \equiv \frac{D(\psi_1,\psi_2,\dots,\psi_{2k-2})}{D(q_3,q_4,\dots,q_k,q_1',q_{(2)}',\dots,q_{(k)}')} \times \frac{D(q_3,q_4,\dots,q_k,q_1',q_{(2)}',\dots,q_{(k)}')}{D(q_3,q_4,\dots,q_k,q_1',q_2',\dots,q_k')}$$

$$= \frac{D(q_{(2)}',\dots,q_{(k)}')}{D(q_2',\dots,q_k')} \equiv \frac{\delta'}{q_1'^{\,(k-1)}} .$$

Then make the change of variable:

$$q_{(2)}'' + \Phi_1\,q_{(2)}' - \Phi_2 = (\beta_2 - \beta_1\,q_{(2)}')\frac{1}{q_1'^{\,2}} .$$

The functions ψ_j become functions ϖ_j of $q_1, q_2, \dots, q_k$, q_1', $q_{(2)}'$, …, $q_{(k)}'$, and one will have:

$$\delta \equiv \frac{D(\varpi_1,\varpi_2,\dots,\varpi_{2k-2})}{D(q_3,\dots,q_k,q_{(2)}'',q_{(2)}',\dots,q_{(k)}')} \times \frac{D(q_3,\dots,q_k,q_{(2)}'',q_{(2)}',\dots,q_{(k)}')}{D(q_3,\dots,q_k,q_1',q_2',\dots,q_k')}$$

$$\equiv \delta'' \times \frac{\partial q_{(2)}''}{q_1'} = -\frac{2\,\delta''(\beta_2 - \beta_1\,q_{(2)}')}{q_1'^{\,2}} .$$

By definition, one has:

$$-\tfrac{1}{2}\delta = \frac{\delta''(\beta_2 - \beta_1\,q_{(2)}')}{q_1'^{\,k+2}} .$$

The expression:

$$\frac{\Delta\, q_1'^{\,k+2}}{[\beta_2 - \beta_1\,q_{(2)}']} ,$$

in which q_1' *is defined by the equality:*

(8) $$q_1'^{\,2} = \frac{\beta_2 - \beta_1\,q_{(2)}'}{q_{(2)}'' + \Phi_1\,q_{(2)}' - \Phi_2} ,$$

is then a multiplier for the differential equations (α) *of the trajectory:*

(α) $$dq_1 = \frac{dq_2}{q_{(2)}'} = \dots = \frac{dq_k}{q_{(k)}'} = \frac{dq_2'}{q_{(2)}''} = \frac{dq_{(2)}''}{f_2} = \frac{dq_{(3)}''}{f_3} = \dots = \frac{dq_{(k)}''}{f_k} .$$

If (A) and (A_1) are correspondents then the two expressions:

$$\frac{\Delta\left(\dfrac{dq_1}{dt}\right)^{k+2}}{[\beta_2 - \beta_1 q'_{(2)}]} \quad \text{and} \quad \frac{\Delta\left(\dfrac{dq_1}{dt_1}\right)^{k+2}}{[\beta'_2 - \beta'_1 q'_{(2)}]}$$

will be two multipliers of the same equations (α).

Since one has, on the one hand:

$$\frac{\beta'_1}{\beta_1} \equiv \frac{\beta'_2}{\beta_2} \equiv \ldots \equiv \frac{\beta'_k}{\beta_k} \equiv \mu ,$$

the equality:

(9) $$\frac{\Delta}{\Delta_1}\mu\left(\frac{dt_1}{dt}\right)^{k+2} = \text{const.}$$

will define a first integral of (α). In (9), dt_1 / dt represents the ratio of the quantities $\dfrac{dq_1}{dt}$ and $\dfrac{dq_1}{dt_1}$ when it is expressed as a function of $q''_{(2)}, q''_{(3)}, \ldots, q''_{(k)}$ (and the q_i). *The equality* (9), *in which the same ratio is expressed as a function of* $q'_1, q'_2, \ldots, q'_k$ (*and the* q_i), *will then define a first integral of* (A).

Now, we have:

$$\left(\frac{dt_1}{dt}\right)^2 = \frac{\dfrac{d\sigma^2}{dt^2}}{\beta'_1} + \frac{\beta_1}{\beta'_1} ,$$

so the first integral (9) of (A) will then be:

$$\left(\frac{\Delta}{\Delta_1}\frac{\beta'_1}{\beta_1}\right)^{\frac{2}{k+3}}\left(\frac{\dfrac{d\sigma^2}{dt^2}}{\beta'_1} + \frac{\beta_1}{\beta'_1}\right) \equiv t - V = \text{const.},$$

in which t denotes a quadratic form in $q'_1, q'_2, \ldots, q'_k$ (whose coefficients depend upon q_i), and V is a simple function of the q_i .

The results that we just obtained are thus summarized as:

When two systems (A) *and* (A_1) *(in which the forces are not zero) are correspondents, the coefficients* $\beta_1, \beta_2, \ldots, \beta_k$ *, and* $\beta'_1, \beta'_2, \ldots, \beta'_k$ *are necessarily proportional, and one can pass from* (A) *to* (A_1) *by a unique transformation of the form:*

$$\beta_1' \, dt_1^2 - \beta_1 \, dt^2 = d\sigma^2 .$$

The integral:

$$(a) \qquad \left(\frac{\Delta}{\Delta_1} \frac{\beta_1'}{\beta_1} \right)^{\frac{2}{k+3}} \left(\frac{d\sigma^2}{\beta_1'} + \frac{\beta_1 \, dt^2}{\beta_1'} \right) = C \, dt^2$$

is a first integral of (A), *and the equality:*

$$(a') \qquad \left(\frac{\Delta_1}{\Delta} \frac{\beta_1}{\beta_1'} \right)^{\frac{2}{k+3}} \left(\frac{d\sigma^2}{\beta_1} + \frac{\beta_1' \, dt_1^2}{\beta_1} \right) = C \, dt^2$$

is a first integral of (A_1).

Nevertheless, observe that the preceding argument shows only that the left-hand side of (a) will remain constant for any motion of (A). It is therefore not impossible *a priori* that the left-hand side reduces to an absolute constant [in which case, the equality (a) will no longer represent a first integral of (A)], but it cannot reduce to a simple function of the q_i without being a constant. In other words, equations (A) will admit a first integral that is independent of the velocities. Hence, if the left-hand side of (a) is not a constant then it will be a second-degree integral of (A). Having made those remarks, we shall list the different cases that can present themselves.

3. HYPOTHESIS I. – *The left-hand side of* (a) *is an absolute constant.*

In this case, the relation between dt and dt_1 has the form:

$$\frac{dt_1}{dt} = C_0 \left(\frac{\Delta_1}{\Delta} \frac{\beta_1}{\beta_1'} \right)^{\frac{1}{k+3}} = \lambda \, (q_1, q_2, \ldots, q_k) ,$$

in which C_0 is a certain number. The correspondence is then of the type that was studied in Chapter II, and all of the properties that were proved in Section II of that Chapter will apply. In particular, *the geodesics of* ds^2 *and* ds_1^2 *will coincide.*

Conversely, if the ratio dt_1 / dt is a function λ of q_i then the left-hand side of (a) will be an absolute constant, and λ will have the value:

$$(b) \qquad C_0 \left(\frac{\Delta_1}{\Delta} \frac{\beta_1}{\beta_1'} \right)^{\frac{1}{k+3}} .$$

Moreover, since one has:

$$d\sigma \equiv 0 \qquad \text{and} \qquad \frac{dt_1^2}{dt^2} = \frac{\beta_1}{\beta_1'} ,$$

in this case, the value of λ^2 must coincide with β_1 / β_1', which will immediately give:

$$\frac{\beta_1}{\beta_1'} = C_0^{\frac{2(k+3)}{k+1}} \left(\frac{\Delta_1}{\Delta}\right)^{\frac{1}{k+1}},$$

and as a result:

$$\frac{dt_1}{dt} = C_0^{\frac{k+3}{k+1}} \left(\frac{\Delta_1}{\Delta}\right)^{\frac{1}{k+1}}.$$

Those equalities agree quite well with the ones that were obtained in Chapter II (*see* pp. 40). Indeed, we saw that if we can pass from (*A*) to (A_1) by a change of variables $dt_1 / dt = \lambda (q_1, q_2, \ldots, q_k)$ then we will necessarily have:

$$(b') \qquad \frac{dt_1}{\Delta_1^{\frac{1}{k+1}}} = \frac{C\,dt}{\Delta^{\frac{1}{k+1}}} \qquad \text{and} \qquad \beta_i \, \Delta^{\frac{2}{k+1}} = C^2 \beta_i' \, \Delta_1^{\frac{2}{k+1}} ,$$

which are equalities that will be no different from the preceding ones when one sets $C_0 = C^{\frac{k+1}{k+2}}$.

