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On the mechanics of rods

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PREFACE

The systematic study of the statics of rods was initiated by KIRCHHOFF (*). Since the general equations of elasticity are applicable to only infinitesimal deformations, but rods are bodies that are susceptible to finite deformations, after supposing that the rod was divided into infinitesimal slices (whose dimensions were all of the same order of magnitude), KIRCHHOFF introduced the fundamental principle that the equations of elasticity could be applied to each element of the system. With that criterion, he succeeded in determining the general equations of equilibrium for a rod. With those equations, one can study all of the statics problem that are inherent to the rod itself and have been treated by various authors: In particular, there is the problem of the planar elastic curve.

No less than to statics, the attention of the mathematicians returned to the dynamics of rods, with the immediate applications to acoustics in mind. One could then study the various types of vibrations of rectilinear or circular rods with various methods (**).

We do not know if anyone has unified the various methods in such a way that the solutions of all problems that refer to the mechanics of rods will depend upon a single principle and give entirely general equations.

We would like to attempt such a unification in the present work.

With KIRCHHOFF, we start from the hypothesis that the rod is divided into infinitesimal slices, each of which can be regarded as rigid from the standpoint of kinematics and endowed with a certain potential energy. After thus schematizing our system, we will then deduce the general equations of the mechanics of rods by starting from a single principle: the principle of virtual work.

We will recover the equations of statics along that path, but with greater generality in comparison to those of KIRCHHOFF, insofar as we will not specify the form of the potential energy.

We will then establish the general equations that govern the motion of the rod and deduce the equations that relate to the vibrations about an equilibrium configuration from those equations.

(*) G. KIRCHHOFF, *Vorlesungen über mathematische Physik*, *Mechanik*, 2nd ed., Leipzig, Teubner, 1877. Lectures 28 and 29.

(**) Cf., e.g., LORD RAYLEIGH, *Theory of Sound*, 2nd ed., London, Macmillan, 1894, V.I; cf. also.
A. LOVE, *The Mathematical Theory of Elasticity*, 2nd ed., Cambridge, the University Press, 1906, Chap. XXI.

The study of those equations in the general case does not seem to be too easy from the analytical viewpoint, and therefore we shall limit ourselves to recovering the formulas that refer to the various cases that have been studied before.

It would be good to add that the stability of the motion considered in the proximity of a minimum-energy state is assured (with a passage to the limit that is conceptually immediate) by the general theorem that is due to RAYLEIGH and has completed by LEVI-CIVITA (*).

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§ I. – KINEMATICAL AND KINETIC CHARACTERISTICS OF OUR SYSTEM

1. Definition of the system. – In order to discuss the problem of the statics and dynamics of the elastic rod (which is imagined to be a body with two dimensions that are small compared to the third), in this first paragraph we will define those material systems that can include rods and establish all of the elements that characterize them in a way that will be as complete as possible.

We then begin by supposing that the rod is divided into infinitesimal slices (whose dimensions must all have the same order of magnitude) and then associate each of them with its center of gravity P. All of the points P constitute a continuous curve that is called the *director* of the rod.

Let s denote the arc-length of the curve when it is counted by starting from one extremity of the rod. If l is its length then s, the length of the director, will increase from 0 to l.

We suppose that each elementary slice can rotate freely around the tangent to the director with respect to the center of gravity independently of the contiguous elements. Obviously, any slice of the system will then have four degrees of freedom.

2. Elements that characterize our system. – Here is how we can specify the four elements (which are functions of s) that specify our system.

In the first place, choose a tri-rectangular trihedron O (X, Y, Z) to be a fixed reference and associate each point P on the director with a moving trihedron P (x, y, z) whose origin is at P and whose z-axis points along the tangent to the director at P with its positive sense pointing in the direction of increasing s. The pair (x, y) is then chosen (in the normal plane to z at P) in such a way that it will constitute a left-handed tri-rectangular triad with z that is congruent to the fixed triad O(X, Y, Z).

The coordinates X, Y, Z, as well as the x, y, z, of the points on the director must be considered to be functions of the parameter s.

We then say *the natural state* of the rod to mean the configuration that the system goes to when all of its internal forces are zero. *The deformed state* is the one that corresponds to a generic state of elastic coaction.

Given that, consider a well-defined element of the rod and fix its position, either in relation to the natural state of that rod or in relation to a generic deformed state. Then let ϖ_1 and ϖ_2 be the

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(*) T. LEVI-CIVITA, "Sullo spostamento dell'equilibrio," Atti del R. Istituto Veneto di Scienze, Lettere ed Arti, **71** (1911), 241-249.

planes of the normal sections that pass through the center of gravity of the element in the two different positions considered. In general, the planes ϖ_1 and ϖ_2 intersect in a line, and it is clear that when one fixes a sense of circulation, there will be only one rotation around that line (the rotation will degenerate when ϖ_1 and ϖ_2 are parallel to each other) for which the two planes will overlap. After that rotation, the *z*-axes (always relative to two different positions of the element) will be parallel to each other, while the pair (x, y), which now lies in the same plane, will be found to have been rotated with respect to each other through a certain angle *f* that is non-zero, in general. If the element of the rod is displaced without experiencing any rotation around its own axes during the deformation then obviously one will have $f = 0$.

Now fix any equilibrium configuration and call a state of the rod *atorsional* when it can be reached by starting from its natural state by simply bending the rod without producing any torsion, or more precisely, the state of the rod that belongs to the same director line of the deformed state and in which the orientation of each element around its own axis does not differ from the one that pertained to when the rod was found in its natural state.

Call the angle that is defined between the planes ϖ_1 and ϖ_2 the *angle of deviation*.

It results from those considerations that the angle *f*, as defined previously, should be considered to be independent of the deformation of the director and a function of only the arc-length *s*. It then follows that one can assume that the Lagrangian coordinates of each slice of the system are *the three coordinates of the point P and the angle f*. Obviously, the three coordinates determine the location of the point *P* along the director line uniquely, and therefore the position of the corresponding element of the rod, and the angle *f* determines the orientation around that director.

3. Kinematical elements and their analytical expression. – From now on, let $ds^{(n)}$ denote the element of arc-length of the director relative to the natural state of the rod and let *ds* denote that linear element that corresponds to a deformed state and relative to an equilibrium configuration. One will then have:

$$ds = (1 + \lambda)\; ds^{(n)}, \tag{1}$$

in which λ denotes the coefficient of linear dilatation. It is important to now assume, with KIRCHHOFF, that although the rod is a body that can submit to finite deformations, the linear dilatation will not cease to be infinitesimally small.

Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be three unitary values that point along the moving axes *Px*, *Py*, *Pz*, respectively.

Let $\mathbf{h}$ denote the vector that represents the rotation that moving axes experience under a displacement by *ds* of their origin along the director line, and let *p*, *q*, *r* be its components with respect to those axes. It is known that *p* and *q* represent the components of the curvature of the director at *P*, *r* represents its torsion.

As a result, we must consider vectorial derivatives with respect to either the fixed axes or the moving axes, and we will use the notation d'/ds for the former, while reserving the usual symbolism for the latter. The two derivatives are related in the known way:

$$\frac{d'}{ds} = \frac{d}{ds} + \mathbf{h} \wedge\ , \tag{2}$$

and in particular, if $\mathbf{n}_i$ is a generic vector that is fixed with respect to the moving axes then one will have POISSON's formula (†):

$$\frac{d'\mathbf{n}_i}{ds} = \mathbf{h} \wedge \mathbf{n}_i \qquad (i = 1, 2, 3). \tag{3}$$

It is known that this formula defines the vector $\mathbf{h}$ and provides the expressions for its three components p, q, r:

$$p = \mathbf{n}_3 \times \frac{d'\mathbf{n}_2}{ds}, \qquad q = \mathbf{n}_1 \times \frac{d'\mathbf{n}_3}{ds}, \qquad r = \mathbf{n}_2 \times \frac{d'\mathbf{n}_1}{ds}.$$

One can add another formula to those, namely:

$$\frac{d'P}{ds} = \mathbf{n}_3 , \tag{4}$$

which expresses the idea that s is not an arbitrary parameter but is belongs to the arc-length of the curve if one so wishes.

4. Variation of the kinematical elements for a virtual displacement of the system. – We now suppose that an arbitrary virtual displacement has been attributed to our system, under which each point of the director in the equilibrium configuration will pass from the position P to the positive $P + \delta P$, which will then give rise to the varied configuration.

