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Dynamical trajectories of holonomic systems with three degrees of freedom

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In one of his valuable works ([1]) in holonomic systems with two degrees of freedom, Birkhoff posed a theorem that is very expressive for the dynamical trajectories of such systems in the case in which the corresponding Lagrangian function also contains terms that are linear in the components of the velocity (although they are explicitly independent of time). The problem in that case is called *irreversible*.

When I learned about that theorem, it compelled me to recall an analogy with holonomic systems with three degrees of freedom that are always subject to the hypothesis of the irreversibility of the motion. In this note, I propose to communicate the result to which I arrived.

In no. **1**, I shall pose the comprehensive equation for the trajectories for the motion in question. We shall arrive at the equation in a simpler way thanks to a very elegant criterion that was proposed by Levi-Civita ([2]) that will allow us to pass (in the ordinary mechanics of holonomic systems that move under the action of conservative forces) from Hamilton's principle to that of least action.

In no. **2**, we shall move on to the dynamical interpretation of that principle.

1. The principle of least action for dynamical systems with three degrees of freedom. – Let S be a dynamical system with three degrees of freedom. Let q_1, q_2, q_3 denote the coordinates of any one of its points, and let t denote time. The equations of motion are known to be included in the variational principle:

$$\delta \int L\,dt = 0\,, \tag{1}$$

in which L indicates the Lagrangian function of the system.

([1]) **G. Birkhoff**, "Dynamical systems with two degrees of freedom," Trans. Amer. Math. Soc. **18** (1917), 199-300.

([2]) **T. Levi-Civita**, these Rendiconti, **26** (1917), 458-470.

L is a quadratic function in the derivatives $\dot{q}$ of the q with respect to time. We suppose that L also contains terms that are linear in the $\dot{q}$, or that the constraints that the point of the system are subject to also depend upon time, as when, e.g., they are referred to uniformly-rotating axes. L will then have the form:

$$L = \tfrac{1}{2}\sum_{r,s=1}^{3} a_{rs}\,\dot{q}_r\,\dot{q}_s + \sum_{r=1}^{3} b_r\,\dot{q}_r + c \qquad (a_{rs} = a_{sr}). \tag{2}$$

The equations of motion that are produced by L obviously do not prove to be invariant under the change from t into $-t$, as they would be if they were produced by an ordinary Lagrangian function that does not contain linear terms. In the ordinary case, the problem of motion is called *reversible* and *irreversible* in the contrary case. The problem that we would like to treat is therefore the irreversible one: For linguistic convenience, we shall call the system itself irreversible.

We then suppose that the first term on the right-hand side of (2) constitutes a positive-definite quadratic form and that the various coefficients of L, i.e., a_{rs}, b_r, and c, are functions of q_1, q_2, q_3, but are explicitly independent of time.

By virtue of the last hypotheses, the Lagrange equation of motion [which are included in (1)] will admit the known integral:

$$\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = h\,. \tag{3}$$

The existence of (3) will permit one to pass from (1) to the *principle of least action* for our problem as a result of Levi-Civita's criterion that was pointed out above and which consists of eliminating time from the equation:

$$\delta\int (L+h)\,dt = 0 \tag{4}$$

[which is essentially equivalent to (1)] by means of (3).

The material operations then follow with great ease. With the special form of the Lagrangian function, (3) will take on the appearance of:

$$\tfrac{1}{2}\sum_{r,s=1}^{3} a_{rs}\,\dot{q}_r\,\dot{q}_s - c = h\,, \tag{5}$$

from which, if one sets:

$$dl^2 = \sum_{r,s=1}^{3} a_{rs}\,dq_r\,dq_s$$

then one will get:

$$\frac{dl^2}{dt^2} = 2\,(c+h) \qquad \text{and therefore} \quad dt = \frac{dl}{\sqrt{2\,(c+h)}}\,. \tag{6}$$

On the other hand, due to (5) and the convention:

$$\sum_{r=1}^{3} b_r \frac{dq_r}{dl} = M$$

one will have:

$$L + h = M \frac{dl}{dt} + 2\,(c + h) .$$

If we substitute that in (4) then by virtue of (6), we will get:

$$\delta \int \{M + \sqrt{2(c+h)}\}\, dl = 0 , \tag{7}$$

which constitutes the equation that we seek.