In the case that we are studying, the equality (*a*) will not provide an integral of (*A*). However, we already know that *the system* (*A*), *in which one annuls the forces* Q_i, *possesses a quadratic integral that is distinct from the vis viva*.

HYPOTHESIS II. – *The quadratic integral that is defined by* (*a*) *coincides with that of vis viva*. (This case can present itself only when the Q_i are derived from a potential *U*.)

In this case, the relation between *dt* and dt_1 has the form:

$$dt_1^2 = C_0^2 \left(\frac{\Delta_1}{\Delta} \frac{\beta_1}{\beta_1'}\right)^{\frac{2}{k+3}} [ds^2 - (U+a)\,dt^2] = \lambda [ds^2 - (U+a)\,dt^2].$$

Introduce the Darboux transformation and replace (*A*) with the corresponding system (*A′*):

$$(A') \qquad \frac{d}{dt'}\left(\frac{\partial T'}{\partial q_i'}\right) - \frac{\partial T'}{\partial q_i} = \frac{\partial U'}{\partial q_i}, \qquad \frac{dq_i}{dt'} = q_i' \qquad (i = 1, 2, \ldots, k),$$

in which:

$$T' \equiv (U+a)\frac{ds^2}{dt'^2} \equiv \frac{ds'^2}{dt'^2}, \qquad U' \equiv \frac{1}{U+a} .$$

We know the relation between dt and dt' (*see* Chapter I, Section IV, pp. 28), namely:

$$dt'^2 = (U+a)^2 [ds^2 - (U+a)\, dt^2] \ .$$

The systems (A') and (A_1) will then be two corresponding systems *such that one can pass from one to the other by the change of variables:*

$$\frac{dt_1^2}{dt'^2} = \frac{\lambda^2}{(U+a)^2} = \mu^2 (q_1, q_2, \ldots, q_k) \ .$$

We then return to the correspondence that was studied in Chapter II: *The geodesics of* ds'^2 *and* ds^2 *coincide*. Moreover, we know that:

$$\mu \equiv C\left(\frac{\Delta_1}{\Delta'}\right)^{\frac{1}{1+k}} \equiv C\left[\frac{\Delta_1}{\Delta (U+a)^k}\right]^{\frac{1}{1+k}},$$

and thus we know a simpler value for λ:

$$\lambda \equiv \mu\, (U+a) \equiv C\left[\frac{\Delta_1 (U+a)}{\Delta}\right]^{\frac{1}{1+k}} .$$

One likewise sees that $\dfrac{-\beta_i\, \Delta^{\frac{2}{k+1}}}{(U+a)^{1+\frac{2}{k+1}}}$ is equal to $C^2 \beta_i'\, \Delta_1^{\frac{2}{k+1}}$, which can be further written as:

$$\frac{\beta_i}{\beta_i'} = -\, C^2 \left[\frac{\Delta_1^2\, (U+a)^{k+3}}{\Delta^2}\right]^{\frac{1}{1+k}} .$$

The first value of λ coincides with the second one for $C_0 = C^{\frac{k+1}{k+3}}$.

We have assumed that (A_1) is not an *ordinary* correspondent of (A). Under those conditions, ds'^2 will not agree with $C\, ds_1^2$ (C is a number). In other words, one will also have $U' = c\, U_1$, i.e., at the same time:

$$ds_1^2 = \frac{U+a}{C}\, ds^2 , \qquad U_1 = \frac{1}{c\,(U+a)},$$

and (A_1) is deduced from (A) by a Darboux transformation.

The systems $\left[(U+a)\dfrac{ds^2}{dt'^2}, Q_i = 0\right]$ *and* $\left[\dfrac{ds_1^2}{dt_1^2}, Q'_i = 0\right]$, *respectively, thus admit a quadratic integral that is distinct from vis viva*.

HYPOTHESIS III. – *The quadratic integral* (*a*) *is distinct from that of vis viva*.

That hypothesis (which is the most general) is always realized when the left-hand side of (*a*) does not reduce to a constant, since *the forces* Q_i *are not derived from a potential*.

Let:

$$\frac{1}{\lambda^2}\left(\frac{dt_1}{dt}\right)^2 \equiv \tau - V = \text{const.}$$

be that integral. From a well-known theorem, τ = const. is an integral of the motion without forces. One cannot have $\tau \equiv \mu\,(q_1, q_2, \ldots, q_k)\,T$, because μ = const. will be an integral of the geodesics of T. Moreover, one does not have $\tau \equiv C\,T$, since otherwise the integral (*a*) would be that of *vis viva* ([1]). The only case in which $d\sigma^2$ is equal to $\mu\,ds^2$ in the equality:

$$dt_1^2 = \frac{d\sigma^2}{\beta'_1} - \frac{\beta_1}{\beta'_1}\,dt^2$$

then corresponds to hypothesis II, in which:

$$dt_1^2 = \lambda^2\,[ds^2 - (U+a)\,dt^2]\,, \qquad \text{with} \qquad \lambda \equiv C\left[\frac{\Delta_1(U+a)}{\Delta}\right]^{\frac{1}{k+1}}.$$

For the same reason, *the only case in which* $d\sigma^2$ *is equal to:*

$$\mu_1\,(q_1, q_2, \ldots, q_k)\,ds_1^2$$

is the one in which the Q'_i *are derived from a potential* U_1, *so one has:*

$$dt^2 = \lambda_1^2\,[ds_1^2 - (U_1+a_1)\,dt_1^2]\,, \qquad \lambda_1 = C'\left[\frac{\Delta\,(U_1+a_1)}{\Delta_1}\right]^{\frac{1}{k+1}},$$

i.e.:

$$dt_1^2 = \frac{1}{(U_1+a_1)}\left[ds_1^2 - \frac{dt^2}{\lambda_1^2}\right].$$

([1]) Indeed, one will have: $\sum \dfrac{\partial V}{\partial q_i}\,dq_i \equiv C \sum Q_i\,dq_i$. The Q_i will then admit the potential $U = V / C$, and the integral (*a*) can be written $C\,(T - U)$ = const.

In the latter case, upon replacing the system (A_1) with the system (A_1') :

$$(A_1') \qquad \frac{d}{dt_1'}\left(\frac{\partial T_1'}{\partial q_i'}\right) - \frac{\partial T_1'}{\partial q_i} = \frac{\partial U_1'}{\partial q_i}, \qquad q_i' = \frac{dq_i}{dt_1'},$$

in which:

$$T_1' = (U_1 + a_1)\,\frac{ds_1^2}{dt_1'^2} = \frac{ds_1'^2}{dt_1'^2} \quad \text{and} \quad U_1' = \frac{1}{U_1 + a_1},$$

one will come back to hypothesis I. One passes from (A) to (A_1') by the transformation:

$$\frac{dt^2}{dt_1'^2} = \frac{\lambda_1^2}{(U_1 + a_1)^2} .$$

We finally point out one last particular case, which is *the one in which the forces Q_i and Q_i' are derived from potentials U and U_1 so the relation between dt and dt_1 has the form:*

$$dt_1^2 - \frac{ds_1^2}{U_1 + a_1} = \mu^2\,(q_1, q_2, \ldots, q_k)\left[dt^2 - \frac{ds^2}{U + a}\right] .$$

Upon replacing (A) with the system $\left[T', \frac{1}{U_1 + a_1}\right]$, in which $T' = (U + a)\,\frac{ds^2}{dt'^2}$, and replacing ($A_1$) with the system $\left[T_1', \frac{1}{U_1 + a_1}\right]$, in which $T_1' = (U_1 + a_1)\,\frac{ds_1^2}{dt_1'^2}$, *one will get back to the first hypothesis*. The geodesics of T' and T_1' coincide, and the two new systems transform into each other by the change of variables $\frac{dt_1'}{dt'} = \left[\frac{\Delta_1\,(U_1 + a_1)^k}{\Delta\,(U + a)^k}\right]^{\frac{1}{k+1}}$. As for the function μ, it will necessarily have the form:

$$\mu \equiv C_0\left(\frac{U + a}{U_1 + a_1}\right)^{\frac{1}{2}}\left(\frac{\Delta_1}{\Delta}\,\frac{\beta_1}{\beta_1'}\right)^{\frac{1}{k+3}} \equiv C\left(\frac{\Delta_1}{\Delta}\right)^{\frac{1}{k+1}}\left(\frac{U + a}{U_1 + a_1}\right)^{\left(\frac{1}{2} + \frac{1}{k+1}\right)},$$

and one will have:

$$\frac{\beta_i\,\Delta^{\frac{2}{k+1}}}{(U + a)^{1 + \frac{2}{k+1}}} = \frac{C^2 \beta_i'\,\Delta^{\frac{2}{k+1}}}{(U_1 + a_1)^{1 + \frac{2}{k+1}}} \qquad (i = 1, 2, \ldots, k),$$

moreover, in which the number C_0 is equal to $C^{\frac{k+1}{k+3}}$.