Let $\mathfrak{h}$ denote the vector that represents the elementary rotation that might realize the virtual displacement that is attributed to our system. The three vectors $\mathbf{n}_i$ (which are fixed with respect to the moving axes) will submit to a variation $\delta\mathbf{n}_i$ with respect to the fixed axes that is defined by:

$$\delta\mathbf{n}_i = \mathfrak{h} \wedge \mathbf{n}_i ,$$

which will permit one to put $\mathfrak{h}$ into the form:

$$\mathfrak{h} = \mathbf{n}_3 \wedge \delta\mathbf{n}_3 + (\mathbf{n}_2 \times \delta\mathbf{n}_1)\, \mathbf{n}_3 . \tag{5}$$

During the passage from the equilibrium configuration to the varied configuration, the vector $\mathbf{h}$ will also submit to an increment $\delta\mathbf{h}$ that is defined by:

$$\delta\mathbf{h} = \frac{d'\mathfrak{h}}{ds} - \mathbf{h}\frac{d\,\delta s}{ds} . \tag{6}$$

(†) Translator: Palatini is using the "∧" for the vector cross product and the "×" for the scalar product.

Indeed, in the ordinary configuration, the elementary rotation that the moving axes submit to as their center moves along the director from the position s to the position $s + ds$ is expressed by $\mathbf{h}\, ds$. In the varied configuration, a final discrepancy of $d\,\mathfrak{h}$ will intercede between the orientations of the moving axes relative to the two positions s and $s + ds$. That increment is referred to the fixed axes and is then specified in the form of $\frac{d'\mathfrak{h}}{ds} ds$. It then follows that:

$$\delta(\mathfrak{h}\, ds) = \frac{d'\mathfrak{h}}{ds} ds ,$$

and that will give (6) immediately (*). Q. E. D.

Now determine the variation δf of the angle f. It is known that δf can be interpreted as the variation of the angle, which is initially right, of the pair (x, y) that effects the virtual displacement. It then follows that:

$$\text{ang}\,(\mathbf{n}_1, \mathbf{n}_2 + \delta\,\mathbf{n}_2) = \frac{\pi}{2} + \delta f ,$$

from which:

$$\cos \text{ang}\,(\mathbf{n}_1, \mathbf{n}_2 + \delta\,\mathbf{n}_2) = -\sin \delta f ,$$

and from that and the definition of the scalar product, one will have:

$$\delta f = \mathbf{n}_2 \times \delta\,\mathbf{n}_1 , \tag{7}$$

up to infinitesimals of order higher than the first.

When one once more adds that to (1), one will get:

$$\frac{\delta\lambda}{1+\lambda} = \frac{d\,\delta s}{ds} , \tag{8}$$

and from (4):

$$\frac{d'\,\delta P}{ds} = \delta\mathbf{n}_3 + \frac{d\,\delta s}{ds}\mathbf{n}_3 .$$

When that is scalar multiplied by $\mathbf{n}_3$, thanks to the preceding, one will then have:

$$\delta\lambda = (1 + \lambda)\,\mathbf{n}_3 \times \frac{d'\,\delta P}{ds} , \tag{9}$$

and if one then vector multiplies that on the left by $\mathbf{n}_3$ then one will get:

(*) In order to establish (6), I have availed myself of a procedure that was already adopted by T. LEVI-CIVITA, "Forma mista di equazioni del moto, etc.," Rendiconti della R. Accademia dei Lincei, (5), II Sem. **24** (1915), 235-248., pp. 236.

$$\mathbf{n}_3 \wedge \delta \mathbf{n}_3 = \mathbf{n}_3 \wedge \frac{d' \delta P}{ds} .$$

In accordance with that relation and (7), (5) will assume the form:

$$\mathfrak{h} = \mathbf{n}_3 \wedge \frac{d' \delta P}{ds} + \delta f \, \mathbf{n}_3 . \tag{10}$$

It might be useful to add that one can fix the displacements δP and δf arbitrarily for any value of the arc-length *s*, and thus that $\mathfrak{h}$ can also be considered to be arbitrary for any value of *s*.

5. Elastic energy. – Let *e* denote the elastic energy per unit length in the deformed rod for an elementary slice of the rod: *e* must be considered to be a function of the deformation of the system that is unknown *a priori*. We introduce the hypothesis that *e* depends upon only the characteristics *p*, *q*, *r*, and the unit elongation λ (*). With that, the elastic energy of deformation will be assumed to be independent of the variation of the angle of deviation along the rod. We justify this hypothesis:

1. By keeping in mind the schematization of our system that was made (that we included it in systems with four degrees of freedom).

2. By recalling that in the classical study surrounding our subject, one also presupposes a quadratic dependency on the four aforementioned parameters. For the sake of generality, we shall not specify that dependency.

We let *E* denote the elastic energy of the entire rod (whose length we have denoted by *l*), which will be:

$$E = \int_l e(p, q, r, \lambda) \, ds . \tag{11}$$

6. Act of motion. – Let *Q* be a generic point of the rod, and let *P* be the barycenter of the slice that it belongs to. As is known, the expression for the velocity **v** of *Q* is:

$$\mathbf{v} = \mathbf{v}_P + \Omega \wedge (Q - P) ,$$

in which $\mathbf{v}_P$ is the velocity of *P*, and Ω is the angular velocity of the moving trihedron.

Let $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ denote the components of the vector Ω along the axes *P* (*x*, *y*, *z*). If one recalls the significance of the angle *f* then one will have:

(*) One can suppose that the elastic energy also depends explicitly upon the arc-length *s*, no matter how many times the transverse section of the rod varies. Our calculations will not be altered substantially by that, but we suppose, for simplicity, that the transverse section is uniform and that, as we have specified, *e* does not depend upon the arc-length *s* explicitly.

$$\mathfrak{R} = \frac{df}{dt}, \tag{12}$$

by which the vector Ω can be considered to be defined by (12) and the formula:

$$\frac{d'\mathbf{n}_3}{dt} = \Omega \wedge \mathbf{n}_3, \tag{13}$$

in which $\mathbf{n}_3$ is defined by

$$\mathbf{n}_3 = \frac{d'P}{ds'}, \tag{14}$$

instead of (4), in which ds' denotes the element of arc-length along the director in any of its configurations in the dynamical state.

(12) and (13) allow one to determine Ω from:

$$\Omega = \mathbf{n}_3 \wedge \frac{d'\mathbf{n}_3}{dt} + \frac{df}{dt}\mathbf{n}_3 .$$

That equation, along with (14), will allow one to know $\mathfrak{R}$ perfectly, in addition to $\mathfrak{P}$ and $\mathfrak{Q}$.

Now let u, v, w be the components of the vector $\mathbf{v}_P$, and let T denote the *vis viva* of the element of the rod that corresponds to the point P. As is known, T is a homogeneous quadratic form in the six characteristics u, v, w, $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$ with constant coefficients, so it must then be considered to be a known function, due to the preceding considerations.

If we now consider each element of the rod to be isolated then we will have $\mathfrak{P} = \mathfrak{Q} = 0$, insofar as it is capable of only rotations around the z-axis. If we now move on to the calculation of the kinetic energy of the entire system then that hypothesis cannot be maintained rigorously, because we can always take into account the contribution from the *vis viva* of the variation of the angle of deviation along the rod. That is because if we confine ourselves to considering very small vibrations then we can, in the first approximation, once more consider the hypothesis $\mathfrak{P} = \mathfrak{Q} = 0$ to be valid. We can find a logical justification for that hypothesis by recalling what one does by analogy with elastic energy, and a factual justification because under ordinary circumstances, the contribution to the kinetic energy that is due to the variation of the angle of deviation is negligible in comparison to the one that is due to the translatory motion. That hypothesis can coincide with the one that says that the *rotatory inertia* is negligible during its motion, which is a hypothesis RAYLEIGH (*), for example, made in his study of vibrations of rectilinear rod.

We will then believe that:

$$\mathfrak{P} = 0, \qquad \mathfrak{Q} = 0,$$

approximately.

(*) *Loc. cit.* [footnote (**) on pp. 1], pp. 260. Also cf. KIRCHHOFF, *loc. cit.* [footnote (*) on page 1], lect. 28, § **4**.

7. Kinetic energy. – Having said that, recall that the point *P* coincides with the barycenter of the transverse section, and therefore one will have the following expression for *T* :

$$2T = (u^2 + v^2 + w^2 + \mathfrak{r}^2\mathfrak{R}^2)\, dm\,, \tag{15}$$

in which *dm* is the mass of the truncated element of the rod, and $\mathfrak{r}$ is the radius of gyration of the transverse section with respect to the barycenter.