We then conclude that:

The dynamical trajectories of the points of an irreversible system with three degrees of freedom are characterized by the variational principle (7)*, which corresponds to the ordinary principle of least action for reversible systems.*

2. Dynamical interpretation. – In order to now move on to the interpretation of the variational principle that was deduced in the preceding section, consider a particular system S^* with three degrees of freedom that is composed of three isolated points P_1, P_2, P_3 *that move in a conservative field* whose force function will be denoted by U.

Let Ω, ξ, η, ζ be a system of orthogonal Cartesian coordinates that is fixed in space. Let Ω, x, y, z be a second system whose origin and z-axis coincide with the origin and ζ-axis of the previous system. Therefore, suppose that the system Let Ω, x, y, z rotates uniformly around the z-axis.

Let (x_i, y_i, z_i) denote the coordinates of the P_i with respect to the rotating system, and let q_1, q_2, q_3 denote the Lagrangian parameters of the system S^* that is composed of three points P_1, P_2, P_3. One will have:

$$x_i = x_i\,(q_1, q_2, q_3)\,, \qquad y_i = y_i\,(q_1, q_2, q_3)\,, \qquad z_i = z_i\,(q_1, q_2, q_3) \qquad (i = 1, 2, 3). \tag{8}$$

The constraints that realize the system prove to be independent of time in that way, but only when referred to the rotating axes. In general, they will also depend upon t with respect to the fixed axes.

Let V_i denote the absolute velocity of the point P_1, and let T denote the *vis viva* of the system when expressed in terms of the coordinates ξ, η, ζ. In addition, suppose that the masses of the points are equal to each other and equal to unity, so one will have:

$$T = \tfrac{1}{2}\sum_{i=1}^{3} V_i^2 .$$

Let ω denote the (constant) angular velocity of the system Ω, x, y, z, and let v_i denote the velocity of P_i with respect to that system, so one will have:

$$V_i = v_i + \omega \wedge (P_i - \Omega) ,$$

from which:

$$V_i^2 = v_i^2 + 2\omega(x_i\,\dot{y}_i - y_i\,\dot{x}_i) + \omega^2(x_i^2 + y_i^2) .$$

The *vis viva* $\mathcal{T}$ of the system S^* in coordinates x, y, z will then be:

$$\mathcal{T} = \tfrac{1}{2}\sum_{i=1}^{3}\left\{v_i^2 + 2\omega\,(x_i\,\dot{y}_i - y_i\,\dot{x}_i) + \omega^2\,(x_i^2 + y_i^2)\right\} .$$

Having thus determined the *vis viva* $\mathcal{T}$, one can easily write down the equations of motion of our system S^*. They are the Lagrange equations that are produced by the Lagrangian function:

$$L^* = \mathcal{T} + U.$$

Now express L^* as a function of the Lagrangian parameters q_1, q_2, q_3. From (8), one has:

$$\dot{x}_i = \sum_{r=1}^{3}\frac{\partial x_i}{\partial q_r}\,\dot{q}_r ,$$

with two analogous ones for $\dot{y}_i$ and $\dot{z}_i$, for which one sets:

$$(9) \qquad A_{rs} = A_{sr} = \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial q_r}\frac{\partial x_i}{\partial q_s} + \frac{\partial y_i}{\partial q_r}\frac{\partial y_i}{\partial q_s} + \frac{\partial z_i}{\partial q_r}\frac{\partial z_i}{\partial q_s}\right), \qquad B_r = \sum_{i=1}^{3}\left(x_i\,\frac{\partial y_i}{\partial q_r} - y_i\,\frac{\partial x_i}{\partial q_r}\right),$$

and one will then find that:

$$\sum_{i=1}^{3} v_i^2 = \sum_{r,s=1}^{3} A_{rs}\,\dot{q}_r\,\dot{q}_s , \qquad \sum_{i=1}^{3}(x_i\,\dot{y}_i - y_i\,\dot{x}_i) = \sum_{r=1}^{3} B_r\,\dot{q}_r .$$