4. – It would be appropriate to complete those remarks with some *converses*. Under hypothesis II, the system (A) will possess a force function U, and the *geodesics of* ds_1^2 will coincide with a *natural congruence* $h = a$ of (A). Conversely, if the Q_i are derived from a potential U and the *natural congruence* $h = a$ of (A) coincides with the *geodesics* of ds_1^2 then one will necessarily find oneself under hypothesis II. Indeed, the system (A_1) and the system (A'), in which $T' \equiv (U+a)\,ds^2 / dt'^2$, $U' = 1 / (U + a)$ are two correspondents whose geodesics coincide. One then passes from one to the other by a transformation such that $dt_1^2 / dt'^2 = \lambda\,(q_1, \ldots, q_k)$, and since, on the other hand, $dt'^2 = (U+a)[ds^2 - (U+a)\,dt^2]$, the relation between dt^2 and dt_1^2 will indeed have the form:

$$dt_1^2 = \mu^2\, ds^2 + \nu\, dt^2,$$

which is true only under hypothesis II.

The same observation will apply to the case in which the forces Q' are derived from a potential U_1, the geodesics of ds^2 will coincide with a natural congruence $h_1 = a_1$ of (A_1). Indeed, it will suffice to permute (A) and (A_1) in order to return to the preceding case. Finally, pass to the latter particular case that I indicated in hypothesis III. In that case, (A) and (A_1) will possess force functions U and U_1, and a natural congruence $h = a$ of (A) will coincide with a natural congruence $h_1 = a_1$ of (A_1). Conversely, if that condition is fulfilled then the systems $\left[(U+a)\dfrac{ds^2}{dt'^2}, \dfrac{1}{U+a}\right]$ and $\left[(U_1+a_1)\dfrac{ds^2}{dt_1'^2}, \dfrac{1}{U_1+a_1}\right]$ will be two correspondents whose geodesics coincide. One will then have a relation such as $dt_1' = \lambda\,(q_1, \ldots, q_k)\; dt'$ between dt' and dt_1', and as a result, a relation such as:

$$\left(dt_1^2 - \frac{ds_1^2}{U_1+a_1}\right) = \mu^2\left(dt^2 - \frac{ds^2}{U+a}\right)$$

between dt_1 and dt. That is, in fact, the relation that characterizes the particular case in question.

5. – Before summarizing the results that were just obtained, I shall infer some further consequence of the form of the relation that exists between dt and dt_1. I shall then write that relation as:

$$(m) \qquad \left(\frac{dt_1}{dt}\right)^2 = \lambda^2\left(\frac{d\sigma^2}{dt^2} - V\right), \qquad \lambda = \left(\frac{\Delta_1}{\Delta}\,\frac{\beta_1}{\beta_1'}\right)^{\frac{1}{k+3}},$$

so the equality:

$$(n) \qquad \frac{d\sigma^2}{dt^2} - V = c$$

will define a first integral of (A). *Any first integral of* (A_1), *say:*

$$f\left(q_1, q_2, \ldots, q_k, \frac{dq_1}{dt_1}, \frac{dq_2}{dt_1}, \ldots, \frac{dq_k}{dt_1}\right) = C,$$

corresponds to a first integral of (A) that one calculates by replacing dt_1 as a function of dt in f using formula (m). However, it is remarkable that *any entire algebraic integral of* (A_1) *corresponds to an integral of* (A) *that is entire, algebraic, and of the same degree.* That is obvious in the case where the geodesics of T and T_1 coincide, i.e., in which dt_1 / dt reduces to a simple function of the q_i. It will then suffice to prove that in the general case.

Let ds_n^n denote a homogeneous form of degree n in $dq_1, dq_2, \ldots, dq_k$. The integral considered in (A) will have the form:

$$ds_n^n + dt_1^2\, ds_{n-2}^{n-2} + \cdots + dt_1^n\, s_0 = C\, dt_1^n \qquad \text{(if } n \text{ is even)},$$

and:

$$ds_n^n + dt_1^2\, ds_{n-2}^{n-2} + \cdots + dt_1^{n-1}\, s_1 = C\, dt_1^n \qquad \text{(if } n \text{ is odd)}.$$

Replace the powers of dt_1^2 with the powers of $\lambda^2(d\sigma^2 - V\, dt^2)$ in the left-hand side and replace dt_1^n with $dt^n\, \lambda^n\, c^{n/2}$ in the right-hand side. One will then obtain an integral of (A) that is entire and of degree n.

The same remark and the same proof apply to the *rational* integrals.

In particular, if (A_1) possesses a linear integral, say:

$$\sum a_i\, dq_i = C\, dt_1,$$

then (A) will admit the integral ([1]):

$$\frac{1}{\lambda}\sum a_i\, dq_i = C'\, dt.$$

If (A_1) possess a quadratic integral, say $dS^2 - W\, dt_1^2 = C\, dt_1^2$, then ($A$) will admit the integral:

$$\frac{dS^2}{\lambda^2} - W\, d\sigma^2 + W\, V\, dt^2 = C'\, dt^2.$$

Apply the last remark to the case *in which the forces in one of the corresponding systems* (A) *and* (A_1) *are defined from a potential.* For example, let U_1 be the force function for Q_i'. In the first place, *if the geodesics of T and T_1 coincide* then (A) will admit the quadratic integral:

([1]) Since any linear integral defines an infinitesimal transformation of the system (A), two corresponding systems (A) and (A_1) (in particular, two corresponding ds^2) will admit the same number of infinitesimal transformations.

$$\left(\frac{\Delta}{\Delta_1}\right)^{\frac{2}{1+k}} (ds_1^2 - U_1\, dt^2) = h_1'\, dt^2 .$$

If the Q_i also admit a potential then (A) and (A_1) will each admit a quadratic integral other than the *vis viva*.

In the general case, the *vis viva* integral of (A_1) will give the following integral for (A):

$$\text{(}p\text{)} \qquad \frac{ds_1^2}{\lambda^2} - U_1\, d\sigma^2 + U_1 V\, dt^2 = h_1'\, dt^2 .$$

It is important to see that this integral is distinct from the quadratic integral (a) and that of *vis viva* of (A) (when the latter exists). In order to discuss that point, we first place ourselves under the hypothesis in which the Q_i are not derived from a potential. *In order for the integral* (p) *to coincide with the integral* (a), *it is necessary and sufficient that one should have:*

$$\frac{ds_1^2}{\lambda^2} - U_1 (d\sigma^2 - V\, dt^2) \equiv a_1 (d\sigma^2 - V\, dt^2) - b_1\, dt^2$$

(in which a_1 and b_1 are two certain constants), or rather, from (m):

$$ds_1^2 - dt_1^2 (U_1 + a_1) + b_1 \lambda^2\, dt^2 = 0 ,$$

which is an equality of the form:

$$dt_1^2 = \frac{ds_1^2}{U_1 + a} + \mu\, dt^2 ,$$

which characterizes hypothesis II, in which the geodesics of ds^2 coincide with a natural congruence $h_1 = a_1$ of (A_1).