Since one has:

$$\frac{dP}{dt} = \mathbf{v}_P\,,$$

and (12) is true, moreover, one can also put (15) into the form:

$$2T = \left\{\frac{dP}{dt}\times\frac{dP}{dt} + \mathfrak{r}^2\left(\frac{df}{dt}\right)^2\right\} dm\,,$$

and if one calls the total *vis viva* in the system $\mathfrak{T}$ then one will have:

$$\mathfrak{T} = \frac{1}{2}\int\left\{\frac{dP}{dt}\times\frac{dP}{dt} + \mathfrak{r}^2\left(\frac{df}{dt}\right)^2\right\} dm\,, \tag{16}$$

in which the integral is extended over the entire rod.

§ II. – GENERAL EQUATIONS OF THE STATICS OF RODS

1. Applying the principal of virtual work. – In order to establish the static equations of the elastic rod, one can profit from the principle of virtual work.

Let **F** and **N** be the resultant and the resultant moment (with respect to *P*) of the external forces that are exerted upon an infinitesimal element, which are forces and moments per unit length along the director of the rod relative to the deformed state. Then let **F** and **M** be the resultant and resultant moment (with respect to *P*) of the forces that are exerted upon the terminal section of each length of the rod by the portion of the rod that corresponds to a greater value of *s*.

When one recalls the notations that were introduced in § I, an application of the principle of virtual work will immediately produce the equation:

$$\delta E + \int_l \{\mathbf{F}\times\delta P + \mathbf{N}\times\mathfrak{h}\}\, ds + [\Phi\times\delta P + \mathbf{M}\times\mathfrak{h}]_l = 0\,, \tag{17}$$

whose terms in square brackets refer to the limits of the rod. It is pointless to add that this equation must be satisfied identically for any values of the displacements δP and δf.

Now, one has from (11) that:

$$\delta E = \int_l \left\{ \operatorname{grad} e \times \delta\mathfrak{h} + \frac{\partial e}{\partial \lambda} \delta\lambda + e \frac{d\,\delta s}{ds} \right\} ds \ ,$$

in which grad e denotes the vector that has the components $\frac{\partial e}{\partial p}$, $\frac{\partial e}{\partial q}$, $\frac{\partial e}{\partial r}$ with respect to the moving axes P (x, y, z). By virtue of (6) and (9), and when one sets:

$$e_1 = (1 + \lambda) \frac{\partial e}{\partial \lambda} + e - \operatorname{grad} e \times \mathbf{h} \ ,$$

the previous equation will take the form:

$$\delta E = \int_l \left\{ \operatorname{grad} e \times \frac{d'\mathfrak{h}}{ds} + e_1 \, \mathbf{n}_3 \times \frac{d'\,\delta s}{ds} \right\} ds \ ,$$

and therefore, by means of an integration by parts:

$$\delta E = \left[\operatorname{grad} e \times \mathfrak{h} + e_1 \, \mathbf{n}_3 \times \delta P \right]_l - \int_l \left\{ \mathfrak{h} \times \frac{d' \operatorname{grad} e}{ds} + \delta P \times \frac{d'(e_1 \, \mathbf{n}_3)}{ds} \right\} ds \ . \tag{19}$$

Then set:

$$\left. \begin{aligned} H_1 &= \left[(\Phi + e_1 \, \mathbf{n}_3) \times \delta P + (\mathbf{M} + \operatorname{grad} e) \times \mathfrak{h} \right]_l \ , \\ K_1 &= \int_l \left(\mathbf{F} - \frac{d'(e_1 \, \mathbf{n}_s)}{ds} \right) \times \delta P \, ds \ , \end{aligned} \right\} \tag{20}$$

and equation (17) will become:

$$H_1 + K_1 + \int_l \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds} \right) \times \mathfrak{h} \, ds = 0 \ . \tag{21}$$

However, from (10), one has:

$$\int_l \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds} \right) \times \mathfrak{h} \, ds = \int_l \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds} \right) \times \left(\mathbf{n}_3 \wedge \frac{d'\,\delta P}{ds} \right) ds + \int_l \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds} \right) \times \mathbf{n}_3 \, \delta f \, ds \ ,$$

and also, by means of an integration by parts:

$$\int_l \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) \times \mathfrak{h}\, ds = H_2 + K_2 + J ,$$

in which one sets:

$$\left.\begin{aligned} H_2 &= \left[\left\{\left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) \wedge \mathbf{n}_3\right\} \times \delta P\right]_l , \\ K_2 &= -\int_l \delta P \times \frac{d'}{ds}\left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) ds, \\ J &= \int_l \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) \times \mathbf{n}_3\ \delta f\, ds. \end{aligned}\right\} \tag{22}$$

Consequently, equation (21) will take the form:

$$H_1 + H_2 + K_1 + K_2 + J = 0 . \tag{23}$$

2. General static equations. – Now recall that equation (23), to which we have arrived, must be satisfied identically for any virtual displacement considered, which demands that the coefficients of δP and δF must be identically zero. First of all, when we take into account the definitions (20) and (22), that will bring us to the following two equations:

$$\mathbf{F} - \frac{d'}{ds}\left\{e_1 \mathbf{n}_3 + \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) \wedge \mathbf{n}_3\right\} = 0 ,$$

$$\left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) \wedge \mathbf{n}_3 = 0 ,$$

and from the arbitrariness of δP and $\mathfrak{h}$ [see § I, no. **4**] at the limits of the rod, we will arrive at the final equations:

$$\Phi + e_1 \mathbf{n}_3 + \left(\mathbf{N} - \frac{d' \operatorname{grad} e}{ds}\right) \wedge \mathbf{n}_3 = 0 ,$$

$$\mathbf{M} + \operatorname{grad} e = 0 ,$$

which can be considered to be the definitions of the vectors Φ and **M**.

One can get the main equations of equilibrium for the rod from the equations that were just obtained, which are the following ones (*):

(*) Cf., e.g., LOVE, *loc. cit.* [footnote (**), page 1], no. 254.

$$\left.\begin{aligned} \mathbf{F} + \frac{d'\Phi}{ds} = 0, \\ \frac{d'\mathbf{M}}{ds} + \mathbf{N} + \mathbf{n}_3 \wedge \Phi = 0 \ , \end{aligned}\right\} \tag{24}$$

as one confirms immediately.

As a result, it will be useful to consider those static equations of the rod to be the main equations, together with the ones that define Φ and **M**, and note, in particular, that:

$$\Phi \times \mathbf{n}_3 + e_1 = 0 \ ,$$

or that:

$$\Phi_z = - e_1 \ ,$$

as one gets from the expression for Φ.

§ III. – PARTICULAR CASE IN WHICH THE SYSTEM IS NOT SUBJECT TO EXTERNAL FORCES

1. Deducing Kirchhoff's static equations. – If the external forces that are applied to the system are zero then equations (24) for the equilibrium of the elastic rod will assume the simpler form:

$$\left.\begin{aligned} \frac{d'\Phi}{ds} = 0, \\ \frac{d'\mathbf{M}}{ds} + \mathbf{n}_3 \wedge \Phi = 0, \end{aligned}\right\} \tag{25}$$

or also if one prefers:

$$\frac{d'\Phi}{ds} = 0 \ ,$$

in which [taking into account the arguments of the preceding paragraph and (2)], Φ is defined by:

$$\left.\begin{aligned} \frac{d \operatorname{grad} e}{ds} + \mathbf{h} \wedge \operatorname{grad} e = \mathbf{n}_3 \wedge \Phi, \\ \Phi_z + e_1 = 0, \end{aligned}\right\} \tag{26}$$

and e_1 is expressed as in (18).

In that form, the equations that are obtained are, in essence, the known equations that KIRCHHOFF established (*). The formal similarity is not immediately obvious from the form of e_1. That similarity will soon appear as soon as one observes that KIRCHHOFF's elastic energy F is referred to a unit of length of the rod when it is in the natural state, and that, in addition, that is

(*) KIRCHHOFF, *loc. cit.* [footnote (*) on page 1], Lect. 28.

not supposed to depend upon the components p, q, r of the vector **h** that was defined by (3), but upon the components (which KIRCHHOFF also called p, q, r) of a vector **h**′ that is defined by:

$$\frac{d'\mathbf{n}_i}{ds^{(n)}} = \mathbf{h}' \wedge \mathbf{n}_i \,.$$

Since one obviously has:

$$\mathbf{h}\,(1 + \lambda) = \mathbf{h}' \,,$$
$$e\,(1 + \lambda) = F \,,$$

one will immediately confirm that:

$$\text{grad}\; e = \text{grad}\; F \,,$$

and that:

$$e_1 = \frac{\partial F}{\partial \lambda} \,.$$

After that, it remains for us to prove, according to our assertion, is that the equations obtained are those of KIRCHHOFF, but with one ultimate difference, namely, that KIRCHHOFF showed that F was a homogeneous quadratic function of the arguments that it depends upon, but for us, e is a function of the arguments p, q, r, and λ that is arbitrary *a priori*.