Now assume, for simplicity, that $\omega = 1$ and further set:

$$\tfrac{1}{2}\sum_{i=1}^{3}(x_i^2 + y_i^2) + U = C .$$

The function L^*, when expressed in terms of q_1, q_2, q_3, will become:

$$L^* = \tfrac{1}{2}\sum_{r,s=1}^{3} A_{rs}\,\dot{q}_r\,\dot{q}_s + \sum_{r=1}^{3} B_r\,\dot{q}_r + C . \tag{10}$$

Having assumed all of that, we can now include the equations of motion of the system L^* in the variational equation:

$$\delta\int L^*\,dt = 0 ,$$

or also, if it would be convenient, in the equation:

$$\delta\int\left\{L^* + \sum_{r=1}^{3}\frac{\partial\vartheta}{\partial q_r}\,\dot{q}_r\right\}dt = 0 , \tag{11}$$

in which L^* has the expression (10) and ϑ denotes a function of q_1, q_2, q_3 that is arbitrary *a priori*. Obviously, the added term is an exact differential, so it will make no contribution to the integral.

The dynamical interpretation of the problem of the motion of the irreversible systems with three degrees of freedom that we aspire to is obtained by attempting to make the equations that are provided by (11) coincide with the ones that are provided by (1).

That coincidence will be obtained immediately when one succeeds in satisfying the equation:

$$L = L^* + \sum_{r=1}^{3}\frac{\partial\vartheta}{\partial q_r}\,\dot{q}_r ,$$

and when one recalls the expressions that are given to the two functions L and L^*, they will split into:

$$\left\{\begin{aligned} A_{rs} &= a_{rs}, \\ B_r + \frac{\partial\vartheta}{\partial q_r} &= b_r, \qquad (r, s = 1, 2, 3). \\ C &= c. \end{aligned}\right. \tag{12}$$

If one observes the explicit expressions A_{rs}, B_r, and C then one will easily see that U is contained in only $C = c$, which is, on the other hand, expressed in finite terms that one can separate from the other ones and use to one's advantage for precisely the determination of U

when one knows x_i and y_i . The second group of (12) can likewise serve to determine the ϑ as long as the x_i and y_i satisfy the condition $\frac{\partial^2 \vartheta}{\partial q_r \, \partial q_s} = \frac{\partial^2 \vartheta}{\partial q_s \, \partial q_r}$, i.e.:

$$\sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial q_s} \frac{\partial y_i}{\partial q_r} - \frac{\partial x_i}{\partial q_r} \frac{\partial y_i}{\partial q_s} \right) = \frac{1}{2} \left(\frac{\partial b_r}{\partial q_s} - \frac{\partial b_s}{\partial q_r} \right) .$$

In order to integrate (12), it is enough, by definition, to associate those three equations with the six equations $A_{rs} = a_{rs}$, which will then constitute a system of nine equations in the nine unknowns x_i, y_i, z_i. That system will certainly admit one solution. The rigorous proof will possibly require one to use Riquier's method. We propose to return to that argument at another time in order for us to now be able to assert that it is possible to choose the constraints that are imposed upon the three points P_1, P_2, P_3 in such a way that the equations that their motion will coincide with those of the motion of the system S that was considered in no. **1**.

Meanwhile, we can conclude with the following statement:

The trajectories of an irreversible dynamical system with three degrees of freedom coincide with those of an ordinary system that is composed of just three points that are conveniently constrained and move in a conservative field while rotating uniformly around an axis.

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