Now suppose that (A) possesses a force function U. *Under what conditions will the integral* (p) *reduce to a combination of the integral* (a) *and that of vis viva?* It is necessary and sufficient that one should have:

$$\frac{ds_1^2}{\lambda^2} - U_1 (d\sigma^2 - V\, dt^2) \equiv a_1 (d\sigma^2 - V\, dt^2) - b_1 (ds^2 - U\, dt^2) + c_1\, dt^2$$

(in which a_1, b_1, c_1 are certain numbers), i.e.:

$$dt_1^2 - \frac{ds_1^2}{U_1 + a_1} = - \frac{\lambda^2 b_1}{U_1 + a_1} \left[ds^2 - \left(U - \frac{c_1}{b_1} \right) dt^2 \right] ,$$

which is an equality of the form:

$$dt_1^2 - \frac{ds_1^2}{U_1 + a_1} = - \mu^2 \left[dt^2 - \frac{ds^2}{U + a} \right]$$

that characterizes the particular case in which a natural congruence $h = a$ of (A) and a natural congruence $h_1 = a_1$ of (A_1) coincide.

Except for those two cases, the integral (p) will be distinct from the integral (a) and that of *vis viva*.

6. – We are now in a position to state the following conclusions:

When two systems (A) *and* (A_1), *in which the forces are not all zero, correspond* ([1]), *one can always pass from one to the other by a unique transformation of the form:*

$$\frac{dt_1^2}{dt^2} = \lambda^2(q_1, \ldots, q_k) \left(\frac{d\sigma^2}{dt^2} - V \right),$$

in which the parentheses define a quadratic integral of (A), *at least when it they do not reduce to a constant*.

However, there are several cases to be distinguished:

1. *The geodesics of ds^2 and ds_1^2 coincide. This is the case in which $dt_1 = \lambda\,(q_1, \ldots, q_k)\, dt$.* Equations (A') and (A_1'), which are deduced from (A) and (A_1) *upon annulling the forces*, admit a quadratic integral without the same thing being necessarily true for (A) and (A_1). When the force in one of the systems, namely (A_1), is derived from a potential, (A) will admit a quadratic integral. When there exists a force function in the two systems, each of them will admit a second quadratic integral in addition to the *vis viva* integral.

2. *At least one of the two systems, say A_1, admits a potential, and a natural congruence $h_1 = a_1$ for* (A_1) *coincides with the geodesics of ds^2.*

One will get back to the first case by replacing (A_1) with the system:

$$\left[(U_1 + a_1) \frac{ds_1^2}{dt_1'^2}, \frac{1}{U_1 + a_1} \right]$$

([1]) It is clear that one can permute (A) and (A_1) in these statements, since (A) and (A_1) play a symmetric role.

with the aid of a Darboux transformation. *The system* (A) *admits a quadratic integral that is distinct from that of vis viva*. The system:

$$\left[(U_1 + a_1)\frac{ds_1^2}{dt_1'^2}, Q_i = 0\right]$$

will also admit a quadratic integral. Finally, *if* (A) *possesses a potential then* (A_1) *itself will admit a quadratic integral that is distinct from that of vis viva*.

3. *The forces on* (A) *and* (A_1) *are derived from potentials U and U_1, resp., and two natural congruences $h = a$, $h_1 = a_1$ of* (A) and (A_1), *resp., coincide.* One will get back to the first case with the aid of a double Darboux transformation. *The two systems* (A) *and* (A_1) *possess a second quadratic integral, along with that of vis viva*.

4. (*General case*). *None of the preceding particular hypotheses are verified. The two systems* (A) *and* (A_1) *have quadratic integrals that are distinct from that of vis viva.* If the forces of (A_1) are derived from a potential then (A) will admit *two* distinct quadratic integrals. *There also exists a force function for* (A), *so the two systems will admit* ***three*** *distinct quadratic integrals, respectively, when one includes the vis viva.*

By definition, the first three cases come down to the case in which there is a correspondence with preservation of the geodesics. All of the results that were obtained in Chapter II then apply to those two cases. On the contrary, the fourth one is completely distinct from the one that was treated in Chapter II.

II. – COROLLARIES TO THE PRECEDING THEOREMS.

7. – I would like to complete the preceding results with some important remarks.

We have said that if one of the corresponding systems (A) and (A_1) admits a force function then the same thing will not be true for the second one, in general. However, let us accept the hypothesis in which U and U_1 exist simultaneously.

We know (*see* Chapter I, pp. 28) that every natural congruence $h = a$ of (A) (in particular, the congruence of geodesics $h = \infty$) coincides with a natural congruence $h_1 = a_1$ of (A_1). As we shall see, that transformation *is the only one that enjoys that property.* In other words, *if* (A_1) *is not an ordinary correspondent of* (A) *then an arbitrary natural congruence $h = a$ of* (A) *will not coincide with a natural congruence of* (A_1), but in fact with a congruence that is obtained by taking a ($2k$–3)-parameter congruence in each congruence $h_1 = a_1$. One can even go further and show that *there exists no natural congruence $h = a$ of* (A) *that coincides with a natural congruence of* (A_1), *in general, and there never exist more than one.*

First of all, if a congruence $h = a$ exists then one will necessarily find oneself in one of the first three cases that were listed above, and one can always place oneself in the first case, which is the one in which the two natural congruences that coincide are geodesic congruences, $h = \infty$, $h_1 = \infty$, by appealing to the Darboux transformation. I say that there cannot exist a second congruence $h = a$ that coincides with a congruence $h_1 = a_1$ unless (A_1) is not an ordinary correspondent of (A).

Indeed, if that were the case then one would need to have, at the same time:

$$\frac{dt_1}{dt} = \lambda\,(q_1, q_2, \ldots, q_k)$$

and

$$dt_1^2 - \frac{ds_1^2}{U_1 + a_1} = \mu^2\left(dt^2 - \frac{ds^2}{U + a}\right),$$

which are equalities that demand the conditions:

$$\lambda^2 \equiv \mu^2, \qquad \frac{ds_1^2}{U_1 + a_1} = \mu^2 \frac{ds^2}{U + a}$$

if they are to be compatible.

However, as we have seen, if two corresponding ds^2 have the form ds^2 and $\mu'^2\, ds^2$ then μ' will necessarily be a constant, and the two corresponding systems will be two ordinary correspondents. The proposition is thus proved.

By definition, *in the first three cases of no.* ***6****, there exists one and* ***only one*** *natural congruence $h = a$ of* (A) *that is also a natural congruence of* (A_1). *It does not exist in the general case* ([1]).

With that, we return to the problem that consists of recognizing whether $ds'^2 \equiv (U + h)\, ds^2$ admits a (non-similar) correspondent:

$$ds_1'^2 \equiv \sum A'_{ij}\,(q_1, q_2, \ldots, q_k, h)\, dq_i\, dq_j$$

for each value of h.

Is it possible that the system $\left(\dfrac{ds_1'^2}{dt_1^2}, Q'_i = 0\right)$ is attached to a system (A_1) that is independent of h in the same way that the system $\left(\dfrac{ds'^2}{dt'^2}, Q_i = 0\right)$ is attached to (A)? In other words, can $ds_1'^2$ have the form $(U_1 + h_1)\, ds_1^2$, where h_1 is a certain function of h that does not depend upon U_1, ds_1^2? That will never be true, because otherwise the systems $\left(\dfrac{ds^2}{dt^2}, U\right)$ and $\left(\dfrac{ds_1^2}{dt_1^2}, U_1\right)$ would be two non-ordinary correspondents, and *any* natural congruence of the one would be a natural congruence of

([1]) I shall insist upon this point, which derives its importance from the theory of groups of transformations of trajectories and which gave rise to a discussion between Liouville and myself. Liouville thought that he had proved that *for* $k > 2$, *every* natural congruence of (A) is *always* a natural congruence of (A_1) (see *Comptes rendus*, 31 October 1892). In reality, the preceding considerations show that this it *never* true.

the other. The search for $ds_1'^2$ is then completely distinct from the search for correspondents to (A).