2. First integrals. – KIRCHHOFF observed that the final equations that he obtained (from which, he eliminated the parameter λ by means of the equation $\Phi_z = 0$) are analogous to the ones that govern the motion of a heavy body around a fixed center.

The formal analogy is no longer preserved when e keeps its full generality, i.e., one does not specify its dependency upon p, q, r, and λ, and one does not eliminate λ from its expression. at that level of generality, one must always take into account the second equation in (26) in the first (vectorial) equation in (26).

Without making any hypothesis, we would now like to show that the system of equations (26) admits four first integrals, which are precisely the ones that are admitted by the system that governs the cited motion (viz., the three area integrals and the *vis viva* integral).

To that end, let $\mathbf{M}_0$ denote the resultant moment of the forces with respect to the point O (viz., the center of the fixed axes) and recall that **M** is the analogous moment with respect to the point P. The known relation exists between $\mathbf{M}_0$ and **M**:

$$\mathbf{M}_0 = \mathbf{M} + (P - O) \wedge \Phi \,.$$

From that, one gets:

$$\frac{d'\mathbf{M}_0}{ds} = \frac{d'\mathbf{M}_0}{ds} + \mathbf{n}_3 \wedge \Phi = 0$$

with the help of (4) and the first of (25).

It will then follow that $\mathbf{M}_0$ = const. with respect to the fixed axes, or:

$$\mathbf{M} + (P - O) \wedge \Phi = \text{const.}$$

However, $\mathbf{M} = -\operatorname{grad} e$, from which, one will get these first three integrals:

$$\operatorname{grad} e + (O - P) \wedge \Phi = \text{const.}$$

In order to get the fourth integral, recall the principle of virtual work, when conveniently specialized to the displacements by supposing precisely that they are zero at the extremes of the rod and that, in addition, one has:

$$\delta P = K\,\mathbf{n}_3\,,$$
$$\delta f = K\,r\,,$$

with K completely arbitrary. With that specialization of (10), it will result that:

$$\mathfrak{h} = K\,\mathbf{h}\,,$$

and (when one recalls that the external forces are zero) from (19):

$$\int_l K\left\{\mathbf{h}\times\frac{d\operatorname{grad} e}{ds} + \mathbf{n}_3\times\frac{d\,(e_1\,\mathbf{n}_3)}{ds}\right\} ds = 0\,,$$

and that will give:

$$\mathbf{h}\times\frac{d\operatorname{grad} e}{ds} + \frac{de_1}{ds} = 0\,,$$

from the arbitrariness of the K.

If one recalls the expression (18) for e_1 then that equation can be altered into another one:

$$\frac{d}{ds}(1+\lambda)^2\,\frac{\partial e}{\partial\lambda} = 0\,,$$

from which, the desired fourth integral is:

$$(1+\lambda)^2\,\frac{\partial e}{\partial\lambda} = \text{const.} \tag{27}$$

Naturally, that integral can also be obtained from the general equations (26), and one succeeds in doing that by eliminating Φ from them.

It is important to note that if e presents itself as the sum of two terms, one of which depends upon p, q, r, and the other of which depends upon λ, then the last integral that was obtained will lend itself to this physical interpretation: The unit elongation of a rod in an equilibrium configuration is uniform along the entire rod.

3. Elastic plane curve. – Let us now study, in particular, the equilibrium state of a rod whose director line belongs completely to a plane ϖ. Naturally, the z-axis of the moving system belongs to that plane, and if we choose the y-axis to also belong to ϖ that we will have:

$$q = 0\,,\;\; r = 0$$

and e will prove to be a function of only the two arguments p and λ.

The equations of equilibrium then reduce to the following ones:

$$\left.\begin{aligned} &\frac{d\Phi_y}{ds} - p\,\Phi_z = 0, \\ &\frac{d\Phi_z}{ds} + p\,\Phi_y = 0, \end{aligned}\right\} \tag{28}$$

with

$$\left.\begin{aligned} &\Phi_y = -\frac{d}{ds}\frac{\partial e}{\partial p}, \\ &\Phi_z = -e_1. \end{aligned}\right\} \tag{29}$$

In the usual treatment of that classical problem, one supposes that the extremity of the rod is subject to a moment of flexure M_x that is proportional to the variation of the curvature between the equilibrium state and the natural state, i.e., one of the type:

$$M_x = A\,(p - p_n)\,,$$

where p_n denotes the curvature of the rod in the natural state. A is a coefficient that depends upon the nature and form of the rod and is equal to precisely the YOUNG's modulus times the moment of inertia of the transverse section, so it must therefore be regarded as constant when the rod is homogeneous, and its transverse section is uniform.

Normally, one makes the final hypothesis that the rod is straight in the natural state. We suppose that $p_n = c =$ const. (which will give the classical case when $c = 0$), which presupposes that in its natural state, the rod is laid along an arc of a circumference (or, in particular, along a straight line).

Before proceeding, recall [cf., the first of (25)] that one has $\Phi =$ const. (with respect to the fixed axes), so if ϑ denotes the angle that the tangent to the curve forms with the line of action of the vector Φ then one will have:

$$p = \frac{d\vartheta}{ds},$$

$$\Phi_y = \Phi \sin \vartheta\,, \qquad \Phi_z = \Phi \cos \vartheta\,,$$

with Φ constant.

Now by virtue of the first hypothesis that was specified, the first of (29) will give:

$$\Phi_y = A\,\frac{dp}{ds}, \tag{30}$$

which is an equation that will be converted into the following one:

$$A\,\frac{d^2\vartheta}{ds^2} - \Phi \sin\,\vartheta = 0\,, \tag{31}$$

from the observations that were made above.

One then recovers the classical formula that determines the equilibrium form for an elastic plane curve, which is formula that then also extends to a rod that is bent along an arc of a circumference in its natural state.

One deduces the known first integral from (31) in an obvious way:

$$\tfrac{1}{2}A\left(\frac{d\vartheta}{ds}\right)^2 + \Phi\cos\,\vartheta = a \qquad\qquad (a \text{ is an integration constant})\,,$$

which one can also write in the form:

$$\tfrac{1}{2}A\,p^2 + \Phi_z = a\,, \tag{32}$$

if one so desires.

Let us make one last observation:

Let $p = 0$ in the equilibrium position. From (30), one will then get $\Phi_y = 0$ and from (28), $\Phi_z =$ const. We conclude that a rod can be kept rectilinear in an equilibrium position only by purely tangential forces. Conversely, if the rod in its equilibrium configuration is subject to only (non-zero) tangential forces then one will get that $p = 0$ from (28), or the rod is necessarily laid along a straight line.

Let $p =$ const. (which is different from zero) in the equilibrium configuration. From (30), one has $\Phi_y = 0$, and from the first of (28), $\Phi_z = 0$. Since, by definition, the natural state of the rod is the one that it assumes when the internal stresses are zero, one concludes that a rod (in the presence of only forces on the ends) cannot be found along an arc of a circumference when it is in an equilibrium state unless it is in its natural state.

§ IV. EQUATIONS THAT GOVERN THE MOTION OF AN ELASTIC ROD

In order to obtain the equations that govern the motion of an elastic rod, one appeals to HAMILTON's principle, which is, as one knows, expressed by the equation:

$$\int_{t_1}^{t_2} dt\,(\delta\mathfrak{T} + \delta E + U) = 0\,, \tag{33}$$

in which $\delta\mathfrak{T}$ and δE are the variations of the *vis viva* and the elastic energy of the system, and U is the work done by a virtual displacement of the system by the externals forces and the stresses that act at the extremities of the rod.

It is almost pointless to add that (33) must be satisfied identically, i.e., for a virtual displacement that is arbitrary, but zero at the extremities of the time interval (t_1, t_2).