8. – In this chapter, we built upon some propositions that were obtained previously, and in particular this one:

In order for two systems (A) *and* (A_1) *to be correspondents with preservation of the geodesics, it is necessary and sufficient that one can pass from* (A) *to* (A_1) *with the aid of the transformation:*

$$\frac{dt_1}{dt} = \lambda\,(q_1, q_2, \ldots, q_k)\,.$$

It is quite easy to prove that proposition by appealing to the relation that exists in any case between dt and dt_1 when (A) and (A_1) are two correspondents for which the forces are not zero. That relation can be written:

$$\frac{dt_1^2}{dq_1^2} - M\,\frac{dt^2}{dq_1^2} = \frac{d\sigma'^2}{dq_1^2}\ .$$

We know that the geodesics of ds^2 define a $(2k – 2)$-parameter congruence of trajectories of (A) that satisfies the condition that $dt / dq_1 = 0$ (or that $\frac{d^2q_2}{dq_1^2} + \Phi_1\frac{dq_2}{dq_1} - \Phi_2 = 0$). In order for the geodesics of ds^2 and ds_1^2 to coincide, it is therefore necessary and sufficient that the conditions $\frac{dt_1}{dq_1} = 0$ and $\frac{dt}{dq_1} = 0$ are equivalent. Since one has:

$$\frac{dt_1^2}{dq_1^2} - \frac{d\sigma'^2}{dq_1^2} = 0$$

for $\frac{dt}{dq_1} = 0$, it is necessary and sufficient that $\frac{d\sigma'^2}{dq_1^2}$ should be identically zero, and as a result, that $\left(\frac{dt_1}{dt}\right)^2 = M\,(q_1, q_2, \ldots, q_k)$. We will then know (*see* no. **3** of this Chapter, pp. 55) that $M \equiv C_0^2\left(\frac{\Delta_1}{\Delta}\right)^{\frac{2}{k+1}}$. Q. E. D.

That will permit us to recover the theorem that was proved in the first chapter and to which we have often had recourse: *If the systems* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(C\,\frac{ds^2}{dt_1^2}, Q_i'\right)$ *are correspondents then one* *will necessarily have* $Q'_1 = c\, Q_1, \ldots, Q'_k = c\, Q_k$. Indeed, the geodesics of ds^2 and $C\, ds^2$ will coincide, so one will have:

$$\frac{dt_1}{dt} = C_0 \left(\frac{\Delta_1}{\Delta}\right)^{\frac{1}{k+1}} = C_0\, C^{\frac{k}{k+1}},$$

and on the other hand:

$$\beta_i\, \Delta^{\frac{2}{k+1}} = C_0^2\, \beta'_i\, \Delta_1^{\frac{2}{k+1}},$$

so

$$\beta_i = \beta'_i\, C_0^2\, C^{\frac{2k}{k+1}},$$

and since:

$$\beta_i = \sum_j a_{ij}\, Q_j \quad \text{and} \qquad \beta'_i = \frac{1}{C} \sum_j a_{ij}\, Q'_j ,$$

one will find that:

$$Q'_i = \frac{1}{C_0^2\, C^{\frac{k-1}{k+1}}} Q_i , \qquad Q'_i = c\, Q_i ,$$

and if one replaces C_0 with a function of C and c in dt_1 / dt then:

$$\frac{dt_1}{dt} = \sqrt{\frac{C}{c}} .$$

Those are, in fact, the results that were obtained before.

9. – We likewise proved in Chapter II that *two systems* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(\frac{ds_1^2}{dt_1^2}, Q'_i\right)$ *cannot be correspondents for two distinct systems of associated forces, say,* Q_i *and* Q'_i *, on the other hand, and* (Q_i) *and* (Q'_i) *, on the other, without the geodesics of* ds^2 *and* ds_1^2 *coinciding* (at least, when k exceeds 2).

Here is a new proof of that theorem that gives us, at the same time, some results for $k = 2$.

Assume that the geodesics ds^2 and ds_1^2 do not coincide. For the systems $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(\frac{ds_1^2}{dt_1^2}, Q'_i\right)$, we have the relation (*see* pp. 50):

$$\text{(1)} \qquad dt^2(\beta_i\, dq_1 - \beta_1\, dq_i) - dt_1^2(\beta'_i\, dq_1 - \beta'_1\, dq_i) = (\Pi_1 - \Pi'_1)\, dq_i - (\Pi_i - \Pi'_i)\, dq_1 \equiv S_i ,$$

in which the Π are quadratic forms in dq_1, dq_2, …, dq_k that depend upon only ds^2 and ds_1^2. We know, moreover, that one necessarily has:

$$\frac{\beta_i}{\beta_i'} = \frac{\beta_1}{\beta_1'},$$

that the right-hand side S_i of (1) is divisible by:

$$(\beta_i\, dq_1 - \beta_1\, dq_i),$$

and that the quotient, namely $d\sigma^2$, is the same for all i. Similarly, one has from $\left[\frac{ds^2}{dt^2}, (Q_i)\right]$ and $\left[\frac{ds_1^2}{dt_1^2}, (Q_i')\right]$ that:

$$dt^2(\gamma_i\, dq_1 - \gamma_1\, dq_i) - dt_1^2(\gamma_i'\, dq_1 - \gamma_1'\, dq_i) = (\Pi_1 - \Pi_1')\, dq_i - (\Pi_i - \Pi_i')\, dq_1,$$

and analogous remarks will apply to that equality.

Having recalled that, I shall now say that *if k exceeds* 2 then one will necessarily have:

$$\frac{\beta_1}{\gamma_1} = \frac{\beta_i}{\gamma_i} \qquad (i = 1, 2, \ldots, k).$$

Indeed, let $\frac{\beta_1}{\gamma_1} \neq \frac{\beta_2}{\gamma_2}$. At least one of those ratios will be different from $\frac{\beta_3}{\gamma_3}$, say, $\frac{\beta_1}{\gamma_1}$. The binomial $(\gamma_2\, dq_1 - \gamma_1\, dq_2)$ divides S_2 [viz., the right-hand side of (1) for $i = 2$], and since it is prime for $(\beta_2\, dq_1 - \beta_1\, dq_2)$, it must divide $d\sigma^2$. For the same reason, $(\gamma_3\, dq_1 - \gamma_1\, dq_3)$ also divides $d\sigma^2$. One must then have:

$$S_2 = M(q_1, \ldots, q_k)\,(\beta_2\, dq_1 - \beta_1\, dq_2)\,(\gamma_2\, dq_1 - \gamma_1\, dq_2)\,(\gamma_3\, dq_1 - \gamma_1\, dq_3),$$

and similarly:

$$S_2 = N(q_1, \ldots, q_k)\,(\beta_2\, dq_1 - \beta_1\, dq_2)\,(\gamma_2\, dq_1 - \gamma_1\, dq_2)\,(\gamma_3\, dq_1 - \gamma_1\, dq_3).$$

That double equality is possible only if $\frac{\beta_3}{\beta_1} = \frac{\gamma_3}{\gamma_1}$, which is contrary to the hypothesis. One will then have:

$$\frac{\beta_1}{\gamma_1} \equiv \frac{\beta_2}{\gamma_2} \equiv \ldots \equiv \frac{\beta_k}{\gamma_k}.$$

On the other hand, the expression:

$$\left(\frac{\Delta}{\Delta_1}\frac{\beta_1'}{\beta_1}\right)^{\frac{2}{k+3}}\left(\frac{dt_1}{dt}\right)^2$$

is a first integral of (A), so the expression:

$$\left(\frac{\Delta}{\Delta_1}\frac{\beta_1'}{\beta_1}\right)^{\frac{2}{k+3}}\frac{S_2}{\beta_1\left(\dfrac{\beta_2}{\beta_1}dq_1 - dq_2\right)} \equiv \left(\frac{\Delta}{\Delta_1}\frac{\beta_1'}{\beta_1}\right)^{\frac{2}{k+3}}\frac{d\sigma'^2}{\beta_1}$$

is an integral of the system $\left(\dfrac{ds^2}{dt^2}, Q_i = 0\right)$. The same thing is true for the expression:

$$\left(\frac{\Delta}{\Delta_1}\frac{\gamma_1'}{\gamma_1}\right)^{\frac{2}{k+3}}\frac{d\sigma'^2}{\gamma_1}.$$

That is possible only if one has:

$$\frac{1}{\beta_1'}\left(\frac{\beta_1}{\beta_1'}\right)^{\frac{2}{k+3}} \equiv \frac{C}{\gamma_1}\left(\frac{\gamma_1'}{\gamma_1}\right)^{\frac{2}{k+3}}.$$

Upon permuting the systems (A) and (A_1), one will likewise find that:

$$\frac{1}{\beta_1'}\left(\frac{\beta_1}{\beta_1'}\right)^{\frac{2}{k+3}} \equiv \frac{C'}{\gamma_1'}\left(\frac{\gamma_1}{\gamma_1'}\right)^{\frac{2}{k+3}},$$

and one infers from those two equalities that:

$$\gamma_1 = c\,\beta_1, \qquad \gamma_1' = c'\,\beta_1'.$$

Finally, since:

$$\beta_i = \sum_j a_{ij}\, Q_j \qquad \text{and} \qquad \gamma_i = \sum_j a_{ij}\,(Q_j),$$

one will have by definition:

$$\frac{(Q_1)}{Q_1} = \frac{(Q_2)}{Q_2} = \ldots = \frac{(Q_k)}{Q_k} = c, \qquad \frac{(Q_1')}{Q_1'} = \frac{(Q_2')}{Q_2'} = \ldots = \frac{(Q_k')}{Q_k'} = c'.$$

The systems of forces (Q_i) and (Q_i') are not *distinct* from the systems Q_i and Q_i'. Q.E.D.

The last part of the proof persists when $k = 2$. As a result, when $\frac{\beta_1}{\gamma_1} = \frac{\beta_2}{\gamma_2}$ the common value of those two ratios is necessarily a constant. However, one can no longer establish, as before, that those ratios are identical. The argument that was employed shows only that:

$$S_2 \equiv M\,(E\,dq_1 + F\,dq_2)\,(\beta_2\,dq_1 - \beta_1\,dq_2)\,(\gamma_2\,dq_1 - \gamma_1\,dq_2)\,.$$

Thus, the system $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ *necessarily admits two* ***distinct*** *quadratic integrals, in addition to that of vis viva*, namely, two integrals of the form:

$$\lambda\,(E\,dq_1 + F\,dq_2)\,(\gamma_1\,dq_2 - \gamma_2\,dq_1) = C\,dt^2$$

and

$$\mu\,(E\,dq_1 + F\,dq_2)\,(\beta_1\,dq_2 - \beta_2\,dq_1) = C'\,dt^2\,.$$

Similarly, the system $\left(\frac{ds_1^2}{dt_1^2}, Q_i' = 0\right)$ admits two integrals of the same form. It follows from this that the systems $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ and $\left(\frac{ds_1^2}{dt_1^2}, Q_i' = 0\right)$ cannot correspond for more than *three* systems of distinct forces. If three such systems exist, say, Q_i and Q_i', (Q_i) and (Q_i'), $[(Q_i)]$ and $[(Q_i')]$, then the systems $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ and $\left(\frac{ds_1^2}{dt_1^2}, Q_i' = 0\right)$ will admit three distinct linear integrals, respectively, *and as a result, ds^2 and ds_1^2 will be the ds^2 for a surface of constant curvature.* I shall conclude my discussion of the particular case of two parameters here, since its study presents no further difficulties and will be developed completely in another article.

10. – As the last corollary to the general theorem, we shall finally prove this important proposition:

Let $\left[\frac{ds^2}{dt^2}, Q_i\right]$ *and* $\left[\frac{ds_1^2}{dt_1^2}, Q_i'\right]$ *be two non-ordinary correspondents. It is impossible for the condition* $ds_1^2 \equiv \mu\,(q_1, \ldots, q_k)\,ds^2$ *to be fulfilled.*

First of all, observe that this condition is realized for the ordinary correspondents because one will then have either $ds_1^2 \equiv C\,ds^2$ or $ds_1^2 \equiv C\,(U + h)\,ds^2$, in which U denotes the potential for Q_i. The theorem says that those correspondents are the only ones that enjoy that property.

That will be obvious when the geodesics coincide, as we have remarked before, because $\mu(q_1, q_2, \ldots, q_k)$ = const. must be an integral of the geodesics, and as a result, will reduce to a constant. Here is a proof that embraces all of the other cases.

Assume that ds_1^2 is equal to $\mu\, ds^2$, and write out one of equations (A) and one of equations (A_1):

$$(A)\qquad \sum_{j=1}^{k} A_{ij}\frac{d^2 q_j}{dt^2} + N_i = Q_i \qquad (i = 1, 2, \ldots, k)$$

and

$$(A_1)\qquad \sum_{j=1}^{k} A_{ij}\frac{d^2 q_j}{dt^2} + N_i + \frac{\partial (T)}{\partial (q_i')}\frac{d\nu}{dt_1} - (T)\frac{\partial \nu}{\partial q_i} = \frac{Q_i'}{\mu}\ .$$

ν represents log μ, and (T) is what T will become when one replaces q_i' with $(q_i') \equiv dq_i / dt_1$. It follows from this that if one lets a_{ij} denote the minor of Δ relative to the element A_{ij} and divides by Δ then one will have:

$$\frac{d^2 q_j}{dt_1^2} = (P_i) - \frac{d\nu}{dt_1}\sum_{j=1}^{k} a_{ij}\frac{\partial (T)}{\partial (q_i')} + (T)\sum_{j=1}^{k} a_{ij}\frac{\partial \nu}{\partial q_i} + \beta_i'\ .$$

However, from the equalities $p_i = \partial T / \partial q_i'$, one will have precisely:

$$q_i' = \sum_j a_{ij}\, p_j\ , \qquad \text{thus} \qquad q_i' \equiv \sum_j a_{ij}\frac{\partial T}{\partial q_j'}\ ,$$

in such a way that upon setting $B_i = \sum_j a_{ij}\dfrac{\partial \nu}{\partial q_j}$, one can write:

$$\frac{d^2 q_j}{dt_1^2} = (P_i) - (q_i')\frac{d\nu}{dt_1} + (T)\, B_i + \beta_i'\ .$$

The relation between dt and dt_1 here is then:

$$dt^2(\beta_2\, dq_1 - \beta_1\, dq_2) - dt_1^2(\beta_2'\, dq_1 - \beta_1'\, dq_2)$$

$$= (\Pi_1 - \Pi_1 + d\nu\, dq_1 - B_1\, ds^2)\, dq_2 - (\Pi_2 - \Pi_2 + d\nu\, dq_2 - B_2\, ds^2)\, dq_1$$
$$= ds^2\,(B_2\, dq_1 - B_1\, dq_2)\ .$$

The binomial ($\beta_2\, dq_1 - \beta_1\, dq_2$) must divide the right-hand side ([1]). It cannot divide ds^2 (when k is greater than 2) because its discriminant is not zero. The relation between dt_1 and dt then has the form:

$$dt_1^2 = M\,(ds^2 - V\,dt^2)\,,$$

which characterizes hypothesis II, in which the geodesics of $(U + a)\, ds^2$ and those of ds_1^2 coincide, where U denotes the force function of (A), which necessarily exists then, and a is a certain finite constant.

Upon permuting the two systems, one will likewise see that the geodesics of ds^2 coincide with the geodesics of $(U_1 + a_1)\, ds_1^2$, where U_1 denote the force function of (A), which necessarily exists, and a_1 is a certain finite number. That double circumstance (from the theorem of no. **7**) can present itself only under the Darboux transformation. Q.E.D.

For the two-parameter systems, it is not impossible that $\beta_2\, dq_1 - \beta_1\, dq_2$ might divide ds^2. I observe only that the ds^2 will then be necessarily the ds^2 of an imaginary surface. One sees immediately that ds^2 (and ds_1^2) are two of Lie's ds^2, because if one reduces those ds^2 to the form $\lambda\, dq_1\, dq_2$ then the system will possess a quadratic integral such as $dq_1\,(m\, dq_1 + n\, dq_2) = C\,dt^2$, which characterize Lie's ds^2. I shall confine myself to those indications in the case of two parameters, which is very easy to treat directly and which I shall discuss in a later article.

III. – SUFFICIENT CONDITIONS FOR A SYSTEM (A) TO ADMIT CORRESPONDENTS. GENERAL EQUATIONS FROM THE CALCULUS OF VARIATIONS.