Now, if one recalls the expression for $\mathfrak{T}$ that one got in § I [form. (16)] then one will have:

$$\int_{t_1}^{t_2} \delta\mathfrak{T}\,dt = \int_{t_1}^{t_2} dt \int \left\{ \frac{d\,\delta P}{dt} \times \frac{dP}{dt} + \mathfrak{r}^2 \frac{d\,\delta f}{dt}\frac{df}{dt} \right\} dm ,$$

or also, with an integration by parts:

$$\int_{t_1}^{t_2} \delta\mathfrak{T}\,dt = -\int_{t_1}^{t_2} dt \int \left\{ \frac{d^2 P}{dt^2} \times \delta P + \mathfrak{r}^2 \frac{d^2 f}{dt^2}\,\delta f \right\} dm .$$

Now let κ denote the linear density of the rod in the deformed state and recall that ds' denotes the element of arc-length along the director in an arbitrary state of motion. One will have:

$$dm = \kappa\,ds' ,$$

and consequently:

$$\int_{t_1}^{t_2} \delta\mathfrak{T}\,dt = -\int_{t_1}^{t_2} dt \int_l \kappa \left\{ \frac{d^2 P}{dt^2} \times \delta P + \mathfrak{r}^2 \frac{d^2 f}{dt^2}\,\delta f \right\} ds' .$$

In order to get the general equations for the dynamics of the rod, it is now sufficient to express the idea that the coefficients of δP and δf in (33) are zero. In order to write them down quickly, it is enough to compare (33) with the equations (17) that determine the equations of statics: The single difference consists of the presence of kinetic energy, whose variation was just calculated, but with the caveat that one must substitute ds' for ds everywhere.

One then gets the following equations:

$$\mathbf{F} - \frac{d'}{ds'}\left\{ e_1\,\mathbf{n}_3 + \left(\mathbf{N} - \frac{d'\,\mathrm{grad}\,e}{ds'} \right) \wedge \mathbf{n}_3 \right\} = \kappa\,\frac{d^2 P}{dt^2} ,$$

$$\left(\mathbf{N} - \frac{d'\,\mathrm{grad}\,e}{ds'} \right) \wedge \mathbf{n}_3 = \kappa\,\mathfrak{r}^2 \frac{d^2 f}{dt^2} ,$$

with the two equations that define Φ and $\mathbf{M}$:

$$\Phi + e_1\,\mathbf{n}_3 + \left(\mathbf{N} - \frac{d'\,\mathrm{grad}\,e}{ds'} \right) \wedge \mathbf{n}_3 = 0 ,$$

$$\mathbf{M} + \mathrm{grad}\,e = 0 .$$

In the case where the external forces are zero, one has simply:

$$\frac{d'\Phi}{ds'} = \kappa \frac{d^2 P}{dt^2} , \tag{34}$$

$$-\frac{d' \operatorname{grad} e}{ds'} \times \mathbf{n}_3 = \kappa \mathfrak{r}^2 \frac{d^2 f}{dt^2} , \tag{35}$$

in which Φ has the expression:

$$\Phi = \frac{d' \operatorname{grad} e}{ds'} - e_1 \mathbf{n}_3 . \tag{36}$$

§ V. – VIBRATIONS AROUND AN EQUILIBRIUM CONFIGURATION

1. Specifying the problem. – We shall now address the problem of establishing the equations that determine the small vibrations of an elastic rod around one of its stable equilibrium conditions in the absence of external forces.

Each characteristic of the phenomenon can then be represented as the sum of two terms, the first of which represents the determination that belongs to the equilibrium position, and the second of which can be interpreted as the alteration that the characteristic experiences as a result of the motion when starting from the equilibrium configuration.

Since we propose to study small vibrations around a stable equilibrium position, the second addend must be regarded as infinitesimal. Therefore, if we confine ourselves to studying the phenomenon in the first approximation then in the process of establishing the equations, we can ignore all terms that contain infinitesimals of order higher than the first, and the final equations will prove to be linear and homogeneous in the unknown variables in that way.

The determination that belongs to a generic characteristic of the system in an equilibrium configuration will be indicated with the index 0. Consequently, any letter that is endowed with the index 0 must be regarded as independent of time and known to correspond to solutions of the static equations that were established in the preceding paragraphs.

2. Orientation of the moving triad with respect to the static triad. – Fix a generic equilibrium configuration of the rod, so a single triad will be specified at each of its points P_0 that must be regarded as varying from point to point but fixed in time. Let P_0 (x, y, z) denote that triad, and let $P(x', y', z')$ denote the corresponding triad that vibrates everywhere along the rod: Let us exhibit their reciprocal orientation.

Set:

$$\mathbf{n}_i = \mathbf{n}_i^{(0)} + \boldsymbol{\nu}_i \qquad (i = 1, 2, 3) \tag{37}$$

and let α_i, β_i, γ_i denote the components of the vector $\boldsymbol{\nu}_i$ with respect to the axes P_0 (x, y, z). Note, first of all, that α_1, β_2, γ_3 can be regarded as zero in the first approximation. Indeed, the direction cosines (for example, the components of the vector $\mathbf{n}_i$) are $1 + \alpha_1$, β_1, γ_1, for which one has the relation:

$$(1+\alpha_1)^2 + \beta_1^2 + \gamma_1^2 = 1 ,$$

i.e.:

$$-2\,\alpha_1 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 .$$

That implies that α_1 is infinitesimal of order two and can then be ignored under our hypothesis.

One can show that β_2 and γ_3 are zero analogously.

The nine direction cosines of the triad x', y', z' with respect to the triad x, y, z are given by the following matrix:

	x	y	z
x'	1	β_1	γ_1
y'	α_2	1	γ_2
z'	α_3	β_3	1

(38)

From the known properties of the determinant that is defined by the nine direction cosines of two left-handed tri-rectangular triads, one will get that:

$$\beta_1 = -\,\alpha_2 , \qquad \gamma_1 = -\,\alpha_3 , \qquad \gamma_2 = -\,\beta_3 .$$

If one takes:

$$f = f_0 + \varphi \tag{39}$$

then it will be easy to show that:

$$\beta_1 = \varphi .$$

Indeed, one takes (7) under consideration and supposes that the virtual variation of the angle f that appears in it coincides with its dynamic alteration φ : With that, one will also have $\delta\mathbf{n}_1 = \boldsymbol{\nu}_1$ and it will result that:

$$\varphi = \mathbf{n}_2 \times \boldsymbol{\nu}_1 ,$$

or:

$$\varphi = \mathbf{n}_2^{(0)} \times \boldsymbol{\nu}_1 = \beta_1 ,$$

up to second-order infinitesimals. Q. E. D.

From the arguments that were made above, when we write α and β in place of α_3 and β_3, for brevity, the matrix (38) can be specified as follows:

	x	y	z
x'	1	φ	$-\alpha$
y'	$-\varphi$	1	$-\beta$
z'	α	β	1

and the components of the vectors $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$, $\boldsymbol{\nu}_3$ will prove to be equal to:

$$\begin{matrix} 0 & \varphi & -\alpha \\ -\varphi & 0 & -\beta \\ \alpha & \beta & 0 \end{matrix} \tag{40}$$

respectively.

3. Complete determination of the kinematic elements. – Together with (37) and (39), set:

$$\left.\begin{matrix} \mathbf{h} = \mathbf{h}_0 + \omega, \quad \lambda = \lambda_0 + \lambda', \\ P = P_0 + \sigma. \end{matrix}\right\} \tag{41}$$

In that, σ is the vector that realizes the dynamical displacement that the point P_0 experiences.

Let p_0, q_0, r_0 (π, χ, ρ, resp.) denote the components of the vectors $\mathbf{h}_0$ and $\boldsymbol{\omega}$ *with respect to the triad* P_0 (x, y, z). It is then known that, from the arguments in the preceding section, the components p, q, r of the vector $\mathbf{h}$ with respect to the triad $P(x', y', z')$ can be specified as follows:

$$\left.\begin{matrix} p = p_0 + q_0\,\varphi - r_0\,\alpha + \pi \; = p_0 + \pi^*, \\ q = q_0 - q_0\,\varphi - r_0\,\beta + \chi = q_0 + \chi^*, \\ r = r_0 + p_0\,\alpha + r_0\,\beta + \rho \; = r_0 + \rho^*, \end{matrix}\right\} \tag{42}$$

with the obvious significance for the auxiliary variables π^*, χ^*, ρ^*.

Since we have set $\lambda = \lambda_0 + \lambda'$ (up to second-order infinitesimals), we will have:

$$ds' = (1 + \lambda')\,ds, \tag{43}$$

in which it is good to not forget that ds denotes the arc-length element of the director of the rod when it is found in the equilibrium configuration.