11. – From the preceding, when a system (A) or $\left[\dfrac{ds^2}{dt^2}, Q_i\right]$ possesses a non-ordinary correspondent $\left[\dfrac{ds_1^2}{dt_1^2}, Q_i'\right]$, the two systems $\left[\dfrac{ds^2}{dt^2}, Q_i = 0\right]$ and $\left[\dfrac{ds_1^2}{dt_1^2}, Q_i' = 0\right]$, respectively, admit a quadratic integral. Nonetheless, in certain cases, one must substitute the system $\left[(U+a)\dfrac{ds^2}{dt'^2}, Q_i = 0\right]$ in the statement of one of those systems – the first one, for example. The existence of a quadratic integral for $\left[\dfrac{ds^2}{dt^2}, Q_i = 0\right]$, or even $\left[\dfrac{ds^2}{dt^2}, Q_i\right]$, is not a *sufficient* condition for (A) to possess a correspondent, moreover. That is why the system (A), which Jacobi encountered, in which ds^2 is equal to $\sum_{i=1}^{k} \dfrac{F'(q_i)}{f(q_i)}\, dq_i^2$ (*see* Chap. II, pp. 45) and $U = \sum \dfrac{\psi_i(q_i)}{F'(q_i)}$,

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([1]) If the right-hand side is identically zero then one will have $dt_1^2 = \lambda^2\, dt^2$, and the geodesics will coincide.

admits a complete system of quadratic integrals without possessing non-ordinary correspondents, in general.

In order to form sufficient conditions, one procedure consists of expressing the idea that one can pass from (A) to (A_1) by a transformation of the form:

$$dt_1^2 = d\sigma^2 - w\,dt^2 .$$

Those conditions take a form that is much more complicated than in the case of zero forces. However, one can simplify them considerably by *immediately* taking into account the necessary conditions that are already known:

1. $\dfrac{\beta_1'}{\beta_1} \equiv (\Pi_1 - \Pi_1')\,dq_i$.

2. The expression $(\Pi_1 - \Pi_1')\,dq_i - (\Pi_i - \Pi_i')\,dq_1$ is divisible by $\dfrac{\beta_i}{\beta_1}dq_1 - dq_i$, and the quotient is independent of i .

3. $\left(\dfrac{\Delta}{\Delta_1}\dfrac{\beta_1'}{\beta_1}\right)^{\frac{2}{k+3}}\left[\dfrac{-d\sigma^2}{\beta_1'} + \dfrac{\beta_1}{\beta_1'}dt^2\right]$ is an integral of (A), and $\left(\dfrac{\Delta_1}{\Delta}\dfrac{\beta_1}{\beta_1'}\right)^{\frac{2}{k+3}}\left[\dfrac{d\sigma^2}{\beta_1} + \dfrac{\beta_1'}{\beta_1}dt_1^2\right]$ is an integral of (A_1).

In order for there to be a correspondence that preserves geodesics, it is necessary that $d\sigma^2 \equiv 0$, i.e., that one must have:

$$(\Pi_1 - \Pi_1')\,dq_i - (\Pi_i - \Pi_i')\,dq_1 \equiv 0 ,$$

which are new conditions that must be added to the preceding ones. However, one can demand to know whether *those conditions are not necessarily consequences of the former ones*. In other words, whether the correspondence with preservation of geodesics (or at least, of a natural congruence) is not *the only one possible*. That is not the case, and in order to assure oneself of that by a simple means, one can construct an example. I shall cite only the following one: The two systems:

$$[T, U] \quad \text{and} \quad [T_1, U_1],$$

in which:

$$T = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \qquad U = g\,z ,$$

and in which:

$$T_1 \equiv \frac{1}{x^4}\left[\left(\frac{dx}{dt}\right)^2\left(1+y^2+\frac{4z^2}{x^2}\right)+2\frac{dx}{dt_1}\frac{dy}{dt_1}x\,y-4\frac{dx}{dt_1}\frac{dy}{dt_1}\frac{z}{x}+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right],$$

$$U_1 = \frac{g\,z}{x^2}$$

are correspondents. Their trajectories are parabolas with their axes parallel to Oz. No natural congruence $h = a$ of (A) is a natural congruence $h_1 = a_1$ of the second system. Observe that those two systems are, at the same time, *homologous*. One passes from one to the other by changing x into $1 / x$, y into y / x, z into z / x^2, and setting $t = t_1$. That change of variables will transform parabolic trajectories into themselves.

If one then forms the *sufficient* conditions that I just indicated for (A) to admit a correspondent, the systems that correspond to those conditions will include the ones that satisfy the conditions that the geodesics (or natural congruence) are preserved, moreover, as *special systems*. On the other hand, it is very easy to find correspondent ds^2 as a result of the systems (A) that admit correspondents that have the same geodesics. Finally, one effortlessly deduces correspondents that possess a common natural congruence from those systems. The four cases that we have listed in no. **6** can indeed present themselves then.

12. – I shall return to the sufficient conditions in question elsewhere. I shall conclude these generalities by remarking that they can be extended to *arbitrary equations that are provided by the calculus of variations.* Consider a function:

$$f\,(q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k')$$

that is subject to only the condition that its Hessian relative to the q_i' is not zero, and write the equations:

$$(\alpha) \qquad \frac{d}{dt}\frac{\partial f}{\partial q_i'} - \frac{\partial f}{\partial q_i} = Q_i\,(q_1, q_2, \ldots, q_k)\,, \qquad \frac{dq_i}{dt} = q_i' \qquad (i = 1, 2, \ldots, k).$$

If a second analogous system (α_1) defines the same trajectories then one can pass from (α_1) to (α) by a transformation:

$$\frac{dt_1}{dt} = \varphi(q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'),$$

so one can deduce a first integral of (α) [and (α_1)], in general. In the case where that integral reduces to an absolute constant, one will, in general, know an integral of the system (α) when one has annulled the Q_i. When f is homogeneous in q_1', q_2', …, q_k', and of degree m ($m \neq 0$ and 1), the analogy with the Lagrange equations will be almost complete: It is then appropriate to further distinguish two cases according to whether all of the Q_i, Q_i' are or are not zero. The latter case decomposes into four other ones, according to the classification of no. **6**.

IV. – GENERAL CONSEQUENCES AND PARTICULAR APPLICATIONS OF THE PRECEDING THEOREMS.

13. – I shall confine myself here to briefly indicating some of the most important consequences of the theorems that I just established, and also some applications that will show the ease by which they follow from those generalities.

First of all, we know that a non-ordinary correspondent (A_1) to (A) is attached to a certain quadratic integral of either (A) or one of the systems $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ or $\left[(U+a)\,\frac{ds^2}{dt^2}, Q_i = 0\right]$. Once that integral is known, the calculation of the correspondent (A_1) will offer no further difficulty. Now, under the most favorable hypothesis, the determination of the quadratic integrals of (A) will depend upon a complete linear differential system. *The determination of the correspondents of a given system* (A) *will never require the integration of a linear equation then*.

In particular, if one would like to solve the problems I and II in the Introduction then one must distinguish those two of the ds_1^2 that are *homologues* of ds^2 and calculate the transformations of passage from ds^2 to ds_1^2. *That search introduces only linear equations, moreover.* In particular, *the calculation of the transformations* $q_i = \varphi_i\,(r_1, \ldots, r_k)$ *that present the trajectories of a given system will never demand the integration of a complete linear system.*

14. – We shall now insist upon the particular problem that was stated at the beginning of the Introduction: *If a system* (A) *is given then does there exist a system* (A_1) *that defines the same* ***motion****?* In order for that to be true, it is necessary and sufficient that there should exist a relation of the form $dt_1 / dt = 0$ between dt and dt_1. If the forces Q_i, Q_i' are zero then in order for the two motions to coincide, it is therefore necessary and sufficient that (A) and (A_1) should be two correspondents for which $\Delta \equiv C\,\Delta_1$, where C is a constant. If the forces Q_i, Q_i' are not all zero then it is necessary and sufficient that (A) and (A_1) should be two correspondents whose geodesics coincide and are such that $\Delta \equiv C\,\Delta_1$, moreover. One will then have $\beta_i = \beta_i'$ and $dt_1 / dt = 1$ for one of the systems $[T_1, c\,Q_i']$.

It follows from this that in order to find all of the desired systems (A_1), *one must determine all of the* ds^2, *say* dS^2, *that have the same geodesics and the same discriminant as* ds^2. *Any system of forces* Q_i *will correspond to a system of forces* Q_i' *(and only one) such that the two motions that are defined by* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *and* $\left(C\,\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ *will coincide.* (C is an arbitrarily-chosen number.)