The vectorial derivative with respect to the triad P_0 (x, y, z) (for which we adopt the usual symbolism) will be coupled to the absolute derivative by the relation:

$$\frac{d'}{ds'} = \frac{d}{ds'} + \mathbf{h}_0 \wedge\ ,$$

or also, if the vectors being derived are infinitesimal, by the relation:

$$\frac{d'}{ds} = \frac{d}{ds} + \mathbf{h}_0 \wedge\ . \tag{44}$$

Now, before we proceed with the determination of the equations of motion, we would like to specify the expression for π, χ, ρ. To that end, start from the kinematical relation:

$$p = \mathbf{n}_3 \times \frac{d'\mathbf{n}_2}{ds'}, \qquad q = \mathbf{n}_1 \times \frac{d'\mathbf{n}_3}{ds'}, \qquad r = \mathbf{n}_2 \times \frac{d'\mathbf{n}_1}{ds'} .$$

From the conventions that were made in regard to the first of those, one has:

$$p_0 + \pi^* = \mathbf{n}_3^{(0)} \times \frac{d'\mathbf{n}_2^{(0)}}{ds} \cdot \frac{1}{1+\lambda'} + \boldsymbol{\nu}_3 \times \frac{d'\mathbf{n}_2^{(0)}}{ds} + \mathbf{n}_3^{(0)} \times \frac{d'\boldsymbol{\nu}_2}{ds},$$

and when one applies that to the last term in (44) and sets:

$$p_0 = \mathbf{n}_3^{(0)} \times \frac{d'\mathbf{n}_2^{(0)}}{ds}, \qquad \frac{d'\mathbf{n}_2^{(0)}}{ds} = \mathbf{h}_0 \wedge \mathbf{n}_2^{(0)},$$

one will get:

$$\pi^* = -p_0\,\lambda' + \boldsymbol{\nu}_3 \times (\mathbf{h}_0 \wedge \mathbf{n}_2^{(0)}) + \mathbf{n}_3^{(0)} \times \left(\frac{d\boldsymbol{\nu}_2}{ds} + \mathbf{h}_0 \wedge \boldsymbol{\nu}_2\right) .$$

Keeping in mind the significance of π^* [see form. (42)] and the matrix (40), it will follow that:

$$\pi = -p_0\,\lambda' - \frac{d\beta}{ds} . \tag{45}$$

If one operates on the expressions for q and r in a perfectly-analogous way then one will find, by definition [while repeating (45)]:

$$\left.\begin{aligned} \pi &= -p_0\,\lambda' - \frac{d\beta}{ds}, \\ \chi &= -q_0\,\lambda' + \frac{d\alpha}{ds}, \\ \rho &= -r_0\,\lambda' + \frac{d\varphi}{ds}. \end{aligned}\right\} \tag{46}$$

We now see how to change the relation:

$$\frac{d'P}{ds'} = \mathbf{n}_3 .$$

From (41), one has:

$$\frac{d'P_0}{ds} + \frac{d'\sigma}{ds} = (1+\lambda')(\mathbf{n}_3^{(0)} + \boldsymbol{\nu}_3) ,$$

from which if one sets $\frac{d'P_0}{ds} = \mathbf{n}_3^{(0)}$ and applies (44):

$$\frac{d\sigma}{ds} + \mathbf{h}_0 \wedge \boldsymbol{\sigma} = \boldsymbol{\nu}_3 + \mathbf{n}_3^{(0)} \lambda' . \tag{47}$$

4. Equations of vibration. – We finally come to the determination of the equations that govern the motion of an elastic rod around one of its equilibrium configurations.

Let $\mathfrak{D}$ denote the vectorial homography whose coefficients are the second derivatives of *e* with respect to *p*, *q*, *r*, when taken in the equilibrium configuration, and set:

$$\left.\begin{aligned} \mathbf{g} &= \left(\operatorname{grad} \frac{\partial e}{\partial \lambda}\right)_0 , \\ \boldsymbol{\varepsilon} &= \mathfrak{D}\,\omega^* + \mathbf{g}\,\lambda', \\ \mu &= \boldsymbol{\omega}^* \times \mathbf{g} + \left(\frac{\partial^2 e}{\partial \lambda^2}\right)_0 \lambda', \end{aligned}\right\} \tag{48}$$

in which $\boldsymbol{\omega}^*$ means the vector whose components are π^*, χ^*, ρ^* [see form. (42)].

If one then develops grad *e* in a TAYLOR series around an equilibrium configuration then one will have:

$$\operatorname{grad} e = \operatorname{grad} e_0 + \boldsymbol{\varepsilon} ,$$

up to higher-order infinitesimals, and analogously:

$$\frac{\partial e}{\partial \lambda} = \left(\frac{\partial e}{\partial \lambda}\right)_0 + \mu ,$$

and therefore:

$$e_1 = e_1^{(0)} + \tau ,$$

with:

$$\tau = 2\left(\frac{\partial e}{\partial \lambda}\right)_0 \lambda' + (1 + \lambda_0)\,\mu - \boldsymbol{\varepsilon} \times \mathbf{h}_0 .$$

From the last of (48), one will have just:

$$\tau = \left\{2\left(\frac{\partial e}{\partial \lambda}\right)_0 + (1+\lambda_0)\left(\frac{\partial^2 e}{\partial \lambda^2}\right)_0\right\}\lambda' + (1+\lambda_0)\,\boldsymbol{\omega}^* \times \mathbf{g} - \boldsymbol{\varepsilon} \times \mathbf{h}_0 ,$$

or also when one sets:

$$L = -\left\{2\left(\frac{\partial e}{\partial \lambda}\right)_0 + (1+\lambda_0)\left(\frac{\partial^2 e}{\partial \lambda^2}\right)_0\right\}$$

and recalls that λ_0 is also infinitesimal:

$$\tau = \boldsymbol{\omega}^* \times \mathbf{g} - \boldsymbol{\varepsilon} \times \mathbf{h}_0 - L\lambda' \quad . \tag{49}$$

It follows that Φ has the expression [form. (36)]:

$$\Phi = \Phi^{(0)} + \psi ,$$

in which $\Phi^{(0)}$ is the determination of Φ that belongs to the equilibrium position and ψ has the following expression:

$$\psi = \frac{d' \operatorname{grad} e_0}{ds} \wedge \boldsymbol{\nu}_3 + \left(\frac{d'e}{ds} - \lambda' \frac{d' \operatorname{grad} e_0}{ds} \right) \wedge \mathbf{n}_3^{(0)} - e_1^{(0)} \boldsymbol{\nu}_3 - \tau\, \mathbf{n}_3^{(0)} \quad .$$

Keeping in mind (44) and (26), the vector ψ can be specified by the following three components:

$$\left.\begin{aligned} \psi_x &= \frac{d\varepsilon_y}{ds} + r_0\, \varepsilon_x - p_0\, \varepsilon_z - \Phi_x^{(0)} \lambda' + \Phi_z^{(0)} \alpha, \\ \psi_y &= -\left(\frac{d\varepsilon_x}{ds} + q_0\, \varepsilon_x - r_0\, \varepsilon_y \right) - \Phi_y^{(0)} \lambda' + \Phi_z^{(0)} \beta, \\ \psi_z &= -\Phi_x^{(0)} \alpha - \Phi_y^{(0)} \beta - \tau . \end{aligned}\right\} \tag{50}$$

Having assumed all of that, when one takes into account the static equations, equations (34) and (35) will change into the following ones in the case of small vibration around an equilibrium configuration:

$$\left.\begin{aligned} &\frac{d\psi}{ds} + \mathbf{h}_0 \wedge \psi = \kappa \frac{\partial^2 \sigma}{\partial t^2}, \\ &\mathbf{n}_3^{(0)} \times \frac{d'\varepsilon}{ds} + \boldsymbol{\nu}_3 \times \frac{d' \operatorname{grad} e_0}{ds} = -\kappa\, \mathfrak{r}^2\, \frac{\partial^2 \varphi}{\partial t^2}, \end{aligned}\right\} \tag{51}$$

in which ψ is defined by (50).

Equations (51) are four in number and involve essentially ten unknowns, which are the three components of the vectors ω and σ, and the scalar quantities α, β, φ, and λ'. Equations (46) and (47) of the preceding section complete the system.

§ VI. – VIBRATIONS OF AN ELASTIC PLANE CURVE

1. General case. – We now come to the particular study of equations (51) for the vibration of an elastic plane curve whose elements in the equilibrium configuration are determined by the equations of § III, no. **3**.

In that case, one will have $q_0 = r_0 = 0$, and if one lets ξ, η, ζ denote the components of the vector σ with respect to the axes P_0 (x, y, z) then equations (51) will simplify into the following ones:

$$\left.\begin{aligned} &\frac{d\psi_x}{ds} = \kappa\frac{d^2\xi}{dt^2}, \\ &\frac{d\psi_y}{ds} - p_0\,\psi_x = \kappa\frac{d^2\eta}{dt^2}, \\ &\frac{d\psi_z}{ds} + p_0\,\psi_y = \kappa\frac{d^2\zeta}{dt^2}, \\ &\frac{d\varepsilon_x}{ds} + p_0\,\varepsilon_y - \Phi_y^{(0)}\,\alpha = -\kappa\,\mathfrak{r}^2\,\frac{d^2\varphi}{dt^2}, \end{aligned}\right\} \tag{52}$$

and the components of the vector ψ in those equations have the expressions:

$$\left.\begin{aligned} \psi_x &= \frac{d\varepsilon_y}{ds} - p_0\,\varepsilon_x + \Phi_z^{(0)}\,\alpha, \\ \psi_y &= -\frac{d\varepsilon_x}{ds} - \Phi_z^{(0)}\,\lambda' + \Phi_z^{(0)}\beta, \\ \psi_z &= -\Phi_y^{(0)}\beta - \tau. \end{aligned}\right\} \tag{53}$$

We complete the system of (52) and (53) with (46) and (47), which simplify as follows in the present case:

$$\left.\begin{aligned} \pi &= -\frac{d\beta}{ds} - p_0\,\lambda', \\ \chi &= \frac{d\alpha}{ds}, \\ \rho &= \frac{d\varphi}{ds}, \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} &\frac{d\xi}{ds} = \alpha, \\ &\frac{d\eta}{ds} - p_0\,\zeta = \beta, \\ &\frac{d\zeta}{ds} + p_0\,\eta = \lambda'. \end{aligned}\right\} \tag{56}$$

We agree to call the vibrations of the rod *oblique* when they happen outside the plane of static equilibrium and *planar* when they happen inside of that plane. Now, we know that if we cannot specify the form of the elastic energy then the system of equations (52, 53, 54, 55) will be a simultaneous system, and therefore the planar vibrations cannot be studied separately from the oblique ones. In the next section, we will see that this can be achieved when we attribute a convenient form to *e*.

2. Specifying the form of the elastic energy. – Let us now make the hypothesis that the rod is rectilinear in the equilibrium state and assumes the form of an arc of a circumference and that the elastic energy *e* has the form:

$$e = -\tfrac{1}{2}\{A(p-c)^2 + Bq^2 + Cr^2\} + \Lambda(\lambda) \ .$$

let *c* denote the curvature (constant, and in particular, equal to zero) of the rod in the natural state and let Λ (λ) denote of a function of only λ. *A* and *B* are equal to the YOUNG modulus times the moments of inertia of the transverse section with respect to the *x* and *y* axes, respectively. However, *C* is the torsional rigidity and is equal to the rigidity of the material times a quantity of degree four in the linear dimensions of the transverse section. The three coefficients *A*, *B*, *C* must be regarded as constants if, as we will suppose, the material of the rod is homogeneous, and its transverse section is uniform.

In that case, it will result that [see form. (48)]:

$$\mathbf{g} = 0 \ ,$$

and the coefficients of the vectorial homography $\mathfrak{O}$ will all be zero, except for the following three:

$$\left(\frac{\partial^2 e}{\partial p^2}\right)_0 = -A \ , \qquad \left(\frac{\partial^2 e}{\partial q^2}\right)_0 = -B \ , \qquad \left(\frac{\partial^2 e}{\partial r^2}\right)_0 = -C \ .$$

Consequently, the vector $\boldsymbol{\varepsilon}$ will have the components:

$$\begin{aligned} \varepsilon_x &= -A\pi^* = -A\,\pi \ , \\ \varepsilon_y &= -B\,\chi^* = -B\,(\chi - p_0\,\varphi) \ , \\ \varepsilon_z &= -C\,\rho^* = -C\,(\rho + p_0\,\alpha) \ , \end{aligned}$$

and in addition [see form. (49)]:

$$\tau = A\,p_0\,\pi - L\,\lambda' \ .$$

With that, (53) will assume the form:

$$\left.\begin{aligned}
\psi_x &= -B\frac{d}{ds}(\chi - p_0\varphi) + C\,p_0(\rho + p_0\,\alpha) + \Phi_z^{(0)}\,\alpha,\\
\psi_y &= A\frac{d\pi}{ds} - \Phi_z^{(0)}\lambda' + \Phi_z^{(0)}\,\beta,\\
\psi_z &= -\Phi_z^{(0)}\beta - A\,p_0\,\pi + L\,\lambda'.
\end{aligned}\right\}\tag{56}$$

Naturally, (54) and (55), and the first three of (52) will not change, while the fourth of them will take the form:

$$C\frac{d}{ds}(\rho + p_0\,\alpha) + B\,p_0\,(\chi - p_0\,\varphi) + \Phi_y^{(0)}\,\alpha \;=\; \kappa\,\mathfrak{r}^2\,\frac{d^2\varphi}{dt^2}\,.\tag{57}$$

A simple inspection of the formulas obtained this section will permit one to split the general problem of the vibrations of a planar elastica into two, i.e., one can study the planar vibrations and the oblique ones separately, as was established in the preceding section.

It is known that the elements (in particular, p_0) that determine the equilibrium configuration of an elastic curve are expressed in terms of elliptic functions. That is the root of the analytical difficulty in confronting the problem of vibrations in full generality. Therefore, we shall confine ourselves to confirming that all of the equations that relate to the various types of vibrations that are studied can be derived from our general equations.

3. Vibrations of a straight rod in the natural state. – In this case, one can keep $p_0 = 0$, $\Phi_y^{(0)} = \Phi_z^{(0)} = 0$, and therefore the preceding section will imply:

1. The equations for the *planar vibrations:*

$$\left.\begin{aligned}
&\frac{d\psi_y}{ds} = \kappa\frac{d^2\eta}{dt^2}, \quad \frac{d\psi_z}{ds} = \kappa\frac{d^2\zeta}{dt^2},\\
&\psi_y = A\frac{d\pi}{ds}, \qquad \psi_z = L\,\lambda',\\
&\pi = -\frac{d\beta}{ds}, \qquad \frac{d\eta}{ds} = \beta, \qquad \frac{d\zeta}{ds} = \lambda'.
\end{aligned}\right\}\tag{58}$$

That will separately give the classical equations (*):

for the *transverse vibrations:* $\qquad A\dfrac{d^4\eta}{ds^4} = -\,\kappa\dfrac{d^2\eta}{dt^2}\,,$

(*) Cf., e.g., LORD RAYLEIGH (*loc. cit.*), page 242, *et seq.*

for the *longitudinal vibrations:* $L\frac{d^4\zeta}{ds^4} = \kappa\frac{d^2\zeta}{dt^2}$.

It is known that one can deduce from the expression for ψ_z (in which λ' is a pure number) that L has the dimension of a force and that L / κ will then have the dimension of the square of a velocity, as it must.

2. The equations for the *oblique vibrations:*

$$C\frac{d\rho}{ds} = \kappa\mathfrak{r}^2\frac{d^2\varphi}{dt}, \qquad \rho = \frac{d\varphi}{ds}.$$

We consider the normal vibrations to the equilibrium plane of the rod, which are obviously identical to the transverse planar vibrations, and we will thus have the known equation for the *torsional vibrations:*

$$C\frac{d^2\varphi}{ds^2} = \kappa\mathfrak{r}^2\frac{d^2\varphi}{dt}.$$

4. Vibrations of a straight line that is subject to longitudinal tension. – In the present case, we can once more keep $p_0 = 0$ and $\Phi_y^{(0)} = 0$, while $\Phi_z^{(0)}$ will be non-zero, as well as constant, from the conclusions of no. **3**, § III.

The equations that determine the planar vibrations in this case are once more (58), although with:

$$\psi_y = A\frac{d\pi}{ds} + \Phi_z^{(0)}\beta, \qquad \psi_z = L\lambda'.$$

One then gets the known equation (*) for the transverse vibrations:

$$A\frac{d^4\eta}{ds^4} - \Phi_z^{(0)}\frac{d^2\eta}{ds^2} + \kappa\frac{d^2\eta}{dt^2} = 0,$$

while the equation of longitudinal vibrations does not change from the one that was considered in the preceding section.

Likewise, the oblique vibrations do not change.

5. Vibrations of a circular rod. – If the rod is found to take the form of an arc of a circumference then, as we saw in § III, no. **3**, it must necessarily be found in the natural state, and therefore not subject to the any terminal stresses. With that, the equations that govern the motion of our rod will be the following ones:

(*) Cf., LORD RAYLEIGH (*loc. cit.*), page 296.