One easily sees that the ds^2 of the surfaces of constant curvature will admit such correspondents ds_1^2, and that the same thing will be true for the ds^2 of the surface of constant curvature in $(k + 1)$-dimensional space. For $k = 2$, there exist no other ds^2 that enjoy that property, but for $k > 2$, that is no longer true. Liouville ([1]) has determined *all of the ds^2 in three variables such that the motion that is defined by* $\left(\frac{ds^2}{dt^2}, Q_i = 0\right)$ *can also be defined by a second system* $\left(\frac{ds_1^2}{dt_1^2}, Q_i' = 0\right)$ *and are such that the discriminants Δ and Δ_1 of ds^2* and ds_1^2*, resp., are identical, moreover.* From the preceding, the latter condition is useless and will revert to the former. The ds^2 that Liouville determined thus constitute all of the ds^2 in three variables such that the *motion* that is defined by a system $\left(\frac{ds^2}{dt^2}, Q_i\right)$ can also be defined by another system $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$ that is distinct from the first one.

15. – We also return to problem II in the Introduction: "If one is given the system (A) or $\left(\frac{ds^2}{dt^2}, Q_i\right)$ then determine the forces R_i' such that if one substitutes them for the Q_i, the new trajectories will be deduced from the first one by changing q_i into φ_i $(q_1, q_2, \dots, q_k)$." The trajectories of $\left(\frac{ds^2}{dt^2}, Q_i\right)$ and $\left(\frac{ds^2}{dt_1^2}, R_i'\right)$ include a common congruence, namely, the geodesics of ds^2. Two cases must be distinguished according to whether that congruence does or does not transform into itself when one changes q_i into φ_i $(q_1, q_2, \dots, q_k)$. We shall deal with only the former case.

The transformation $q_i = \varphi_i$ and its inverse will then replace ds^2 with two homologous ds^2, say ds'^2 and ds_1^2 ([2]), whose geodesics coincide with those of ds^2. As a result, any system $\left(\frac{ds^2}{dt^2}, Q_i\right)$ will possess a correspondent of the form $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$. The homologue that is deduced from that correspondent by changing the q_i into φ_i $(q_1, q_2, \dots, q_k)$ has the form $\left[\frac{ds^2}{dt_1^2}, R_i'\right]$, and the forces R_i' will be appropriate to the problem.

([1]) See the *Comptes rendus*, April 1891.

([2]) ds_1^2 can coincide with ds^2, moreover.

Therefore, if the geodesics of ds^2 *admit a transformation* $q_i = \varphi_i$ *into themselves then any system of forces* Q_i *will correspond to forces* R_i' *such that the trajectories that are defined by* $\left(\frac{ds^2}{dt_1^2}, R_i'\right)$ *are deduced from the trajectories that are defined by* $\left(\frac{ds^2}{dt^2}, Q_i\right)$ *by changing* q_i *into* φ_i $(q_1, q_2, \ldots, q_k)$.

16. – Regardless of whether the geodesics are or are not preserved, is it possible that the transformation $q_i = \varphi_i$ might be *conformal*, I would like to say that it changes ds^2 into $\mu\,(q_1, q_2, \ldots, q_k)\, ds^2$? If that were true then the inverse of that transformation would replace the system $\left(\frac{ds^2}{dt_1^2}, R_i'\right)$ with a correspondent to *A*, say $\left(\frac{ds_1^2}{dt_1^2}, Q_i'\right)$, in which $ds_1^2 \equiv \mu\, ds^2$. It would then be necessary that one should have either $ds_1^2 = C\, ds^2$ and $Q_i' = c\, Q_i$ or rather $ds_1^2 = (a\, U + b)\, ds^2$ and $U_1 = C / (a\, U + b)$. One thus determines all of the ds^2 from the form $C\, ds^2$ or the form $(a\,U + b)\, ds^2$ (when U exists), which are *homologues* of ds^2, and all of the transformations of passage $q_i = \varphi_i$. If one takes $R_i' = C \sum_{j=1}^{k} \frac{\partial \varphi_j}{\partial q_i} Q_j(\varphi_1, \ldots, \varphi_k)$ for the first case, and if one changes the q_i into φ_i $(q_1, q_2, \ldots, q_k)$ in $C / (a\, U + \beta)$ for the second one then one will get forces R_i' or a force function U' such that the trajectories of $\left(\frac{ds^2}{dt_1^2}, R_i'\right)$ or $\left(\frac{ds^2}{dt_1^2}, U'\right)$ are deduced from those of the first by a conformal transformation.

For example, let $ds^2 = dx^2 + dy^2$. In addition to the common transformation that is defined by the equality $x + i\, y = (A + i\, B)\,(x_1 \pm i\, y_1) + C + i\, D$, there will exist other conformal transformations that make the trajectories of $\left(\frac{ds^2}{dt^2}, Q_i\right)$ pass to the trajectories of $\left(\frac{ds^2}{dt_1^2}, R_i'\right)$ if U exists and satisfies the condition $\frac{\partial^2 \log U}{\partial x^2} + \frac{\partial^2 \log U}{\partial y^2} \equiv 0$. Then set:

$$- \log \tfrac{1}{2} U + i\, V = \log \frac{df_1}{dz} + A + i\, B\,,$$

in which f_1 represents an analytic function of $z = x + i\, y$. The equality $z_1 = f_1\,(z)$ defines two functions $x = \varphi\,(x_1, y_1)$, $y = \psi\,(x_1, y_1)$. Let U' be what the expression C / U will become when one changes x and y in it into $\varphi\,(x, y)$ and $\psi\,(x, y)$, resp. The trajectories of the system $\left(\frac{ds^2}{dt_1^2}, U'\right)$ are deduced from those of the system by changing x and y into $\varphi(x, y)$ and $\psi(x, y)$, resp. That is a theorem of Goursat (1).

17. – An application of some of the preceding remarks will allow one to effortlessly recover all of the results that were obtained by the various authors that dealt with the particular corresponding systems. For example, the geodesics of $ds^2 \equiv dx^2 + dy^2$ admit the group of *homographic transformations* in two variables. From the theorem in no. **15**, each system of forces Q_i will correspond to forces R_i' such that the trajectories of $\left(\frac{ds^2}{dt^2}, Q_i\right)$ and $\left(\frac{ds^2}{dt_1^2}, R_i'\right)$ are deduced from each other by an arbitrary homographic transformation that was given in advance. The formulas of passage result immediately from the formulas that were established in Chapter II (pp. 40) on the correspondences that preserve geodesics. One will then recover the well-known results of Appell.

It is clear that the same conclusions will apply to the ds^2 in two variables whose geodesics admit the most general homographic transformation. What are those ds^2 ? Since their geodesics are straight lines $A\,q_1 + B\,q_2 + C = 0$, those ds^2 will be correspondents to $dx^2 + dy^2$, and as a result will possess three infinitesimal transformations into themselves (*see* Chap. III, Section V, pp. 61): From a theorem of Lie, those are the ds^2 of surfaces of constant curvature. On the other hand, one sees immediately on the sphere and the pseudosphere that any surface of constant curvature is geodesically representable on the plane. It results from that single remark (Chap. II, Section VI, pp. 39) that *any motion on a surface of constant curvature corresponds to a planar motion*, and that the surfaces of constant curvature are the only ones that enjoy that property. One knows of the work of Paul Serret, Appell, and Dautheville on that question.

All of those remarks can be repeated without modification for the ds^2 that have the form $dx_1^2 + \cdots + dx_k^2$ and the ds^2 of the surfaces of constant curvature in $(k + 1)$-dimensional spaces.

However, the generalities that were developed in this article include many other applications that are entirely new. That is how they permit one to completely explain the question of correspondents *for* $k = 2$ by forming all of the types [with the help of some recent work of Kœnigs (2)], and as a result, to determine the groups of transformations of the trajectories. The latter study can be performed by a direct procedure, moreover. In a later article, at the same time that I will return to the sufficient conditions for there to exist correspondents, I will treat the most important applications in detail, and especially the ones that are concerned with systems with two and three parameters.

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(1) See the *Comptes rendus* (April 1889) and the note by Darboux that followed that of Goursat.
(2) See the *Annales de la Faculté des Sciences de Toulouse* (1893).