1. *Planar vibrations:*

$$\frac{d\psi_y}{ds} - c\,\psi_z = \kappa\,\frac{d^2\eta}{dt^2}\,, \qquad \frac{d\psi_z}{ds} + c\,\psi_y = \kappa\,\frac{d^2\xi}{dt^2}\,, \tag{59}$$

with

$$\psi_y = A\,\frac{d\pi}{ds}\,, \qquad \psi_z = L\,\lambda' - A\,c\,\pi\,,$$

and to those equations, one must add the following ones:

$$\pi = -\frac{d\beta}{ds} - c\,\lambda'\,,$$
$$\beta = \frac{d\eta}{ds} - c\,\zeta\,,$$
$$\lambda' = \frac{d\zeta}{ds} + c\,\eta\,.$$

We shall treat two distinct cases:

a) *Vibrations of an inextensible circular rod.*

In order to examine this question, it is necessary to take a step back.

The vibrations of a circular ring were studied for the first time by HOPPE (*) and more recently by RAYLIEGH (**), who treated the question by starting from the principle of the conservation of energy, like us. However, he supposed that the director line was inextensible, which led him to assume that the energy of the system was independent of λ. We cannot therefore recover HOPPE's vibrations from our formulas as such because they always reflected the hypothesis that the director line is extensible. However, one can see that if one recalls the question right from the start then the hypothesis of inextensibility (or, if one prefers, the hypothesis that the elastic energy is independent of λ) will translate into the fact that the tangential stress Φ_z will remain indeterminate. Indeed, if the director line is inextensible then one can replace equation (9) with the following one:

$$\mathbf{n}_3 \times \frac{d'\,\delta P}{ds} = 0\,,$$

which should warn us that the components of the displacement δP are not mutually independent. If Γ denotes a LAGRANGE multiplier, and if the calculations of § II are repeated then we confirm that the equation $\Phi_z + e_1 = 0$ must be replaced with the following one:

$$\Phi_z = \Gamma\,,$$

(*) R. HOPPE, "Vibrationen eines Rings in seiner Ebene," J. reine. angew. Math. **73** (1871), 158-170.
(**) LORD RAYLEIGH, *loc. cit.*, pp. 383, *et seq.*

which will justify the statement that was made above precisely.

In order to recover HOPPE's vibrations, it is then necessary to eliminate the ψ_z from (59), as LOVE suggested (*), and to suppose that $\lambda' = 0$, in addition. When one performs the indicated operations, one will get the equation:

$$A\left(\frac{1}{c}\frac{d^6\zeta}{ds^6}+2c\frac{d^4\zeta}{ds^4}+c^3\frac{d^2\zeta}{ds^2}\right) = \kappa\frac{d^2}{dt^2}\left(c\,\zeta-\frac{1}{c}\frac{d^2\zeta}{ds^2}\right),$$

or if $d\vartheta$ denotes the angle at the center that corresponds to the arc ds, so:

$$c\,ds = d\vartheta, \tag{60}$$

the equation:

$$A\,c^4\left(\frac{d^6\zeta}{d\vartheta^6}+2\frac{d^4\zeta}{d\vartheta^4}+\frac{d^2\zeta}{d\vartheta^2}\right) = \kappa\frac{d^2}{dt^2}\left(\zeta-\frac{d^2\zeta}{d\vartheta^2}\right). \tag{61}$$

If the rod is comprised of a complete ring then one can suppose that λ depends sinusoidally on time and the angle ϑ, i.e., it will have the form:

$$\zeta = a\,e^{i\,(kt+n\vartheta)} \qquad (a = \text{constant}),$$

and equation (61) will then provide the relation that exists between the frequency constant k and n in the form (**):

$$k^2 = \frac{Ac^4(n^2-1)^2n^2}{\kappa(n^2+1)}.$$

b) *Vibrations of an extensible circular rod.*

If we abandon the hypothesis of inextensibility in the director line then the equations of planar vibrations of the circular rod will be the following two:

$$A\,c^4\left(\frac{d^4\zeta}{d\vartheta^4}+2\frac{d^2\zeta}{d\vartheta^2}+\eta\right)+c^2L\left(\frac{d\zeta}{d\vartheta}+\eta\right) = -\kappa\frac{d^2\eta}{dt^2},$$

$$c^2L\left(\frac{d^2\zeta}{d\vartheta^2}+\frac{d\eta}{d\vartheta}\right) = \kappa\frac{d^2\zeta}{dt^2},$$

in the writing of which we have taken into account the convention (60).

(*) LOVE (*loc. cit.* on pp. 1), pp. 431.
(**) Cf., LOVE, *loc. cit.*, page 431, or RAYLEIGH, *loc. cit.*, page 71.

We have already pointed out that L has the dimension of a force: However, we now add that A [cf., no. **2** of this section] has the dimension of a force per square length. If we imagine (as would be natural for a rod, moreover) that the radius of gyration of the transverse section is small compared to the radius $1 : c$ of the circumference to which the director of rod belongs then we can neglect terms in the first of the equations that were written above that contain A as a coefficient and thus see that the vibrations in question are straight from the equations:

$$c^2 L \left(\frac{d\zeta}{d\vartheta} + \eta \right) = -\kappa \frac{d^2\zeta}{dt^2}, \qquad c^2 L \left(\frac{d^2\zeta}{d\vartheta^2} + \frac{d\eta}{d\vartheta} \right) = \kappa \frac{d^2\zeta}{dt^2} .$$

If the ring is complete then one can set:

$$\eta = a\, e^{i\,(kt + n\vartheta)}, \qquad \zeta = b\, e^{i\,(kt + n\vartheta)}, \qquad (a \text{ and } b \text{ constants})$$

and then find that one must set:

$$a\, n\, i + b = 0 ,$$

and that the following relation exists between k and n (*):

$$k^2 = \frac{c^2 L}{\kappa} (1 + n^2) .$$

2. *Oblique vibrations:*

$$\frac{d\psi_x}{ds} = \kappa \frac{d^2\xi}{dt^2}, \qquad \psi_x = -B \left(\frac{d\chi}{ds} - c \frac{d\varphi}{ds} \right) + c\, C\, (\rho + c\,\alpha) ,$$

$$C \left(\frac{d\rho}{ds} + c \frac{d\alpha}{ds} \right) + B\, c\, (\chi - c\,\varphi) = \kappa\, \mathfrak{r}^2 \frac{d^2\varphi}{dt^2} ,$$

with

$$\chi = \frac{d\alpha}{ds}, \qquad \rho = \frac{d\varphi}{ds}, \qquad \alpha = \frac{d\xi}{ds}.$$

Those equations can be summarized in these two:

$$\left. \begin{aligned} c^2 (B + C) \frac{d^2\varphi}{d\vartheta^2} - B\, c^2 \frac{d^4\xi}{d\vartheta^4} + C\, c^4 \frac{d^2\xi}{d\vartheta^2} &= \kappa \frac{d^2\xi}{dt^2}, \\ c^2 C \frac{d^2\varphi}{d\vartheta^2} - c^2 B\, \varphi + c^3 (B + C) \frac{d^2\xi}{d\vartheta^2} &= \kappa\, \mathfrak{r}^2 \frac{d^2\varphi}{dt^2}, \end{aligned} \right\} \tag{62}$$

(*) Cf., LOVE, *loc. cit.*, pp. 433.

which will permit one to study the vibrations normal to the plane of equilibrium for the rod separately from the torsional ones.

a) *Vibrations normal to the plane of the circular rod.*

Suppose, as LOVE did (*), that the angular acceleration is negligible and the rod is comprised of a complete ring. One can suppose that ξ and φ take the form:

$$\xi = a\, e^{i\,(kt + n\vartheta)}, \qquad \varphi = b\, e^{i\,(kt + n\vartheta)}, \qquad (a \text{ and } b \text{ constants})$$

and one will then find the following relation between the frequency constant k and n :

$$k^2 = \frac{B\,C\,c^4\,n^2(n^2-1)^2}{\kappa\,(C\,n^2+B)}\,.$$

b) *Torsional vibrations:*

If one supposes that ξ is small compared to $\varphi : c$ then (62) will become simply (**):

$$c^2\left(C\,\frac{d^2\varphi}{d\vartheta^2} - B\,\varphi\right) = \kappa\,\mathfrak{r}^2\,\frac{d^2\varphi}{dt^2}\,,$$

so if the rod constitutes a complete ring and φ has the form:

$$\varphi = b\, e^{i\,(kt + n\vartheta)} \qquad (b \text{ constant})$$

then one will get the relation:

$$k^2 = \frac{c^2\,(C\,n^2+B)}{\kappa\,\mathfrak{r}^2}\,.$$

Padua, July 1917.

(*) Cf., LOVE, *loc. cit.*, pp. 432.
(**) Cf., LOVE, *loc. cit.*, pp. 432.